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THE SHRINKABILITY OF BING–WHITEHEAD DECOMPOSITIONS

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1. INTRODUCTION

THIS PAPER is a study of a special class of toroidal decompositions of 3-manifolds called Bing–Whitehead decompositions. It is well known that a Bing–Whitehead decomposition of a 3-manifold is shrinkable if all successive stages are Bing nested; but it is not shrinkable if all successive stages are Whitehead nested. (See Figs 1 and 2.) Consider a Bing–Whitehead decomposition of a 3-manifold which is defined by b_1 successive Bing nested stages, followed by 1 Whitehead nested stage, followed by b_2 successive Bing nested stages, followed by 1 Whitehead nested stage, ... The principal result of this paper is that this decomposition is shrinkable if and only if

$$\sum_{i=1}^{\infty} \frac{b_i}{2^i} = \infty.$$

This result clarifies an issue raised in the proof of M. H. Freedman's Disk Theorem for Four Dimensional Manifolds ([4], p. 652).

We recall the relevant definitions. A *decomposition* G of a 3-manifold M is a collection of pairwise disjoint non-empty subsets of M whose union is M. G is *upper semicontinuous* if each element of G is compact and if the quotient map $M \to M/G$ is a closed map. A *defining sequence* for G is a sequence $\{X_i: i \geq 0\}$ of compact 3-manifolds in $\text{int}(M)$ such that $X_i \subset \text{int}(X_{i-1})$ for $i \geq 1$ and such that the non-singleton elements of G coincide with the non-singleton components of $\cap \{X_i: i \geq 0\}$. A defining sequence $\{X_i\}$ is *toroidal* if each component of each X_i is a solid torus. If G has a toroidal defining sequence then G is called a *toroidal* decomposition. G is *cell-like* if for each element C of G, the inclusion of C into each of its neighborhoods in M is null-homotopic. Toroidal decompositions need not be cell-like; however, the special class of toroidal decompositions of concern here—the Bing–Whitehead decompositions—are, in fact, cell-like.

Let T be a solid torus. A *ramification* of T is a finite family $\{T_1, \ldots, T_k\}$ of solid tori in T such that the $(k+1)$-tuple $(T, T_1, \ldots, T_k)$ is homeomorphic to the $(k+1)$-tuple $(S^1 \times B, S^1 \times B_1, \ldots, S^1 \times B_k)$ where S^1 is a circle and $\{B_1, \ldots, B_k\}$ is a pairwise disjoint family of disks in the interior of the disk B.

Suppose G is a toroidal decomposition of a 3-manifold M, and $\{X_i\}$ is a toroidal defining sequence for G. Let $i \geq 1$. X_i is *Bing nested* in X_{i-1} if for each component T of X_{i-1}, there is a ramification $\{T_1, \ldots, T_k\}$ of T such that $T \cap X_i \subset \cup \{\text{int}(T_j): 1 \leq j \leq k\}$ and for each j, $1 \leq j \leq k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair $(U, V \cup W)$

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where U, V and W are as in Fig. 1. X_i is *Whitehead nested* in X_{i-1} if for each component T of X_{i-1}, there is a ramification $\{T_1, \ldots, T_k\}$ of T such that $T \cap X_i \subset \cup\{\text{int}(T_j): 1 \le j \le k\}$, and for each j, $1 \le j \le k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair (U, V) where U and V are as in Fig. 2.

Again suppose G is a toroidal decomposition of a 3-manifold M, and $\{X_i\}$ is a toroidal defining sequence for G. If for each $i \ge 1$, either X_i is Bing nested in X_{i-1}, or X_i is Whitehead nested in X_{i-1}, then $\{X_i\}$ is called a *Bing–Whitehead* defining sequence, and G is caled a *Bing–Whitehead* decomposition.

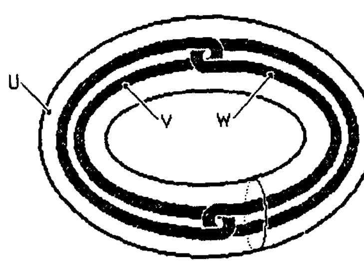
Now suppose G is a Bing–Whitehead decomposition of a 3-manifold M, and $\{X_i\}$ is a Bing–Whitehead defining sequence for G. If X_i is Bing nested in X_{i-1} for each $i \ge 1$, then $\{X_i\}$ is called a *pure Bing* defining sequence, and G is called a *pure Bing* decomposition. If X_i is Whitehead nested in X_{i-1} for each $i \ge 1$, then $\{X_i\}$ is called a *pure Whitehead* defining sequence, and G is called a *pure Whitehead* decomposition. If for each $i \ge 1$, X_i is Bing nested in X_{i-1} when i is odd, and X_i is Whitehead nested in X_{i-1} when i is even, then $\{X_i\}$ is called a *strictly alternating* Bing–Whitehead defining sequence, and G is caled a *strictly alternating* Bing–Whitehead decomposition.

Again let G be a Bing–Whitehead decomposition of a 3-manifold M, and let $\{X_i\}$ be a Bing–Whitehead defining sequence for G. A sequence $\{b_i\}$ of positive integer is called an *index* of $\{X_i\}$ and of G if it has the following two properties:

(1) X_j is Bing nested in X_{j-1} for $1 \le j < b_1$, $b_1 < j < b_1 + b_2$, $b_1 + b_2 < j < b_1 + b_2 + b_3, \ldots$.

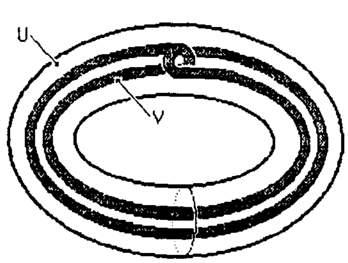
(2) X_j is Whitehead nested in X_{j-1} for $j = b_1, j = b_1 + b_2, j = b_1 + b_2 + b_3, \ldots$.

Thus, $\{X_i\}$ begins with $(b_i - 1)$ Bing nestings; and for $i > 1$, there are $(b_i - 1)$ Bing nestings between the $(i-1)$th and ith Whitehead nestings. Observe that $\{X_i\}$ is pure Bing if and



Bing nesting

Fig. 1.



Whitehead nesting

Fig. 2.

only if $b_1 = \infty$; $\{X_i\}$ is pure Whitehead if and only if each $b_i = 1$; and $\{X_i\}$ is strictly alternating Bing–Whitehead if and only if each $b_i = 2$.

A decomposition G of a 3-manifold M is *shrinkable* if the quotient map $q: M \to M/G$ can be approximated by homeomorphisms. Thus, G is shrinkable if and only if for every open cover $\mathscr{U}$ of M/G, there is a homeomorphism h: M/G such that $\{\{q(x), h(x)\}: x \in M\}$ refines $\mathscr{U}$.

R. H. Bing proved in [1] that each pure Bing decomposition of a 3-manifold is shrinkable. If follows indirectly from J. H. C. Whitehead's results in [5] that no pure Whitehead decomposition is shrinkable. These results suggest the question: Is a strictly alternating Bing–Whitehead decomposition shrinkable? More generally one can ask: Which Bing–Whitehead decompositions are shrinkable? These questions are settled by the principal result of this paper, which we now state.

THEOREM. *A Bing–Whitehead decomposition of a 3-manifold with index $\{b_i\}$ is shrinkable if and only if*

$$\sum_{i=1}^{\infty} \frac{b_i}{2^i} = \infty.$$

COROLLARY. *No strictly alternating Bing–Whitehead decomposition of a 3-manifold is shrinkable.*

The Corollary follows from the Theorem because a strictly alternating Bing–Whitehead decomposition has index $(2, 2, 2, \ldots\}$ and

$$\sum_{i=1}^{\infty} \frac{2}{2^i} = 2 < \infty.$$

We briefly describe the contents of the rest of this paper. In section 2, we state a shrinking criterion for toroidal decompositions which plays a central role in both directions of the proof of the Theorem. The proof of the Theorem occupies sections 3 and 4. In section 3, we show that the divergence of $\sum b_i/2^i$ implies the shrinkability of a Bing–Whitehead decomposition with index $\{b_i\}$. Section 4 contains the proof of the converse: convergence implies non-shrinkability. The proof in section 4 relies on a Technical Lemma concerning the way in which a solid torus in $\mathbb{R}^3$ can intersect a family of parallel planes. The proof of this Technical Lemma is the subject of section 5. We end this introductory section with an account of the source of the questions which led to these results.

M. H. Freedman's proof of the 4-dimensional Poincaré conjecture [3] relies on an analysis of the structure of *Casson handles*. A Casson handle is the 4-dimensional thickening of an infinite tower of immersed disks. The frontier of a Casson handle is the quotient space of a pure Whitehead decomposition of a solid torus. Thus, the frontier of a Casson handle is never a manifold, a fact which injects technical complications into the analysis of Casson handles.

Freedman's Disk Theorem for 4-Manifolds [4] extends the results of [3] to certain non-simply connected situations. In the proof of the Disk Theorem, Casson handles are replaced by objects which have come to be called *Freedman handles*. The virtue of Freedman handles is that they can be constructed to have manifold frontiers. This makes one aspect of the proof of [4] simpler than [3]. (Any such simplification is welcome, because other aspects of the proof in [4] are significantly more complicated than in [3].)

A Freedman handle is the 4-dimensional thickening of a *cope*. A cope is an infinite tower of embedded surfaces and immersed disks. The frontier of a Freedman handle is the quotient space of a Bing–Whitehead decomposition of a solid torus. Each Bing nested stage in the Bing–Whitehead decomposition corresponds to a layer of embedded surfaces in the cope, and each Whitehead nested stage corresponds to a layer of immersed disks. To insure that the Freedman handle has the desired property that its frontier is a manifold, one must arrange that the associated Bing–Whitehead decomposition is shrinkable. According to the Theorem stated above, this can be achieved by having sufficiently long sequences of Bing nestings between successive Whitehead nestings. This corresponds to having long sequences of embedded surface layers in the cope between successive immersed disk layers. Fortunately, the process of constructing the cope is sufficiently flexible to accommodate the insertion of arbitrarily long sequences of embedded surface layers between successive immersed disk layers. Thus, Freedman handles with manifold frontiers are available when needed.

At the Santa Barbara 4-Manifolds Conference in August, 1983, Freedman described the results of [4], and the question arose as to which Freedman handles have manifold frontiers (or equivalently, which Bing–Whitehead decompositions are shrinkable). At that time, it was conjectured that a strictly alternating Bing–Whitehead decomposition is shrinkable. During the course of the conference, Ancel outlined an argument which showed this conjecture to be false. However, that argument was not completed until Starbird contributed the idea for the proof of the Technical Lemma which appears in section 5 of this paper. Also, at the time of the conference, it was recognized that a Bing–Whitehead decomposition is shrinkable if the number of Bing nested stages between successive Whitehead nested stages grows sufficiently rapidly; e.g., on page 652 of [4], Freedman remarks that $b_i = 4^i$ suffices. However, the precise formula of the above stated Theorem was not known at that time; it was discovered by Ancel only in 1985.

The issue of the shrinkability of a strictly alternating Bing–Whitehead decomposition independently occurred to R. J. Daverman (also in connection with some 4-dimensional topology problems). Daverman and D. G. Wright have independently given proofs of the above Corollary by a somewhat different method.

2. A SHRINKING CRITERION FOR TOROIDAL DECOMPOSITIONS

In this section, we state a shrinking criterion for toroidal decompositions. This criterion plays a central role in both directions of the proof of the Theorem, which is given in sections 3 and 4.

TOROIDAL SHRINKING THEOREM. *A toroidal decomposition G of a 3-manifold M with a toroidal defining sequence $\{X_i\}$ is shrinkable if and only if it satisfies the following toroidal shrinking criterion. For every $\varepsilon > 0$ and for every $i \geq 0$, there is a homeomorphism h of M which restricts to the identity on $M - X_i$ such that* $\operatorname{diam} h(C) < \varepsilon$ *for each $C \in G$.*

Remarks about the proof. It is easy to establish that for toroidal decompositions, the toroidal shrinking criterion is equivalent to the Bing shrinking criterion. The Bing shrinking criterion is a well known necessary and sufficient condition for the shrinkability of a decomposition. It was originally introduced in [1]. A succinct formulation and proof can be found on page 120 of [2]. □

3. DIVERGENCE IMPLIES SHRINKABILITY

In this section, we prove that $\Sigma b_i/2^i = \infty$ is a sufficient condition for the shrinkability of a Bing–Whitehead decomposition with index $\{b_i\}$.

Suppose that G is a Bing–Whitehead decomposition of a 3-manifold M, $\{X_i\}$ is a Bing–Whitehead defining sequence for G, $\{b_i\}$ is the index of $\{X_i\}$, and $\sum b_i/2^i = \infty$. We shall now prove that G is shrinkable by verifying the toroidal shrinking criterion.

Throughout the proof, we shall reposition subsets of some X_i by homeomorphisms of M which restrict to the identity on $M - X_i$. To minimize notation, we shall not name the repositioning homeomorphisms, and we shall use the same name for a set and for its repositioned image.

Lemmas 1 and 2 below record the progress toward satisfying the toroidal shrinking criterion that can be made by simple repositioning moves. The following terminology helps us to describe this progress. Let S^1 denote the unit circle and B^2 the unit disk in $\mathbb{R}^2$. Define the covering max exp: $\mathbb{R} \to S^1$ by $\exp(t) = e^{2\pi i t}$. Suppose T is a solid torus in the 3-manifold M. Let $\varepsilon > 0$, and let n be a positive integer. If there is a homeomorphism $h\colon S^1 \times B^2 \to T$ such that diam $h(\{p\} \times B^2) < \varepsilon$ for every $p \in S^1$, then we say that T is *ε-thin*. If there is a homeomorphism $h\colon S^1 \times B^2 \to T$ such that

$$\operatorname{diam} h\left(\exp\left[\frac{i-1}{n}, \frac{i}{n}\right] \times B^2\right) < \varepsilon$$

for $1 \leq i \leq n$, then we say that *the ε-length of T is $\leq n$*; and, for $1 \leq i \leq n$, we call each set

$$h\left(\exp\left[\frac{i-1}{n}, \frac{i}{n}\right] \times B^2\right)$$

an *ε-compartment* of T. Observe that if T is ε-thin, then its ε-length is $\leq n$ for some $n \geq 1$. Conversely, if the ε-length of T is $\leq n$, then T is ε-thin.

LEMMA 1. *Suppose X_i is Bing nested in X_{i-1}. Suppose that $\varepsilon > 0$ and n is a positive integer such that each component of X_{i-1} has ε-length $\leq 2n$. Then there is a homeomorphism of M which restricts to the identity on $M - X_{i-1}$ and which repositions the components of X_i so that each has ε-length* $\leq \max\{2n-2, 1\}$.

Proof. Let T be a component of X_{i-1}. There is a ramification $\{T_1, \ldots, T_k\}$ of T so that $T \cap X_i \subset \cup\{\operatorname{int}(T_j)\colon 1 \leq j \leq k\}$, and so that for each j, $1 \leq j \leq k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair $(U, V \cup W)$ where U, V and W are as in Fig. 1. We position $T_1 \cup T_2 \cup \ldots \cup T_k$, by a homeomorphism of M which restricts to the identity on $M - T$, so that the ε-length of each T_j is $\leq 2n$. Now, for each, j, $1 \leq j \leq k$, there is a homeomorphism of M which restricts to the identity on $M - T_j$, and which positions $T_j \cap X_i$ so that the ε-length of each component of $T_j \cap X_i$ is $\leq \max\{2n-2, 1\}$. The correct positioning of the components of $T_j \cap X_i$ is shown in Fig. 3. When $n \geq 2$, each component of $T_j \cap X_i$ has $2(n-2)$ *straight* ε-compartments and two *U-shaped* ε-compartments at top and bottom, giving a total of $2(n-2) + 2 = 2n - 2$ ε-compartments. □

We remark that Lemma 1 alone is adequate to verify the toroidal shrinking criterion for a *pure Bing* decomposition. We sketch this argument as a warm-up for the more complicated general case which appears below. This proof is close in spirit to Bing's archetypal shrinking argument in [1]. Suppose $\{X_i\}$ is a pure Bing defining sequence for the decomposition G. Let $\varepsilon > 0$ and let $i \geq 1$. We squeeze the components of X_{i+1} toward their cores, by a homeomorphism of M which restricts to the identity on $M - X_i$, so that each

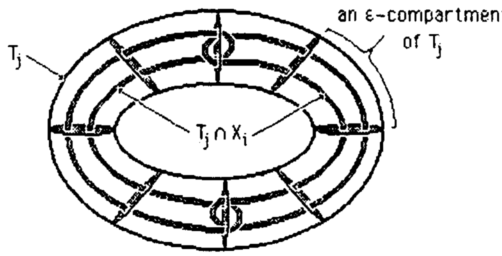


Fig. 3.

component of X_{i+1} becomes ε-thin. Then there is a positive integer n such that the ε-length of each component of X_{i+1} is $\leq 2n$. We now apply Lemma 1 repeatedly: first to X_{i+2}, then to X_{i+3}, then to $X_{i+4}, \ldots$, and finally to X_{i+n+1}. This will produce a homeomorphism of M which restricts to the identity on $M - X_i$, and which repositions the components of X_{i+2}, $X_{i+3}, \ldots, X_{i+n+1}$ so that the each component of X_{i+2} has ε-length $\leq 2n-2$, each component of X_{i+3} has ε-length $\leq 2n-4, \ldots$, each component of X_{i+n} has ε-length ≤ 2, and each component of X_{i+n+1} has ε-length ≤ 1. Consequently, the homeomorphism shrinks each element of G to a set of diameter $< \varepsilon$, thereby verifying the toroidal shrinking criterion.

LEMMA 2. *Suppose X_i is Whitehead nested in X_{i-1}. Suppose that $\varepsilon > 0$ and n is a positive integer such that each component of X_{i-1} has ε-length $\leq n$. Then there is a homeomorphism of M which restricts to the identity on $M - X_{i-1}$ and which repositions the components of X_i so that each has ε-length $\leq \max\{2n-2, 1\}$.*

Proof. Let T be a component of X_{i-1}. There is a ramification $\{T_1, \ldots, T_k\}$ of T so that $T \cap X_i \subset \cup\{\text{int}(T_j): 1 \leq j \leq k\}$, and so that for each j, $1 \leq j \leq k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair (U, V) where U and V are as in Fig. 2. We position $T_1 \cup T_2 \cup \ldots \cup T_k$, by a homeomorphism of M which restricts to the identity on $M - T$, so that the ε-length of each T_j is $\leq n$. Now, for each j, $1 \leq j \leq k$, there is a homeomorphism of M which restricts to the identity on $M - T_j$, and which positions $T_j \cap X_i$ so that the ε-length of each component of $T_j \cap X_i$ is $\leq \max\{2n-2, 1\}$. The correct positioning of the components of $T_j \cap X_i$ is shown in Fig. 4. When $n \geq 2$, each component of $T_j \cap X_i$ has $2(n-2)$ *straight* ε-compartments and two *U-shaped* ε-compartments at the top, giving a total of $2(n-2)+2 = 2n-2$ ε-compartments. □

We are now ready to verify the toroidal shrinking criterion under the hypothesis $\sum b_i/2^i = \infty$. Let $\varepsilon > 0$ and let $i \geq 1$. Our first step is to squeeze the components of

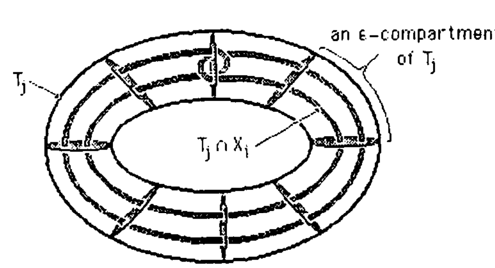


Fig. 4.

X_{i+1} toward their cores, by a homeomorphism of M which restricts to the identity on $M - X_i$, so that each component of X_{i+1} becomes ε-thin. Then there is a positive integer n such that the ε-length of each component of X_{i+1} is $\leq 2n$.

For $j \geq 0$, set $Y_j = X_{i+1+j}$. Then $\{Y_j\}$ is a Bing–Whitehead defining sequence for G, and each component of Y_0 has ε-length $\leq 2n$. Let $\{c_j\}$ be the index of $\{Y_j\}$. Then $\{b_i\}$ and $\{c_j\}$ have identical tails. So $\sum c_j/2^j = \infty$. Hence, there is a positive integer r such that

$$\sum_{j=1}^{r} c_j/2^j > n.$$

We now apply Lemma 1 $c_1 - 1$ times in succession to $Y_1, Y_2, \ldots, Y_{c_1-1}$; and then we apply Lemma 2 once to Y_{c_1}. This produces a homeomorphism of M which restricts to the identity on $M - X_i$, and which repositions the components of $Y_1, Y_2, \ldots Y_{c_1}$, so that each component of Y_{c_1} has ε-length $\leq$

$$\max\{2(2n-(c_1-1))-2, 1\} = \max\{4n-2c_1, 1\}.$$

Next, we apply Lemma 1 $c_2 - 1$ times in succession to $Y_{c_1+1}, Y_{c_1+2}, \ldots, Y_{c_1+c_2-1}$, followed by a single application of Lemma 2 to $Y_{c_1+c_2}$. This results in a homeomorphism of M which restricts to the identity on $M - X_i$, and which repositions the components of $Y_{c_1+1}, Y_{c_1+2}, \ldots, Y_{c_1+c_2}$ so that each component of $Y_{c_1+c_2}$ has ε-length $\leq$

$$\max\{2(4n-2c_1-(c_2-1))-2, 1\} = \max\{8n-4c_1-2c_2, 1\}.$$

We repeat this procedure r times. In the final run, Lemma 1 is applied $c_r - 1$ times in succession, followed by a single application of Lemma 2. This yields a homeomorphism of M which restricts to the identity on $M - X_i$, and which repositions the components of Y_j, $c_1 + \ldots + c_{r-1} + 1 \leq j \leq c_1 + \ldots + c_{r-1} + c_r$, so that if $k = c_1 + c_2 + \ldots + c_r$, then each component of Y_k has ε-length $\leq$

$$\max\{2^{r+1}n - 2^r c_1 - 2^{r-1}c_2 - \ldots - 2c_r, 1\} = \max\left\{2^{r+1}\left(n - \sum_{j=1}^{r} c_j/2^j\right), 1\right\} = 1$$

The net result is a homeomorphism of M which restricts to the identity on $M - X_i$, and which shrinks each element of G to a set of diameter $< \varepsilon$. We have now verified the toroidal shrinking criterion. It follows that G is shrinkable. □

4. CONVERGENCE IMPLIES NON-SHRINKABILITY

In this section, we prove that if $\sum b_i/2^i < \infty$, then no Bing–Whitehead decomposition with index $\{b_1\}$ is shrinkable.

We shall need the following terminogy. Let X be a compact 3-manifold each component of which is a solid torus. A *meridian* of X is a simple closed curve in ∂X which bounds a disk in X but not in ∂X. A *meridianal disk* of X is a disk D in X such that $D \cap \partial X = \partial D$ is a meridian of X. A finite sequence $D_1, D_2, \ldots, D_n$ of pairwise disjoint meridianal disks of X is *in cyclic order* on X if $D_1, D_2, \ldots, D_n$ lie in a single component T of X and if $1 \leq i < j < k < l < n$ implies that D_i and D_k lie in different components of $T - (D_j \cup D_l)$, and D_j and D_l lie in different components of $T - (D_i \cup D_k)$.

For the remainder of this section, we suppose that G is a Bing–Whitehead decomposition of a 3-manifold M, $\{X_i\}$ is a Bing–Whitehead defining sequence for G, $\{b_i\}$ is the index of $\{X_i\}$ and $\sum b_i/2^i < \infty$.

We shall prove that G is not shrinkable by demonstrating the failure of the toroidal shrinking criterion. Let us assume that the toroidal shrinking criterion is valid. We shall obtain a contradiction.

Let n be a positive integer so large that $n/2 > \sum b_i/2^i$. This choice of n insures that for each integer $r \geq 0$, the following inequality holds:

$$2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r > b_{r+1} \qquad \ldots (*)$$

Now, let $D_1, E_1, D_2, E_2, \ldots, D_n, E_n$ be a sequence of $2n$ pairwise disjoint meridianal disks of X_0 in cyclic order on X_0. Set $D = D_1 \cup D_2 \cup \ldots \cup D_n$ and $E = E_1 \cup E_2 \cup \ldots \cup E_n$. Let ε denote the distance from D to E. Then the toroidal shrinking criterion provides a homeomorphism h of M which restricts to the identity on $M - X_0$ such that diam $h(C) < \varepsilon$ for each $C \in G$. It follows that for some integer $i \geq 1$, each component of $h(X_i)$ is of diameter $< \varepsilon$. Thus, no component of $h(X_i)$ intersects both D and E. Consequently, no component of X_i intersects both $h^{-1}(D)$ and $h^{-1}(E)$. We observe that $h^{-1}(D_1)$, $h^{-1}(E_1)$, $h^{-1}(D_2)$, $h^{-1}(E_2), \ldots, h^{-1}(D_n)$, $h^{-1}(E_n)$ is a sequence of pairwise disjoint meridianal disks of X_0 in cyclic order on X_0. Hence, Lemma 3, below, implies that some component of X_i must intersect both $h^{-1}(D)$ and $h^{-1}(E)$. We have reached a contradiction. □

LEMMA 3. *Suppose G is a Bing–Whitehead decomposition of a 3-manifold M, $\{X_i\}$ is a Bing–Whitehead defining sequence for G, $\{b_i\}$ is the index of $\{X_i\}$, and $\sum b_i/2^i < \infty$. Let n be a positive integer so large that $n/2 > \sum b_i/2^i$. Suppose $D_1, E_1, D_2, E_2, \ldots, D_n, E_n$ is a sequence of pairwise disjoint meridianal disks of X_0 in cyclic order on X_0. Set $D = D_1 \cup D_2 \cup \ldots \cup D_n$ and $E = E_1 \cup E_2 \cup \ldots \cup E_n$. Then for each $i \geq 0$, some component of X_i intersects both D and E.*

Proof. We begin by modifying $D \cup E$ so that it intersects each ∂X_i nicely. Specifically, we adjust $D \cup E$ by ambient isotopy, keeping it away from any component of any X_i which it already misses, to achieve two objectives. First, we make $D \cup E$ transverse to each ∂X_i. Second, for each $i \geq 1$, we remove the components of $(D \cup E) \cap (\partial X_i)$ that are not meridians of X_i. To attain the second objective, we consider a component T of X_i. We remove the components of $(D \cup E) \cap (\partial T)$ which bound disks in ∂T by "cutting off" $D \cup E$ on ∂T. Then we consider a component J of $(D \cup E) \cap (\partial T)$ which is innermost in $D \cup E$. Since J bounds a disk in X_0 whose interior is disjoint from ∂T, then either J is a meridian of T or J is "parallel" to a spine of T. In the latter case, T can be isotoped to a "thin" solid torus near J but disjoint from $D \cup E$. The inverse of this isotopy moves $D \cup E$ off T. This leaves the case in which J is a meridian of T. Since all the other components of $(D \cup E) \cap (\partial T)$ are simple closed curves in ∂T which are disjoint from J and which don't bound disks in ∂T, then they must all be meridians as well, and our objective is achieved. We repeat this process for each component of each X_i. Now we may assume that for each $i \geq 1$, $D \cup E$ is transverse to ∂X_i and each component of $(D \cup E) \cap (\partial X_i)$ is a meridian of X_i.

To prove that each X_i intersects both D and E, we shall actually prove that a stronger relationship than just intersection with D and E is preserved as we descend through the X_i's. *An interlacing of order m* on X_i shall mean a sequence $D'_1, E'_1, D'_2, E'_2, \ldots, D'_m, E'_m$ of pairwise disjoint meridianal disks of X_i in cyclic order on X_i such that $D'_1 \cup D'_2 \cup \ldots \cup D'_m \subset D$ and $E'_1 \cup E'_2 \cup \ldots \cup E'_m \subset E$. The following two lemmas allow us to find interlacings of positive order as we descend through the X_i's.

LEMMA 4. *Suppose X_i is Bing nested in X_{i-1}. If $m > 1$ and there is an interlacing of order m on X_{i-1}, then there is an interlacing of order $m-1$ on X_i.*

LEMMA 5. *Suppose X_i is Whitehead nested in X_{i-1}. If $m \geq 1$ and there is an interlacing of order m on X_{i-1}, then there is an interlacing of order $2m-1$ on X_i.*

We postpone the proofs of Lemmas 4 and 5 until we finish the proof of Lemma 3. Continuing with the proof of Lemma 3: we are given an interlacing of order n on X_0. Inequality (*) implies that $n > b_1$. So we can apply Lemma 4 $b_1 - 1$ times in succession to find an interlacing of order $n-i$ on X_i, for $1 \leq i < b_1$. In particular, there is an interlacing of order $n-(b_1-1)$ on X_{b_1-1}. We then apply Lemma 5 to find an interlacing of order $2(n-(b_1-1))-1$ on X_{b_1}. Since $2(n-(b_1-1))-1 > 2n-2b_1$, then there is an interlacing of order $2n-2b_1$ on X_{b_1}.

We repeat the procedure of the previous paragraph infinitely often. In the general step, r is a non-negative integer, set $j = b_1 + b_2 + \ldots + b_r$, and assume there is an interlacing of order $2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r$ on X_j. Set $k = b_1 + b_2 + \ldots + b_r + b_{r+1}$. The inequality (*) insures that we can apply Lemma 4 $b_{r+1} - 1$ times in succession to find an interlacing of order $2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r - i$ on X_{j+i}, for $1 \leq i < b_{r+1}$. In particular, since $j + (b_{r+1} - 1) = k - 1$, then there is an interlacing of order $2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r - (b_{r+1} - 1)$ on X_{k-1}. We then apply Lemma 5 to find an interlacing of order $2(2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r - (b_{r+1} - 1)) - 1$ on X_k. Since

$$2(2^r n - 2^r b_1 - 2^{r-1} b_2 - \ldots - 2b_r - (b_{r+1} - 1)) - 1 > 2^{r+1} n - 2^{r+1} b_1 - 2^r b_2 - \ldots - 2^2 b_r - 2b_{r+1},$$

then there is an interlacing of order $2^{r+1}n - 2^{r+1}b_1 - 2^r b_2 - \ldots - 2^2 b_r - 2b_{r+1}$ on X_k.

It follows inductively that the procedure of the previous paragraph can be repeated ad infinitum. Hence, each X_i has an interlacing of positive order. We conclude that each X_i intersects both D and E. □

We now give proofs of Lemmas 4 and 5. In these arguments, a crucial role is played by the Technical Lemma which is the subject of Section 5 of this paper.

Proof of Lemma 4. Suppose $D'_1, E'_1, D'_2, E'_2, \ldots, D'_m, E'_m$ is an interlacing of order m on X_{i-1}. Set $D' = D'_1 \cup D'_2 \cup \ldots, D'_m$ and $E' = E'_1 \cup E'_2 \cup \ldots \cup E'_m$. Let T be the component of X_{i-1} which contains $D' \cup E'$. There is a ramification $\{T_1, T_2, \ldots, T_k\}$ of T such that $T \cap X_i \subset T_1 \cup T_2 \cup \ldots, \cup T_k$ and for each j, $1 \leq j \leq k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair $(U, V \cup W)$ where U, V and W are as in Fig. 1. Let R and S denote the two solid torus components of $T_1 \cap X_i$. Since the pair (T, T_1) is homeomorphic to the pair $(S^1 \times B, S^1 \times B_1)$ where S^1 is a circle and B_1 is a disk in the interior of the disk B, it follows that the pair $(T, R \cup S)$ is homeomorphic to the pair $(U, V \cup W)$ where U, V and W are as in Fig. 1.

Each D'_j and each E'_j must intersect either R or S. Indeed, the link consisting of a spine of R, a spine of S, and $\partial D'_j$ (or $\partial E'_j$) as a copy of the *Borromean rings*; and no component of this well known link bounds a disk in the complement of the other two components. Thus, one of R and S, say R, intersects at least m of the $2m$ disks $D'_1, E'_1, D'_2, E'_2, \ldots, D'_m, E'_m$.

There is a covering map $\pi: \mathbb{R}^3 \to \text{int}(T)$ such that $\pi^{-1}(D' \cup E') = \cup \{P_j : j \in \mathbb{Z}\}$ where $\{P_j : j \in \mathbb{Z}\}$ is a discrete family of parallel planes in $\mathbb{R}^3$ such that P_k separates P_j from P_l whenever $j < k < l$, $P_j \subset \pi^{-1}(D')$ when j is odd, and $P_j \subset \pi^{-1}(E')$ when j is even. Let $\tilde{R}$ be a component of $\pi^{-1}(R)$. Then $\tilde{R}$ is a solid torus, $\partial \tilde{R}$ is transverse to $\pi^{-1}(D' \cup E')$, each component of $(\partial \tilde{R}) \cap \pi^{-1}(D' \cup E')$ is a meridian of $\tilde{R}$, $\pi | \tilde{R} : \tilde{R} \to R$ is a homeomorphism, and $\tilde{R}$ must intersect at least m distinct P_j's. Thus there is a $k \in \mathbb{Z}$ such that $\tilde{R}$ intersects P_{k+1},

$P_{k+2}, \ldots, P_{k+m}$. We now invoke the Technical Lemma of section 5 to obtain disk components $A_1, A_2, \ldots, A_{2m}$ of $\tilde{R} \cap (P_{k+1} \cup P_{k+2} \cup \ldots \cup P_{k+m})$ which are meridianal disks of $\tilde{R}$ in cyclic order on $\tilde{R}$ such that $A_j \cup A_{2m+1-j} \subset P_{k+j}$ for $1 \leq j \leq m$. (See Fig. 5.) Hence $\pi(A_1), \pi(A_2), \ldots, \pi(A_{2m})$ is a sequence of meridianal disks of R in cyclic order on R such that $\pi(A_j) \cup \pi(A_{2m+1-j}) \subset D'$ when $k+j$ is odd, and $\pi(A_j) \cup \pi(A_{2m+1-j}) \subset E'$ when $k+j$ is even. To complete the proof of Lemma 4, we observe that in the case that k is even, the sequence $\pi(A_1), \pi(A_2), \ldots, \pi(A_{m-1}), \pi(A_{m+1}), \pi(A_{m+2}), \ldots, \pi(A_{2m-1})$ is an interlacing of order $m-1$ on X_i; and in the case that k is odd, the sequence $\pi(A_2), \pi(A_3), \ldots, \pi(A_m), \pi(A_{m+2}), \pi(A_{m+3}), \ldots, \pi(A_{2m})$ is an interlacing of order $m-1$ on X_i. □

Proof of Lemma 5. This proof is very similar to the previous one. Again suppose $D'_1, E'_1, D'_2, E'_2, \ldots, D'_m, E'_m$ is an interlacing of order m on X_{i-1}. Set $D' = D'_1 \cup D'_2 \cup \ldots, D'_m$ and $E' = E'_1 \cup E'_2 \cup \ldots \cup E'_m$. Let T be the component of X_{i-1} which contains $D' \cup E'$. There is a ramification $\{T_1, T_2, \ldots, T_k\}$ of T such that $T \cap X_i \subset T_1 \cup T_2 \cup \ldots \cup T_k$ and for each j, $1 \leq j \leq k$, the pair $(T_j, T_j \cap X_i)$ is homeomorphic to the pair (U, V) where U and V are as in Fig. 2. Let R denote the solid torus $T_1 \cap X_i$. Since the pair (T, T_1) is homeomorphic to the pair $(S^1 \times B, S^1 \times B_1)$ where S^1 is a circle and B_1 is a disk in the interior of the disk B, it follows that the pair (T, R) is homeomorphic to the pair (U, V) where U and V are as in Fig. 2.

Each D'_j and each E'_j must intersect R. Indeed, the link consisting of a spine of R and $\partial D'_j$ (or $\partial E'_j$) is a copy of the *Whitehead link*; and no component of this well known link bounds a disk in the complement of the other component. Thus, R intersects each of the $2m$ disks $D'_1, E'_1, D'_2, E'_2, \ldots, D'_m, E'_m$.

Again, there is a covering map $\pi: \mathbb{R}^3 \to \mathrm{int}(T)$ such that $\pi^{-1}(D' \cup E') = \cup \{P_j : j \in \mathbb{Z}\}$ where $\{P_j : j \in \mathbb{Z}\}$ is a discrete family of parallel planes in $\mathbb{R}^3$ such that P_k separates P_j from P_l whenever $j < k < l$, $P_j \subset \pi^{-1}(D')$ when j is odd, and $P_j \subset \pi^{-1}(E')$ when j is even. Let $\tilde{R}$ be a component of $\pi^{-1}(R)$. Proceeding as in Lemma 4, we invoke the Technical Lemma of section 5 to obtain $4m$ disk components $A_1, A_2, \ldots, A_{4m}$ of $\tilde{R} \cap (P_{k+1} \cup P_{k+2} \cup \ldots \cup P_{k+2m})$ such that either the sequence $\pi(A_1), \pi(A_2), \ldots, \pi(A_{2m-1}), \pi(A_{2m+1}), \pi(A_{2m+2}), \ldots, \pi(A_{4m-1})$ or the sequence $\pi(A_2), \pi(A_3), \ldots, \pi(A_{2m}), \pi(A_{2m+2}), \pi(A_{2m+3}), \ldots, \pi(A_{4m})$ is an interlacing of order $2m-1$ on X_i. □

5. THE TECHNICAL LEMMA

We recall the following terminology. Let T be a solid torus. A *meridian* of T is a simple closed curve in ∂T which bounds a disk in T but not in ∂T. A *meridianal disk* of T is a disk D in T such that $D \cap \partial T = \partial D$ is a meridian of T. A finite sequence $D_1, D_2, \ldots, D_n$ of pairwise

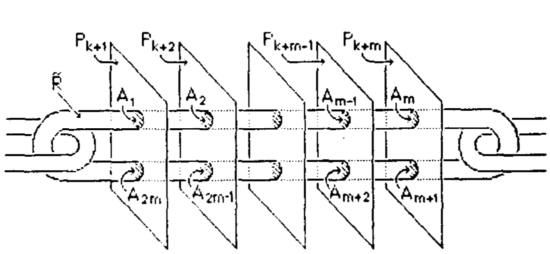


Fig. 5.

disjoint meridianal disks of T is *in cyclic order on* T if $1 \le i < j < k < l \le n$ implies that D_i and D_k lie in different components of $T-(D_j \cup D_l)$ and D_j and D_l lie different components of $T-(D_i \cup D_k)$. We extend this terminology slightly by declaring a pairwise disjoint sequence $J_1, J_2, \ldots, J_n$ of meridians of T to be *in cyclic order* on ∂T if $1 \le i < j < k < l \le n$ implies that J_i and J_k lie in different components of $\partial T-(J_j \cup J_l)$ and J_j and J_l lie in different components of $\partial T-(J_i \cup J_k)$.

THE TECHNICAL LEMMA. *Suppose $P_1, P_2, \ldots, P_m$ is a sequence of parallel planes in $\mathbb{R}^3$ such that if $1 \le i < j < k \le m$, then P_j separates P_i and P_k. Set $P = P_1 \cup P_2 \cup \ldots \cup P_m$. Suppose T is a solid torus in $\mathbb{R}^3$ such that ∂T is transverse to P, each component of $(\partial T) \cap P$ is a meridian of T, and $T \cap P_i \ne \varnothing$ for $1 \le i \le m$. Then there is a sequence $A_1, A_2, \ldots, A_{2m}$ of pairwise disjoint meridianal disks of T in cyclic order on T such that $A_i \cup A_{2m+1-i} \subset P_i$ for $1 \le i \le m$.*

The technical Lemma may strike the reader as an obvious fact, in which case he will find the proof surprisingly complicated. To convince the reader that the proof requires some subtlety, we present the following example. This example shows that the Technical Lemma becomes *false* if, in its hypothesis, parallel planes are replaced by concentric 2-spheres. In Fig. 6 below, S_1, S_2 and S_3 are topologically concentric 2-spheres (i.e., there is a homeomorphism of $\mathbb{R}^3$ which carries S_1, S_2 and S_3 to geometrically concentric round 2-spheres) and S_2 separates S_1 from S_3. T is a solid torus which intersects each S_i, and ∂T intersects each S_i transversely in meridians of T. However, there is *no* sequence $A_1, A_2, \ldots, A_6$ of pairwise disjoint meridianal disks of T in cyclic order on T such that $A_i \cup A_{2m+1-i} \subset S_i$ for $1 \le i \le 3$.

Proof of the Technical Lemma. If J is a simple closed curve in P, let $D(J)$ denote the disk in P bounded by J. Define the *height* of a component J of $(\partial T) \cap P$ to be the maximum of all positive integers k such that there is a sequence $J_1, J_2, \ldots, J_k = J$ of k components of $(\partial T) \cap P$ such that $D(J_j) \subset \text{int}(D(J_{j+1}))$ for $1 \le j < k$.

Observe that if J is a component of $(\partial T) \cap P$ of height 1, then $D(J) \subset T$. Indeed, if J is height 1, then either $D(J) \subset T$ or $\text{int}(D(J)) \cap T = \varnothing$. However, the latter alternative is ruled out by the fact that J links each spine of T.

We shall induct on the number $c(T)$ of components of $(\partial T) \cap P$ of height > 1.

We first consider the case $c(T) = 0$. Here all the components of $T \cap P$ are disks. Let J be a simple closed curve in ∂T which meets each component of $(\partial T) \cap P$ transversely in a single

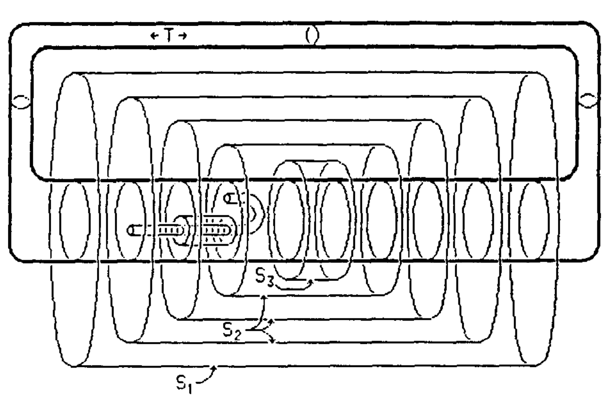


Fig. 6.

point. Choose points $p, q \in J$ so that P_1 separates p from P_2 and P_m separates q from P_{m-1}. J is the union of two arcs K and L such that $\partial K = \partial L = \{p, q\}$. Orient both K and L from p to q. For $1 \leq i \leq m$, let A_i be the component of $T \cap P_i$ which K meets first, and let A_{2m+1-i} be the component of $T \cap P_i$ which L meets first. Then $A_1, A_2, \ldots, A_{2m}$ satisfies the conclusion of the Technical Lemma.

Now let $c > 0$ and inductively assume that the conclusion of the Technical Lemma holds whenever $c(T) < c$. Suppose $c(T) = c$.

Case 1. One of the components of $T \cap P$ is an annulus.

Let E be an annulus component of $T \cap P$. There is a meridianal disk D of T which is disjoint from E. (See Fig. 7.) D is obtained by simplifying the intersection of an arbitrary meridianal disk of T with E. Either one produces D directly, or one obtains a disk D' in $\text{int}(T)$ such that $D' \cap E = \partial D'$ is essential in E. Then D' is transformed to D by sliding $\partial D'$ along E, across ∂E, and into ∂T. Let $N(E)$ be a thin regular neighborhood of E in T which is disjoint from $D \cup ((T \cap P) - E)$. Let U be the component of $cl(T - N(E))$ which contains D. Then U is a solid torus, and D is a meridianal disk of U. Clearly, ∂U is transverse to P and each component of $(\partial U) \cap P$ is a meridian of U. U has a spine J which intersects D transversely in a single point. J must also be a spine of T. Hence, J intersects each P_i. So, U intersects each P_i.

Apparently, $c(U) \leq c(T) - 2$ because $U \cap E = \emptyset$. So by inductive hypothesis, there is a sequence $A_1, A_2, \ldots, A_{2m}$ of pairwise disjoint meridianal disks of U in cyclic order on U such that $A_i \cup A_{2m+1-i} \subset P_i$ for $1 \leq i \leq m$. Clearly $A_1, A_2, \ldots, A_{2m}$ are also meridianal disks of T in cyclic order on T.

Case 2. No component of $T \cap P$ is an annulus.

Since $c(T) > 0$, there must be a component J_0 of $(\partial T) \cap P$ of height 2. J_0 bounds a meridianal disk E_0 of T. We adjust E_0 to make it transverse to P and disjoint from the disk components of $T \cap P$. Then $E_0 \cup D(J_0)$ is a 2-sphere which bounds a 3-ball C_0 in $\mathbb{R}^3$. We now assume that J_0 and E_0 have been chosen to minimize the number of components of $(\partial T) \cap P$ in $\text{int}(C_0)$.

We assert that J_0 is the only height 2 component of $(\partial T) \cap P$ that intersects C_0. For suppose there is a height 2 component J_1 of $(\partial T) \cap P$ which intersects C_0 and is distinct from J_0. Then $J_1 \subset \text{int}(C_0)$, and $D(J_1)$ is the union of a punctured disk whose interior lies outside T and some disk components of $T \cap P$. It follows that $D(J_1)$ is disjoint from both $D(J_0)$ and E_0. Hence, $D(J_1) \subset \text{int}(C_0)$. J_1 bounds a meridianal disk E_1 of T which can be adjusted to be transverse to P and disjoint from the disk components of $T \cap P$ and from E_0. This implies $E_1 \subset \text{int}(C_0)$. Now $E_1 \cup D(J_1)$ is a 2-sphere which bounds a 3-ball C_1 which

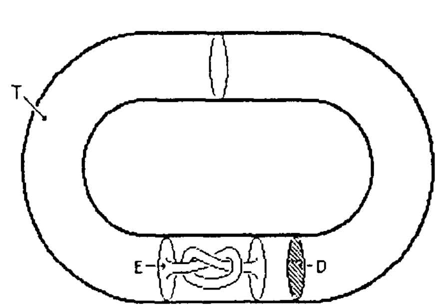


Fig. 7.

must lie in $\operatorname{int}(C_0)$. (See Fig. 8). Since J_1 lies in $\operatorname{int}(C_0)$ but is disjoint from $\operatorname{int}(C_1)$, then $\operatorname{int}(C_1)$ contains fewer components of $(\partial T) \cap P$ than does $\operatorname{int}(C_0)$. This contradicts the choice of J_0 and E_0, and proves our assertion.

Since $\operatorname{int}(E_0) \subset \operatorname{int}(T)$, then there is a homeomorphism, which is supported on a 3-ball that is slightly larger than C_0, and which spreads out a neighborhood of $\operatorname{int}(E_0)$ that initially lies in $\operatorname{int}(T)$ to engulf C_0. More precisely, there is a 3-ball C_0' and a homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that

(1) $C_0 \subset \operatorname{int}(C_0')$,
(2) $C_0' \cap ((\partial T) \cap P) = C_0 \cap ((\partial T) \cap P)$,
(3) h is supported on C_0', and
(4) $h(\operatorname{int}(T)) \supset C_0$ and $h((\partial T) \cap C_0') \cap P = \varnothing$. (See Fig. 9.)

It follows that $h(\partial T) \cap P = ((\partial T) \cap P) - C_0'$, and on this set h is the identity. Therefore, $h(\partial T)$ is transverse to P, and each component of $h(\partial T) \cap P$ is a meridian of $h(T)$. Furthermore, $h(T) \cap P_i \neq \varnothing$ for $1 \leq i \leq m$. To see the last assertion, let $1 \leq i \leq m$ and let J be a component of $(\partial T) \cap P_i$. If J intersects C_0, then $J \subset h(T)$; whereas if J is disjoint from C_0, then $h(J) = J \subset P_i$. Thus, in either case, $J \subset h(T) \cap P_i$.

Since $h(\partial T) \cap P = ((\partial T) \cap P) - C_0'$ and since $J_0 \subset h(\operatorname{int}(T))$, then $c(h(T)) < c(T)$. So, by inductive hypothesis, there is a sequence $A_1, A_2, \ldots, A_{2m}$ of pairwise disjoint meridianal disks of $h(T)$ in cyclic order on $h(T)$ such that $A_i \cup A_{2m+1-i} \subset P_i$ for $1 \leq i \leq m$. According to the next paragraph, each A_i is also a disk component of $T \cap P$. We must show that the A_i's are meridianal disks of T in cyclic order on T. To do this, we observe that ∂A_1, $\partial A_2, \ldots, \partial A_{2m}$ is a sequence of disjoint meridians of $h(T)$ which is in cyclic order on $h(\partial T)$. Since h^{-1} doesn't move the points of $h(\partial T) \cap P = ((\partial T) \cap P) - C_0'$, we conclude that ∂A_1,

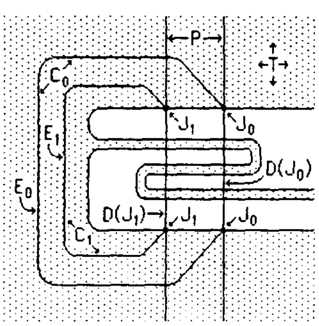


Fig. 8.

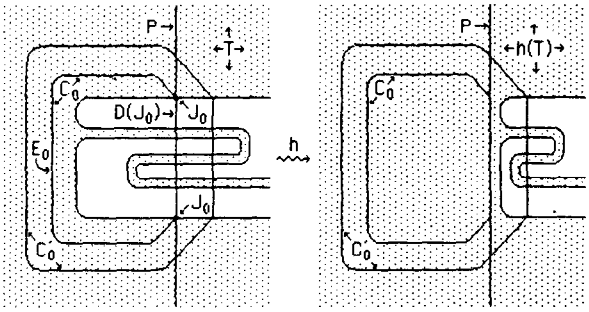


Fig. 9.

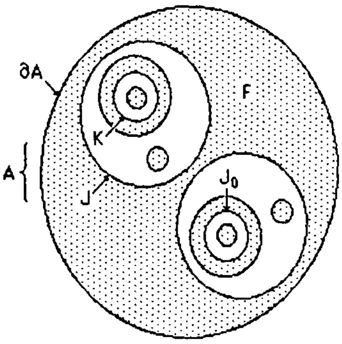


Fig. 10.

$\partial A_2, \ldots, \partial A_{2m}$ is a sequence of disjoint meridians of T which is in cyclic order on ∂T. It follows that $A_1, A_2, \ldots, A_{2m}$ is a sequence of pairwise disjoint meridianal disks of T in cyclic order on T.

It remains to verify the following assertion: if A is a disk component of $h(T) \cap P$, then A is a disk component of $T \cap P$. We proceed by contradiction. Suppose A is a disk component of $h(T) \cap P$ but not of $T \cap P$. Then ∂A is a component of $h(\partial T) \cap P$. Since $h(\partial T) \cap P \subset (\partial T) \cap P$, we conclude that ∂A is a component of $T \cap P$. Let F denote the component of $T \cap P$ which contains ∂A. F must be a proper subset of A, because A is not contained in $T \cap P$. Thus, F is not a disk. Also, the hypothesis of Case 2 prevents F from being an annulus. (This is the only point at which this hypothesis is used.) Hence, $(\partial F) \cap (\text{int}(A))$ has at least two components. Suppose J is a component of $(\partial F) \cap (\text{int}(A))$. Then $D(J)$ is not contained in T; but $\text{int}(D(J))$ must intersect T because J, being a meridian of T, links a spine of T. Hence, J is of height ≥ 2. Thus, either $D(J_0) \subset D(J)$, or $D(J_0) \cap D(J) = \emptyset$. Since $(\partial F) \cap (\text{int}(A))$ has more than one component, we can assume $D(J) \cap D(J_0) = \emptyset$. As J is of height ≥ 2, then $D(J)$ contains a component K of $(\partial T) \cap P$ of height exactly equal to 2. Our choice of J insures that $K \neq J_0$. (See Fig. 10.) In an earlier paragraph, we argued that J_0 is the only height 2 component of $(\partial T) \cap P$ that intersects C_0. So $K \cap C_0 = \emptyset$. Therefore, $K \cap C'_0 = \emptyset$. Hence, $K \subset ((\partial T) \cap P) - C'_0 = h(\partial T) \cap P$. But $K \subset \text{int}(A) \subset h(\text{int}(T))$. We have reached the desired contradiction. □

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