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STATIONARY LOGIC

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We investigate the logic $L(aa)$ which allows the second-order quantifier "$aa s$" meaning "for almost all countable sets s." We prove Completeness, Compactness, and Omitting Types Theorems and develop a Gentzen-style proof theory for this logic, as well as for the infinitary version $L_A(aa)$. Relations with various sublogics like $L(Q)$ are discussed.

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1. Introduction

Shelah, on p. 356 of [24], remarks that while the traditional second-order quantifiers are too strong, "there could be generalized second-order quantifiers which are weak enough for their model theory to be nice, for example by satisfying Löwenheim-Skolem, Compactness or Completeness Theorems." He goes on to make several suggestions as to specific generalized quantifiers to study, one of which is the quantifier "$aa s$" which means "for almost all countable subsets s."

In this paper we study the logics $L_{\omega\omega}(aa)$ and $L_{\omega_1\omega}(aa)$ obtained by augmenting $L_{\omega\omega}$ and $L_{\omega_1\omega}$ by this generalized second-order quantifier aa. We prove Completeness, Compactness and Omitting Types Theorems for $L_{\omega\omega}(aa)$ and the corresponding analogues for $L_{\omega_1\omega}(aa)$.

The logics studied here are closely related to the logic $L(Q)$, where "Qx" means "there exist uncountably many x." In fact Qx is definable in $L(aa)$ so it is only natural that our proof of Completeness should be an extension of the corresponding Completeness Theorem for $L(Q)$ proved in Keisler [15]. Certain rough edges in $L(Q)$ are smoothed out in $L(aa)$. For example, the crucial axiom which Keisler discovered for $L(Q)$ is: $Qy\,\exists x\,\varphi(x, y) \rightarrow \exists x\, Qy\,\varphi(x, y) \vee Qx\,\exists y\,\varphi(x, y)$.

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This cumbersome axiom has effectively blocked a reasonable proof theory for L(Q). In L(aa), this axiom is a consequence of the simpler but stronger axiom:

$$\forall x\ \text{aa}s\ \psi(x, s) \rightarrow \text{aa}s\ \forall x \in s\ \psi(x, s).$$

This removes the obstacle and we are able to develop a Gentzen style proof theory for L(aa) in Section 6.

Let us begin by recalling what is meant by the phrase "almost all countable subsets s of M." Let $P_{\omega_1}(M)$ denote the set of all countable subsets of M. A set $X \subseteq P_{\omega_1}(M)$ is *unbounded* if every $s_0 \in P_{\omega_1}(M)$ is a subset of some $s \in X$. X is *closed* if whenever $s = \bigcup_{n<\omega} s_n$, each $s_n \in X$ and $s_0 \subseteq s_1 \subseteq \cdots \subseteq s_n \subseteq \ldots$, then $s \in X$. A set X is cub if X is closed and unbounded. It is easy to check that the intersection of two cub sets is again cub.

The cub filter on M (actually it's a filter on $P_{\omega_1}(M)$) consists of those $X \subseteq P_{\omega_1}(M)$ which contain some cub subset X_0. This filter was introduced by Kueker [16], [17], and Jech [12]. A property $\varphi(s)$ of elements of $P_{\omega_1}(M)$ *holds for almost all countable* s if $\{s \in P_{\omega_1}(M) \mid \varphi(s)\}$ is in the cub filter.

Kueker [17] Section 1.1c, has a nice description of the filter in terms of an infinite two person game. Namely, a set $X \subseteq P_{\omega_1}(M)$ is in the cub filter on M iff

$$\forall x_1 \in M\ \exists y_1 \in M\ \forall x_2 \in M\ \exists y_2 \in M \cdots \{x_1, x_2, \ldots, y_1, y_2, \ldots\} \in X,$$

where the usual game-theoretic interpretation is given to the infinite string of quantifiers.

1.1. Lemma. *The* cub *filter* $\mathfrak{F}$ *on* $P_{\omega_1}(M)$ *has the following properties:*

(i) $\mathfrak{F}$ *is a countably complete filter,*

(ii) $\mathfrak{F}$ *is closed under diagonalization—i.e., if* $X_a \in \mathfrak{F}$ *for all* $a \in M$ *then the diagonal intersection* $\{s \mid s \in X_a$ *for all* $a \in s\}$ *is in* $\mathfrak{F}$,

(iii) *If* M *is countable then* $\mathfrak{F} = \{X \mid M \in X\}$ *so that in this case* $\mathfrak{F}$ *is a principal filter,*

(iv) *If* M *is uncountable then* $\mathfrak{F}$ *is non-principal and is not an ultrafilter. (This last fact uses* AC *strongly.)*

Proof. See Kueker [17] or Jech [12]. □

Kueker's diagonalization principle (ii) above is a generalization of the Fodor regressive function theorem (cf. Kunen [18], for example) for the case of regressive functions on ω_1. The connection becomes clear when one notices that a set of countable ordinals is a cub set in the usual sense iff the set of all initial segments $\{\beta \mid \beta < \alpha\}$ for $\alpha \in X$ forms a cub in the sense used here.

A set $Y \subseteq P_{\omega_1}(M)$ is *stationary* if $Y \cap X \neq 0$ for all X in the cub filter. Thus Y is stationary iff its complement $\neg Y (= P_{\omega_1}(M) - Y)$ is not in the cub filter. Note that the intersection $X \cap Y$ of a cub set X and a stationary set Y is again stationary.

Given a countable first-order language L, expand it by adding countably many unary predicate variables $s_1, s_2, \ldots$ and a new quantifier aa. Formulas of L(aa) are formed as usual with the new formation rule: *if φ is a formula of L(aa) so is $\mathsf{aa}s\,\varphi$ for each predicate variable s.* A *sentence* of L(aa) is a formula with no free individual variables or predicate variables. Note that while we allow $\forall x$ and $\exists x$ we do not allow $\forall s$ or $\exists s$ as formation rules. That is, $\mathsf{aa}s$ is the only second-order quantifier allowed. We define stat $s\,\varphi(s)$ to be $\neg\mathsf{aa}s\,\neg\varphi(s)$; this is read as "there is a stationary set of countable sets s such that $\varphi(s)$." The infinitary language $L_{\omega_1\omega}(\mathsf{aa})$ is defined analogously.

The meaning of formulas is defined in the obvious way. If $\mathfrak{M}=\langle M, \ldots\rangle$ is a structure for L and $\varphi(x_1 \cdots x_n, s_1 \cdots s_m)$ is a formula of L(aa) then we define $\mathfrak{M}\models\varphi(x_1 \cdots x_n,\ s_1 \cdots s_m)$ for all $x_1 \cdots x_n \in M$, $s_1 \cdots s_m \in P_{\omega_1}(M)$ by induction with the crucial clause being:

$$\mathfrak{M}\models\mathsf{aa}s\,\varphi(s) \quad \text{iff} \quad \mathfrak{M}\models\varphi(s) \quad \text{for almost all countable } s\subseteq M.$$

1.2. Examples. (i) If $\mathfrak{M}$ is countable then $\mathfrak{M}\models\mathsf{aa}s\,\varphi(s)$ iff $\mathfrak{M}\models\varphi(M)$ so that in this case the quantifier is trivial.

(ii) We can define "there exist uncountably many x such that $\varphi(x)$" by the formula

$$\text{stat } s\,\exists x\,(\varphi(x)\wedge\neg s(x)),$$

so that L(aa) contains L(Q), where Q means uncountable. We henceforth use $\mathrm{Q}x\,\varphi(x)$ for the above expression. We will use "L(Q)" to denote all translations into L(aa) of L(Q) formulas, plus subformulas of such translates.

(iii) Let $\mathfrak{M}=\langle M, \approx\rangle$ where $\approx$ is an equivalence relation. It is fairly well known, and not hard to check, that one cannot express the assertion that "$\approx$ has $\leqslant\aleph_0$ equivalence classes" in L(Q). It can be expressed in L(aa) by

$$\mathsf{aa}s\,\forall x\,\exists y\,(x\approx y\wedge s(y)).$$

(iv) Let $\mathfrak{M}=\langle M, <, \ldots\rangle$ be a linear ordering. In L(Q) one can express that $<$ is ω_1-like by the two axioms

$$\mathrm{Q}x\,(x=x),$$
$$\forall y\,\neg\mathrm{Q}x\,(x<y).$$

The following cannot be expressed as a sentence of L(Q) (or even Σ^1_1 in L(Q)):

$$\mathsf{aa}s\,\exists x\,\forall y\,(s(y)\leftrightarrow\ <x). \tag{*}$$

This is equivalent to the existence of a *sup-preserving* function mapping $\langle\omega_1, <\rangle$ cofinally into $\langle M, <\rangle$. (The proof that (*) is not Σ^1_1 in L(Q) can be obtained from Keisler's proof of the Completeness Theorem for L(Q). You build an uncountable model $\mathfrak{M}=\langle M, <, \ldots\rangle=\bigcup_{\alpha<\omega_1}\mathfrak{M}_\alpha$ in stages, using the proof of Keisler's Main Lemma to make sure that at each stage α, there is not a first new element in $\mathfrak{M}_{\alpha+1}$.)

(v) Makowsky and Shelah [19] introduce a logic L^{pos} properly containing $L(Q)$ by the usual formation rules for $\neg$, $\wedge$, $\exists x$, $\forall x$ plus: *if* $\varphi(s_+)$ *is a formula of* L^{pos} *in which* s *occurs positively, then* $\exists s\,\varphi(s_+)$ *is a formula of* L^{pos}. The dual rule is: if $\varphi(s_-)$ is a formula in which s occurs negatively then $\forall s\,\varphi(s_-)$ is a formula. The semantics is given by $\mathfrak{M}\models\exists s\,\varphi(s)$ if there is a countable set s such that $\mathfrak{M}\models\varphi(s)$. This logic is clearly a sublogic of $L(\mathsf{aa})$ since, if $\varphi(s_+)$ is s-positive

$$\exists s\,\varphi(s)\leftrightarrow \mathsf{aa}s\,\varphi(s); \tag{**}$$

for if $\varphi(s_0)$ holds then $\varphi(s)$ holds for all $s\supseteq s_0$. Thus, while we do not in general have the quantifiers $\exists s$, $\forall s$, we do have those instances of them which correspond to the above formation rules. We thus consider L^{pos} as a sublogic of $L(\mathsf{aa})$ by (**). (The sentence (*) in (iv) is also not in L^{pos}.)

(vi) There is a sentence of $L(\mathsf{aa})$ not equivalent to any sentence of $L_{\infty\infty}$. This sentence will be given in Section 5.

(vii) Let φ be a sentence of $L_{\omega_1\omega}$ (not $L_{\omega_1\omega}(\mathsf{aa})$) and write $\varphi^{(s)}$ for the result of relativizing all quantifiers in φ to s. We can express one of Kueker's results in [17] in terms of aa by the fact that $\varphi\leftrightarrow\mathsf{aa}s\,\varphi^{(s)}$ is valid; i.e., true in all models. That is, $\mathfrak{M}\models\varphi$ iff φ holds in almost all countable substructures of $\mathfrak{M}$.

(viii) This example comes from Eklof [8] and is included to show the relevance of the quantifier aa for the theory of abelian groups. Let T be a torsion (abelian) group and let J be torsion free. If J is countable, then Baer's Theorem gives a necessary and sufficient condition that $\mathrm{Ext}(J,T)=0$, a condition that can be expressed in $L_{\omega_1\omega}$. If we assume $V=L$ then Eklof [8] proves that for $|J|\leq\aleph_1$, $\mathrm{Ext}(J,T)=0$ iff

$$\mathsf{aa}s\ \mathrm{Ext}(s,T)=0,$$

$$\mathsf{aa}s\ \mathsf{aa}s'\ (s'\supseteq s\rightarrow \mathrm{Ext}(s'/s,T)=0),$$

so that, for $|J|\leq\aleph_1$, $\mathrm{Ext}(J,T)=0$ is expressible in $L_{\omega_1\omega}(\mathsf{aa})$. (We can assume that s ranges over *subgroups* of J since almost every subset is a subgroup.) On the other hand, Eklof shows, again assuming $V=L$, that this is not expressible in $L_{\infty\omega_1}$. This shows that, assuming $V=L$, $L_{\omega_1\omega}(\mathsf{aa})\not\subseteq L_{\infty\omega_1}$, and is what suggested looking for a sentence as described in (vi) above.

We now turn to the axioms and rules for $L(\mathsf{aa})$.

1.3. Definition. Let T be a set of sentences of $L(\mathsf{aa})$. The set of *theorems of* T in $L(\mathsf{aa})$ is the smallest set containing all the usual axioms of first-order logic, containing all instances of Axioms A0–A5 below, closed under the usual rules of modus ponens and generalization and closed under the rule of aa-generalization.

We write $T \vdash \varphi$ if φ is a theorem of T.

A0. $\mathbf{aa}s_i\, \varphi(s_i) \leftrightarrow \mathbf{aa}s_j\, \varphi(s_j)$;
A1. $\neg \mathbf{aa}s$ (false);
A2. $\mathbf{aa}s\,(x \in s)$, $\mathbf{aa}s_j(s_i \subseteq s_j) \quad (i \neq j)$;
A3. $\mathbf{aa}s\,\varphi \wedge \mathbf{aa}s\,\psi \to \mathbf{aa}s\,(\varphi \wedge \psi)$;
A4 $\mathbf{aa}s\,(\varphi \to \psi) \to (\mathbf{aa}s\,\varphi \to \mathbf{aa}s\,\psi)$;
A5. $\forall x\, \mathbf{aa}s\,\varphi(x, s) \to \mathbf{aa}s\, \forall x \in s\, \varphi(x, s)$.

Rule of $\mathbf{aa}$-generalization: If $T \vdash \eta \to \varphi(s)$ where s is not free in η then $T \vdash \eta \to \mathbf{aa}s\,\varphi(s)$.

Some comments on the axioms are in order.

A0. Here it is assumed that $\varphi(s_i)$ is a formula in which s_j does not occur and $\varphi(s_j)$ is the result of replacing all free occurrences of s_i by s_j. These axioms insure that $\mathbf{aa}s$ really does bind the relation variable s.

A1. Here (false) is some identically false proposition like $\exists x\, (x \neq x)$. The effect of this axiom, with the others, is to allow us to prove $\varphi \leftrightarrow \mathbf{aa}s\,\varphi$ whenever s is not free in φ; see Lemma 2.4 below.

A2. These axioms have *free* variables x, s_i respectively. The expression $s_i \subseteq s_j$ is shorthand for $\forall x\, (s_i(x) \to s_j(x))$. The axioms are sound, i.e., valid in all models for all x and all countable subsets s_i by the definition of the cub filter.

A3, A4. These axioms are sound since the cub filter is indeed a filter.

A5. This axiom is sound by the diagonal intersection property of the cub filter, see Lemma 1.1 (ii). $\forall x \in s\, \varphi$ means $\forall x\, (s(x) \to \varphi)$.

Rule of $\mathbf{aa}$-generalization. The rule preserves soundness; that is, if $\eta \to \varphi(s)$ is true in all models of T for all countable s, then so is $\eta \to \mathbf{aa}s\,\varphi(s)$. Recall here that T is a set of sentences so s is not free in T. It is not free in η by the condition imposed by the rule.

We say that T is *consistent* if it's not the case that $T \vdash$ false. One of the main results of this paper is the following Completeness Theorem. We use the term "standard model" below to emphasize that the model has the intended interpretations of the second-order quantifiers $\mathbf{aa}$, stat.

1.4. Completeness Theorem for L($\mathbf{aa}$). *A set T of sentences of* L($\mathbf{aa}$) *is consistent iff T has a standard model.*

The proof of this will be given in Section 3.

The finitary nature of the rules immediately yield a Compactness Theorem. Recall that our basic language L($\mathbf{aa}$) is always countable.

1.5. Compactness Theorem for L($\mathbf{aa}$). *If T is a set of sentences of* L($\mathbf{aa}$) *and every finite subset $T_0 \subseteq T$ has a standard model then T has a standard model.*

Proof. Immediate from Theorem 1.4. $\square$

The proof of the Completeness Theorem yields a model of power $\leq \aleph_1$ so we get the following downward Löwenheim-Skolem Theorem. Clearly one can't do better since either $\mathsf{Q}x\,(x = x)$ or its negation might be in T.

1.6. Downward Löwenheim-Skolem Theorem for $\mathrm{L}(\mathit{aa})$. *Any set T of sentences of $\mathrm{L}(\mathit{aa})$ which has a standard model has one of power at most $\aleph_1$.*

We now turn to $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$ and its countable fragments. A *fragment* $\mathrm{L}_\mathrm{A}(\mathit{aa})$ of $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$ is defined as usual to be a set of formulas of $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$ closed under subformulas, substitutions and the finitary operations $\forall$, $\wedge$, $\neg$, aa, and the following: if $\bigwedge_{i<\omega} \mathit{aa}s\,\varphi_i$ is in $\mathrm{L}_\mathrm{A}(\mathit{aa})$ so is $\bigwedge_{i<\omega} \varphi_i$.

1.7. Definition. Let $\mathrm{L}_\mathrm{A}(\mathit{aa})$ be a fragment of $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$ and let T be a set of sentences of $\mathrm{L}_\mathrm{A}(\mathit{aa})$. The *consequences of* T form the smallest set of $\mathrm{L}_\mathrm{A}(\mathit{aa})$ formulas containing T, all the usual axioms for L_A (as given in Section III. 4.1 of Barwise [2], e.g.), all $\mathrm{L}_\mathrm{A}(\mathit{aa})$ instances of Axioms A0–A5 above and Axiom A6 below, and closed under the usual rules of modus ponens, generalization, infinitary conjunction and the rule of aa-generalization given above.

A6. $\bigwedge_{i<\omega} \mathit{aa}s\,\varphi_i(s) \to \mathit{aa}s \bigwedge_{i<\omega} \varphi_i(s)$.

The Axiom A6 is valid since the cub filter is countably complete. We say that T is consistent (with respect to $\mathrm{L}_\mathrm{A}(\mathit{aa})$) if false is not a consequence of T.

1.8. Completeness Theorem for $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$. Let $\mathrm{L}_\mathrm{A}(\mathit{aa})$ be a countable fragment of $\mathrm{L}_{\omega_1\omega}(\mathit{aa})$ and let T be a set of sentences of $\mathrm{L}_\mathrm{A}(\mathit{aa})$. *$T$ is consistent iff T has a standard model.*

Again the proof yields a model of power $\leq \aleph_1$.

The usual methods yield the appropriate completeness and compactness results for countable, admissible fragments $\mathrm{L}_\mathrm{A}(\mathit{aa})$. The statements and proofs of these are completely analogous to the usual for L_A. See, e.g., Section III.5 of Barwise [2].

There is an Omitting Types Theorem for $\mathrm{L}_\mathrm{A}(\mathit{aa})$. It will be proved in Section 4. Theorem 1.8 will be a consequence of this Omitting Types Theorem.

2. Some formal consequences of the axioms and rules

In the next section we will be working with certain "weak" models of the axioms and rules. We thus need to get a better feeling for some of the formal consequences of the axioms and rules since these will hold in all weak models. The reader might want to skip ahead to Section 3 and then come back to pick up anything actually needed from this section. The lemmas of this section will apply to $\mathrm{L}(\mathit{aa})$ or any other fragment $\mathrm{L}_\mathrm{A}(\mathit{aa})$. We fix a theory T and write $\vdash \varphi$ for $T \vdash \varphi$ below.

2.1. Lemma. *If* $\vdash \varphi(s) \to \psi(s)$ *then* $\vdash \mathsf{aa}s\varphi(s) \to \mathsf{aa}s\psi(s)$.

Proof. Apply aa-generalization to obtain $\mathsf{aa}s\,(\varphi \to \psi)$ and then use Axiom A4 and modus ponens. □

2.2. Lemma. *If* $\vdash \varphi(s) \to \psi(s)$ *then* $\vdash \text{stat}\, s\,\varphi(s) \to \text{stat}\, s\,\psi(s)$.

Proof. Apply contraposition, then Lemma 2.1, then take contrapositives again to obtain $\vdash \neg\mathsf{aa}s\,\neg\varphi \to \neg\mathsf{aa}s\,\neg\psi$ which is the desired conclusion. □

2.3. Lemma. *If* $\vdash \varphi(s) \to \eta$ *where* s *is not free in* η *then* $\vdash$ $\text{stat}\, s\,\varphi(s) \to \eta$.

Proof. Similar to Lemma 2.2. □

2.4. Lemma. *If* s *is not free in* φ *then* $\vdash \varphi \leftrightarrow \mathsf{aa}s\,\varphi$.

Proof. The implication $\vdash \varphi \to \mathsf{aa}s\,\varphi$ follows directly by aa-generalization. By propositional logic, $\vdash \neg\varphi \to (\varphi \to \text{false})$, so, by aa-generalization, $\vdash \neg\varphi \to \mathsf{aa}s\,(\varphi \to \text{false})$. Using Axiom A4 and modus ponens we obtain $\vdash \neg\varphi \to (\mathsf{aa}s\,\varphi \to \mathsf{aa}s\,(\text{false}))$. But $\vdash \neg\mathsf{aa}s\,(\text{false})$ by Axiom A2, so, by propositional logic we obtain $\vdash \neg\varphi \to \neg\mathsf{aa}s\,\varphi$ or, taking contrapositives, the desired conclusion. □

2.5. Lemma. *Let* ψ_1 *be a formula of the form* $\text{stat}\, s_1 \cdots \text{stat}\, s_n\, \varphi$ *and let* ψ_2 *be* $\text{stat}\, s_1 \cdots \text{stat}\, s_n\, \text{stat}\, t_1 \cdots \text{stat}\, t_k\, \exists y_1 \cdots \exists y_k\, \varphi$, *where the* t_i *and* y_i *are not free in* φ. *Then* $\vdash \psi_1 \leftrightarrow \psi_2$.

Proof. $\vdash (\varphi \leftrightarrow \exists \mathbf{y}\,\varphi)$ so, by Lemma 2.4 applied k times, $\vdash (\varphi \leftrightarrow \text{stat}\, \mathbf{t}\, \exists \mathbf{y}\,\varphi)$. Then, applying Lemma 2.2 n times, we get $\vdash \psi_1 \leftrightarrow \psi_2$. □

We now become even more informal in our proofs, leaving it to the reader to convince himself that they can be formalized.

2.6. Lemma. $\vdash \mathsf{aa}s\,\varphi \wedge \text{stat}\, s\,\psi \to \text{stat}\, s\,(\varphi \wedge \psi)$.

Proof. We assume $\mathsf{aa}s\,\varphi$ and $\neg\text{stat}\, s\,(\varphi \wedge \psi)$ and prove $\neg\text{stat}\, s\,\psi$. In other symbols, we assume $\mathsf{aa}s\,\varphi$ and $\mathsf{aa}s\,\neg(\varphi \wedge \psi)$ and prove $\mathsf{aa}s\,\neg\psi$. Since $\vdash (\varphi \wedge \neg(\varphi \wedge \psi) \to \neg\psi)$ we have

$$\vdash \mathsf{aa}s\,(\varphi \wedge \neg(\varphi \wedge \psi)) \to \mathsf{aa}s\,\neg\psi.$$

But also

$$\vdash \mathsf{aa}s\,\varphi \wedge \mathsf{aa}s\,\neg(\varphi \wedge \psi) \to \mathsf{aa}s\,(\varphi \wedge \neg(\varphi \wedge \psi))$$

by A3, so, by modus ponens, we get $\vdash \mathsf{aa}s\,\neg\psi$. □

2.7. Lemma. *If s is not free in φ then* $\vdash \varphi \wedge \text{stat}\, s\, \psi \to \text{stat}\, s\, (\varphi \wedge \psi)$.

Proof. Combine Lemmas 2.4 and 2.6. □

2.8. Lemma. $\vdash \mathfrak{aa} s\, \varphi(s) \to \text{stat}\, s\, \varphi(s)$.

Proof. Apply Lemma 2.6 with ψ the formula $\neg$(false) and use Axiom A1 and Lemma 2.2. □

2.9. Lemma. $\vdash \forall x\, \mathfrak{aa} s\, \varphi(x, s) \leftrightarrow \mathfrak{aa} s\, \forall x \in s\, \varphi(x, s)$.

Proof. The $\to$ half is Axiom A5. To prove $\leftarrow$, assume $\mathfrak{aa} s\, \forall x \in s\, \varphi(x, s)$. Pick any x. By Axiom A2, $\mathfrak{aa} s\, (x \in s)$. By Axiom A3, $\mathfrak{aa} s\, (x \in s \wedge \forall x \in s\, \varphi(x, s))$. But

$$\vdash x \in s \wedge \forall x \in s\, \varphi(x, s) \to \varphi(x, s),$$

so by Lemma 2.1 and modus ponens we obtain $\mathfrak{aa} s\, \varphi(x, s)$. Thus $\forall x\, \mathfrak{aa} s\, \varphi(x, s)$. □

2.10. Lemma. $\vdash \exists x_1 \cdots \exists x_k\, \text{stat}\, s_1 \cdots \text{stat}\, s_n\, \varphi \to \text{stat}\, s_1 \cdots \text{stat}\, s_n\, \exists x_1 \cdots \exists x_k\, \varphi$.

Proof. By induction on $k + n$. Let us check the basic manipulation

$$\exists x\, \text{stat}\, s\, \varphi \to \text{stat}\, s\, \exists x\, \varphi.$$

Assume $\exists x\, \text{stat}\, s\, \varphi$, i.e. $\neg \forall x\, \mathfrak{aa} s\, \neg \varphi$. Then by Lemma 2.9,

$$\neg \mathfrak{aa} s\, \forall x \in s\, \neg \varphi$$

which is $\text{stat}\, s\, \exists x \in s\, \varphi$ By Lemma 2.2 we get $\text{stat}\, s\, \exists x\, \varphi$. □

These lemmas are what will be needed in the proofs in Sections 3 and 4. The rest of this section is taken up with investigating the role of Axiom A5 in deriving the crucial axiom of Keisler [15] mentioned in the introduction. Notice that in standard models, $\exists^{<\aleph_0} x\, \varphi(x)$ is equivalent to $\text{stat}\, s\, (\varphi \subseteq s)$, i.e., to

$$\text{stat}\, s\, \forall x\, (\varphi(x) \to s(x)),$$

so that $\mathsf{Q} x\, \varphi(x)$ is equivalent to $\mathfrak{aa} s\, \exists x \notin s\, \varphi(x)$. On the other hand, we have *defined* $\mathsf{Q} x\, \varphi(x)$ by $\text{stat}\, s\, \exists x \notin s\, \varphi(x)$, so we should be able to prove these equivalent.

2.11. Lemma. *If s is not free in $\varphi(x)$ then*

$$\vdash \mathfrak{aa} s\, \exists x \notin s\, \varphi(x) \leftrightarrow \text{stat}\, s\, \exists x \notin s\, \varphi(x).$$

Proof. The implication $\rightarrow$ follows from Lemma 2.8. To prove the converse we argue as follows to prove the contrapositive:

$$\text{stat}\, s\, (\varphi \subseteq s) \rightarrow \text{aa} s\, (\varphi \subseteq s).$$

Fix s_0, s two distinct variables. We have $\vdash \varphi \subseteq s_0 \rightarrow \varphi \subseteq s_0$ so, by aa-generalization, $\vdash \varphi \subseteq s_0 \rightarrow \text{aa} s\, (\varphi \subseteq s_0)$. We also have $\vdash \text{aa} s\, (s_0 \subseteq s)$ by Axiom A2. Thus, by Axiom A3 and modus ponens we have

$$\vdash (\varphi \subseteq s_0 \rightarrow \text{aa} s\, (\varphi \subseteq s_0 \wedge s_0 \subseteq s)).$$

But $\vdash (\varphi \subseteq s_0 \subseteq s \rightarrow \varphi \subseteq s)$, so, by Lemma 2.1, $\vdash \varphi \subseteq s_0 \rightarrow \text{aa} s\, (\varphi \subseteq s)$. Now apply Lemma 2.3 and Axiom A0 to obtain

$$\vdash \text{stat}\, s\, (\varphi \subseteq s) \rightarrow \text{aa} s\, (\varphi \subseteq s). \quad \square$$

We now come to the proof of the crucial axiom from Keisler regarding **Q**, or rather, its contrapositive.

2.12. Lemma. $\vdash \forall x\, \exists^{\leq \aleph_0} y\, \varphi(x, y) \wedge \exists^{\leq \aleph_0} x\, \exists y\, \varphi(x, y) \rightarrow \exists^{\leq \aleph_0} y\, \exists x\, \varphi(x, y)$.

Proof. We take $\exists^{\leq \aleph_0} x\, \psi(x)$ to be given by $\text{aa} s\, (\psi \subseteq s)$. The first conjunct of the hypothesis gives (1), the second gives (2), below.

$$\forall x\, \text{aa} s\, \forall y\, (\varphi(x, y) \rightarrow y \in s, \tag{1}$$

$$\text{aa} s\, \forall x\, (\exists y\, \varphi(x, y) \rightarrow x \in s). \tag{2}$$

From (1) and Axiom A5 we obtain

$$\text{aa} s\, \forall x \in s\, \forall y\, (\varphi(x, y) \rightarrow y \in s), \tag{3}$$

which, taken with (2) and Axiom A3 gives

$$\text{aa} s\, [\forall x \in s\, \forall y\, (\varphi(x, y) \rightarrow y \in s) \wedge \forall x\, (\exists y\, \varphi(x, y) \rightarrow x \in s)]. \tag{4}$$

The part of (4) in brackets implies $\forall y\, (\exists x\, \varphi(x, y) \rightarrow y \in s)$ so (4) itself implies $\text{aa} s\, \forall y\, (\exists x\, \varphi(x, y) \rightarrow y \in s)$ which is

$$\exists^{\leq \aleph_0} y\, \exists x\, \varphi(x, y)$$

as desired. $\square$

The proof shows the crucial role of Axiom A5. (We'll see later that a weaker form will suffice.) It will play a pivotal role in the proof of the Completeness Theorem.

Finally, for moral support, we state one last trivial fact. It explains why we do not need to make our weak models extensional in what follows.

2.13. Lemma. *If s_j does not occur in $\varphi(s_i)$ then*

$$\vdash \forall x\, (s_i(x) \leftrightarrow s_j(x)) \rightarrow (\varphi(s_i) \leftrightarrow \varphi(s_j)).$$

Proof. This is proved by induction on formulas. Assume, e.g., that the result is true of $\psi(s_i, \ )$ and $\psi(s_j, s)$ and let $\varphi(s_i)$ be $\text{aa}s\,\psi(s_i, s)$. Assume $\forall x\,[s_i(x) \leftrightarrow s_j(x)]$. Then $\psi(s_i, s) \leftrightarrow \psi(s_j, s)$, so, by aa-generalization we have $\text{aa}s\,[\psi(s_i, s) \leftrightarrow \psi(s_j, s)]$. This, with Axiom A4, gives

$$\text{aa}s\,\psi(s_i, s) \leftrightarrow \text{aa}s\,\psi(s_j, s),$$

as desired. □

3. Weak models and the Completeness Theorem for $L(\text{aa})$.

The proof of the Completeness Theorem given below follows in outline that of Keisler [15] for $L(Q)$. We first define a notion of weak model for $L(\text{aa})$. Next, we prove a weak completeness theorem which shows that every consistent theory has a weak model. We then prove our main lemma, which allows us to extend weak models. A suitable iteration of the main lemma ω_1 times finishes the proof. The main differences in the proof from that of Keisler for $L(Q)$ lie in the much stronger conclusion of our main Lemma and the additional care needed in the iteration of the main Lemma.

For each formula $\varphi(\boldsymbol{x}, s, \boldsymbol{t})$ of $L(\text{aa})$, where $\boldsymbol{x}$ is the sequence of first-order variables in φ, and s, $\boldsymbol{t}$ is the sequence of second-order variables in φ, let $R_{\text{aa}s\,\varphi}$ be a new relation symbol with arguments $\boldsymbol{x}$ and $\boldsymbol{t}$. Actually, it is desirable to exercise a certain amount of caution when introducing the $R_{\text{aa}s\,\varphi}$; see Section 3.9 for details. In our weak models, we hope to recapture properties of the $\text{aa}s$ quantifier with first-order formulas involving the new and old atomic formulas. This device was applied by Ressayre [23] in his study of the language $L(Q)$. Keisler's weak models in Keisler [15] are of a different nature, and their equivalents in our context are what we will call cub-like models; see Section 8.2.

We consider first-order, two-sorted structures of the form

$$\mathfrak{M}^* = (\mathfrak{M}, P, \in, R_{\text{aa}s\,\varphi})_{\varphi \in L(\text{aa})},$$

where $\mathfrak{M} = (M, \ldots)$ is a structure for L and $\in$ is a subset of $M \times P$ (so that P is essentially a collection of subsets of M, with possible repetition). The language of $\mathfrak{M}^*$ is the two-sorted language having two sorts of variables called "first-order" and "second-order", respectively. The first-order variables range over the set M, the second-order ones over the set P. The non-logical symbols of this language are those of L, the places of which are all first-order; the symbol $\in$, the first place of which is fiirst-order, the second second-order, and the symbols $R_{\text{aa}s\,\varphi}$ for $\varphi \in L(\text{aa})$, with the "sorting" of places explained above.

Each formula φ of $L(\text{aa})$ has a counterpart φ^* in the new language. This operation * is defined inductively by: $(s(x))^* = (x \in s)$, $\varphi^* = \varphi$ for other atomic φ, * commutes with everything except $\text{aa}s$ and $(\text{aa}s\,\varphi)^* = R_{\text{aa}s\,\varphi}$.

Thus we can define satisfaction of $L(aa)$ formulas φ with parameters $\boldsymbol{a}$ in $\mathfrak{M}^*$ by: $\mathfrak{M}^* \models \varphi(\boldsymbol{a})$ iff $\mathfrak{M}^* \models \varphi^*(\mathbf{a})$. where the latter is just the first-order notion of satisfaction.

3.1. Definition. A *weak model* for $L(aa)$ is a model

$$\mathfrak{M}^* = (\mathfrak{M}, P, \in, R_{aas\,\varphi})_{\varphi \in L(aa)}$$

(as above) with the following properties: all instances of the axioms of $L(aa)$ are true in $\mathfrak{M}^*$; and whenever $\mathfrak{M}^* \models \varphi(t)$ for all $t \in P$, then $\mathfrak{M}^* \models aas\,\varphi(s)$. (Here φ may have parameters in $M \cup P$.)

A word on this definition. The latter condition is just saying that the aa-generalization rule is valid in $\mathfrak{M}^*$. Since (by Axiom A1) $P \neq \emptyset$, the latter condition can also be thought of as saying that stationary sets are non-empty. In any case, provability is sound with respect to weak models: if $\mathfrak{M}^*$ is a weak model, $\mathfrak{M}^* \models T$, and $T \vdash \theta$ then $\mathfrak{M}^* \models \theta$ for any interpretation of the free variables in θ.

A *countable* weak model $\mathfrak{M}^*$ is one where both M and $P^{\mathfrak{M}^*}$ are countable. (Here, and from now on, we assume that

$$\mathfrak{M}^* = (\mathfrak{M}, P^{\mathfrak{M}^*}, \in^{\mathfrak{M}^*}, R^{\mathfrak{M}^*}_{aas\,\varphi})_{\varphi \in L(aa)}, |\mathfrak{M}| = M.)$$

Given a set S of first- and second-order constants, let $K = L \cup S$. We will need to consider the corresponding expansion $K(aa)$ of $L(aa)$.

A weak model for such a language would look like $(\mathfrak{M}^*, a^{\mathfrak{M}^*})_{a \in S}$. The axioms and rules of inference for $K(aa)$ include all instances of substituting first- and second-order constants for first- and second-order variables (respectively) in the axioms and rules of $L(aa)$. If T is a theory in such an expansion $K(aa)$, we say T is *consistent* if it's consistent with the axioms and rules for $K(aa)$.

3.2. Lemma. (*Weak Completeness Theorem for* $L(aa)$). *Let T be a set of sentences of $K(aa)$, an expansion of $L(aa)$ by at most countably many constants (and possibly none). If T is consistent then there is a countable weak model $\mathfrak{M}^*$ of T. (That is, $\mathfrak{M}^* \models \varphi$ for all $\varphi \in T$.)*

Proof. We expand our language, adding first- and second-order witnessing constants. Let $C = \{c_n : n \in \omega\}$ and $U = \{u_n : n \in \omega\}$ be sets of first- and second-order constants (respectively) disjoint from $K(aa)$. Let $(K(aa))'$ be all sentences obtained from formulas of $K(aa)$ by replacing free variables x_i by c_i and s_i by u_i. We will define by induction on n a finite set T_n of sentences of $(K(aa))'$ and a number k_n. Our inductive hypotheses are as follows.

(1) $i < j \leqslant n \to T_i \subseteq T_j$.

(2) If c_i or u_i occurs in a sentence of T_n, then $i \leqslant k_n$.

(3) $T \cup \{\text{stat } u_0 \text{ stat } u_1 \cdots \text{stat } u_{k_n} \exists c_0 \exists c_1 \cdots \exists c_{k_n} \wedge T_n(\boldsymbol{u}, \boldsymbol{c})\}$ is consistent (where here we abuse notation by regarding the c_i and u_i as variables).

Our goal is to construct the sets T_n so that we can build the desired model in the usual fashion from $\bigcup_{n\in\omega} T_n$, out of the c_i, u_i, and any second-order constants of $K(\mathfrak{a})$.

Let $(\varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n))_{n\in\omega}$ be an enumeration of the sentences of $(K(\mathfrak{a}))'$. We define T_n and k_n, assuming that T_{n-1} and k_{n-1} have been defined. (First, set $T_{-1}=\emptyset$, $k_{-1}=0$.)

Let $k_n = \sup\{k_{n-1}, i : c_i \text{ or } u_i \text{ occurs in } \varphi_n\}+1$. If

$$\text{stat}(u_i : i < k_n)\,\exists(c_i : i < k_n)\,(\wedge T_{n-1} \wedge \varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n))$$

is not consistent with T, then by inductive hypothesis 3, Lemma 2.5, and Axiom A3,

$$\text{stat}(u_i : i \leq k_n)\,\exists(c_i : i \leq k_n)\,(\wedge T_{n-1} \wedge \neg\varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n))$$

is consistent with T; so we can let $T_n = T_{n-1} \cup \{\neg\varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n)\}$, and the inductive hypotheses are preserved. Otherwise, let

$$T_n^0 = T_{n-1} \cup \{\varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n)\}.$$

To get T_n from T_n^0 we sometimes add witnesses. If φ_n is $\exists x\,\psi$ for some ψ, let

$$T_n = T_n^0 \cup \{\psi(\boldsymbol{c}^n, \boldsymbol{u}^n, c_{k_n})\}.$$

If φ_n is stat $s\,\psi(s)$, let

$$T_n = T_n^0 \cup \{\psi(\boldsymbol{c}^n, \boldsymbol{u}^n, u_{k_n})\}.$$

Otherwise, let $T_n = T_n^0$. In any case, it is easy to see that the inductive hypotheses are preserved. For example, if φ_n is stat $s\,\psi(s)$, then by construction and Lemma 2.5,

$$\text{stat}(u_i : i < k_n)\,\exists(c_i : i < k_n)\,(\wedge T_n^0 \wedge \text{stat}\ s\,\psi(s))$$

is consistent with T; therefore, by applying Lemmas 2.7 and 2.10, Axiom A0, and then Lemma 2.5, we have

$$\text{stat}(u_i : \leq k_n)\,\exists(c_i : i \leq k_n)\,(\wedge T_n^0 \wedge \psi(u_{k_n}))$$

is consistent with T, as desired.

The induction is complete. Let $T_\infty = \bigcup_n T_n$; note that $T_\infty \supset T$. We construct a weak model $\mathfrak{M}^*$ in a manner similar to that employed in usual Henkin constructions. Define an equivalence relation $\sim$ on the elements of C by: $c_i \sim c_j$ iff "$c_i = c_j$" $\in T_\infty$. Then

$$M = \{c_i/\!\sim\, : i \in \omega\}.$$

Set $P^{\mathfrak{M}^*} = \{u_i : i < \omega\} \cup \{u : u \text{ is a second-order constant of } K(\mathfrak{a})\}$. (The latter set in the definition of $P^{\mathfrak{M}^*}$ has no counterpart in the definition of M because its analogue is not needed there: $\vdash \exists x\,(x = c)$ for any first-order $c \in K$, and we added witnesses in our construction.) Define $\in^{\mathfrak{M}^*}$ by:

$$c_i/\!\sim\ \in^{\mathfrak{M}^*} u \quad \text{iff} \quad \text{"}u(c_i)\text{"} \in T_\infty;$$

the equality axioms of first-order logic insure that $\in^{\mathfrak{M}^*}$ is well-defined. Define the other relations on $\mathfrak{M}^*$ as follows. For $R \in L$, define $R^{\mathfrak{M}}(c/\sim)$ iff $R(c) \in T_\infty$ (or, if as suggested in Section 8.2 below, relations between first- and second-order objects are allowed in L, $R^{\mathfrak{M}}(u, c/\sim)$ iff $R(u, c) \in T_\infty$). Also, define

$$R^{\mathfrak{M}^*}_{\text{aa} s\, \varphi}(u, c/\sim) \quad \text{iff} \quad \text{aa} s\, \varphi(s, u, c) \in T_\infty.$$

Again, these are well-defined statements. Functions on $\mathfrak{M}$ are defined similarly.

We now establish for φ in $(K(\text{aa}))'$, by induction on the complexity of φ^* (*not* of), that

$$\mathfrak{M} \models \varphi(u, c/\sim) \quad \text{iff} \quad \varphi(u, c) \in T_\infty \tag{1}$$

for all sequences u from U, c from C. When φ^* is atomic it's true by definition. For $\varphi = \neg\psi$, ($\Leftarrow$) is by completeness of T_∞, and ($\Rightarrow$) is by consistency of T_∞ (which follows from the third inductive hypothesis). The case $\varphi = \psi_1 \vee \psi_2$ is also easy to check, as is the ($\Leftarrow$) direction of the case $\varphi = \exists x\, \psi$, using the consistency and completeness of T_∞. Finally, if $\exists x\, \psi \in T_\infty$, then $\psi(c) \in T_\infty$ for some c (by construction of T_∞), so by the inductive hypothesis $\mathfrak{M}^* \models \psi(c/\sim)$ and thus $\mathfrak{M}^* \models \exists x\, \psi$, as desired. Note that the case $\varphi = \text{aa} s\, \psi$ was taken care of at the atomic stage.

What remains is to show that $\mathfrak{M}^*$ is a weak model. Since T_∞ is consistent (with all the axioms of $(K(\text{aa}))'$), we know by (1) above that $\mathfrak{M}^*$ satisfies all instances of the axioms of L(aa) except possibly those in which there are free second-order variables. Consider, for example, the following instance of Axiom A2: $\text{aa} s\, (u_k \subset s)$. Let $n \in \omega$ be such that $\text{aa} s\, (u_k \subset s)$ is φ_n. Now $\vdash \text{aa} t\, \text{aa} s\, (t \subset s)$, so at stage n we could add $\text{aa} s\, (u_k \subset s)$ to T_{n-1}, by Lemma 2.6. So by construction, we did add it, and by (1), $\mathfrak{M}^* \models \text{aa} s (u_k \subset s)$. More generally, a similar argument shows that whenever T proves a sentence

$$\text{aa} s_0\, \text{aa} s_1 \cdots \text{aa} s_m\, \forall x_0 \forall x_1 \cdots \forall x_n\, \varphi,$$

then

$$\mathfrak{M}^* \models \varphi(u_{k_0}, u_{k_1}, \ldots, u_{k_m}, x) \quad \text{for all } k_0 < k_1 < \cdots < k_m.$$

Thus, since $\vdash \text{aa} s\, \forall x\, \varphi$ for each axiom φ, for every ordering s of the free second-order variables in φ, all instances of the axioms are true in $\mathfrak{M}^*$.

Finally, we show that whenever $\mathfrak{M}^* \models \text{stat}\, s\, \varphi(s)$, there is a k such that $\mathfrak{M}^* \models \varphi(u_k)$. This follows because witnesses were added in such cases; in fact, if $\text{stat}\, s\, \varphi(s)$ is φ_n, then we can take k to be k_n. □

To state the main lemma we need the following Definition 3.3 (i). As usual, the point of introducing the notion of elementary submodel is to get something preserved under unions, so we also define this in Definition 3.3(ii).

3.3. Definitions. (i) Let $\mathfrak{M}^*$ and $\mathfrak{N}^*$ be two weak models for L(aa). We say that $\mathfrak{N}^*$ is an L(aa) *elementary extension of* $\mathfrak{M}^*$ and write $\mathfrak{M}^* \prec \mathfrak{N}^*$ (wrt L(aa)) if

$\mathfrak{M}^* \subset \mathfrak{N}^*$ (as first-order structures), and for every $\mathrm{L}(aa)$ formula φ with parameters from $M \cup P^{\mathfrak{M}^*}$, $\mathfrak{M}^* \models \varphi$ iff $\mathfrak{N}^* \models \varphi$.

(ii) Let $\langle \mathfrak{M}^*_\alpha : \alpha < \gamma \rangle$ be an elementary chain of weak models: $\alpha < \beta < \gamma \rightarrow \mathfrak{M}^*_\alpha < \mathfrak{M}^*_\beta$ (wrt $\mathrm{L}(aa)$). The *union* $\mathfrak{M}^* = \bigcup_{\alpha<\gamma} \mathfrak{M}^*_\alpha$ is defined to be the usual first-order union.

One should check that if $\langle \mathfrak{M}^*_\alpha : \alpha < \gamma \rangle$ is an elementary chain of weak models for $\mathrm{L}(aa)$ with union $\mathfrak{M}^*$, then $\mathfrak{M}^*_\alpha < \mathfrak{M}^*$ (wrt $\mathrm{L}(aa)$) for all $\alpha < \gamma$, and that therefore $\mathfrak{M}^*$ is also a weak model for $\mathrm{L}(aa)$. This is a routine proof by induction on formulas.

We can now state the main result needed for the proof of the Completeness Theorem for $\mathrm{L}(aa)$.

3.4. Theorem (Main Lemma). *Let $\mathfrak{M}^*$ be a countable weak model for* $\mathrm{L}(aa)$ *and let* stat $s\,\psi(s)$ *be some formula of* $\mathrm{L}(aa)$ *(with parameters from $M \cup P^{\mathfrak{M}^*}$) which is true in $\mathfrak{M}^*$. There is a countable weak model $\mathfrak{N}^*$ with the following properties:*

(i) $\mathfrak{M}^* < \mathfrak{N}^*$ *(wrt* $\mathrm{L}(aa)$*)*;

(ii) $M \in P^{\mathfrak{N}^*}$; *that is, for some $s_0 \in P^{\mathfrak{N}^*}$, $\{x \in N : \mathfrak{N}^* \models x \in s_0\} = M$ (so that from now on we'll identify s_0 and M)*;

(iii) $\mathfrak{N}^* \models \psi(M)$;

(iv) *for all formulas (with parameters) $aa\,s\,\varphi(s)$ true in $\mathfrak{M}^*$, $\mathfrak{N}^* \models \varphi(M)$; and*

(v) *for every $s \in P^{\mathfrak{M}^*}$, the extension of s in $\mathfrak{N}^*$ is the same as that in $\mathfrak{M}^*$, i.e., for all $x \in N$, $\mathfrak{N}^* \models x \in s$ iff $x \in M$ and $\mathfrak{M}^* \models x \in s$.*

Proof. Let $K(aa)$ be the result of expanding $\mathrm{L}(aa)$ by adding a first-order constant m for each $m \in M$, a second-order constant $\bar{m}$ for each $s \in P^{\mathfrak{M}^*}$, and an additional second-order constant symbol $\bar{M}$. Let T_0 be the set of all $K(aa)$ sentences true in $\mathfrak{M}^*$ (where $\bar{m}$ is interpreted by m, $\bar{s}$ is interpreted by s, and no truth value is assigned to sentences in which $\bar{M}$ occurs). Let T be T_0 together with the following sentences of $K(aa)$:

(1) $\psi(\bar{M})$,

(2) $\varphi(\bar{M})$, whenever $\mathfrak{M}^* \models aa\,s\,\varphi(s)$.

The roles of (1) and (2) are respectively to insure (iii) and (iv) in the statement of the theorem. Also, from (2) we get that for each $m \in M$, "$\bar{m} \in \bar{M}$" $\in T$, and for each $s_0 \in P^{\mathfrak{M}^*}$, "$\bar{s}_0 \subset \bar{M}$" $\in T$, so that (v) will hold. (Actually (v) follows from (i)–(iv).) To insure (ii) we need a model of T plus $\forall x \vee \Sigma(x)$ where $\Sigma(x)$ is

$$\{\neg \bar{M}(x),\, x = \bar{m} : m \in M\}.$$

Any weak model of $T + \forall x \vee \Sigma(x)$ will prove the theorem.

We first prove the following "consistency criterion": for any sentence $\theta(\bar{M})$.

$$T \vdash \theta(\bar{M}) \quad \text{iff} \quad \mathfrak{M}^* \models aa\,s\,(\psi(s) \rightarrow \theta(s)). \tag{$*$}$$

To prove ($\Leftarrow$), suppose $\mathfrak{M}^* \models aa\,s\,(\psi(s) \rightarrow \theta(s))$. Then $\psi(\bar{M}) \rightarrow \theta(\bar{M})$ is in T and so is $\psi(\bar{M})$, so $T \vdash \theta(\bar{M})$.

To prove ($\Rightarrow$), suppose $T \vdash \theta(\bar{M})$. There are a finite number of sentences of forms (1) and (2) above, say $\psi(\bar{M}), \varphi_1(\bar{M}), \ldots, \varphi_k(\bar{M})$, such that T_0 together with these proves $\theta(\bar{M})$. We can write this as

$$T_0 \vdash \bigwedge \varphi_i(\bar{M}) \to (\psi(\bar{M}) \to \theta(\bar{M})).$$

We can replace M by a second-order variable s and then apply Lemma 2.1 to get

$$T_0 \vdash \mathit{aa}s(\bigwedge \varphi_i(s)) \to \mathit{aa}s\,(\psi(s) \to \theta(s)),$$

so this must be true in $\mathfrak{M}^*$ since $\mathfrak{M}^*$ is a weak model of T_0. Since $\mathfrak{M}^* \models \mathit{aa}s\,\varphi_i(s)$ for each i, it follows that

$$\mathfrak{M}^* \models \mathit{aa}s\,(\psi(s) \to \theta(s)),$$

and (*) holds. Note that since

$$\mathfrak{M}^* \models \neg \mathit{aa}s\,(\psi(s) \to \neg \psi(s))$$

(because $\mathfrak{M}^* \vdash \neg \mathit{aa}s\, \neg\psi(s)$), we have that T does not prove $\neg \psi(\bar{M})$—so T is consistent. Note also the contrapositive of (*):

$$T + \varphi(\bar{M}) \text{ is consistent iff } \mathfrak{M}^* \models \text{stat } s\,(\varphi(s) \wedge \psi(s)). \qquad (**)$$

We are now ready to build a model $\mathfrak{N}^*$ of T plus $\forall x \vee \Sigma(x)$. The construction is exactly the proof of the weak completeness theorem (3.2) except for one additional step. After completing all the steps required to get T_n from T_{n-1}, we add one more. We want to guarantee, for each n, that if $c_n/\!\sim$ winds up in $\bar{M}$, then $c_n/\!\sim\, = \bar{m}$ for some $m \in M$. Let $\mathbf{c}$ be the sequence $c_0, c_1, \ldots, c_{n-1}, c_{n+1}, \ldots, c_{k_n}$.

Let T_n^1 represent the version of T_n we have obtained from T_{n-1} so far (as before). Thus

$$\text{stat } \boldsymbol{u}\, \exists c_n\, \exists \mathbf{c} \wedge T_n^1(c_n, \mathbf{c}, \boldsymbol{u}, \bar{M})$$

is consistent with T (where we suppress the parameters from $M \cup P^{\mathfrak{M}^*}$). By the consistency criterion (**),

$$\mathfrak{M}^* \models \text{stat } s\,(\text{stat } \boldsymbol{u}\, \exists c_n\, \exists \mathbf{c} \wedge T_n^1(c_n, \mathbf{c}, \boldsymbol{u}, s) \wedge \psi(s)). \qquad (1)$$

Now if there is an $m \in M$ such that

$$\mathfrak{M}^* \models \text{stat } s\,(\text{stat } \boldsymbol{u}\, \exists \mathbf{c} \wedge T_n^1(m, \mathbf{c}, \boldsymbol{u}, s) \wedge \psi(s)),$$

let $T_n = T_n^1 \cup \{c_n = \bar{m}\}$; then using (**) we see that the inductive hypotheses are preserved, and $c_n/\!\sim\, = m$ in the resulting model $\mathfrak{N}$ since "$c_n = \bar{m}$" $\in T_\infty$. Otherwise, there is no such $m \in M$; so

$$\mathfrak{M}^* \models \forall x\, \mathit{aa}s\,(\psi(s) \to \mathit{aa}\boldsymbol{u}\, \forall \mathbf{c} \neg \wedge T_n^1(x, \mathbf{c}, \boldsymbol{u}, s)).$$

Thus we have, by several uses of the axiom A5 (which we haven't used till now)

$$\mathfrak{M}^* \models \mathit{aa}s\,(\psi(s) \to \mathit{aa}\boldsymbol{u}\, \forall x\, \forall \mathbf{c}\, (\bigwedge_{i \le k_n} x \in u_i \wedge x \in s \to \neg \wedge T_n^1(x, \mathbf{c}, \boldsymbol{u}, s))). \qquad (2)$$

Also, $\vdash aa\, u_i\ (s \subset u_i)$ for all i, so by Lemma 2.5, $\vdash aa\, \boldsymbol{u}\ (s \subset u_i)$ for all i, and thus (by Axiom A4) $\vdash aa\, \boldsymbol{u} \bigwedge_{i \le k_n} s \subset u_i$; so

$$\vdash aa\, s\ aa\, \boldsymbol{u} \bigwedge_{i \le k_n} s \subset u_i$$

by aa-generalization. Thus

$$\mathfrak{M}^* \models aa\, s\ aa\, \boldsymbol{u} \bigwedge_{i \le k_n} s \subset u_i. \tag{3}$$

Taking (2) and (3) together and using Axiom A4 and then Lemma 2.1 and modus ponens, we have

$$\mathfrak{M}^* \models aa\, s\ (\psi(s) \to aa\, \boldsymbol{u}\ \forall x\ \forall \boldsymbol{c}\ (\wedge T_n^1(x, \boldsymbol{c}, \boldsymbol{u}, s) \to \neg x \in s)). \tag{4}$$

Using Lemma 2.6 on (1) and (4) (as well as Lemma 2.2 and modus ponens), we obtain

$$\mathfrak{M}^* \models \text{stat}\ s\ \text{stat}\ \boldsymbol{u}\ \exists x\ \exists \boldsymbol{c}\ (\wedge T_n^1(x, \boldsymbol{c}, \boldsymbol{u}, s) \wedge \neg x \in s \wedge \psi(s)). \tag{5}$$

Thus we can let $T_n = T_n^1 \cup \{\neg c_n \in \bar{M}\}$; (5) and the consistency criterion (**) guarantee that the inductive hypotheses are preserved. Also, since $(c_n \notin \bar{M}) \in T_n$, we have $(c_n \notin M) \in T_\infty$ so that when we're done, $\mathfrak{N}^* \models \neg(c_n/\sim\ \in \bar{M})$.

Thus in any case, for all $n \in \omega$, $\mathfrak{N}^* \models \vee \Sigma(c_n/\sim)$, as desired. This completes the proof of the main lemma. □

The Completeness Theorem for $\mathrm{L}(aa)$ (1.4) follows immediately from the weak completeness theorem and the following result. For any standard model $\mathfrak{N}$, let $\mathfrak{N}^*$ denote the corresponding weak model

$$(\mathfrak{N}, P_{\omega_1}(N), \in, R^{\mathfrak{N}^*}_{aa s \varphi})_{\varphi \in \mathrm{L}(aa)}, \quad \text{where } R^{\mathfrak{N}^*}_{aa s \varphi}(\boldsymbol{a}) \text{ iff } \mathfrak{N} \models aa\, s\ \varphi(s, \boldsymbol{a}).$$

Note that for any φ of $\mathrm{L}(aa)$ with parameters in $N \cup P_{\omega_1}(N)$, $\mathfrak{N} \models \varphi$ iff $\mathfrak{N}^* \models \varphi^*$.

3.5. Theorem. *Let $\mathfrak{M}^*$ be a countable weak model for $\mathrm{L}(aa)$. Let* stat $s\ \psi(s)$ *be some formula true in $\mathfrak{M}^*$. There is a model $\mathfrak{N}$ for* L *so that the corresponding weak model $\mathfrak{N}^*$ has the following properties*:

(i) $\mathfrak{M}^* < \mathfrak{N}^*$ wrt $\mathrm{L}(aa)$,

(ii) $\mathfrak{N}^* \models \psi(M)$,

(iii) $\mathfrak{N}^* \models \varphi(M)$ *for all formulas* $aa\, s\ \varphi(s)$ *true in* $\mathfrak{M}^*$.

Proof. Let C be a set of ω_1 constant symbols, P a set of ω_1 unary relation symbols. There are ω_1 sentences of $\mathrm{L}(aa)$ allowing symbols from $C \cup P$ of the form stat $s\ \psi(s)$. Let I be the set of all such. Partition ω_1 into ω_1 disjoint stationary sets (stationary in the usual sense), $\omega_1 = \bigcup_{i \in I} S_i$. It is well known that this is possible, see e.g. Kunen [18]. We define a sequence $\langle \mathfrak{M}^*_\alpha;\ \alpha < \omega_1 \rangle$ of weak models for $\mathrm{L}(aa)$. Let $\mathfrak{M}^*_0 = \mathfrak{M}^*$, and choose names for everything in M_0 from C, in $P^{\mathfrak{M}_0}$ from P. For limit λ let $\mathfrak{M}^*_\lambda = \bigcup_{\alpha<\lambda} \mathfrak{M}^*_\alpha$. For $\alpha+1$, apply the main lemma (3.4) to $\mathfrak{M}^*_\alpha$ and the unique formula stat $s\,\psi(s)$ such that $\alpha \in S_{\text{stat } s\, \psi}$, if stat $s\,\psi(s)$ is defined and true in $\mathfrak{M}^*_\alpha$, to define $\mathfrak{M}^*_{\alpha+1}$. (Otherwise let $\mathfrak{M}^*_{\alpha+1} = \mathfrak{M}^*_\alpha$.) Choose names for everything in $M_{\alpha+1}$ from C, everything in $P^{\mathfrak{M}^*_{\alpha+1}}$ from P. Let $\mathfrak{N}^+ = \bigcup_{\alpha<\omega_1} \mathfrak{M}^*_\alpha$, and let $\mathfrak{N}$ be the L-reduct of $\mathfrak{N}^+$. For every $s \in P^{\mathfrak{N}^+}$, we have that $s \in P^{\mathfrak{M}_\alpha^*}$ for some $\alpha < \omega_1$, and by part (v) of the main lemma, the extension $\{x \in N : \mathfrak{N}^+ \models x \in s\}$ of s in $\mathfrak{N}^+$ (also denoted by s) is the same as the extension of s in $\mathfrak{M}^*_\alpha$; in particular the extension of s is a countable subset of N. Also, by part (ii) of the main lemma, there is a constant $\tilde{M}_\alpha \in P^{\mathfrak{M}_{\alpha+1}}$ whose extension in $\mathfrak{M}^*_{\alpha+1}$, and hence in $\mathfrak{N}^+$, is M_α, the first-order universe of $\mathfrak{M}^*_\alpha$. $\mathfrak{N}^+$ is a weak model satisfying (i)–(iii) above, but it is not $\mathfrak{N}^*$. We need to verify that for all formulas φ of $L(\mathit{aa})$, for all $\boldsymbol{a} \in N \cup P^{\mathfrak{N}^+}$, $\mathfrak{N}^+ \models \varphi(\boldsymbol{a})$ in the sense of weak models iff $\mathfrak{N}^* \models \varphi(\boldsymbol{a})$ in the standard sense. This is by induction on formulas, the only non-trivial step being:

$$\mathfrak{N}^+ \models \mathit{aa}s\, \varphi(s) \quad \text{iff} \quad \mathfrak{N}^* \models \mathit{aa}s\, \varphi(s).$$

To prove ($\Rightarrow$), note that if $\mathfrak{N}^+ \models \mathit{aa}s\,\varphi(s)$ then $\mathfrak{N}^+ \models \varphi(M_\alpha)$ for all $\alpha > \alpha_0$, some $\alpha_0 < \omega_1$. The set $\{M_\alpha \mid \alpha > \alpha_0\}$ is a cub subset of $P_{\omega_1}(N)$. By the induction hypothesis, $\mathfrak{N}^* \models \varphi(M_\alpha)$ for all $\alpha > \alpha_0$ so $\mathfrak{N}^* \models \mathit{aa}s\,\varphi(s)$.

To prove ($\Leftarrow$), assume $\mathfrak{N}^* \models \mathit{aa}s\,\varphi(s)$. Hence there is a cub set $Y_1 \subseteq P_{\omega_1}(N)$ such that for every $s \in Y_1$, $\mathfrak{N}^* \models \varphi(s)$. But $Y_2 = \{M_\alpha \mid \alpha < \omega_1\}$ is cub so $Y_1 \cap Y_2$ is cub. Thus the set $\{\alpha < \omega_1 \mid \mathfrak{N}^* \models \varphi(M_\alpha)\}$ contains a cub set $X (= Y_1 \cap Y_2)$ of ordinals. Now, toward a contradiction, suppose $\mathfrak{N}^+ \models \text{stat } s\, \neg\varphi(s)$. Let $\alpha_0 < \omega_1$ be chosen so that all parameters in φ are in $\mathfrak{M}^*_{\alpha_0}$, and let $S = \{\alpha > \alpha_0 \mid \alpha \in S_{\text{stat } s\, \neg\varphi}\}$. Then $\mathfrak{N}^+ \models \neg\varphi(M_\alpha)$ for all $\alpha \in S$. The stationary set S and the cub set X must have some ordinal α in common. For this α, $\mathfrak{N}^* \models \varphi(M_\alpha)$, $\mathfrak{N}^+ \models \neg\varphi(M_\alpha)$, which contradicts the inductive hypothesis.

This proves conditions (i) and (iii) of the theorem. To ensure (ii), we need only make sure, when we partition ω_1 into the S_i, that $0 \in S_{\text{stat } s\, \psi}$. □

3.6. Remark. What happens if $\neg\mathbf{Q}x\,(x = x)$ is in our original theory T? Then the weak model $\mathfrak{M}^*_0$ which starts our ω_1-chain of models will satisfy $\neg\mathbf{Q}x\,(x = x)$, i.e., $\mathit{aa}s\, \forall x\,(x \in s)$. Applying the main lemma yields $\mathfrak{M}^*_1$, which must then satisfy $\forall x\,(x \in M_0)$; thus $M_1 = M_0$. It's clear then that $M_\alpha = M_0$ for all α, so that $N = M_0$ as well. Thus our final (standard) model of T is countable, as it should be. The point is that although new countable subsets may be added to $P^{\mathfrak{M}_\alpha^*}$ to get $P^{\mathfrak{M}^*_{\alpha+1}}$, no new elements are added to M_α.

On the other hand, if $\mathbf{Q}x\,(x = x)$ is in T, then each $M_\alpha \models \mathit{aa}s\, \exists x\,(x \notin s)$; so $\mathfrak{M}_{\alpha+1} \models \exists x\,(x \notin M_\alpha)$. That is, at least one new element has been added to M_α to get $M_{\alpha+1}$. Therefore N is really uncountable.

3.7. Remark. Our proof of completeness shows a little more than we stated. Namely, if T is a theory and $\varphi(\mathbf{s})$ is a formula, which may have free variables, then $T \vdash \varphi$ iff for all models $\mathfrak{M}$ of T, $\mathfrak{M}$ satisfies φ for *all* sequences s of countable subsets of M.

The proof also shows (see especially the proof of Lemma 3.2) that the following set of axioms is complete for $L(\mathit{aa})$: $\{\mathit{aa}\, s\, \forall \boldsymbol{x}\, \varphi : \varphi$ is one of the axioms of A0 through A5 or of first-order logic, and all the free variables of φ are included in $s \cup \boldsymbol{x}\} \cup \{\varphi \leftrightarrow \mathit{aa}\, s\, \varphi : s$ is not free in $\varphi\}$. Since any proof in this modified axiom system involves only sentences, the rule of aa-generalization is not necessary in this system.

3.8. Remark. Our proofs of the weak completeness theorem and the main lemma can be modified to work for the following expanded version of $L(\mathit{aa})$: allow existential (and universal) second-order quantification in the formation rules. The Elementary Chain Theorem in 3.3 is also still true. (However, the proof in Theorem 3.5 would give a standard model which is still an elementary extension *only* for the formulas of our usual notion of $L(\mathit{aa})$.) This remark can be used in Kaufmann's proof that interpolation fails in $L(\mathit{aa})$, a result of Shelah discussed in Remark 6.9.

3.9. Remark. The converse of the weak completeness theorem is true. To see this, one shows that if $T \vdash \varphi$ and $\mathfrak{M}^*$ is a weak model of T then $\mathfrak{M}^* \models \varphi$, that is, $\mathfrak{M}^* \models \varphi^*$. There are some subtleties in showing this, though. One way of avoiding the subtleties is to define the $R_{\mathit{aa}\, s\, \varphi}$ in such a way that we can show the following: if ψ is obtained from φ by (proper) substitution, then the same substitution in φ^* gives ψ^*. This property seems nice in its own right, and simplifies some other verifications in our proofs.

We have the following arrangement to ensure this property. Call (temporarily) a formula $\varphi \in L(\mathit{aa})$ *free* if it does not contain (individual) constants and if no variable has two distinct free occurrences in φ. Call two free formulas *equivalent* if one is obtained from the other by a substitution. Now, it is clear that any formula $\psi \in L(\mathit{aa})$ (possibly containing constants) is obtained by a substitution from a free formula φ, called the *skeleton* of ψ; φ is determined by ψ up to equivalence and the substitution is uniquely determined by ψ and φ. With every *equivalence class* $[\varphi]$ of free formulas, we associate a new relation symbol R_φ; for distinct equivalence classes, the R_φ are distinct, but $R_\varphi = R_{\varphi'}$ whenever φ and φ' are equivalent free formulas. With k and l the numbers of first-order and second-order free variables in φ, respectively, we let R_φ be a $k+l$-place relation symbol in which the first k places are (to be filled in by) *first-order* (variables or constants) and the last l places are *second-order*. In other words, we are dealing with a two-sorted language, with two sorts of variables.

If $\mathit{aa}\, s\, \varphi$ is any formula in $L(\mathit{aa})$ starting with an $\mathit{aa}\, s$ quantifier, $R_{\mathit{aa}\, s\, \varphi}$ denotes $R_{\mathit{aa}\, s\, \varphi_0}$, where $\mathit{aa}\, s\, \varphi_0$ is the skeleton of $\mathit{aa}\, s\, \varphi$. Given $\mathit{aa}\, s\, \varphi$, we define $(\mathit{aa}\, s\, \varphi)^*$ to be $R_{\mathit{aa}\, s\, \varphi}(x, t)$ where $\mathit{aa}\, s\, \varphi = (\mathit{aa}\, s\, \varphi_0)(x, t)$ for the skeleton $\mathit{aa}\, s\, \varphi_0$ of $\mathit{aa}\, s\, \varphi$, and for t the k- and l-tuples of first- and second-order variables or constants, respectively, whose substitution in the given order in $\mathsf{aa}s\,\varphi_0$ yields $\mathsf{aa}s\,\varphi$. In particular, $\mathsf{aa}s\,\varphi$ and $(\mathsf{aa}s\,\varphi)^*$ have the same free variables and constants.

Notice that whenever $\varphi \in \mathrm{L}(\mathsf{aa})$ is an instance of a valid first-order axiom scheme, then φ^* is an instance of the same scheme (hence in particular, it is valid). This is so essentially because now * "respects substitution" in the obvious sense. Now it is clear that our notion of satisfaction "$\mathfrak{M}^* \models \varphi(\boldsymbol{a})$", $\varphi \in \mathrm{L}(\mathsf{aa})$, obeys all the laws of first-order logic, which was the aim of our somewhat formalistic procedure.

3.10. Remark. It is useful to see what $\mathsf{aa}s_1\, \mathsf{aa}s_2 \cdots \mathsf{aa}s_n\, \varphi$ means in the model $\mathfrak{N}$ constructed in Theorem 3.5. For example, suppose $\mathsf{aa}s_1\, \mathsf{aa}s_2\, \mathsf{aa}s_3\, \varphi$ is a sentence of $\mathrm{L}(\mathsf{aa})$ with parameters in $\mathfrak{M}^*_{\alpha_0}$. Then

$$\mathfrak{N} \models \mathsf{aa}s_1\, \mathsf{aa}s_2\, \mathsf{aa}s_3\, \varphi(s_1, s_2, s_3) \quad \text{iff} \quad \mathfrak{N} \models \varphi(M_{\alpha_1}, M_{\alpha_2}, M_{\alpha_3})$$

Whenever $\alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \omega_1$.
This observation may be helpful in understanding Section 4.

4. Omitting types in $\mathrm{L}_A(\mathsf{aa})$

In this section we prove an Omitting Types Theorem which is similar to the one in Keisler [15] (4.9), but is to some extent a second-order theorem. Out of this will come the completeness of the axioms and rules for $\mathrm{L}_{\omega_1\omega}(\mathsf{aa})$.

We've already done an omitting types argument. The "M-type" $\Sigma(x)$ in the main lemma (more precisely, $\{\neg\sigma(x) \mid \sigma \in \Sigma(x)\}$) had to be omitted. The use of the diagonal intersection axiom A5 in that argument illustrates much of the idea behind the proof of the Omitting Types Theorem in this section.

What we want is a useful sufficient condition for a theory T to have a model of some sentence of the form $\forall x \bigvee_{n<\omega} \neg\sigma_n(x)$, or $\mathsf{aa}s \bigvee_{n<\omega} \neg\sigma_n(s)$, or say $\mathsf{aa}s\, \mathsf{aa}t\, \forall x \bigvee_n \neg\sigma_n(x, s, t)$. Note while reading the following that the omitting types condition for the latter is different than the one for $\mathsf{aa}t\, \mathsf{aa}s\, \forall x \bigvee_n \neg\sigma_n(x, s, t)$. Let's look first at the first-order case. Fix a countable fragment $\mathrm{L}_A(\mathsf{aa})$. We want a notion of "strongly omits" similar to that of Keisler [15]. Let T be a set of sentences of $\mathrm{L}_A(\mathsf{aa})$, and let $\Sigma(\boldsymbol{x})$ be a set of formulas of $\mathrm{L}_A(\mathsf{aa})$ in the finite string $\boldsymbol{x}$ of variables. Let's say that T *strongly omits* Σ if whenever a sentence $\mathrm{stat}\, \boldsymbol{s}\, \exists \boldsymbol{x}\, \varphi(\boldsymbol{x}, \boldsymbol{s})$ is consistent with T, there is a $\sigma \in \Sigma$ such that $\mathrm{stat}\, \boldsymbol{s}\, \exists \boldsymbol{x}\, (\varphi(\boldsymbol{x}, \boldsymbol{s}) \wedge \neg\sigma(\boldsymbol{x}))$ is consistent with T.

Omitting Types Theorem (first-order version). *Let T be a set of sentences, and $\Sigma(x)$ a set of formulas, of $\mathrm{L}_A(\mathsf{aa})$. Suppose that T strongly omits Σ. Then T has a (standard) model $\mathfrak{N}$ which omits Σ; that is, $\mathfrak{N} \models \forall \boldsymbol{x} \bigvee_{\sigma\in\Sigma} \neg\sigma(\boldsymbol{x})$.*

Since our notion of strongly omits uses the syntactic definition of consistent, the Completeness Theorem is a special case of the above.

It might be instructive for the reader to prove the above before we prove the full Omitting Types Theorem, where we'll consider sets $\Sigma(x_1, \ldots, x_n, t_1, \ldots, t_m)$ of formulas (where the x_i and t_j are first- and second-order variables, respectively). Fix such a set $\Sigma(\boldsymbol{x}, \boldsymbol{t})$ of formulas of $\mathrm{L}_{\mathrm{A}}(\mathit{aa})$. We'll use the symbol S_Σ to range over strings of quantifiers resulting from the string stat t_1 stat $t_2 \cdots$ stat t_m, by inserting any number of distinct quantifiers stat s_i. More formally, S_Σ can represent any string of the form

$$\text{stat } s_{11} \text{ stat } s_{12} \cdots \text{stat } s_{1k_1} \text{ stat } t_1 \text{ stat } s_{21} \cdots \text{stat } s_{2k_2}$$

$$\text{stat } t_2 \cdots \text{stat } t_m \text{ stat } s_{m+1,1} \cdots \text{stat } s_{m+1,k_{m+1}},$$

where some or all of the k_i might be 0.

4.1. Definition. Let T be a set of sentences, and let $\Sigma(\boldsymbol{x}, \boldsymbol{t})$ be a set of formulas of $\mathrm{L}_{\mathrm{A}}(\mathit{aa})$. *$T$ strongly omits Σ* if whenever S is an S_Σ as above and $S\,\exists \boldsymbol{x}\,\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t})$ is consistent with T then for some $\sigma \in \Sigma$, $S\,\exists \boldsymbol{x}\,(\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t}) \wedge \neg\sigma(\boldsymbol{x}, \boldsymbol{t}))$ is consistent with T. A weak model $\mathfrak{M}^*$ is said to *strongly omit* Σ if its complete $\mathrm{L}(\mathit{aa})$ theory does (with all parameters from $M \cup P^{\mathfrak{M}^*}$).

We remark that the phrase "T strongly omits Σ" is always used with respect to a fixed ordering $\boldsymbol{t}$ of the free second-order variables in Σ.

4.2. Omitting Types Theorem. *Let T be a consistent set of sentences of $\mathrm{L}_{\mathrm{A}}(\mathit{aa})$. Suppose T strongly omits Σ_n for all $n \in \omega$. Then T has a standard model $\mathfrak{N}$ such that for all $n \in \omega$,*

$$\mathfrak{N} \models \mathit{aa}\, t_1 \cdots \mathit{aa}\, t_{k_n} \forall \boldsymbol{x} \bigvee_{\sigma \in \Sigma_n} \neg\sigma(\boldsymbol{x}, \boldsymbol{t}).$$

(We say $\mathfrak{N}$ omits Σ_n in this case.)

4.3. Remarks. (A) A similar theorem has been proved for $\mathrm{L}(\mathrm{Q})$ by Kaufmann.

(B) Here's an alternate definition of "T strongly omits $\Sigma(\boldsymbol{x}, \boldsymbol{t})$." Let S denote strings of quantifiers as before except that $\exists y_i$ for various i may also occur. Then *T strongly omits Σ* if whenever $S\,\exists \boldsymbol{x}\,\varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t})$ is consistent with T, so is $S\,\exists \boldsymbol{x}(\varphi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{s}, \boldsymbol{t}) \wedge \neg\sigma(\boldsymbol{x}, \boldsymbol{t}))$ for some $\sigma \in \Sigma$. It's not hard to see that the two definitions are equivalent. For example, suppose T strongly omits $\Sigma(x_1, x_2, t_1, t_2)$ in the original sense and that

$$\text{stat } s_1\, \exists y_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists y_2\, \exists y_3\, \exists \boldsymbol{x}\, \varphi$$

is consistent with T. Then so is

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists \boldsymbol{x}\, \exists y_1\, \exists y_2\, \exists y_3$$

$$(y_1 \in t_1 \wedge y_1 \in t_2 \wedge y_1 \in s_2 \wedge \varphi)$$

by the converse of the diagonal intersection axiom (see Lemma 2.9). By the original strong omitting condition, there is a $\sigma \in \Sigma$ such that

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat, } t_2 \text{ stat } s_2 \, \exists \boldsymbol{x} \, (\exists y_1 \, \exists y_2 \, \exists y_3 \\ (y_1 \in t_1 \wedge y_1 \in t_2 \wedge y_1 \in t_3 \wedge \varphi) \wedge \neg \sigma(\boldsymbol{x}, \boldsymbol{t}))$$

is consistent with T. By manipulating the existential quantifiers and applying the contrapositive of the diagonal intersection axiom, we conclude that

$$\text{stat } s_1 \, \exists y_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2 \, \exists y_2 \, \exists y_3 \;\; \exists \boldsymbol{x} \, (\varphi \wedge \neg \sigma)$$

is consistent with T, as desired. So we can use our second strong omitting condition whenever we assume the first. And that's what we'll do!

4.4. Remark. For complete theories T, the converse of the Omitting Types Theorem is true. For suppose T has a (standard) model $\mathfrak{N}$ omitting $\Sigma(\boldsymbol{x}, \boldsymbol{t})$. Suppose $S \, \exists \boldsymbol{x} \, \varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t})$ is consistent with T. Then by completeness of T,

$$\mathfrak{N} \models S \, \exists \boldsymbol{x} \, \varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t});$$

but

$$\mathfrak{N} \models \text{aa}\, \boldsymbol{t} \, \forall \boldsymbol{x} \bigvee_{\sigma \in \Sigma} \neg \sigma(\boldsymbol{x}, \boldsymbol{t}).$$

So

$$\mathfrak{N} \models S \, \exists \boldsymbol{x} \, (\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t}) \wedge \bigvee_{\sigma \in \Sigma} \neg \sigma(\boldsymbol{x}, \boldsymbol{t}));$$

$$\mathfrak{N} \models S \exists \boldsymbol{x} \bigvee_{\sigma \in \Sigma} (\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t}) \wedge \neg \sigma(\boldsymbol{x}, \boldsymbol{t})));$$

$$\mathfrak{N} \models \bigvee_{\sigma \in \Sigma} S \, \exists \boldsymbol{x} \, (\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t}) \wedge \neg \sigma(\boldsymbol{x}, \boldsymbol{t})).$$

Thus for some $\sigma \in \Sigma$ $\mathfrak{N} \models S \, \exists \boldsymbol{x} \, (\varphi \wedge \neg \sigma)$. So $S \, \exists \boldsymbol{x} \, (\varphi \wedge \neg \sigma)$ is consistent with T.

4.5. In this subsection we prove the Omitting Types Theorem for the case $T \subset \mathrm{L}_{\omega\omega}(\text{aa})$. In Lemma 4.5.1 it is shown that if T strongly omits Σ_n for all $n \in \omega$, then T has a weak model strongly omitting each Σ_n. We want to use that model as the start of an elementary chain of length ω_1 of weak models $\mathfrak{M}^*_\alpha (\alpha < \omega_1)$ each strongly omitting each Σ_n. Thus we have to get from one such weak model to another. Lemma 4.5.2 shows (as a special case) that the particular theory T from the main lemma for $\mathfrak{M}^* = \mathfrak{M}^*_\alpha$ strongly omits any set of formulas strongly omitted by $\mathfrak{M}^*_\alpha$; we can then apply Lemma 4.5.1 to get a model $\mathfrak{M}^*_{\alpha+1}$ for T which strongly omits each Σ_n. In fact, Lemma 4.5.3 shows that this T strongly omits the "M_α-type" (called Σ in the proof of the main lemma), which means that we can further require $\mathfrak{M}^*_{\alpha+1}$ to strongly omit the M_α-type, so that indeed $\mathfrak{M}^*_{\alpha+1} \succ \mathfrak{M}^*_\alpha$ (wrt $\mathrm{L}(\text{aa})$) subject to the requirements in the conclusion of the main lemma.

With this basic plan and Lemmas 4.5.1, 2 and 3 behind us, we'll construct our ω_1-chain with the aforementioned properties (subject to a further inductive hypothesis) which ultimately will yield the conclusion of the theorem, at least for $T \subset L_{\omega\omega}(aa)$. Finally, we'll deduce the general case of $T \subset L_A(aa)$ in Section 4.6.

Lemma 4.5.1. *Let T be a consistent set of sentences of some expansion $K(aa)$ of $L(aa)$ by at most countably many constants, and suppose T strongly omits sets Σ_n ($n \in \omega$) of formulas of $L(aa)$. Then T has a countable weak model $\mathfrak{M}^*$ strongly omitting each Σ_n.*

Proof. The proof is an expansion of the proof of the weak completeness theorem (3.2) much as the proof of the main lemma (3.4) was such an expansion. Let $\langle \varphi_n(\boldsymbol{c}^n, \boldsymbol{u}^n), i_n, S_n \rangle_{n<\omega}$ enumerate all triples $\langle \varphi, i, S \rangle$ such that $i < \omega$; $\varphi = \varphi(\boldsymbol{c}, \boldsymbol{u}) \in (K(aa))'$; and S is, if possible, an $S_{\Sigma_i}(\boldsymbol{x}, \boldsymbol{t})$ as in Definition 4.1 such that φ is $S\,\exists\boldsymbol{x}\,\psi$, some ψ — otherwise S is some new object like "slash." Note that S is uniquely determined by φ and i. Now proceed as in the proof of the weak completeness theorem except for one additional step. Let T_n^1 be the T_n constructed (in the aforementioned proof) from T_{n-1}. If S_n is "slash" or if $\varphi_n \notin T_n^1$, let $T_n = T_n^1$. Otherwise, φ_n is $S_n\,\exists\boldsymbol{x}\,\psi$, some ψ, and

$$\text{stat } \boldsymbol{u}\,\exists\boldsymbol{c}\,S_n\exists\boldsymbol{x}\,(\wedge T_n^1 \wedge \psi)$$

is consistent with T (by Lemmas 2.6 and 2.2). Choose a $\sigma \in \Sigma_{i_n}$ such that

$$\text{stat } \boldsymbol{u}\,\exists\boldsymbol{c}\,S_n\,\exists\boldsymbol{x}\,(\wedge T_n^1 \wedge \psi \wedge \neg\sigma)$$

is consistent with T; this is possible because T strongly omits Σ_{i_n}. Thus

$$\text{stat } \boldsymbol{u}\,\exists\boldsymbol{c}\,(\wedge T_n^1 \wedge S_n\,\exists\boldsymbol{x}\,(\psi \wedge \neg\sigma))$$

is consistent with T. We let $T_n = T_n^1 \cup \{S_n\,\exists\boldsymbol{x}\,(\psi \wedge \neg\sigma)\}$; then the inductive hypotheses are preserved.

Now form T_∞ and $\mathfrak{M}^*$ as before. As in Section 3.2, $\mathfrak{M}^*$ is a weak model of T. We need only show that $\mathfrak{M}^*$ strongly omits each Σ_i to complete the proof of Lemma 4.5.1.

Fix i, and let $S\,\exists\boldsymbol{x}\,\varphi$ be the sort of sentence we have to consider. Choose n so that $\langle S\,\exists\boldsymbol{x}\,\varphi, i, S \rangle = \langle \varphi_n, i_n, S_n \rangle$. We must have added $S\,\exists\boldsymbol{x}\,\varphi$ at stage n since we're assuming that $S\,\exists\boldsymbol{x}\varphi$ is in $\mathfrak{M}^*$, and thus consistent with T_∞ by (1) in the proof of Lemma 3.2. But then by construction, for some $\sigma \in \Sigma$, $S\exists x\,(\varphi \wedge \neg\sigma) \in T_n \subset_\infty$, so again by (1) of Lemma 3.2, $\mathfrak{M}^* \models S\,\exists\boldsymbol{x}\,(\varphi \wedge \neg\sigma)$, as desired. This concludes the proof of Lemma 4.5.1. □

Lemma 4.5.2. *Let $\mathfrak{M}^*$ be any countable weak model for $L(aa)$, $\mathfrak{M} \models \text{stat } s\,\psi(s)$, and let T be the associated theory from the proof of the main lemma. For any $\Sigma(x_1, \ldots, x_n, t_1, \ldots, t_m)$, if $\mathfrak{M}^*$ strongly omits Σ, then T strongly omits each of the* *following*:

(a) $\Sigma(x_1, \ldots, x_n, t_1, \ldots, t_m)$;
(b) $\Sigma(x_1, \ldots, x_n, \bar{M}, t_2, \ldots, t_m)$.

Proof. Though (a) can be deduced more or less from (b), we'll check them separately.

(a) Assume $S\,\exists\boldsymbol{x}\,\varphi(\boldsymbol{x}, s, \bar{M}, \boldsymbol{t})$ is consistent with T. By the same Consistency Criterion used in the proof of the main lemma, we see that $\mathfrak{M}^*$ is a model of

$$\text{stat}\ u\,[S\,\exists\boldsymbol{x}\,\varphi(\boldsymbol{x}, s, u, \boldsymbol{t}) \wedge \psi(u)].$$

so for some $\sigma \in \Sigma$,

$$\mathfrak{M}^* \models \text{stat}\ u\,S\,\exists\boldsymbol{x}\,(\varphi \wedge \neg\sigma \wedge \psi),$$

since $\mathfrak{M}^*$ strongly omits Σ. Thus, $S\,\exists\boldsymbol{x}\,(\varphi(\boldsymbol{x}, s, \bar{M}, \boldsymbol{t}) \wedge \neg\sigma(\boldsymbol{x}, \boldsymbol{t}))$ is consistent with T (by the other direction of the consistency criterion).

(b) Let's use $\hat{t}$ to denote $(t_2, \ldots, t_n)$. If $S\,\exists\boldsymbol{x}\,\varphi(\boldsymbol{x}, s, \bar{M}, \hat{t})$ is consistent with T, then (using the Consistency Criterion)

$$\mathfrak{M}^* \models \text{stat}\ u\,(S\,\exists\boldsymbol{x}\,\varphi(\boldsymbol{x}, s, u, \hat{t}) \wedge \psi(u)).$$

By renaming u as t_1 and playing with quantifiers, we obtain

$$\mathfrak{M}^* \models \text{stat}\ t_1\,S\,\exists\boldsymbol{x}\,(\varphi(\boldsymbol{x}, s, t_1, \hat{t}) \wedge \psi(t_1)).$$

So, since $(t_1, \hat{t})$ is $(t_1, t_2, \ldots, t_n) = \boldsymbol{t}$, and since (stat t_1 S) is a string of quantifiers of the right form for $\Sigma(\boldsymbol{x}, \boldsymbol{t})$, there is a $\sigma \in \Sigma$ such that

$$\mathfrak{M}^* \models \text{stat}\ t_1\,S\,\exists\boldsymbol{x}(\varphi(\boldsymbol{x}, s, \boldsymbol{t}) \wedge \neg\sigma(\boldsymbol{x}, \boldsymbol{t}) \wedge \psi(t_1))$$

So, as desired, the following is consistent with T:

$$S\,\exists\boldsymbol{x}\,(\varphi(\boldsymbol{x}, s, \bar{M}, \hat{t}) \wedge \neg\sigma(\boldsymbol{x}, \bar{M}, \hat{t})).\quad \square$$

Lemma 4.5.3. *Let $\mathfrak{M}^*$ be a weak model, $\mathfrak{M}^* \models$ stat $s\,\psi(s)$, and let T come from $\mathfrak{M}^*$ as in the main lemma. Then T strongly omits the M-type $\Sigma(x) = \{x \in \bar{M}, x \neq \bar{m} : m \in M\}$.*

Proof. The proof of Lemma 4.5.3 is implicit in the proof of the main lemma, but we include it here for completeness. Suppose stat $\boldsymbol{s}\,\exists x\,\varphi(x, \boldsymbol{s}, \bar{M})$ is consistent with T. Then, using the Consistency Criterion (as we will implicitly from now on), we see that $\mathfrak{M}^*$ is a model of:

$$\text{stat}\ u\,(\psi(u) \wedge \text{stat}\ \boldsymbol{s}\,\exists x\,\varphi(x, \boldsymbol{s}, u)). \tag{1}$$

To reach a contradiction, let's assume T does not strongly omit Σ. Then we have a formula stat $\boldsymbol{s}\,\exists x\,\varphi(x, \boldsymbol{s}, \bar{M})$ such that $\mathfrak{M}^*$ is a model of (1) as well as (2) and $(3)_m$ below, for all $m \in M$.

$$\textit{aa}\,u\,(\psi(u) \rightarrow \neg\text{stat}\ \boldsymbol{s}\,\exists x(\varphi(x, \boldsymbol{s}, u) \wedge x \notin u)); \tag{2}$$

$$\textit{aa}\,u\,(\psi(u) \rightarrow \neg\text{stat}\ \boldsymbol{s}\,\exists x\,(\varphi(x, \boldsymbol{s}, u) \wedge x = m)). \tag*{$(3)_m$}$$

Using these we see that $\mathfrak{M}^* \models (2'), (3')$ below:

$$\mathsf{aa} u\, (\psi(u) \to \mathsf{aa} s\, \forall x\, (\varphi(x, s, u) \to x \in u)); \tag{2'}$$

$$\forall y\, \mathsf{aa} u\, (\psi(u) \to \mathsf{aa} s\, \neg\varphi(y, s, u)). \tag{3'}$$

But (3′) yields (4) by two uses of the Diagonal Intersection Axiom A5:

$$\mathsf{aa} u\, \forall y \in u\, (\psi(u) \to \mathsf{aa} s\, \neg\varphi(y, s, u)),$$

$$\mathsf{aa} u\, (\psi(u) \to (\forall y \in u)\, \mathsf{aa} s\, \neg\varphi(y, s, u)),$$

$$\mathsf{aa} u\, (\psi(u) \to \mathsf{aa} s\, \forall y\, (\textstyle\bigwedge_i y \in s_i \wedge y \in u \to \neg\varphi(y, s, u)). \tag{4}$$

Intersecting the sets of u's represented by (1), (2′) and (4) (i.e., use Lemma 2.6 and Axiom A3) yields $\mathfrak{M}^* \models \mathrm{stat}\, u\, [(5) \wedge (6) \wedge (7)]$:

$$[\psi(u) \wedge \mathrm{stat}\, s\, \exists x\, \varphi(x, s, u)], \tag{5}$$

$$[\psi(u) \to \mathsf{aa} s\, \forall x\, (\varphi(x, s, u) \to x \in u)], \tag{6}$$

$$[\psi(u) \to \mathsf{aa} s\, \forall y\, (\textstyle\bigwedge_i y \in s_i \wedge y \in u \to \neg\varphi)]. \tag{7}$$

So

$$\mathfrak{M}^* \models \mathrm{stat}\, u[\mathrm{stat}\, s\, \exists x\, \varphi \wedge \mathsf{aa} s\, \forall x (\varphi \to x \in u) \wedge$$
$$\mathsf{aa} s\, \forall x\, (\textstyle\bigwedge_i x \in s_i \wedge x \in u \to \neg\varphi)].$$

Intersecting the above sets of s and x, again (by Lemma 2.6 and Axiom A3) we obtain

$$\mathfrak{M}^* \models \mathrm{stat}\, u\; \mathrm{stat}\, s\, \exists x (\varphi \wedge (\varphi \to x \in u) \wedge (\textstyle\bigwedge_i x \in s_i \wedge x \in u \to \neg\varphi)),$$

and thus

$$\mathfrak{M}^* \models \mathrm{stat}\, u\; \mathrm{stat}\, s\, \exists x\, (x \in u \wedge \neg \textstyle\bigwedge_i x \in s_i). \tag{8}$$

But

$$\mathfrak{M}^* \models \mathsf{aa} u\; \mathsf{aa} s (\textstyle\bigwedge_i u \subset s_i), \tag{9}$$

as we saw before; and (8) and (9) together imply $\mathfrak{M}^* \models \mathrm{stat}\, u\; \mathrm{stat}\, s$ (false), a contradiction. □

Proof of theorem for $\mathrm{L}(\mathsf{aa})$. Let T be a consistent theory of $\mathrm{L}(\mathsf{aa})$ which strongly omits Σ_n for each $n \in \omega$. By Lemma 4.5.1, let $\mathfrak{M}_0^*$ be a weak model of T strongly omitting each Σ_n. Partition ω_1 into ω_1 disjoint stationary subsets as in the proof of completeness of $\mathrm{L}(\mathsf{aa})$. We'll apply Lemmas 4.5.1–3 to get a chain $(\mathfrak{M}_\alpha^* : \alpha < \omega_1)$ with the following properties:

(a) $(\mathfrak{M}_\alpha^* : \alpha < \omega_1)$ satisfies all the conditions of the chain constructed in proving completeness of $\mathrm{L}(\mathsf{aa})$. Then we can take the union, as in that proof, and get all the properties we got in that proof.

(b) For all $\alpha < \omega_1$, $\mathfrak{M}^*_\alpha$ strongly omits each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \bar{M}_{\beta_2}, \ldots, \bar{M}_{\beta_i}, t_{i+1}, \ldots, t_{k_n})$$
$$\text{for all} \quad \beta_1 < \beta_2 < \cdots < \beta_i < \alpha, 0 \leqslant i \leqslant k_n.$$

At limits, we simply take unions. It's easy to see that (a) and (b) are preserved at limits.

So now assume (a) and (b) are true up to stage α. To get $\mathfrak{M}^*_{\alpha+1}$, let T^*_α be the theory given by the main Lemma at stage α (from $\mathfrak{M}^*_\alpha$ and the appropriate ψ). Now $\mathfrak{M}^*_\alpha$ strongly omits each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_i}, t_{i+1}, \ldots, t_{k_n}) \quad \text{for} \quad \beta_1 < \beta_2 < \cdots < \beta_i < \alpha, 0 \leqslant i \leqslant k_n,$$

by the inductive hypothesis. Hence, by Lemma 4.5.2, T^*_α strongly omits each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_i}, t_{i+1}, \ldots, t_{k_n})$$

and each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_i}, \bar{M}_\alpha, t_{i+2}, \ldots, t_{k_n}) \quad \text{for} \quad \beta_1 < \cdots < \beta_n < \alpha, 0 \leqslant i \leqslant k_n.$$

So by reindexing, T^*_α strongly omits each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_i}, t_{i+1}, \ldots, t_{k_n}) \quad \text{for} \quad \beta_1 < \cdots < \beta_i < \alpha + 1, 0 \leqslant i \leqslant k_n.$$

Also, by Lemma 4.5.3, T^*_α strongly omits the M_α-type. So by Lemma 4.5.1, let $\mathfrak{M}^*_{\alpha+1} \models T^*$ be such that $\mathfrak{M}^*_{\alpha+1}$ strongly omits and thus omits the M_α-type and each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_i}, t_{i+1}, \ldots, t_{k_n}) \quad \text{for} \quad \beta_1 < \cdots < \beta_i < a + 1, 0 \leqslant i \leqslant k_n.$$

Then the inductive hypotheses are maintained.

Let $\mathfrak{N}^+ = \bigcup_\alpha \mathfrak{M}^*_\alpha$. Then $\mathfrak{N}^+$ strongly omits each

$$\Sigma_n(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_{k_n}}) \quad \text{for} \quad \beta_1 < \cdots < \beta_{k_n} < \omega_1.$$

Since for any $\boldsymbol{c}$, $\exists \boldsymbol{x}\, (\boldsymbol{x} = \boldsymbol{c})$ is true in $\mathfrak{N}^+$, there is some $\sigma \in \Sigma_n$ such that

$$\exists \boldsymbol{x}\, (\boldsymbol{x} = \boldsymbol{c} \wedge \neg\varphi(\boldsymbol{x}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_{k_n}}))$$

is true in $\mathfrak{N}^+$. Thus $\mathfrak{N}^+ \models \neg\sigma(\boldsymbol{c}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_{k_n}})$ for some $\sigma \in \Sigma$. Thus, we can form $\mathfrak{N}^*$ as before, and $\mathfrak{N}^* \models \neg\sigma(\boldsymbol{c}, \bar{M}_{\beta_1}, \ldots, \bar{M}_{\beta_{k_n}})$. This shows that for all $\beta_1 < \beta_2 < \beta_3 < \cdots < \beta_n$,

$$\mathfrak{N}^* \models \forall \boldsymbol{x} \bigvee_{\sigma \in \Sigma_n} \neg\sigma(\boldsymbol{x}, \bar{M}_{\beta_1}, \bar{M}_{\beta_2}, \ldots, \bar{M}_{\beta_n}).$$

Thus (as in Remark 3.10)

$$\mathfrak{N} \models \mathsf{aa} t_1 \cdots \mathsf{aa} t_{k_n} \forall \boldsymbol{x} \bigvee_{\sigma \in \Sigma_n} \neg\sigma(\boldsymbol{x}, \boldsymbol{t}),$$

as desired. This completes the proof for the case $T \subset L_{\omega\omega}(\mathsf{aa})$.

4.6. The infinitary case.

Now we consider the case $T \subset \mathrm{L}_A(\mathit{aa})$. To reduce to finitary logic, we replace each infinitary conjunction $\bigwedge\Phi(\boldsymbol{x}, \boldsymbol{s})$ by an atomic formula $R_{\bigwedge\Phi}(\boldsymbol{x}, \boldsymbol{s})$. The exact procedure is similar to the reduction in Section 3, but here we are *not* turning $\mathit{aa}s\,\varphi$ into an atomic formula. Consider the following transformation "prime" (′) from formulas of $\mathrm{L}_A(\mathit{aa})$ to those of $K_{\omega\omega}(\mathit{aa})$, where here

$$K = \mathrm{L} \cup \{R_{\bigwedge\Phi}(\boldsymbol{s}, \boldsymbol{x}) : \bigwedge\Phi(\boldsymbol{s}, \boldsymbol{x}) \in \mathrm{L}_A(\mathit{aa})\}.$$

φ' is defined by induction on complexity of φ so that "prime" commutes with everything except: $(\bigwedge\Phi)' = R_{\bigwedge\Phi}$. Also note that "prime" has an inverse ($^-$) so that $(\varphi')^- = \varphi$ and $(\psi^-)' = \psi$.

Let $T' = \{\varphi' : T \models \varphi$ and $\varphi \in \mathrm{L}_A(\mathit{aa})\}$, and let $\Sigma' = \{\sigma' : \sigma \in \Sigma\}$.

Claim 4.6.0. *For any $\varphi \in \mathrm{L}_A(\mathit{aa})$, $T \vdash \varphi$ iff $T' \vdash \varphi'$.*

Proof. ($\Rightarrow$) is by the very definition of T'. To prove ($\Leftarrow$), one argues by induction on proofs, that any proof of φ' from T' in $K(\mathit{aa})$ can be inverted into a proof from T of φ in $\mathrm{L}_A(\mathit{aa})$. The Axioms of T' invert to theorems of T and the finitary rules of inference available in $K(\mathit{aa})$ invert to thenselves in $\mathrm{L}_A(\mathit{aa})$. $\square$

Claim 4.6.1. *If T strongly omits $\Sigma(\boldsymbol{x}, \boldsymbol{t})$ then T' strongly omits Σ'.*

Proof. Suppose $S\,\exists\boldsymbol{x}\,\varphi$ is consistent with T'. Then by Claim 4.6.0, $S\,\exists\boldsymbol{x}\,\varphi^-$ is consistent with T. So for some $\sigma \in \Sigma$, $S\,\exists\boldsymbol{x}\,(\varphi^- \wedge \neg\sigma)$ is consistent with T. So (again, by Claim 4.6.0), $S\,\exists\boldsymbol{x}\,(\varphi \wedge \neg\sigma')$ is consistent with T'. $\square$

Claim 4.6.2. *T strongly omits each $\Sigma_{\bigwedge\Phi} = \{\neg\bigwedge\Phi\} \cup \{\varphi : \varphi \in \Phi\}$ for $\bigwedge\Phi \in \mathrm{L}_A(\mathit{aa})$, with any order $\boldsymbol{t}$ of the second-order variables in $\bigwedge\Phi = \bigwedge\Phi(\boldsymbol{x}, \boldsymbol{t})$.*

Proof. This is a consequence of the Axiom A6. Suppose that S is an S_Σ for $\Sigma = \Sigma_{\bigwedge\Phi}$ and $S\,\exists\boldsymbol{x}\,\psi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t})$ is consistent with T. Also suppose that $T \vdash \neg S\,\exists\boldsymbol{x}\,(\psi \wedge \neg\varphi)$ for every $\varphi \in \Phi$ (otherwise we are done). This means that $T \vdash S^*\,\forall\boldsymbol{x}\,(\psi \to \varphi)$ for every $\varphi \in \Phi$, hence

$$T \vdash \bigwedge_{\varphi \in \Phi} S^*\,\forall\boldsymbol{x}\,(\psi \to \varphi),$$

where S^* results by replacing each "stat" by "aa" and each "$\exists$" by "$\forall$". Using Axiom A6 and the first-order valid version of Axiom A6 with "$\mathit{aa}s$" replaced by "$\forall x$", we obtain that $T \vdash S^*\,\forall\boldsymbol{x}\,(\psi \to \bigwedge\Phi)$. Essentially by Lemma 2.6, we have that the latter sentence and $S\,\exists\boldsymbol{x}\,\psi$ (which is consistent with T) imply $S\,\exists\boldsymbol{x}\,(\psi \wedge \bigwedge\Phi)$. Thus, the last sentence is consistent with T, proving the claim by the definition of $\Sigma_{\bigwedge\Phi}$. $\square$

Now suppose $T \subset L_A(aa)$ strongly omits Σ_n for each $n < \omega$. So T' strongly omits each Σ'_n ($= \{\sigma' : \sigma \in \Sigma_n\}$) and each $\Sigma_{\wedge\Phi}$($= \{\sigma' : \sigma \in \Sigma_{\wedge\Phi}\}$) by Claims 4.6.1 and 4.6.2. Let $\mathfrak{N}$ be the (standard) model of T' omitting each Σ_n and $\Sigma_{\wedge\Phi}$ from the proof in Section 4.5. We will show that $\mathfrak{N} \models T$ and that $\mathfrak{N}$ omits each Σ_n. For the latter, we show (as before) that $\mathfrak{N}$ strongly omits each

$$\Sigma_n(M_{\beta_1}, \ldots, M_{\beta_k}, \boldsymbol{x}), \quad \beta_1 < \beta_2 < \cdots < \beta_k.$$

By (*), Section 4.5, it suffices to prove the following claim.

Claim 4.6.3. *For any φ in $L_A(aa)$ with parameters in $\{\bar{M}_\alpha : \alpha < \omega_1\} \cup |\mathfrak{N}|$,*

$$\mathfrak{N} \models \varphi(\boldsymbol{m}, \bar{\boldsymbol{M}}) \quad \text{iff} \quad \mathfrak{N} \models \varphi'(\boldsymbol{m}, \bar{\boldsymbol{M}}).$$

Proof. The only steps which need doing are $\bigwedge$ and aas in our induction on complexity of φ. We treat $\bigwedge$ first.

If $\mathfrak{N} \models (\bigwedge \Psi(\boldsymbol{m}, \bar{\boldsymbol{M}}))'$, i.e., $\mathfrak{N} \models R_{\wedge\Psi}(\boldsymbol{m}, \boldsymbol{M})$, then (since $\mathfrak{N} \models T'$) $\mathfrak{N} \models \psi'(\boldsymbol{m}, \boldsymbol{M})$ for each $\psi \in \Psi$. So by induction, $\mathfrak{N} \models \psi(\boldsymbol{m}, \boldsymbol{M})$, all $\psi \in \Psi$; so $\mathfrak{N} \models \bigwedge \Psi(\boldsymbol{m}, \boldsymbol{M})$. For the other direction, suppose $\mathfrak{N} \models \bigwedge \Psi(\boldsymbol{m}, \boldsymbol{M})$. Then $\mathfrak{N} \models \psi(\boldsymbol{m}, \boldsymbol{M})$, all $\psi \in \Psi$; so $\mathfrak{N} \models \psi'(\boldsymbol{m}, \boldsymbol{M})$ for all $\psi \in \Psi$, by induction. Now we've already seen that T' strongly omits each $\Sigma'_{\wedge\Psi(\boldsymbol{x},\boldsymbol{t})}$. So by construction of $\mathfrak{N}$, $\mathfrak{N}$ omits $\Sigma'_{\wedge\Psi(\boldsymbol{x},\boldsymbol{M})'}$ and therefore $\mathfrak{N} \models (\bigwedge \Psi(\boldsymbol{m}, \boldsymbol{M}))'$ (i.e. $\mathfrak{N} \models R_{\wedge\Psi}(\boldsymbol{m}, \boldsymbol{M})$), as desired.

Now for the aa-step. If $\mathfrak{N} \models aas\, \psi(\boldsymbol{m}, \boldsymbol{M}, s)$, then for some cub $C \subseteq \omega_1$, we have

$$\forall \alpha \in C \quad \mathfrak{N} \models \psi(\boldsymbol{m}, \boldsymbol{M}, M_\alpha).$$

So $\mathfrak{N} \models \psi'(\boldsymbol{m}, \boldsymbol{M}, M_\alpha)$ for all $\alpha \in C$; thus $\mathfrak{N} \models aas\, \psi'(\boldsymbol{m}, \boldsymbol{M}, s)$.
That is, $\mathfrak{N} \models (aas\, \psi(\boldsymbol{m}, \boldsymbol{M}, s))'$; as desired. The converse is just as easy. □

4.7. There is a more general version of our Omitting Types Theorem. In the version just proved, the "almost all" quantifiers precede the universal (first-order) quantifiers. In the general version, we allow any ordering of the quantifiers. Corresponding to that ordering, we have notions of "T strongly omits Σ" and "$\mathfrak{A}$ omits Σ" entirely analogous to the notions we had before. Then the general theorem is stated just as the original theorem was stated; however, any ordering of the quantifiers is allowed, and the notions used are the ones that go with that ordering.

To simplify notation we will present only a special case of the generalization. Fix a set T of sentences of $L_A(aa)$ and a set $\Sigma(x_1, x_2, t_1, t_2)$ of formulas of $L_A(aa)$. Consider the ordering $<$ of the variables of Σ given by $x_1 < t_1 < x_2 < t_2$. Let S refer to any sequence of quantifiers which results from inserting various stat s_i into the sequence $\exists x_1$ stat $t_1\, \exists x_2$ stat t_2.

Let's say that T *strongly omits Σ with respect to* $<$(wrt $<$) if, whenever $S\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t})$ is consistent with T (S as above), there is a $\sigma \in \Sigma$ such that $S\,(\varphi(\boldsymbol{x}, \boldsymbol{s}, \boldsymbol{t}) \wedge \neg\sigma(\boldsymbol{x}, \boldsymbol{t}))$ is consistent with T. Let's say that a model $\mathfrak{A}$ *omits*

Σ(wrt $<$) if

$$\mathfrak{A} \models \forall x_1\, \mathsf{aa} t_1\, \forall x_2\, \mathsf{aa} t_2 \bigvee_{\sigma\in\Sigma} \neg\sigma(x_1, x_2, t_1, t_2).$$

Claim. *If T strongly omits Σ (wrt $<$), then T has a (standard) model $\mathfrak{A}$ which omits Σ (wrt $<$).*

Proof. This could be proved by modifying the original proof. Instead, we'll show how it follows from the previous theorem. Let

$$\Gamma(x_1, x_2, t_1, t_2) = \Sigma(x_1, x_2, t_1, t_2) \cup \{t_1(x_1), t_2(x_1), t_2(x_2)\}.$$

First we note that T strongly omits Γ in the original sense. To see this, suppose for example that

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists x_1\, \exists x_2\, \varphi(\mathbf{x}, \mathbf{s}, \mathbf{t})$$

is consistent with T. If some

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists x_1\, \exists x_2\, (\varphi \wedge \neg t_i(x_j))$$

is consistent with T, for $(i, j) = (1, 1), (2, 1)$ or $(2, 2)$, then we're done. Otherwise,

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists x_1\, \exists x_2\, (\varphi \wedge t_1(x_1) \wedge t_2(x_1) \wedge t_2(x_2))$$

is consistent with T. Since

$$\vdash \mathsf{aa} s_1\, \mathsf{aa} t_1\, \mathsf{aa} t_2\, \mathsf{aa} s_2\, (t_1 \subseteq s_2 \wedge t_2 \subseteq s_2),$$

we have that

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists x_1\, \exists x_2$$
$$(\varphi \wedge t_1(x_1) \wedge t_2(x_1) \wedge s_2(x_1) \wedge t_2(x_2) \wedge s_2(x_2))$$

is consistent with T. By the contrapositive of the diagonal intersection principle,

$$\text{stat } s_1\, \exists x_1 \text{ stat } t_1\, \exists x_2 \text{ stat } t_2 \text{ stat } s_2\, \varphi$$

is consistent with T. But by hypothesis, T strongly omits Σ (wrt $<$). So let $\sigma \in \Sigma$ such that

$$\text{stat } s_1\, \exists x_1 \text{ stat } t_1\, \exists x_2 \text{ stat } t_2 \text{ stat } s_2\, (\varphi \wedge \neg\sigma)$$

is consistent with T. Then also

$$\text{stat } s_1 \text{ stat } t_1 \text{ stat } t_2 \text{ stat } s_2\, \exists x_1\, \exists x_2\, (\varphi \wedge \neg\sigma)$$

is consistent with T, as desired.

Since T strongly omits Γ (in the original sense), there is a model $\mathfrak{A}$ of T which omits Γ (in the original sense). Thus

$$\mathfrak{A} \models \mathsf{aa} t_1\, \mathsf{aa} t_2\, \forall x_1\, \forall x_2 \left(\neg(x_1 \in t_1 \wedge x_1 \in t_2 \wedge x_2 \in t_2) \vee \bigvee_{\sigma\in\Sigma} \neg\sigma\right).$$

Hence

$$\mathfrak{A} \models \text{aa} t_1\, \text{aa} t_2 \forall x_1 \in t_1 \cap t_2 \forall x_2 \in t_2 \left(\bigvee_{\sigma \in \Sigma} \neg\sigma \right).$$

So by the converse of the Diagonal Intersection Principle,

$$\mathfrak{A} \models \forall x_1\, \text{aa} t_1 \forall x_2\, \text{aa} t_2 \bigvee_{\sigma \in \Sigma} \neg\sigma,$$

as desired. □

5. A sentence φ of $L(\text{aa})$ which is not expressible in $L_{\infty\infty}$

The definition of φ is based on the fact that there is no sentence of $L_{\infty\infty}$ saying that two arbitrary well-orderings are isomorphic. (See Malitz [22].) A suggestion by Kunen has been used to simplify our original example somewhat.

The kind of model of φ we have in mind is illustrated in Fig. 1. Here $P^U = P_{\omega_1}(U)$, $P^V = P_{\omega_1}(V)$, and $(U, <)$ and $(V, <)$ are disjoint isomorphic well-orderings, with $R(x, y)$ saying that x and y have isomorphic order types under the restrictions of $<$ to U and V, respectively.

We take φ to be (abbreviating slightly)

$$\text{aa} s\, \exists x\, \exists y\, [P^U(x) \wedge P^V(y) \wedge x = s \cap U \wedge y = s \cap V \wedge R(x, y)].$$

To achieve a contradiction, suppose that φ has the same models as some φ' in $L_{\infty\infty}$. Let ψ (in $L_{\omega_1\omega_1}$) be the conjunction of the following:

(1) $(U, <)$ and $(V, <)$ are disjoint well-orderings and

$$\forall x\, \forall y\, (U(x) \wedge V(y) \rightarrow \neg x < y \wedge \neg y < x).$$

(2) $\in$ is an extensional relation which is a subset of $U \times P^U \cup V \times P^V$. So we can think of the elements of P^U and P^V as being subsets of U and V, respectively.

(3) $P^U = P_{\omega_1}(U)$ and $P^V = P_{\omega_1}(V)$ (in the sense explained in (2) above). For example, one part says

$$\forall x\, (P^U(x) \leftrightarrow \exists y_1 y_2 \cdots \forall z\, [z \in x \leftrightarrow z = y_1 \vee \cdots \vee z = y_n \vee \cdots]).$$

(4) $\forall x\, \forall y\, [R(x, y) \leftrightarrow P^U(x) \wedge P^V(y) \wedge (x, < \upharpoonright x) \cong (y, < \upharpoonright y)]$.

This can be expressed in $L_{\omega_1\omega_1}$ much like (3) above.

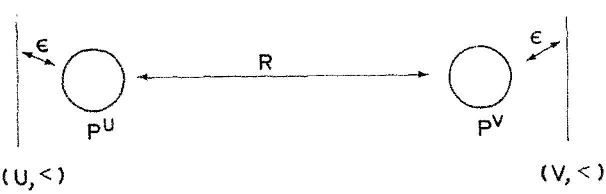


Fig. 1

We now show that for any $\mathfrak{A}$ satisfying ψ,

$$\mathfrak{A} \models \varphi' \quad \text{iff} \quad (U^{\mathfrak{A}}, <^{\mathfrak{A}} \restriction U^{\mathfrak{A}}) \cong (V^{\mathfrak{A}}, <^{\mathfrak{A}} \restriction V^{\mathfrak{A}}). \tag{*}$$

Suppose first that $j:(U,<) \cong (V,<)$ (abbreviating somewhat). Then

$$\begin{aligned}\{s: R(s\cap U, s\cap V)\} &= \{s:(s\cap U, < \restriction s\cap U) \cong (s\cap V, < \restriction s\cap V)\}\\ &\supset \{s: j''(s\cap U) = s\cap V\},\end{aligned}$$

which is cub.

For the other direction, suppose $\mathfrak{A} \models \psi \wedge \varphi'$, but that $(U,<) \not\cong (V,<)$. Without loss of generality let j map $(U,<)$ isomorphically onto a proper initial segment of $(V,<)$, and let c be the $<$-least element of $V\backslash\text{image}(j)$. Then it's easy to see that

$$\mathfrak{A} \models aas\, R(s\cap U, s\cap V)$$

(since $\mathfrak{A} \models \varphi$);

$$\mathfrak{A} \models aas\, (c \in s);$$

and

$$\mathfrak{A} \models aas\, R(s\cap U, s\cap \{x: x<c\}),$$

the latter following just as the other direction of this implication did. Intersecting these cub sets and using the definition of R given by condition (4) of ψ,

$$\begin{aligned}\mathfrak{A} \models aas\, [&(s\cap U,<) \cong (s\cap V,<) \wedge c\in s \wedge \\ &(s\cap U,<) \cong (s\cap \{x: x<c\},<)].\end{aligned}$$

But any such s would have $s\cap V$ isomorphic to a proper initial segment of itself (since $c\in s\cap V$), a contradiction.

The proof that no sentence φ' of $L_{\infty\omega}$ can satisfy (*) is much like the corresponding proof in Malitz [22]. Let $\lambda \geq \omega_1$ be sufficiently large so that φ' is in $L^{\lambda}_{\infty\omega}$, the set of formulas of $L_{\infty\omega}$ of quantifier rank $<\lambda$. Let κ be such that every sentence of $L^{\lambda}_{\infty\omega}$ is equivalent to one of $L_{\kappa\kappa}$. Let α be large enough so that $(\alpha, P_{\omega_1}(\alpha), \in)$ has a proper $L_{\kappa\kappa}$-elementary submodel $(\beta, P_{\omega_1}(\beta), \in)$. Then we define $U^{\mathfrak{A}} = V^{\mathfrak{A}} = U^{\mathfrak{B}} = \alpha$ and $V^{\mathfrak{B}} = \beta$. $<$ is defined from the usual ordering on ordinals, where $< = < \restriction U \cup < \restriction V$ in each model $\mathfrak{A}, \mathfrak{B}$. P^U and P^V are defined to be $P_{\omega_1}(U)$ and $P_{\omega_1}(V)$ in each model. Then by well-known methods using back-and-forth systems, one can easily see that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent with respect to $L^{\lambda}_{\infty\omega}$. Moreover, we can use (4) in the definition of ψ to define $R^{\mathfrak{A}}$ and $R^{\mathfrak{B}}$; then $(\mathfrak{A}, R^{\mathfrak{A}}) \equiv_{L^{\lambda}_{\infty\omega}} (\mathfrak{B}, R^{\mathfrak{B}})$ since R has the same definition in $\mathfrak{A}$ as it has in $\mathfrak{B}$. Also, each of these structures satisfies ψ. But then $(\mathfrak{A}, R^{\mathfrak{A}}) \models \varphi'$ and $(\mathfrak{B}, R^{\mathfrak{B}}) \models \neg\varphi'$, by (*). This contradicts the $L^{\lambda}_{\infty\omega}$-equivalence of $(\mathfrak{A}, R^{\mathfrak{A}})$ and $(\mathfrak{B}, R^{\mathfrak{B}})$, since φ' is in $L^{\lambda}_{\infty\omega}$.

6. A proof theory for $L(aa)$

This section can be read after Section 1 and with little detailed knowledge of proof theory.

Alan Anderson often argued that every reasonable formal system has both a Hilbert-style notion of proof (like that given in Section 1) and a Gentzen-style notion of proof. While this may overstate the case a bit, it is certainly true that a Gentzen-style approach, with its emphasis on rules, rather than on axioms, lays bare the laws of thought inherent in any given logic in a way not done by a Hilbert-style system. Thus it is satisfying that stationary logic does have a reasonable Gentzen-style complete set of rules and, moreover, that one can prove a Cut-elimination Theorem for it.

The cut-free system has the usual subformula property so that we obtain, as corollaries, Gentzen systems for L(Q) and the intermediate logic L^{pos}.

The crucial rule in our Gentzen system for L(aa) is the rule ($s\vdash$) below. However, as far as L(Q) and L^{pos} are concerned, there is no real way to use the full power of this rule. A weaker rule wk-($s\vdash$) suffices. Considering this rule, we are led in the next section to the study of other meanings of the quantifier "almost all." Thus, for a change, there is a real interaction between model theory and proof theory, the interaction going both ways.

We begin by restricting ourselves to the finitary case and assume that our language L has no function or constant symbols. We will discuss the extensions later. We treat equality as a non-logical symbol. We assume in this section that all logical symbols are defined in terms of the primitives $\wedge$, $\neg$, $\forall$, aa.

We use $\Gamma, \Delta', \Gamma, \Delta', \Gamma_1, \ldots$ to range over finite sets of L(aa) formulas. We write Γ, φ for $\Gamma \cup \{\varphi\}$ without assuming that $\varphi \notin \Gamma$. We write Γ, Δ for $\Gamma \cup \Delta$.

In ordinary logic a sequent is simply an ordered pair $\langle \Gamma, \Delta \rangle$, written $\Gamma \vdash \Delta$ since all the free variables are treated equally. In stationary logic, however, we must somehow set up a system which allows us to derive

$$\mathsf{aa}s_0\, \mathsf{aa}s_1\, (s_0 \subseteq s_1),$$

which is valid, but which does not allow us to derive

$$\mathsf{aa}s_1\, \mathsf{aa}s_0\, (s_0 \subseteq s_1),$$

which only holds in countable models. Thus we define a *sequent for* L(aa) as a triple $\langle q, \Gamma, \Delta \rangle$ where q is a finite *sequence* (not set) of variables of both kinds, first- and second-order. We write a sequent $\langle q, \Gamma, \Delta \rangle$ as $q: \Gamma \vdash \Delta$. A variable is *completely free* in $q: \Gamma \vdash \Delta$ if it does not appear in q.

To explain the intuitive meaning of $q: \Gamma \vdash \Delta$, write $\tilde{q}$ for the result of putting $\forall$ in front of each first-order variable x_i, aa in front of each second-order variable s_i. A derivation of $q{:}4\Gamma \vdash \Delta$ is intended to demonstrate that, for all structures $\mathfrak{M}$ and *all* assignments of the *completely* free variables, $\mathfrak{M} \models \tilde{q}(\bigwedge\Gamma \to \bigvee\Delta)$. Thus, we call a sequent $q: \Gamma \vdash \Delta$ *valid* if this is the case; that is, if for all $\mathfrak{M}$ and *all* assignments to the *completely* free variables, $\mathfrak{M} \models \tilde{q}(\bigwedge\Gamma \to \bigvee\Delta)$. In stating our axioms and rules, we must make sure that they preserve validity. This will be verified in the proof of the Soundness Theorem.

Axioms.

(Ax) $\quad q: \Gamma, \varphi \vdash \Delta, \varphi$,

$(\vdash s)$ $\quad q: \Gamma \vdash \Delta, s(x)$.

In the Axiom $(\vdash s)$, it is required that s occur in q and that x does not occur after s in the sequence q.

Rules.

$$(\wedge\vdash) \qquad \frac{q: \Gamma, \varphi, \psi \vdash \Delta}{q: \Gamma, \varphi \wedge \psi \vdash \Delta},$$

$$(\vdash\wedge) \qquad \frac{q: \Gamma \vdash \Delta, \varphi \quad q: \Gamma \vdash \Delta, \psi}{q: \Gamma \vdash \Delta, (\varphi \wedge \psi)},$$

$$(\neg\vdash) \qquad \frac{q: \Gamma \vdash \Delta, \varphi}{q: \Gamma, \neg\varphi \vdash \Delta},$$

$$(\vdash\neg) \qquad \frac{q: \Gamma, \varphi \vdash \Delta}{q: \Gamma \vdash \Delta, \neg\varphi},$$

$$(\forall\vdash) \qquad \frac{q: \Gamma, \varphi(x) \vdash \Delta}{q: \Gamma, \forall y\, \varphi(y) \vdash \Delta},$$

$$(\vdash\forall) \qquad \frac{q: \Gamma \vdash \Delta, \varphi(x)}{q: \Gamma \vdash \Delta, \forall y\, \varphi(y)}.$$

In the Rule $(\vdash\forall)$, it is required that x not be free in $\Gamma \cup \Delta$ *and* that x be the last variable in the string q to be free in $\Gamma \cup \Delta \cup \{\varphi(x)\}$.

$$(\mathrm{aa}\vdash) \qquad \frac{q: \Gamma, \varphi(s') \vdash \Delta}{q: \Gamma, \mathrm{aa}s\, \varphi(s) \vdash \Delta},$$

$$(\vdash\mathrm{aa}) \qquad \frac{q: \Gamma \vdash \Delta, \varphi(s')}{q: \Gamma \vdash \Delta, \mathrm{aa}s\, \varphi(s)}.$$

In both these rules it is required that s' be the last variable in q to be free in $\Gamma \cup \Delta \cup \{\varphi(s')\}$. In Rule $(\vdash\mathrm{aa})$ it is also required that s' is not free in $\Gamma \cup \Delta$.

We now come to the most important rule, $(s\vdash)$, and the weaker form wk-$(s\vdash)$. The Rule $(s\vdash)$ is valid by the Diagonal Intersection Property of the cub filter.

$$(s\vdash) \qquad \frac{q_0 x q_1 q_2: \Gamma \vdash \Delta}{q_0 q_1 x q_2: \Gamma, s(x) \vdash \Delta},$$

where s is a second-order variable not in q_2 and, if it is in q_1, then it must be the very first variable in q_1. (Other possibilities are where s is in q_0 or else s in none of q_0, q_1, q_2.)

The rule wk-($s\vdash$) is just like the rule ($s\vdash$) except that s is allowed to be in neither q_1 nor q_2. To check its validity one only needs the countable completeness of the cub filter.

We need a trivial variable rule.

(D) $$\frac{q\colon \Gamma\vdash\Delta}{q_0\colon \Gamma\vdash\Delta}.$$

In the rule (D) ("D" is for "delete"), q_0 results from q by deleting some variables in q not free in $\Gamma\cup\Delta$.

Finally, we state the Gentzen-style analogue of modus ponens, the Cut Rule.

(Cut) $$\frac{q\colon \Gamma,\varphi\vdash\Delta \quad q\colon \Gamma\vdash\Delta,\varphi}{q\colon \Gamma\vdash\Delta}.$$

6.1. Definition. A sequent $q\colon \Gamma\vdash\Delta$ is *$L(aa)$-derivable* if it is in the smallest set of sequents containing the Axioms (Ax), ($\vdash s$) and closed under the rules above. It is *$L(aa)$-derivable without cut* if it is in the smallest set containing the axioms and closed under all the rules except the cut rule.

We make obvious modifications on this definition. For example, we say that a sequent is *L^{pos}-derivable* if it is in the smallest set Y of sequents $q\colon \Gamma\vdash\Delta$ with $\Gamma\cup\Delta\subseteq L^{pos}$ such that Y contains the axioms (Ax), ($\vdash s$) and is closed under the rules above. We say that $q\colon \Gamma\vdash\Delta$ is $L(aa)$-derivable using wk-($s\vdash$) if every instance of the rule ($s\vdash$) is actually an instance of the rule wk-($s\vdash$).

6.2. Lemma. *The set of $L(aa)$-derivable sequents is closed under the following rules. So is the set of sequents $L(aa)$-derivable without cut. The same holds with Rule ($s\vdash$) replaced by Rule wk-($s\vdash$).*

(Add) $$\frac{q\colon \Gamma\vdash\Delta}{q_0q\colon \Gamma,\Gamma'\vdash\Delta,\Delta'}$$ (q_0q *a sequence of distinct variables*),

(ML) $$\frac{q_0q_1xq_2\colon \Gamma\vdash\Delta}{q_0xq_1q_2\colon \Gamma\vdash\Delta},$$

(ML′) $$\frac{q_1xq_2\colon \Gamma\vdash\Delta}{q_1q_2\colon \Gamma\vdash\Delta},$$

($\vdash\vee$) $$\frac{q\colon \Gamma\vdash\Delta,\varphi,\psi}{q\colon \Gamma\vdash\Delta,(\varphi\vee\psi)},$$

($\vee\vdash$) $$\frac{q\colon \Gamma,\varphi\vdash\Delta \quad q\colon \Gamma,\psi\vdash\Delta}{q\colon \Gamma,(\varphi\vee\psi)\vdash\Delta},$$

($\vdash\rightarrow$) $$\frac{q\colon \Gamma,\varphi\vdash\Delta,\psi}{q\colon \Gamma\vdash\Delta,(\varphi\rightarrow\psi)},$$

($\rightarrow\vdash$) $$\frac{q\colon \Gamma,\varphi\vdash\Delta \quad q\colon \Gamma\vdash\Delta,\psi}{q\colon \Gamma,\psi\rightarrow\varphi\vdash\Delta}.$$

Proof. The first three are by induction on the notion of derivable, and are easy. The others follow from the definitions of $\bigvee$, $\rightarrow$ from $\neg$, $\bigwedge$. □

Given these derived rules, let's write down some derivations of some formulas that will be of use to us.

6.3. Examples. (i) A derivation of Axiom A2, $\mathsf{aa}s\,(s_0 \subseteq s)$.

$$\begin{array}{ll}
s: \vdash s(x) & (\vdash s),\\
x, s: \vdash s(x) & (\text{Add}),\\
s, x: s_0(x) \vdash s(x) & \text{wk-}(s\vdash),\\
s, x: \vdash s_0(x) \rightarrow s(x) & (\vdash \rightarrow),\\
s: \vdash \forall x\,(s_0(x) \rightarrow s(x)) & (\vdash\forall), (\text{D}),\\
\vdash \mathsf{aa}s\,\forall x\,(s_0(x) \rightarrow s(x)) & (\vdash \mathsf{aa}), (\text{D}).
\end{array}$$

This derivation is cut-free and uses only wk-$(s\vdash)$.

(ii) A derivation of $\mathsf{aa}s\,(\varphi \rightarrow \psi)$, $\mathsf{aa}s\,\varphi \vdash \mathsf{aa}s\,\psi$. First, one uses the propositional rules to get a derivation of $s: (\varphi \rightarrow \psi),\ \varphi \vdash \psi$. Then:

$$\begin{array}{ll}
s: (\varphi \rightarrow \psi),\ \varphi \vdash \psi, & \\
s: \mathsf{aa}s\,(\varphi \rightarrow \psi),\ \mathsf{aa}s\,\varphi \vdash \psi & (\mathsf{aa}\vdash)\ \text{twice},\\
\mathsf{aa}s\,(\varphi \rightarrow \psi),\ \mathsf{aa}s\,\varphi \vdash \mathsf{aa}s\,\psi & (\vdash \mathsf{aa}), (\text{D}).
\end{array}$$

(iii) A derivation of $\forall x \in s_0\, \mathsf{aa}s\, \varphi(x, s) \vdash \mathsf{aa}s\, \forall x \in s_0\, \varphi(x, s)$ using only wk-$(s\vdash)$. To derive this, we first derive

$$x, s: \mathsf{aa}s\, \varphi(x, s),\ s_0(x) \vdash \varphi(x, s),$$

which is easy. This, with $x, s: s_0(x) \vdash s_0(x)$, yields, by $(\rightarrow\vdash)$,

$$x, s: s_0(x) \rightarrow \mathsf{aa}s\, \varphi(x, s),\ s_0(x) \vdash \varphi(x, s).$$

Now we use the rule wk-$(s\vdash)$, with the variable s_0, to obtain the first line of the following:

$$\begin{array}{ll}
s, x: s_0(x) \rightarrow \mathsf{aa}s\, \varphi(x, s),\ s_0(x) \vdash \varphi(x, s), & \\
s, x: \forall x\,(s_0(x) \rightarrow \mathsf{aa}s\, \varphi(x, s)) \vdash s_0(x) \rightarrow \varphi(x, s) & (\vdash \rightarrow), (\forall\vdash),\\
s: \forall x \in s_0\, \mathsf{aa}s\, \varphi(x, s) \vdash \forall x\,(s_0(x) \rightarrow \varphi(x, s)) & (\vdash\forall), (\text{D}),\\
\forall x \in s_0\, \mathsf{aa}s\, \varphi(x, s) \vdash \mathsf{aa}s\, \forall x \in s_0\, \varphi(x, s) & (\vdash \mathsf{aa}), (\text{D}).
\end{array}$$

(iv) We contrast (iii) with a derivation of Axiom A5,

$$\forall x\, \mathsf{aa}s\, \varphi(x, s) \vdash \mathsf{aa}s\, \forall x \in s\, \varphi(x, s).$$

To derive this we must use the full Rule ($s\vdash$), not just wk-($s\vdash$).

$$\begin{array}{ll} x, s: \varphi(x, s) \vdash \varphi(x, s) & (\mathrm{Ax}), \\ x, s: \forall x\, \mathsf{aa} s\, \varphi(x, s) \vdash \varphi(x, s) & (\mathsf{aa}\vdash), \text{ then } (\forall\vdash), \\ s, x: \forall x\, \mathsf{aa} s\, \varphi(x, s), s(x) \vdash \varphi(x, s) & (s\vdash), \\ s: \forall x\, \mathsf{aa} s\, \varphi(x, s) \vdash \forall x\, (s(x) \rightarrow \varphi(x)) & (\vdash\rightarrow), (\vdash\forall), (\mathrm{D}), \\ \forall x\, \mathsf{aa} s\, \varphi(x, s) \vdash \mathsf{aa} s\, \forall x \in s\, \varphi(x) & (\vdash\mathsf{aa}), (\mathrm{D}). \end{array}$$

We now state the main results for this system.

6.4. Soundness Theorem. *Every* L(aa)*-derivable sequent is valid.*

6.5. Completeness Theorem. *Every valid sequent is* L(aa)*-derivable.*

6.6. Cut-elimination Theorem. *Every* L(aa)*-derivable sequent is* L(aa)*-derivable without cut. The same holds with Rule* ($s\vdash$) *replaced by Rule* wk-($s\vdash$).

These are proved in Sections 6.10–6.12 below.

As corollaries of these three results we obtain complete, cut-free Gentzen systems for L(Q) and $\mathrm{L}^{\mathrm{pos}}$.

6.7. Corollary. *Let* $q: \Gamma \vdash \Delta$ *be an* L(Q) (*or* $\mathrm{L}^{\mathrm{pos}}$) *sequent. Then* $q: \Gamma \vdash \Delta$ *is valid iff it is* L(Q)*-derivable* ($\mathrm{L}^{\mathrm{pos}}$*-derivable*) *without cut.*

Proof. This is an immediate consequence of Theorems 6.4, 6.5 and 6.6 together with the observation that a cut-free derivation of an L(Q)-sequent can only involve L(Q)-formulas, by the subformula property. Similarly for $\mathrm{L}^{\mathrm{pos}}$. □

If one examines proofs of sequents from L(Q) or $\mathrm{L}^{\mathrm{pos}}$, e.g. Example 6.8 below, one sees that one never really uses the full Rule ($s\vdash$), rather, only the weaker Rule wk-($s\vdash$). One can give a direct proof that any $\mathrm{L}^{\mathrm{pos}}$-derivation of an $\mathrm{L}^{\mathrm{pos}}$ sequent $q: \Gamma \vdash \Delta$ can be transformed into one that only uses the weaker rule. Instead, we will ask a more general question. *Exactly which valid sequents of* L(aa) *are provable using the weaker rule?* This leads us to the countably complete filter logic $\mathrm{L}^{\mathrm{Flt}}(\mathsf{aa})$ of the next section.

6.8. Example. A cut free L(Q)-derivation of Keisler's crucial axiom for L(Q). Our aim is to derive:

$$\forall x\, \exists^{\leq\aleph_0} y\, \varphi(x, y), \exists^{\leq\aleph_0} x\, \exists y\, \varphi(x, y) \vdash \exists^{\leq\aleph_0} y\, \exists x\, \varphi(x, y).$$

This is shorthand for

$$\begin{array}{l} \forall x\, \mathsf{aa} s\, \forall y\, [\varphi(x, y) \rightarrow s(y)], \mathsf{aa} s\, \forall x\, [\exists y\, \varphi(x, y) \rightarrow s(x)] \\ \quad \vdash \mathsf{aa} s\, \forall x\, \forall y\, [\varphi(x, y) \rightarrow s(y)]. \end{array}$$

The trick in all such proofs is to fix a countable s_0 on the left at the first opportunity. For us, this means we derive the above from the following sequent

$$s_0 : \forall x\ \mathsf{aa}s\ \forall y\,[\varphi(x,y) \rightarrow s(y)], \forall x\,[\exists y\,\varphi(x,y) \rightarrow s_0(x)]$$
$$\vdash \mathsf{aa}s\ \forall x\ \forall y\,[\varphi(x,y) \rightarrow s(y)]$$

by using Rules ($\mathsf{aa}\vdash$) and (D). This, in turn, comes from

$$s_0, s, x_0 : \mathsf{aa}s\ \forall y\,[\varphi(x_0,y) \rightarrow s(y)], \exists y\,\varphi(x_0,y) \rightarrow s_0(x_0)$$
$$\vdash \forall y\,[\varphi(x_0,y) \rightarrow s(y)]$$

by using Rule ($\forall\vdash$) twice, then Rule ($\vdash\forall$) to get rid of x_0, then Rule ($\vdash\mathsf{aa}$) on the variable s, then Rule (D). This is derived from the following two sequents by Rule ($\rightarrow\vdash$).

$$s_0, s, x_0 : \mathsf{aa}s\ \forall y\,[\varphi(x_0,y) \rightarrow s(y)] \vdash \exists y\,\varphi(x_0,y), \forall y\,[\varphi(x_0,y) \rightarrow s(y)],$$

$$s_0, s, x_0 : \mathsf{aa}s\ \forall y\,[\varphi(x_0,y) \rightarrow s(y)], s_0(x_0) \vdash \forall y\,[\varphi(x_0,y) \rightarrow s(y)].$$

The first of these is just first-order logic. The second comes from the following by the Rule wk-($s\vdash$).

$$s_0, x_0 s : \mathsf{aa}s\ \forall y\,[\varphi(x_0,y) \rightarrow s(y)] \vdash \forall y\,[\varphi(x_0,y) \rightarrow s(y)].$$

This, of course, is just the result of applying Rule ($\mathsf{aa}\vdash$) to the Axiom

$$s_0, x_0, s : \forall y\,[\varphi(x_0,y) \rightarrow s(y)] \vdash \forall y\,[\varphi(x_0,y) \rightarrow s(y)]. \quad \square$$

6.9. Remark. Before turning to the proofs of Theorems 6.4–6.6, we pause to discuss the question of the interpolation Theorem, or Craig's Theorem. Shelah pointed out over breakfast with the first author that considerations involving Aronszajn trees show that interpolation fails for L(aa). His proof, to appear in Makowsky–Shelah [19] shows that there are sentences $\varphi(X)$, $\psi(f, <, \mathbb{Q})$ of L(Q) such that $\varphi(X) \rightarrow \neg\psi(f, <, \mathbb{Q})$ is valid, but there is no interpolant even in $\mathrm{L}_{\omega_1\omega}(\mathsf{aa})$. ($\varphi(X)$ says X is an uncountable branch through a tree, $\psi(f, <, \mathbb{Q})$ says that f is an order preserving map of the tree into the rationals $\langle \mathbb{Q}, < \rangle$.) A different proof of this, one which uses the ideas of Section 3, will appear in Kaufmann [14]. If one takes the trouble to see where the usual proof of "cut elimination $\Rightarrow$ interpolation" breaks down, it is in the Rule ($s\vdash$), or even with the Rule wk-($s\vdash$). The lack of symmetry between the Axiom ($\vdash s$) and the Rule ($s\vdash$) seems to block interpolation.

6.10. Proof of the Soundness Theorem. This is by induction on the notion of L(aa)-derivable. One shows that all the axioms are valid and that the rules preserve validity. We list below the non-trivial axioms and rules next to the corresponding properties of the cub filter.

$(\vdash s)$	$\{s \mid x \in s\}$ is cub.
$(\vdash\bigwedge)$	The cub filter is closed under intersection.
$(\mathsf{aa}\vdash)$	The cub filter is indeed a filter.
wk-$(s\vdash)$	The cub filter is countably complete.
$(s\vdash)$	Diagonal Intersection Property.
(Cut)	The cub filter is closed under intersection.

We leave it to the reader to check the details.

6.11. Proof of the Completeness Theorem. Call a formula φ derivable if $\vdash\varphi$ is derivable.

(1) Every instance of a propositional tautology is derivable.
In fact, you can do (1) with q empty using just Rules (Ax), $(\bigwedge\vdash)$, $(\vdash\bigwedge)$, $(\neg\vdash)$, $(\vdash\neg)$ by the usual proof. See Barwise [3] for example.

(2) $\forall x\, \varphi(x) \rightarrow \varphi(y)$ is derivable, for all φ.
You just apply Rules $(\forall\vdash)$ to $\varphi(y)\vdash\varphi(y)$.

(3) Axioms (A0)–(A5) of Section 1 are derivable.
We've done the more interesting cases in Example 6.3.

(4) (*Modus ponens*) If $\varphi, (\varphi \rightarrow \psi)$ are derivable, so is ψ.
This follows from cut. For we have (using Rule $(\rightarrow\vdash)$) $\varphi, (\varphi \rightarrow \psi)\vdash\psi$; $\vdash\varphi$; and $\vdash(\varphi \rightarrow \psi)$ derivable; two cuts give us $\vdash\psi$.

(5) (*Generalization*) If $\eta \rightarrow \varphi(v)$ is derivable, and v is not free in η then $\eta \rightarrow \forall v\, \varphi(v)$ is derivable.
First, using $\eta \rightarrow \varphi(v)$, $\eta\vdash\varphi(v)$ and then cut, we get $\eta\vdash\varphi(v)$. Then use Rule $(\vdash\forall)$ and then Rule $(\vdash\rightarrow)$.

(6) (aa*-generalization*) If $\eta \rightarrow \varphi(s)$ is derivable, and s is not free in η then $\eta \rightarrow \mathsf{aa}s\, \varphi(s)$ is derivable.
Just like (5), except you must use Rule (Add) to go from $\eta\vdash\varphi(s)$ to $s\colon \eta\vdash\varphi(s)$, then on to $\eta\vdash\mathsf{aa}s\, \varphi(s)$.

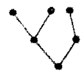
If we combine (1)–(6) and the Completeness Theorem of Section 1, we see that if φ is valid then $\vdash\varphi$ is derivable. Thus, if $q\colon \Gamma\vdash\Delta$ is a valid sequent, we have $\vdash\tilde{q}(\bigwedge\Gamma \rightarrow \bigvee\Delta)$ is derivable. But what we want is $q\colon \Gamma\vdash\Delta$. This follows from (7) and several applications of cut.

(7) The following is derivable: $q\colon \tilde{q}\varphi\vdash\varphi$.
The proof is by induction on the length of q. We do a typical case:

$$s_1, x, s_2\colon \mathsf{aa}s_1\, \forall x\, \mathsf{aa}s_2\, \varphi\vdash\varphi.$$

Begin with $\varphi\vdash\varphi$, then apply Rule $(\mathsf{aa}\vdash)$ to s_2, then Rule $(\forall\vdash)$ to x, then Rule $(\mathsf{aa}\vdash)$ to s_1. $\square$

6.12. Proof of the Cut Elimination Theorem. To prove the theorem we will need a notion of $L(\mathfrak{a})$-*derivation* to go along with the notion of $L(\mathfrak{a})$-derivable used above. A derivation $\mathscr{D}$ of $q\colon \Gamma \vdash \Delta$ is a finite tree like



of sequents which has axioms at the upper nodes and such that each node not at the top follows from the one or two nodes directly above it by one of the rules, and such that $q\colon \Gamma \vdash \Delta$ is the unique bottom node. We take it that anyone who has read this far can formalize this imprecise definition. (Otherwise look in Feferman [9], Takeuti [27], or Barwise [1].) We prove the result for the system with Rule ($s\vdash$) and the one with Rule wk-($s\vdash$) at the same time. {Any remarks needed for the weaker system are put in braces.} The theorem follows by induction from the following lemma.

6.13. Lemma. *If $q_0\colon \Gamma_0, \varphi \vdash \Delta_0$ and $q_1\colon \Gamma_1 \vdash \Delta_1, \varphi$ have cut-free derivations $\mathscr{D}_0$ and $\mathscr{D}_1$ respectively, and if q results from each q_i by deleting some variables not free in $\Gamma_0 \cup \Gamma_1 \cup \Delta_0 \cup \Delta_1 \cup \{\varphi\}$, then there is a cut-free derivation $\mathscr{D}$ of*

$$q\colon \Gamma_0, \Gamma_1 \vdash \Delta_0 \Delta_1.$$

We refer to this rule as cut in the proof that follows. Notice that it is a derived rule of the system with cut, using Rules (Add), (D), and what we called (Cut) before.

Proof. The proof of this lemma follows the usual pattern of cut elimination arguments. We will set up the double induction and outline the proof, filling in details only in those cases where unfamiliar considerations arise from $\mathfrak{a}$.

Let $c(\varphi)$ be the *complexity* of φ measured by length, number of symbols, some such thing let $r(\mathscr{D}, \varphi)$, *the rank of φ in $\mathscr{D}$*, be defined for cut-free $\mathscr{D}$ as follows, by induction. If $\mathscr{D}$ is an axiom $q\colon \Gamma \vdash \Delta$ then $r(\mathscr{D}, \varphi) = 1$ for all $\varphi \in \Gamma \cup \Delta$, $r(\mathscr{D}, \varphi) = 0$ for all other φ. If $\mathscr{D}$ ends in a step

$$\begin{array}{c} \mathscr{D}_0 \\ \downarrow \\ q_0\colon \Gamma_0 \vdash \Delta_0 \\ \hline q_1\colon \Gamma_1 \vdash \Delta_1 \end{array},$$

then for $\varphi \in \Gamma_1 \cup \Delta_1$, $r(\mathscr{D}, \varphi) = r(\mathscr{D}_0, \varphi) + 1$. For $\varphi \notin \Gamma_1 \cup \Delta_1$, $r(\mathscr{D}, \varphi) = 0$. If $\mathscr{D}$ ends in

$$\begin{array}{c} \begin{array}{cc} \mathscr{D}_0 & \mathscr{D}_1 \\ \downarrow & \downarrow \\ S_0 & S_1 \end{array} \\ \hline q\colon \Gamma \vdash \Delta \end{array},$$

then for $\varphi \in \Gamma \cup \Delta$, $r(\mathscr{D}, \varphi) = r(\mathscr{D}_0, \varphi) + r(\mathscr{D}_1, \varphi) + 1$. For $\varphi \notin \Gamma \cup \Delta$, $r(\mathscr{D}, \varphi) = 0$. Thus $r(\mathscr{D}, \varphi)$ is the size of the largest subtree of sequents in which φ occurs. If $\mathscr{D}$ is as in Lemma 6.13, then define $r(\mathscr{D}, \varphi) = r(\mathscr{D}_0, \varphi) + r(\mathscr{D}_1, \varphi) + 1$.

We will prove that for all natural numbers c and all natural numbers r, if $\mathscr{D}$ is any derivation which ends with a cut of the form

$$\frac{\begin{array}{cc} \mathscr{D}_0 & \mathscr{D}_1 \\ \downarrow & \downarrow \\ q_0\colon \Gamma_0, \varphi \vdash \Delta_0 & q_1\colon \Gamma_1 \vdash \Delta_1, \varphi \end{array}}{q\colon \Gamma_0, \Gamma_1 \vdash \Delta_0, \Delta_1},$$

where $\mathscr{D}_0, \mathscr{D}_1$ and the sequents are as in the lemma, if $c(\varphi) = c$ and $r(\mathscr{D}, \varphi) = r$ then there is a cut-free derivation of $q\colon \Gamma_0, \Gamma_1 \vdash \Delta_0, \Delta_1$.

The proof is by induction on c and then on r. Thus we may fix $\varphi, \mathscr{D}$ as above and assume that the lemma holds for all $\varphi^*, \mathscr{D}^*$ with $c(\varphi^*) < c = c(\varphi)$, or with $c(\varphi^*) \leqslant c$ and $r(\mathscr{D}^*, \varphi^*) < r = r(\mathscr{D}, \varphi)$. Note that $r \geqslant 3$ since φ occurs in the end sequents of both $\mathscr{D}_0$ and $\mathscr{D}_1$.

In the following we let $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Delta = \Delta_0 \cup \Delta_1$. We let S_0 be the sequent $q_0\colon \Gamma_0, \varphi \vdash \Delta_0$, S_1 be the sequent $q_1\colon \Gamma_1 \vdash \Delta_1, \varphi$ and S be the end sequent $q\colon \Gamma \vdash \Delta$. Thus $\mathscr{D}$ ends with the cut

$$\frac{\begin{array}{cc} \mathscr{D}_0 & \mathscr{D}_1 \\ \downarrow & \downarrow \\ S_0 & S_1 \end{array}}{S},$$

where both $\mathscr{D}_0$ and $\mathscr{D}_1$ are cut-free. The proof breaks into cases (A) where $r = 3$ and (B) where $r > 3$.

Case (A). $r = 3$. This means that φ only appears in the end sequents S_0 and S_1, not from the sequents just above them in $\mathscr{D}_0$ and $\mathscr{D}_1$. Thus each S_i $(i = 0, 1)$ is either an axiom or else φ was introduced into S_i in the very last step of $\mathscr{D}_i$. This gives us three subcases:

(A1) S_0 is an axiom,

(A2) S_0 is not an axiom and S_1 is an axiom,

(A3) neither S_0 nor S_1 is an axiom.

(A1) S_0 is an axiom. If $q_0\colon \Gamma_0 \vdash \Delta_0$ is still an axiom, we can use Rules (D), and (Add) to get S. Otherwise, the appearance of φ on the left must be crucial, so S_0 must be of the form $q_0\colon \Gamma_0, \varphi \vdash \Delta_0$ where $\varphi \in \Delta_0$. But then $\varphi \in \Delta = \Delta_0 \cup \Delta_1$ so we obtain S by weakening S_1, just apply Rules (D) and (Add).

(A2) S_0 is not an axiom and S_1 is an axiom. If S is an (Ax)-Axiom, or if $q_1\colon \Gamma_1 \vdash \Delta_1$ is still an axiom, then proceed as in (A1). The other possibility which might arise at first glance is where φ is $s(x)$ and S_1 is of the form $q_1\colon \Gamma_1 \vdash \Delta_1, s(x)$, where q_1 is a string of variables with s in it and x not after s in q_1. Look at $\mathscr{D}_0$ to see how S_0 was derived. Since $r = 3$, $r(\mathscr{D}_0, s(x)) = 1$ so the last step in $\mathscr{D}_0$ must introduce $s(x)$ by Rule $(s\vdash)$ {or wk-$(s\vdash)$}:

$$\frac{q_0\colon \Gamma_0 \vdash \Delta_0}{q_0\colon \Gamma_0, s(x) \vdash \Delta_0}.$$

But then either s is not in q_0 or else x is after s in q_0. Thus, there is no way to get q from both q_0 and q_1 by deleting variables not free in $\Gamma \cup \Delta \cup \{s(x)\}$. In other words, there is no way to use Axiom $(\vdash s)$ to cut out an $s(x)$ on the left just after it is introduced by Rule $(s\vdash)$ {or wk$(s\vdash)$}.

(A3) Neither S_0 nor S_1 is an axiom. Thus both final inferences in $\mathscr{D}_0$ and $\mathscr{D}_1$ introduce φ. Furthermore φ cannot be of the form $s(x)$ since there is no way to introduce s on the right except as an axiom. Thus φ is not an atomic formula. There are four cases depending on whether φ begins

(a) with $\neg$,
(b) with $\bigwedge$,
(c) with $\forall$ or
(d) with aa.

Cases (a) and (b) are like the usual proofs. You apply the induction hypothesis on $c(\varphi)$ to the next to the last sequents in $\mathscr{D}_0, \mathscr{D}_1$ to obtain the desired result.

(A3c) Suppose φ is $\forall x\, \psi(x)$. The derivations $\mathscr{D}_0, \mathscr{D}_1$ end as follows.

$\mathscr{D}_0$ ends with an application of Rule $(\forall\vdash)$.

$$\frac{q_0\colon \Gamma_0, \psi(y) \vdash \Delta_0}{q_0\colon \Gamma_0, \forall x\, \psi(x) \vdash \Delta_0}.$$

$\mathscr{D}_1$ ends with Rule $(\vdash\forall)$.

$$\frac{q_1\colon \Gamma_1 \vdash \Delta_1, \psi(z)}{q_1\colon \Gamma_1 \vdash \Delta_1, \forall x\, \psi(x)}.$$

Here z is the last variable in q_1 free in $\Gamma_1 \cup \Delta_1 \cup \{\psi(z)\}$ and z is not free in $\Gamma_1 \cup \Delta_1$. We may assume that y does not occur after z in q_1, for if it does we simply change it to a y' not in $\mathscr{D}_0$ or $\mathscr{D}_1$. The idea, as usual, is to change z to y and then apply cut by the induction hypothesis on complexity. But we must keep the string q_1 the same so that q_0 and q_1 will give rise to q. Thus we need the following lemma.

Sublemma. *If $q, z, q'\colon \Gamma(z) \vdash \Delta(z)$ has a cut-free derivation $\mathscr{D}$, and if y is not in q' then there is a cut-free derivation $\mathscr{D}'$ of $q, z, q'\colon \Gamma(y) \vdash \Delta(y)$.*

The sublemma is proved by induction on $\mathscr{D}$. The hypothesis $y \notin q'$ is needed for Axiom $(\vdash s)$.

Given the sublemma we get a derivation $\mathscr{D}'$ of $q_1\colon \Gamma_1 \vdash \Delta_1, \psi(y)$. We now apply the induction hypothesis (and possibly Rule (D)) to get $q\colon \Gamma \vdash \Delta$.

(A3d) Suppose φ is $\mathsf{aa}s\,\psi(s)$. The derivations $\mathscr{D}_0$ and $\mathscr{D}_1$ end as follows, with applications of Rules ($\mathsf{aa}\vdash$), ($\vdash\mathsf{aa}$) respectively.

$$\frac{q_0: \Gamma_0, \psi(s_0) \vdash \Delta_0}{q_0: \Gamma_0, \mathsf{aa}s\,\psi(s) \vdash \Delta_0},$$

$$\frac{q_1: \Gamma_1 \vdash \Delta_1, \psi(s_1)}{q_1: \Gamma_1 \vdash \Delta_1, \mathsf{aa}s\,\psi(s)},$$

where s_0 is the last variable in q_0 free in $\Gamma_0, \Delta_0, \psi(s_0)$ and s_1 is the last variable in q_1 free in $\Gamma_1, \Delta_1, \psi(s_1)$. Further, s_1 is not free in $\Gamma_1 \cup \Delta_1$, but it might be free in Γ_0. Δ_0 so it might or might not occur in q. If s_1 (resp. s_0) does not occur in q, then change s_0 to s_1 in $\mathscr{D}_0$ (resp. s_1 to s_0 in $\mathscr{D}_1$) and apply the induction hypothesis. Thus, we assume s_0 and s_1 both occur in q. If s_0 and s_1 are the same variable, we may apply the induction hypothesis directly. The only case left is where s_0, s_1 are distinct, and both in q, hence both in each of q_0 and q_1. If s_0 precedes s_1 in q, then apply the following sublemma to $\mathscr{D}_0$. If s_1 precedes s_0 in q then apply it to $\mathscr{D}_1$. In either case we can then apply the induction hypothesis to get $q: \Gamma \vdash \Delta$.

Sublemma. *If $q, s, q': \Gamma(s) \vdash \Delta(s)$ has a cut-free derivation $\mathscr{D}$ and no variable in q' is free in $\Gamma(s) \cup \Delta(s)$, and if $s' \in q'$ then $q, s, q': \Gamma(s') \vdash \Delta(s')$ also has a cut-free derivation.*

This is easily seen using Rule (D) and its obvious converse, and finishes case (A3d), hence case (A3), and thus case (A).

Case (B). $r > 3$. This means $r(\mathscr{D}_0, \varphi) \geqslant 2$ or $r(\mathscr{D}_1, \varphi) \geqslant 2$ or both. We assume that $\varphi \notin \Gamma_1 \cup \Delta_0$, since otherwise we may obtain S from one of S_0 or S_1 by Rule (Add).

(B1) $r(\mathscr{D}_0, \varphi) \geqslant 2$. We need to consider two possibilities here,

(a) where φ is not actively involved in the last inference, and

(b) where φ is actively involved, or as it is called in the literature, where φ is the principal formula of the last inference.

(B1a) Assume φ is not the principal formula in the last inference in $\mathscr{D}_0$. As usual, what we want to do is to apply the induction hypothesis to the derivation(s) of the next to last sequent(s) in $\mathscr{D}_0$ (with smaller rank) and then perform the last step used in $\mathscr{D}_0$ to get S. This works as long as the last step in $\mathscr{D}_0$ doesn't alter the string of variables in front. For example, suppose $\mathscr{D}_0$ ends with Rule ($\vdash\bigwedge$) applied to $(\psi_1 \wedge \psi_2)$:

$$\frac{q_0: \Gamma_0, \varphi \vdash \Delta'_0, \psi_1 \quad q_0: \Gamma_0, \varphi \vdash \Delta'_0, \psi_2}{q_0: \Gamma_0, \varphi \vdash \Delta'_0, (\psi_1 \wedge \psi_2)}.$$

The induction hypothesis gives us cut-free derivations of $q: \Gamma_0, \Gamma_1 \vdash \Delta'_0, \psi_1, \Delta_1$ and $q: \Gamma_0, \Gamma_1 \vdash \Delta'_0, \psi_2, \Delta_1$. Then we apply Rule ($\vdash\bigwedge$) to get

$$q: \Gamma_0, \Gamma_1 \vdash \Delta'_0, \Delta_1, (\psi_1 \wedge \psi_2),$$

which is S. So we need only check the rules which alter variable strings, the Rules ($s\vdash$) {or wk-($s\vdash$)} and (D). The Rule (D) causes no trouble because we have built it into cut in the lemma. So suppose $\mathscr{D}_0$ ends with

$$\frac{q', x, q'', q''': \Gamma_0', \varphi \vdash \Delta_0}{q', q'', x, q''': \Gamma_0', \varphi, s(x) \vdash \Delta_0}.$$

Let q_0' be q', x, q'', q''' and let q_1' be the result of moving x left to a corresponding spot in q_1, and define q' similarly. We apply Rule (ML) to $\mathscr{D}_1$ to get a derivation $\mathscr{D}_1'$ of $q_1': \Gamma_1 \vdash \Delta_1, \varphi$. (Actually, we need to check that we can get a $\mathscr{D}_1'$ with $r(\mathscr{D}_1, \varphi) = r(\mathscr{D}_1', \varphi)$, but there is no trouble with this.) This puts us in a position to apply the inductive hypothesis to get a cut-free derivation of $q': \Gamma_0', \Gamma_1 \vdash \Delta_0, \Delta_1$. Then apply Rule ($s\vdash$) {or wk-($s\vdash$)} to get $q: \Gamma_0', \Gamma_1, s(x) \vdash \Delta_0, \Delta_1$, which is S.

(B1b) Suppose now that φ is the principal formula in $\mathscr{D}_0$. We now have five possibilities, depending on where φ is of the form $s(x)$, $(\psi_1 \wedge \psi_2)$, $\neg\psi$, $\forall x\, \psi(x)$ or $\text{\ae} s\, \psi(s)$.

(B1bi) Suppose φ is $s(x)$ so that we are trying to eliminate the final cut from the following picture. Recall that $r(\mathscr{D}_0, s(x)) > 1$.

$$\frac{\dfrac{\begin{array}{c}\mathscr{D}_0'\\ \downarrow\\ q', x, q'', q''': \Gamma_0, s(x) \vdash \Delta_0\end{array}}{q'q'', x, q''': \Gamma_0, s(x) \vdash \Delta_0} \qquad q_1: \Gamma_1 \vdash \Delta_1, s(x)}{q: \Gamma_0, \Gamma_1 \vdash \Delta_0, \Delta_1}.$$

We apply the following sublemma to $\mathscr{D}_0'$ to get a derivation $\mathscr{D}_0''$ of S_0 with $r(\mathscr{D}_0'', s(x)) = r(\mathscr{D}_0', s(x)) < r(\mathscr{D}_0, s(x))$. Then we can apply the induction hypothesis.

Sublemma. *If $\mathscr{D}$ is a cut-free derivation of*

$$q', x, q'', q''': \Gamma, s(x) \vdash \Delta$$

where s is not in q''' and, if in q'', is the first variable in q'', then there is a cut-free derivation $\mathscr{D}'$ of

$$q', q'', x, q''': \Gamma, s(x) \vdash \Delta$$

with $r(\mathscr{D}, s(x)) = r(\mathscr{D}', s(x))$.

This is proved by induction on length of $\mathscr{D}$. The only interesting case is where $\mathscr{D}$ is the Axiom ($\vdash s$), in which case $\mathscr{D}'$ is an (Ax)-Axiom.

(B1bii) if φ is $\neg\psi$ or $(\psi_1 \wedge \psi_2)$, the proof is just like in the first-order case, and is simpler than the following two subcases.

(B1biii) If φ is $\forall x\, \psi(x)$ then we are faced with the following picture:

$$\frac{\dfrac{q_0: \Gamma_0, \forall x\, \psi(x), \psi(y) \vdash \Delta_0}{q_0: \Gamma_0, \forall x\, \psi(x) \vdash \Delta_0} \qquad q_1: \Gamma_1 \vdash \Delta_1, \forall x\, \psi(x)}{q: \Gamma_0, \Gamma_1 \vdash \Delta_0, \Delta_1}.$$

It is annoying, but the variable y may or may not appear in q_0, and if it appears in q_0 it may or may not appear in q. We treat the case where y is in q_0 but not in q. Thus y is not free in $\Gamma \cup \Delta$. Let q_1' be the result of putting y in q_1 in a spot compatible with q_0, and similarly with q. Let $\mathscr{D}_1'$ be a cut-free derivation of q_1': $\Gamma_1 \vdash \Delta_1, \forall x\, \psi(x)$ with $r(\mathscr{D}_1, \forall x\, \psi(x)) = r(\mathscr{D}_1', \forall x\, \psi(x))$. (This calls for another obvious sublemma, which we omit.) Apply the induction hypothesis to $\mathscr{D}_0, \mathscr{D}_1'$ to get a cut-free derivation $\mathscr{D}^*$ of q': $\Gamma, \psi(y) \vdash \Delta$. Let $\mathscr{D}^{**}$ be a cut-free derivation of q_1': $\forall x\, \psi(x) \vdash \psi(y)$ with $r(\mathscr{D}^{**}, \forall x\, \psi(x)) = 1$. The induction hypothesis applied to $\mathscr{D}^{**}$ and $\mathscr{D}_1'$ gives a cut free derivation $\mathscr{D}^{***}$ of q_1': $\Gamma_1 \vdash \Delta_1, \psi(y)$. The induction hypotheses on c applied to $\mathscr{D}^*$ and $\mathscr{D}^{***}$ yield a cut-free derivation of q': $\Gamma \vdash \Delta$. Then we apply Rule (D) to get q: $\Gamma \vdash \Delta$.

(B1biv) The situation with $\varphi = \mathsf{aa}s\, \psi(s)$ is similar but a little different because of the ($\mathsf{aa}\vdash$)-rule's extra condition on variables. We are faced with the following situation.

$$\dfrac{\dfrac{\begin{array}{c}\mathscr{D}_0' \\ \downarrow \\ q's'q''\colon \Gamma_0, \mathsf{aa}s\, \psi(s), \psi(s') \vdash \Delta_0\end{array}}{q's'q''\colon \Gamma_0, \mathsf{aa}s\, \psi(s) \vdash \Delta_0} \qquad q_1\colon \Gamma_1 \vdash \Delta_1, \mathsf{aa}s\, \psi(s)}{q\colon \Gamma \vdash \Delta}.$$

The situation is straightforward if q contains the variable s'. One uses a derivation $\mathscr{D}^{**}$ of q_1: $\mathsf{aa}s\, \psi(s) \vdash \psi(s')$ of rank 1 as before. If s' is not in q then it is not free in $\Gamma \cup \Delta$ so we can modify $\mathscr{D}_1$ to get a cut-free derivation $\mathscr{D}_1'$ of the same rank (i.e. $r(\mathscr{D}_1', \varphi) = r(\mathscr{D}_1, \varphi)$) of q_1': $\Gamma_1 \vdash \Delta_1, \mathsf{aa}s\, \psi(s)$, where q_1' has s' inserted at the right place in q_1. Then we again argue as in part (B1biii). This finishes (B1).

(B2) $r(\mathscr{D}_0, \varphi) = 1$ and $r(\mathscr{D}_1, \varphi) \geq 2$. Again we have cases

(a) where φ is not the principal formula in the last inference in $\mathscr{D}_1$ and

(b) where it is.

Case (a) is just as in (B1a).

Case (B2b). This time we do not need to consider $\varphi = s(x)$ since this cannot be the principal formula in $\mathscr{D}_1$. So we are left with φ of the forms $\neg\psi$, $(\psi_1 \wedge \psi_2)$, $\forall x\, \psi(x)$ and $\mathsf{aa}s\, \psi(s)$. These are all entirely analogous to the corresponding cases in (B1b), so we leave them to the interested reader. $\square$

6.14. Remarks. To extend to languages with terms we modify Axiom ($\vdash s$) to

$$(\vdash s) \qquad q\colon \Gamma \vdash \Delta, s(\tau)$$

with the condition that s is in q but no variable in the term τ should occur after s in q. Rule ($\forall\vdash$) is modified as usual. To extend to the infinitary case, $\mathrm{L}_A(\mathsf{aa})$, one follows Feferman [9].

7. Other meanings of "almost all"

The difference between the Rule ($s\vdash$) and the weaker Rule wk-($s\vdash$) tempts us to look for a different meaning of aa, one where the wk-($s\vdash$)-Rule would give us exactly the valid sequents. This logic should still contain L^{pos}, and hence L(**Q**).

Let $\mathfrak{F} \subseteq P_{\omega_1}(M)$ be any filter. We can interpret aa in the expanded structure $(\mathfrak{M}, \mathfrak{F})$ by

$$(\mathfrak{M}, \mathfrak{F}) \models \mathsf{aa}\, s\, \varphi\,(s, \boldsymbol{a}) \quad \text{iff} \quad \{s \in P_{\omega_1}(M) \mid (\mathfrak{M}, \mathfrak{F}) \models \varphi(s, \boldsymbol{a})\} \in \mathfrak{F}.$$

For $\mathfrak{F} = \mathfrak{F}_{\mathrm{cub}}$ = the cub filter, this is the old definition. What other $\mathfrak{F}$'s are there to try?

7.1. Definition. The *eventual filter* on $P_{\omega_1}(M)$ is the filter generated by sets of the form

$$X_{s_0} = \{s \mid s_0 \subseteq s\}$$

for $s_0 \in P_{\omega_1}(M)$.

Since $\bigcap_{i<\omega} X_{s_i} = X_{\bigcup_{i<\omega} s_i}$, the eventual filter is countably complete. With this meaning of "almost all", all of the rules of our Gentzen system are valid, including Rule wk-($s\vdash$), but not Rule ($s\vdash$). In terms of Section 1, this meaning of "almost all" satisfies all of Axioms (A0)–(A4) and our rule of aa-generalization, but not Axiom (A5).

There is, however, no Completeness Theorem or Compactness Theorem for this meaning of almost all. The set of valid sentences is not arithmetical (or even Π^1_1). These negative results follow from the following result (and obvious strengthenings).

7.2. Proposition. *There is a sentence $\varphi(N, <, \cdots)$ of L(aa) such that, when aa is reinterpreted by the eventual filter, every model $\mathfrak{M}$ of φ has $\langle N^{\mathfrak{M}}, <^{\mathfrak{M}}\rangle \cong \langle \omega, <\rangle$.*

Proof. φ can be defined once we observe that a certain class of well-ordered structures can be characterized in this logic. Let ψ say that $<$ is an ω_1-like linear ordering in which every element has an immediate successor, such that some final segment is well-ordered. It's not hard to see that this can be said as follows:

$$\exists s_0\, \forall s \supseteq s_0\, (s \text{ has a sup}), \text{ i.e. } \mathsf{aa}\, s(s \text{ has a sup}).$$

Let θ say that given any interval $[0, z)$, and an x, then $f(x, z, \cdot)$ takes $[0, z)$ in an order-preserving way to $[x, y)$ for some y. Clearly any model of $\psi \wedge \theta$ has order type ω_1. φ is

$$\psi \wedge \theta \wedge \forall x(N(x) \leftrightarrow \forall y \leq x\, (y = 0 \text{ or } y \text{ has an immediate predecessor})). \quad \square$$

Once we give up on the eventual filter, where do we go? There aren't a whole lot of filters on $P_{\omega_1}(M)$ that have names, that is, that are defined in a uniform way on all M. So, we punt and leave $\mathfrak{F}$ undetermined.

7.3. Definition. *Filter logic* $L^{Flt}(aa)$, more accurately called $\aleph_0$-complete filter logic, has the same syntax as $L(aa)$. A structure, however, consist of a pair $(\mathfrak{M}, \mathfrak{F})$, where $\mathfrak{F}$ is a countably complete filter containing the eventual filter. Satisfaction is defined as above. $L^{Flt}_{\omega_1\omega}(aa)$ and its countable fragments $L^{Flt}_A(aa)$ are defined analogously.

There is a Completeness Theorem for filter logic by using Rule wk-($s\vdash$) rather than Rule ($s\vdash$). In terms of a Hilbert-style system, this amounts to replacing Axiom A5 by:

$$\forall x \in s_0\; aa s\; \varphi(x, s) \rightarrow aa s\; \forall x \in s_0\; \varphi(x, s). \qquad \text{wk-(A5)}$$

Notice that Axioms (A0)–(A4) + wk-(A5) are valid in all our structures, as are the axioms and rules of the Gentzen system with Rule wk-($s\vdash$). Since we have already shown how to prove Axiom wk-(A5) in the Gentzen system using only Rule wk-($s\vdash$), we will get completeness of both systems by showing completeness of the Hilbert style system. This proof also gives compactness for filter logic.

7.4. Theorem. *The Compactness, Completeness and Omitting Types Theorems proved earlier for* $L(aa)$ *and* $L_{\omega_1\omega}(aa)$ *are also true for the filter logics* $L^{Flt}(aa)$ *and* $L^{Flt}_{\omega_1\omega}(aa)$.

7.5. Corollary. *A sequent* $q : \Gamma \vdash \Delta$ *is provable in the weak Gentzen system* (Ax), ($\vdash s$), ($\vdash\wedge$), ($\wedge\vdash$), ($\vdash\neg$), ($\neg\vdash$), (D), (P) *and* wk-($s\vdash$) *iff it holds in all structures* $(\mathfrak{M}, \mathfrak{F})$ *for filter logic.*

7.6. Corollary. *Any valid sequent* $q: \Gamma \vdash \Delta$ *of* L^{pos} *is derivable using only the Rule* wk-($s\vdash$) (*as opposed to Rule* ($s\vdash$)). *Hence the same holds for* $L(\mathbf{Q})$.

Proof. Let $(\mathfrak{M}, \mathfrak{F}_1)$, $(\mathfrak{M}, \mathfrak{F}_2)$ be two structures for $L^{Flt}(aa)$ with the same $\mathfrak{M}$. An easy inductive proof shows that for any φ of L^{pos}, $(\mathfrak{M}, \mathfrak{F}_1) \models \varphi$ iff $(\mathfrak{M}, \mathfrak{F}_2) \models \varphi$, for all substitutions of the free variables. I.e., the formulas of L^{pos} are invariant under changes of filters. Thus, if they are valid under their intended interpretations, they are valid in the sense of filter logic. $\square$

Notice that in the Gentzen system we have Rule ($\vdash s$), which corresponds to only the first half of Axiom (A2). The second half $aa s\,(s_0 \subseteq s)$ is derivable using Rule wk-($s\vdash$). This corresponds to the fact that we can derive $aa s\,(s_0 \subseteq s)$ in the Hilbert system for filter logic from Axiom wk-(A5) and $aa s\,(x \in s)$.

The rest of this section is devoted to sketching the proofs needed to establish Theorem 7.4. We again use weak models. A weak model is just as before except that it satisfies all instances of Axiom wk-(A5) but not necessarily of Axiom (A5). The relevant definitions (satisfaction, elementary extension) are as before.

We'll start by outlining a proof of completeness for $L^{Flt}(aa)$. First note that Axiom (A5) was never actually used in the proof of the Weak Completeness

Theorem for $L(\mathit{aa})$, except to get a model satisfying Axiom (A5). Thus that proof is essentially also a proof of weak completeness for $L^{Flt}(\mathit{aa})$, although a slight bit of care should be taken to insure that all instances of Axiom wk-(A5) are true in the weak model which is constructed.

Next, a version for $L^{Flt}(\mathit{aa})$ of the main lemma of $L(\mathit{aa})$ (Section 3.4) can be seen to be true. This time we do not require that $M \in P^{\mathfrak{N}^*}$. Instead, we require that M is a "subset" of some element U_M of $P^{\mathfrak{N}^*}$ such that: $\mathfrak{N}^* \models \psi(U_M)$ for the appropriate ψ, as before; and $\mathfrak{N}^* \models \varphi(U_M)$ for all φ such that $\mathfrak{M}^* \models \mathit{aa}\, s\, \varphi(s)$. We also require that the elements of $P^{\mathfrak{M}^*}$ be fixed; that is, if $s \in P^{\mathfrak{M}^*}$ and $\mathfrak{N}^* \models n \in s$, then $n \in M$. The proof is analogous to that of the main lemma for $L(\mathit{aa})$. For example, the same consistency criterion holds. The main difference is that Axiom wk-(A5) is used to guarantee that the elements of $P^{\mathfrak{M}^*}$ are fixed, where before axiom (A5) was used to guarantee that M was "fixed".

Finally, to prove the Completeness Theorem for $L^{Flt}(\mathit{aa})$, a result can be proved which is analogous to the one in Theorem 3.5. This time, divide ω_1 into ω_1 disjoint unbounded subsets (not necessarily stationary). Then apply the main lemma for $L^{Flt}(\mathit{aa})$ ω_1 times, just as we did for $L(\mathit{aa})$ in Theorem 3.5, to get a standard model $\mathfrak{N}$. Now define a filter $\mathfrak{F}$ on $P_{\omega_1}(N)$ by

$$X \in \mathfrak{F} \quad \text{iff} \quad \exists \alpha_0 \forall \alpha > \alpha_0 (U_{M_\alpha} \in X).$$

It's easy to see that $\mathfrak{F}$ is countably complete and contains each set of the form $\{s \in P_{\omega_1}(N) \colon s_0 \subseteq s\}$. It's also easy to check (by induction on complexity, as usual) that for all $\alpha < \omega_1$, and all φ with parameters in $M_\alpha \cup P\,\mathfrak{M}^*_\alpha$,

$$\mathfrak{M}^*_\alpha \models \varphi^* \quad \text{iff} \quad (\mathfrak{N}, \mathfrak{F}) \models \varphi.$$

In particular, this is true for $\alpha = 0$, so $(\mathfrak{N}, \mathfrak{F})$ has the desired properties.

The Omitting Types Theorem (OTT) for $L^{Flt}_{\omega_1\omega}(\mathit{aa})$ is exactly the same as before except that we must use the second notion of "strongly omits" (Section 4.3) rather than the simpler, first definition (Section 4.1). This is because we used Axiom (A5) (in Section 4.3) to show that the first definition implies the second — but we actually used the second definition in the proof of the OTT for $L_A(\mathit{aa})$.

There are some modifications required to transform the proof of the OTT for $L_A(\mathit{aa})$ into one for $L^{Flt}_A(\mathit{aa})$. Let's start first with the finitary case ($T \subset L^{Flt}_{\omega\omega}(\mathit{aa})$), which was handled for $L_{\omega\omega}(\mathit{aa})$ in Section 4.5. Lemma 4.5.1 from that section, which gave weak models which strongly omit types, has an analogue for $L^{Flt}(\mathit{aa})$; moreover, the proofs are similar, just as the proofs of the weak Completeness Theorems are similar. Recall that Lemma 4.5.2 showed that the theory from the main Lemma strongly omits any type which is strongly omitted by $\mathfrak{M}^*$ (from the main Lemma), and then some. Lemma 4.5.2 has an analogue which has the same statement and proof. Lemma 4.5.3 said that the theory from the main Lemma strongly omits the "M-type". Its analogue for $L^{Flt}(\mathit{aa})$ is that the "s-types" are strongly omitted for all $s \in P^{\mathfrak{M}^*}$, so that $\mathfrak{M}^*$ can be expanded while the elements of

$P^{\mathfrak{M}*}$ remain fixed. Finally, the rest of the proof of the OTT for $L^{Flt}(aa)$ can be obtained from the one for $L(aa)$, much as the proof of the analogue of the "final result" 3.5 could be obtained from the proof of Theorem 3.5.

There's really no difference between the way the proof of the OTT for $L(aa)$ was extended to $L_A(aa)$ and the way the proof of the OTT for $L^{Flt}(aa)$ can be extended to $L_A^{Flt}(aa)$. However, the general case discussed in Section 4.7 does not follow for $L_A^{Flt}(aa)$ in the same way that it followed for $L_A(aa)$. The problem is that full use was made of Axiom (A5) in Section 4.7. Nevertheless, the general case is true for $L_A^{Flt}(aa)$. To see this one has to modify directly the proof of the OTT for $L_A^{Flt}(aa)$. The main modification is in condition (b) of (the analogue of) Lemma 4.5.2. Suppose the variables of Σ are v, $\boldsymbol{w}$, where v is the $<$-least variable, let $\lhd$ be $< \restriction \boldsymbol{w}$. Then the modified condition (b) is that if T strongly omits Σ (wrt$<$), then T strongly omits

$$\Sigma(\bar{m}, \boldsymbol{w})(\text{wrt} \lhd), \quad \text{for each} \quad m \in M, \quad \text{if } v \text{ is first-order,}$$

$$\Sigma(\bar{\boldsymbol{M}}, \boldsymbol{w})(\text{wrt} \lhd), \qquad \text{if } v \text{ is second-order.}$$

The latter of these was essentially verified before, and the first is actually easier. Now one proceeds with the proof of the OTT as outlined above, but uses the modified version of Lemma 4.5.2 when constructing the ω_1-chain of models.

8. Concluding remarks

8.1. Relations with the literature. The Completeness Theorem presented here has three principal ancestors: Hutchinson [11], Keisler [15] and Shelah [24]. The debt to Keisler's paper is obvious in Sections 3 and 4.

Shelah's paper, where $L_{\omega\omega}(aa)$ is introduced (under the notation $L(Q^{ss}_{\aleph_1})$) does not state any results about $L_{\omega\omega}(aa)$. It is possible, however, to derive compactness and abstract completeness (the set of valid sentences is r.e.) of $L_{\omega\omega}(aa)$ from similar results about a different logic contained in Theorem 2.14 of Shelah [24]. Thus, our contribution is the explicit analysis of the laws of logic needed for completeness. As far as $L_{\omega_1\omega}(aa)$ is concerned, the only reference seems to be in Shelah [25], where he remarks that $L(aa)$, which he now calls $L(Q^{st})$ (confusingly, since this means something else in Shelah [24]), "is very similar to $L(Q)$ for models of power $\aleph_1$, and in fact also $L_{\omega_1\omega}(Q^{st})$ is very similar to $L_{\omega_1\omega}(Q)$" where Q is "there exist uncountably many".

Hutchinson's [11] has not been mentioned before, but it is where our interest in stationary logic began. His main result is that if $\mathfrak{M}$ is a countable model of set theory and κ is a regular cardinal of $\mathfrak{M}$, then there is an $\mathfrak{N} > \mathfrak{M}$ where there is a first new ordinal $< \kappa$ greater than all ordinals $\alpha \in \mathfrak{M}$, $\alpha < \kappa$. This result is a version of our main lemma, stated for models of set theory. In fact, Hutchinson's Theorem is a special case of our main lemma if one restates the main lemma to apply to a "relativized" aa quantifier, say $\mathsf{aa}_\psi s\, \varphi(s)$. This relativized form reads "for almost all countable $s \subseteq \{x \mid \psi(x)\}$, $\varphi(s)$", where almost all is taken in the sense of the cub filter on $\{x \mid \psi(x)\}$. In standard models this can be *defined* by

$$\mathsf{aa}_\psi s\, \varphi(s) \leftrightarrow \mathsf{aa} s\, \varphi(\{x \in s \mid \psi(x)\}).$$

But there are interesting applications to weak models when one allows aa_ψ but not aa. For example, to obtain Hutchinson's theorem, let $\psi(x)$ be $(x < \kappa)$. You can turn the countable model of set theory into a weak model $\mathfrak{M}^* = (\mathfrak{M}, P, \ldots)$ in the obvious way. Fodor's Theorem is used to verify Axiom (A5). Then Hutchinson's Theorem is a consequence of the relativized main Lemma.

The paper Dubiel [7] is also a descendant of Hutchinson [11] and Keisler [15], but goes in other directions.

The Completeness Theorems for $L(\mathsf{aa})$ and $L_A(\mathsf{aa})$ were announced in the abstract Barwise and Makkai [4], but there was a gap in the infinitary case. The Omitting Types Theorem and the example of Section 5 were announced in the abstract Kaufmann [13], and is part of the second author's Ph.D. Thesis. The OTT filled in the gap in the proof of the infinitary case of the Completeness Theorem. In another recent abstract, Bruce [6] discusses an extension of forcing to $L(\mathsf{aa})$. He uses this to give a different proof of completeness and mentions that an Omitting Types Theorem for basic formulas can be proved. Bruce has also shown how one could give a proof of the Compactness Theorem for $L(\mathsf{aa})$ via definable ultrapowers.

The third author has observed that one can combine the argument given here with the Magidor–Malitz [21] Completeness Theorem for $L(Q^1, Q^2, Q^3, \ldots)$ to get a Compactness and abstract Completeness Theorem for $L(\mathsf{aa}, Q^2, Q^3, \ldots)$ (with the corresponding results for countable, admissible $L_A(\mathsf{aa}, Q^2, Q^3, \ldots)$). This, together with Section 5, gives an example of a countably compact logic which strictly contains $L(Q^1, Q^2, \ldots)$.

8.2. Cub-like models. There is an alternate notion of weak model which parallels more closely both Keisler's notion of weak model and the notion of filter model from Section 7. Let $\mathfrak{M}^\# = (\mathfrak{M}, P, \mathfrak{F})$, where $P \subset P_{\omega_1}(M)$ and $\mathfrak{F} \subset \mathrm{Power}(P)$. We define $\mathfrak{M}^\# \models \varphi$ by induction on complexity of φ as usual, where

$$\mathfrak{M}^\# \models \mathsf{aa} s\, \varphi \quad \text{iff} \quad \{s \in P \colon \mathfrak{M}^\# \models \varphi(s)\} \in \mathfrak{F}.$$

We call $\mathfrak{M}^\#$ a *cub-like model* if $P \in \mathfrak{F}$, and all instances of the axioms of $L(\mathsf{aa})$ are true in $\mathfrak{M}^\#$.

The main lemma is true for cub-like models, and the proof is similar (although some extra care must be taken). Alternatively, it can be derived from the main lemma for weak models (Section 3.4) by setting up a correspondence between the extensional weak models, and the cub-like models in which the elements of $\mathfrak{F}$ are definable. The details are left to the reader.

Cub-like models are somewhat more natural than weak models, especially in view of Section 7, and could probably be used to study what additional (or fewer) axioms one might want to put on the "filter" $\mathfrak{F}$. Also, one might be able to find applications of cub-like models similar to those in Keisler [15]. However, our weak models proved more suitable than cub-like models for proving the Omitting Types Theorem, especially the infinitary case. Moreover, the proof shows that we can allow third-order relations $R_i \subset M^{m_i} \times P^{n_i}$ and still have Completeness, Compactness, and Omitting Types Theorems. Thus, for example, we can expand L(aa) to allow a relation defining a bijection between M and $P^{\mathfrak{M}^*}$.

8.3. Shelah [24] defines a qunatifier $Q^{st}_{\aleph_1, X}$, where $X \subset \omega_1$. Implicit in his discussion is a version $L^X(aa)$ of L(aa) relativized to a stationary subset of ω_1. The syntax of $L^X(aa)$ is that of L(aa). For the semantics we first define a relativized notion of almost all.

Let N be any set of cardinality $\aleph_1$. Pick any increasing sequence of countable sets $\langle N_\alpha : \alpha < \omega_1 \rangle$ such that $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$ for limits λ. Let $\mathfrak{F}^X \subset \text{Power}(P_{\omega_1}(N))$ be generated by those $Y \subset P_{\omega_1}(N)$ such that

$$\{\alpha : N_\alpha \in Y\} \cup (\omega_1 \setminus X)$$

is a cub subset of ω_1. Notice that if $Z \subset P_{\omega_1}(N)$, then $P_{\omega_1}(N) \setminus Z \notin \mathfrak{F}^X$ iff $\{\alpha : N_\alpha \in Z\} \cap X$ is stationary in ω_1. This suggests that we think of $\mathfrak{F}^X$ as being the cub fifilter relative to X. We should point out that $\mathfrak{F}^X$ is well-defined; for if we replace $\langle N_\alpha : \alpha < \omega_1 \rangle$ by another increasing sequence of countable sets with unions at limits, then the two sequences will agree on a cub subset of ω_1.

We can define satisfaction for formulas of $L^X(aa)$ using the notation of Section 7. Let $\mathfrak{N}$ be a structure for L of cardinality $\aleph_1$. Then we define $\mathfrak{N} \models \varphi(\boldsymbol{a})$ (wrt $L^X(aa)$) as $(\mathfrak{N}, \mathfrak{F}^X) \models \varphi(\boldsymbol{a})$ (as in Section 7). Then the axioms and rules for L(aa) are easily seen to be valid in the sense of $L^X(aa)$. Eklof has informed us that this notion has come up in some of his work on abelian groups.

Our main observation in this section is that the axioms and rules for L(aa) are complete for $L^X(aa)$, where X is any stationary subset of ω_1. To see this, one can modify the proof of the Completeness Theorem as follows. Everything goes the same until the chain $\langle \mathfrak{M}^*_\alpha : \alpha < \omega_1 \rangle$ is constructed (Section 3.5). First, divide X (rather than ω_1) into ω_1 disjoint stationary subsets. Modify the construction of the chain $\langle \mathfrak{M}^*_\alpha : \alpha < \omega_1 \rangle$ accordingly. It's not hard to check (as before) that in the resulting model $\mathfrak{N} = \bigcup_{\alpha<\omega_1} \mathfrak{M}_\alpha$, the following is true for each $\alpha < \omega_1$, $\boldsymbol{a}$ from $M_\alpha \cup P^{\mathfrak{M}^*_\alpha}$, and formula φ of L(aa):

$$(\mathfrak{N}, \mathfrak{F}^X) \models \varphi(\boldsymbol{a}) \quad \text{iff} \quad \mathfrak{M}^*_\alpha \models \varphi(\boldsymbol{a}).$$

In fact, these are both equivalent to $\mathfrak{N} \models \varphi(\boldsymbol{a})$ in the sense of ordinary L(aa).

As a corollary we have the compactness of $L^X(\text{aa})$. Also, all of this can of course be extended to give the Completeness and Omitting Types Theorems for $L^X_A(\text{aa})$, for every stationary $X \subset \omega_1$.[1]

8.4. Determinacy. We want to consider structures $\mathfrak{M}$ where the cub filter approximates an ultrafilter by satisfying the following scheme:

$$\text{aa}s\, \varphi(s) \vee \text{aa}s\, \neg\varphi(s). \tag{Det}$$

Using Kucker's game theoretic interpretation of the cub filter mentioned in Section 1, we see that $\text{aa}s\, \varphi(s) \vee \text{aa}s\, \neg\varphi(s)$ asserts that the game determined by $\varphi(s)$ is determined; i.e., either $\forall$ or $\exists$ has a winning strategy. Borrowing the idea from Takeuti [27], we call $\mathfrak{M}$ $L_A(\text{aa})$-*determinate* if $\mathfrak{M}$ is a model of all $L_A(\text{aa})$ instances of the Scheme (Det). This has the effect of making stat $s\, \varphi(s)$ equivalent to $\text{aa}s\, \varphi(s)$ for φ in $L_A(\text{aa})$.

Every countable structure $\mathfrak{M}$ is $L_A(\text{aa})$-determinate (since in this case the cub filter really is an ultrafilter) but there are also uncountable $L_A(\text{aa})$-determinate structures. For example, if $\mathfrak{M} = \langle M \rangle$ where M is uncountable, then $\mathfrak{M}$ is $L_{\omega_1\omega}(\text{aa})$-determinate. [Proof. A trivial automorphism argument shows that

$$\mathfrak{M} \models \text{aa}t\, \varphi(s, \boldsymbol{x}, t) \quad \text{iff} \quad \mathfrak{M} \models \varphi(s, \boldsymbol{x}, t)$$

for some (or all) t containing infinitely many elements in addition to $\bigcup_i s_i \cup \{\boldsymbol{x}\}$, from which Scheme (Det) follows.] Another example, noticed in conversation with Kunen, is $\mathfrak{M} = \langle \omega_1, < \rangle$. (See Kaufmann [14] for a proof using a back-and-forth argument.) On the other hand, if we use AC to construct a stationary subset X of ω_1 which does not contain a cub, then $\mathfrak{M} = \langle \omega_1, <, X \rangle$ is not $L(\text{aa})$-determinate.

The Completeness Theorem for $L_A(\text{aa})$ shows that any theory T consistent with Scheme (Det) is true in an $L_A(\text{aa})$-determinate model. However, the proof of the Completeness Theorem uses the same consequence of AC as the above example of a non-determinate structure (to split ω_1 into disjoint stationary sets) so let us consider what happens if we drop AC and assume, instead, the problematic hypothesis AD+DC, the Axiom of Determinacy plus Dependent Choice. An early result of Solovay shows that for M of size $\aleph_1$, AD implies that the cub filter on M is an ultrafilter. (Solovay's result was for $P(\omega_1)$, not $P_{\omega_1}(\omega_1)$, but they are clearly equivalent since almost every countable subset of ω_1 is an ordinal.) Thus, assuming AD, every structure of size $\leq \aleph_1$ is determinate. Further, using DC, we see that Axioms (A0)–(A6) still hold in such structures.

We claim that Axioms (A0)–(A6)+(Det) plus the rule of aa-generalization is a complete system, given AD+DC. The proof is much easier than before, since we no longer have to worry about formulas of the form stat $s\, \psi(s)$. Just iterate the

[1] Our results in Section 8.3 could be derived from our completeness theorem and results in Shelah [24], where he treats relativizations to several stationary sets.

main lemma ω_1 times with ψ some tautology. It is easy to see that the final $\mathfrak{N}$ is indeed a standard model. This also shows that L(aa) is compact in this context. Similar observations hold for Omitting Types.

The Gentzen-style equivalent of Scheme (Det) is the rule:

$$\frac{q\colon \Gamma \vdash \Delta, \varphi(s')}{q\colon \Gamma \vdash \Delta, \mathit{aa}\, s\, \varphi(s)}, \qquad \text{Det-}(\vdash \mathit{aa})$$

as long as s' is the last variable of q free in $\Gamma \cup \Delta \cup \varphi(s')$. In other words, we drop the requirement that s' not be free in $\Gamma \cup \Delta$ needed in Rule ($\vdash \mathit{aa}$). If this seems strange, notice that the above is the derived rule for ($\vdash$stat) and that Scheme (Det) makes stat $s\, \varphi(s)$ imply $\mathit{aa}\, s\, \varphi(s)$. Conversely, using Rule Det-($\vdash \mathit{aa}$), we derive $\mathit{aa}\, s\, \varphi(s) \vee \mathit{aa}\, s\, \neg\varphi(s)$ as follows:

$$s\colon \varphi(s) \vdash \varphi(s),$$
$$s\colon \vdash \varphi(s), \neg\varphi(s),$$
$$s\colon \vdash \mathit{aa}\, s\, \varphi(s), \neg\varphi(s),$$
$$s\colon \vdash \mathit{aa}\, s\, \varphi(s), \mathit{aa}\, s\, \neg\varphi(s),$$

etc.

It is clear that more research is needed to explore the relationship between L(aa) and the "determinate logic" of Takeuti [27].

9. Open questions

9.1. As mentioned earlier (Remark 6.9), interpolation fails for L(aa), and in fact for fragments $L_A(\mathit{aa})$. Moreover, the Suslin–Kleene-property (Δ-interpolation) fails under the assumption MA + $\neg$CH. (See Kaufmann [in 14].) Does the Suslin–Kleene-property fail absolutely? How about Beth-definability?

There is a more general class of questions from abstract model theory which remain open. Some of these questions are raised in Makowsky, Shelah and Stavi [20]. Problem 3.2 of that paper, which they attribute to Feferman, asks if there is an extension of L(Q) which is $\aleph_0$-compact (that is, compact for countable theories), axiomatizable, and has interpolation (or even the Suslin–Kleene-property). In Problem 3.4 they essentially ask if any proper extension of $L_{\omega\omega}$ which is axiomatizable or $\aleph_0$-compact has any interpolation properties (even the Beth- or weak Beth-properties). In a verbal communication Shelah has suggested that perhaps there is no extension of L(Q) which has interpolation, is $\aleph_0$-compact, and satisfies the Downward Lowenheim–Skolem Theorem to ω_1. A weaker conjecture is that there is no such extension of L(Q) which also has a complete cut-free Gentzen system (one with the subformula property). This is suggested by Remark 6.9.

9.2. What are the Hanfnumbers of $L(aa)$ and $L_A(aa)$? Keisler has observed that Vaught's Gap-ω Theorem shows that the Hanfnumber of $L(Q)$ is $\beth_\omega$, and that for admissible sets $A \neq H(\omega)$, the Hanfnumber of $L_A(Q)$ is the same as that of L_A. These seem like reasonable answers for $L(aa)$ and $L_A(aa)$ as well. Note that the well-ordering numbers of $L(aa)$ and $L_A(aa)$ equal those of L and L_A; the proof in Barwise [2] (III.7.3) for L_A carries over directly to $L_A(aa)$.

9.3. Does every[2] (standard) model for $L(aa)$ (or $L_A(aa)$) have an elementary submodel of cardinality $\aleph_1$? (This is stronger than the result of Section 1.5.)

9.4. The main Lemma (Section 3.4) can be modified to hold for weak models of certain fragments F of $L(aa)$ which are not closed under $(aa s)$. F needs only to be closed under the other operations, subformulas, and certain transformations such as

$$\forall x \; aa s\, \varphi \mapsto aa s\, \forall x \in s\, \varphi.$$

The model given by this modified main Lemma would be constructed to be a model of a smaller fragment F': $\varphi \in F'$ iff $aa s\, \varphi \in F$ for some s. Thus we can't hope to iterate the modified main Lemma.

We can, however, consider a special case which is closely related to Hutchinson's paper [11]. Define the *1-fragment* to be the least set F of formulas which has the following properties: it contains those of L; it is closed under the first-order operations; and it has the feature that whenever φ is a formula of L, $aa s\, \varphi$ is a formula of F. Then by the paragraph above, every weak model $\mathfrak{M}^*$ of the 1-fragment has an elementary extension $\mathfrak{N}^*$ in which $\mathfrak{M}$ is first-order definable (namely, $m \in M$ iff $\mathfrak{N}^* \models m \in M$). $\mathfrak{N}^*$ will be a proper extension of $\mathfrak{M}^*$ if $\mathfrak{M}^* \models Qx\,(x = x)$. *Thus any countable structure $\mathfrak{M}$ which can be expanded to a weak model of the 1-fragment plus $Qx\,(x = x)$ has an expansion $\mathfrak{M}'$ (namely $\mathfrak{M}' = \mathfrak{M}^*$) such that for some $\mathfrak{N}' > \mathfrak{M}'$, M is first-order definable with parameters in $\mathfrak{N}'$.*

The problem is to find a reasonable converse of this statement. There is a sort of converse for models of set theory: If $\mathfrak{M} <_{\text{end}} \mathfrak{N}$ and for some $u \in N$, $\{x \in N : \mathfrak{N} \models x \in u\} = M$, then $\mathfrak{M}$ can be expanded to a weak model of the 1-fragment plus $Qx\,(x = x)$.

9.5. We discussed the "positive logic" L^{pos} of Makowsky and Shelah in Section 1.2 and again in Sections 6, 7. They also define in Makowsky–Shelah [19] a "negative logic" L^{neg} with the following new formation rule. If a second-order

[2] The answer to this question turns out to be independent of the axioms of set theory, as Harrington, Kunen and Shelah have pointed out. This question is intimately connected with Jensen's principle E for various $\kappa > \omega_1$. If $V = L$, then there is a structure of size ω_2 with no $L(aa)$-elementary submodel of power w_1. Furthermore, if one collapses a supercompact cardinal to ω_2, then every uncountable structure has an $L(aa)$-elementary submodel of size ω_1. The exact strength of this Downward Löwenheim-Skolem Theorem is not clear. For example, Kunen has asked whether it implies that $0^{\#}$ exists. (See also note added in proof.)

variable X_i occurs negatively in $\varphi(X_i^-)$, and $\varphi(X_i^-)$ is a formula of $\mathrm{L}^{\mathrm{neg}}$, then $\exists X_i\,\varphi(X_i^-)$ is a formula of $\mathrm{L}^{\mathrm{neg}}$. (The dual rule for forming $\forall X_i\,\varphi(X_i^+)$ can be added, or define $\forall X_i$ from $\exists X_i$). The second-order variables X_i range over *uncountable* sets.

$\mathrm{L}^{\mathrm{neg}}$ is rather powerful; for example, all of the Magidor–Malitz quantifiers $\mathbf{Q}^n$ for $n \in \omega$ can be expressed in $\mathrm{L}^{\mathrm{neg}}$. Makowski and Shelah [19] raised the question of whether $\mathrm{L}^{\mathrm{neg}}$ is compact or complete, assuming $\diamond\omega_1$. Let's extend that question by first defining $\mathrm{L}^{\mathrm{neg}}(aa)$ as being the logic one gets by combining in the obvious way $\mathrm{L}^{\mathrm{neg}}$ and $\mathrm{L}(aa)$. Then we ask: is $\mathrm{L}^{\mathrm{neg}}$ compact or complete, assuming $\diamond\omega_1$? Perhaps one can start with a reasonable set of axioms, and then modify the proof of the Completeness Theorem of $\mathrm{L}(aa)$ to get completeness of $\mathrm{L}^{\mathrm{neg}}(aa)$. ($\diamond\omega_1$ will have to be used in an essential way.) (See note added in proof.)

9.6. We've seen in Section 7 that the formulas of $\mathrm{L}^{\mathrm{pos}}$ are invariant for $\mathrm{L}^{\mathrm{Fit}}_{\omega_1\omega}(aa)$. Are these (up to logical equivalence) the only invariant formulas of $\mathrm{L}^{\mathrm{Fit}}_{\omega_1\omega}(aa)$?

9.7. Investigate more notions of "almost all". In one direction, one could look for other filters on $P_{\omega_1}(N)$ besides the cub filter which give rise to a well-behaved logic (with the same syntax as $\mathrm{L}(aa)$ but semantics based on the other filter). For example, we "relativize to a stationary subset of ω_1" in Section 8.3. Another possible modification is to consider "filter models" $(\mathfrak{N}, P, \mathfrak{F})$, where P may be a proper subset of $P_{\omega_1}(N)$. The notion of "sampling" in Benda [5] may be relevant, since he uses this notion to generalize Hutchinson's Theorem in another direction. See also Makowsky, Shelah, and Stavi [20] Section 3, for a treatment of "cofinally invariant logics". In still another direction, one could look at $P_\kappa(N)$ rather than $P_{\omega_1}(N)$. There is a notion of a cub filter on $P_\kappa(N)$ defined in Kueker [17]. One could investigate Completeness, Compactness, and Transfer Theorems. These questions were essentially raised in Shelah [24]. A relevant paper which solves some of these types of questions is Schmerl [26].

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Added in proof (Dec 23, 1977): The results mentioned in footnote 2 also appear in a preprint "On Shelah's Compactness of Cardinals" by S. Ben-David. Back-and-forth systems for $\mathrm{L}(aa)$ appear in [14]. Similar systems have been developed by Caicedo, Makowsky, Seese and Weese. Using these and a construction of Shelah [19], Makowsky (with some details supplied by Kaufmann) has shown that $\neg$Beth $(\mathrm{L}(\mathbf{Q}), \mathrm{L}_{\infty\omega}(aa))$. With regard to 9.5, Stavi has given a negative answer by showing that ω_1 is Σ^1_1- definable in $\mathrm{L}^{\mathrm{neg}}$.

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