# Higher rank numerical ranges and low rank perturbations of quantum channels 

Chi-Kwong Li ${ }^{\text {a }}$, Yiu-Tung Poon ${ }^{\text {b }}$, Nung-Sing Sze ${ }^{\mathrm{c}, *}$<br>${ }^{\text {a }}$ Department of Mathematics, College of William \& Mary, Williamsburg, VA 23185, USA<br>${ }^{\text {b }}$ Department of Mathematics, Iowa State University, Ames, IA 50051, USA<br>${ }^{\text {c }}$ Department of Mathematics, University of Connecticut, Storrs, CT 06269, USA

## ARTICLE INFO

## Article history:

Received 9 January 2008
Available online 14 August 2008
Submitted by Goong Chen

## Keywords:

Hilbert space
Bounded linear operators
Higher rank numerical range
Quantum error correcting codes
Quantum channels


#### Abstract

For a positive integer $k$, the rank- $k$ numerical range $\Lambda_{k}(A)$ of an operator $A$ acting on a Hilbert space $\mathcal{H}$ of dimension at least $k$ is the set of scalars $\lambda$ such that $P A P=\lambda P$ for some rank $k$ orthogonal projection $P$. In this paper, a close connection between low rank perturbation of an operator $A$ and $\Lambda_{k}(A)$ is established. In particular, for $1 \leqslant r<k$ it is shown that $\Lambda_{k}(A) \subseteq \Lambda_{k-r}(A+F)$ for any operator $F$ with $\operatorname{rank}(F) \leqslant r$. In quantum computing, this result implies that a quantum channel with a $k$-dimensional error correcting code under a perturbation of rank at most $r$ will still have a $(k-r)$ dimensional error correcting code. Moreover, it is shown that if $A$ is normal or if the dimension of $A$ is finite, then $\Lambda_{k}(A)$ can be obtained as the intersection of $\Lambda_{k-r}(A+F)$ for a collection of rank $r$ operators $F$. Examples are given to show that the result fails if $A$ is a general operator. The closure and the interior of the convex set $\Lambda_{k}(A)$ are completely determined. Analogous results are obtained for $\Lambda_{\infty}(A)$ defined as the set of scalars $\lambda$ such that $P A P=\lambda P$ for an infinite rank orthogonal projection $P$. It is shown that $\Lambda_{\infty}(A)$ is the intersection of all $\Lambda_{k}(A)$ for $k=1,2, \ldots$ If $A-\mu I$ is not compact for all $\mu \in \mathbb{C}$, then the closure and the interior of $\Lambda_{\infty}(A)$ coincide with those of the essential numerical range of $A$. The situation for the special case when $A-\mu I$ is compact for some $\mu \in \mathbb{C}$ is also studied.


Published by Elsevier Inc.

## 1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with $M_{n}$ if $\mathcal{H}$ has dimension $n$. For $k \leqslant \operatorname{dim} \mathcal{H}$, define the rank-k numerical range of $A \in \mathcal{B}(\mathcal{H})$ by

$$
\Lambda_{k}(A)=\{\lambda \in \mathbb{C}: P A P=\lambda P \text { for some rank- } k \text { orthogonal projection } P \in \mathcal{B}(\mathcal{H})\}
$$

Note that we allow $k=\infty$ if $\operatorname{dim} \mathcal{H}=\infty$. Evidently, $\lambda \in \Lambda_{k}(A)$ if and only if there is an orthogonal basis of $\mathcal{H}$ such that $\lambda I_{k}$ is the leading principal submatrix of the operator matrix of $A$ with respect to the basis; equivalently, there is an isometry $X: \mathbb{C}^{k} \rightarrow \mathcal{H}$ such that $X^{*} A X=\lambda I_{k}$. (For $k=\infty$, we take $X: \ell^{2} \rightarrow \mathcal{H}$.) When $k=1$, this concept reduces to the classical numerical range of $A$ defined by

$$
W(A)=\{\langle A x, x\rangle: x \in \mathcal{H},\langle x, x\rangle=1\}
$$

which is useful in studying operators and matrices; for example see [10].

[^0]The higher rank numerical range was introduced in connection to the construction of quantum error correction code in the study of quantum information theory; see [7]. In quantum computing, information is stored in qubits (quantum bits). Mathematically, the state of a qubit is represented by a $2 \times 2$ rank one Hermitian matrix $Q$ satisfying $Q^{2}=Q$. A state of $N$-qubits $Q_{1}, \ldots, Q_{N}$ is represented by their tensor products in $M_{n}$ with $n=2^{N}$. A quantum channel for states of $N$-qubits corresponds to trace preserving completely positive linear map $\Phi: M_{n} \rightarrow M_{n}$. By the structure theory of completely positive linear map [3], there are $T_{1}, \ldots, T_{m} \in M_{n}$ with $\sum_{j=1}^{m} T_{j}^{*} T_{j}=I_{n}$ such that

$$
\begin{equation*}
\Phi(X)=\sum_{j=1}^{m} T_{j} X T_{j}^{*} \tag{1.1}
\end{equation*}
$$

Let $\mathbf{V}$ be a subspace of $\mathbb{C}^{n}$ and $P_{\mathbf{V}}$ the orthogonal projection of $\mathbb{C}^{n}$ onto $\mathbf{V}$. Then $\mathbf{V}$ is a quantum error correction code for $\Phi$ if there exists a trace preserving completely positive linear map $\Psi: M_{n} \rightarrow M_{n}$ such that $\Psi \circ \Phi(A)=A$ for all $A \in P_{\mathbf{V}} M_{n} P_{\mathbf{V}}$. This happens if and only if there are scalars $\gamma_{i j}$ with $1 \leqslant i, j \leqslant m$ such that

$$
P_{\mathbf{v}} T_{i}^{*} T_{j} P_{\mathbf{v}}=\gamma_{i j} P_{\mathbf{v}}, \quad 1 \leqslant i, j \leqslant m
$$

see $[7,11]$. It turns out that even for a single matrix $A$, determining $\Lambda_{k}(A)$ is highly non-trivial, and the results are useful in quantum computing, say, in constructing binary unitary channels; see [5]. In a sequence of papers [4-7,9,12,13,16], researchers studied the set $\Lambda_{k}(A)$ for $A \in \mathcal{B}(\mathcal{H})$. Many interesting results (see (P1)-(P8) below) were obtained.

In the study of operator theory and applications, it is often useful to study the properties of an operator which are stable under different kinds of perturbation. For example, the essential numerical range of an infinite dimensional operator $A \in \mathcal{B}(\mathcal{H})$ can be defined as

$$
\begin{equation*}
W_{e}(A)=\bigcap\{\mathbf{C l}(W(A+F)): F \in \mathcal{B}(\mathcal{H}) \text { has finite rank }\} \tag{1.2}
\end{equation*}
$$

which captures many important properties of $A$ (see $[1,8,15,17]$ ). Here $\mathbf{C l}(S)$ denotes the closure of the set $S$. In fact, one can include all compact operators $F$ in $\mathcal{B}(\mathcal{H})$ on the right-hand side of (1.2). If $\mathcal{K}$ is the algebra of compact operators in $\mathcal{B}(\mathcal{H})$ and if $\psi: \mathcal{B}(\mathcal{H}) \mapsto \mathcal{B}(\mathcal{H}) / \mathcal{K}$ is the canonical homomorphism of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H}) / \mathcal{K}$, then $W_{e}(A)$ is the closure of the numerical range of $\psi(A)$. In [1, Theorem 4], it was also proven that

$$
\begin{equation*}
\Lambda_{\infty}(A)=\bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { has finite rank }\} \tag{1.3}
\end{equation*}
$$

In this paper, we study the change of the higher rank numerical range of an operator under low rank perturbation. For instance, we show in Theorem 3.1 that for $1 \leqslant r<k<\infty$, if $A, F \in \mathcal{B}(\mathcal{H})$ with $\operatorname{rank}(F) \leqslant r$, then

$$
\begin{equation*}
\Lambda_{k}(A) \subseteq \Lambda_{k-r}(A+F) \tag{1.4}
\end{equation*}
$$

In Theorem 5.1, we refine the set equalities (1.2) and (1.3) by using smaller sets of operators $F$ for the intersection on the right-hand side of the equalities.

It is worth noting that the inclusion (1.4) has the following implication in the theory of quantum computing. Suppose $A \in M_{n}$ corresponds to a quantum channel with a $k$-dimensional error correcting code (realized as a subspace of $\mathbb{C}^{n}$ ), then for any perturbation of the channel $A$ by an operator $F$ of rank bounded by $r$, the resulting channel $A+F$ will have a ( $k-r$ )-dimensional error correcting code. More generally, if the matrices $T_{1}, \ldots, T_{m}$ correspond to quantum channel (1.1) with a $k$-dimensional error correcting code, and if $T_{j}$ is changed to $T_{j}+F_{j}$ such that the sum of the range spaces of

$$
\left(T_{i}+F_{i}\right)^{*}\left(T_{j}+F_{j}\right)-T_{i}^{*} T_{j}=T_{i}^{*} F_{j}+F_{i} T_{j}^{*}+F_{i} F_{j}^{*}, \quad 1 \leqslant i, j \leqslant n
$$

has dimension bounded by $r$, then the resulting quantum channel will still have a $(k-r)$-dimensional error correcting code.
Our paper is organized as follows. First, we study $\Lambda_{k}(A)$ for $A \in \mathcal{B}(\mathcal{H})$ when $k$ is finite in Sections 2-4. In Section 2, we give a complete description of the closure and interior of $\Lambda_{k}(A)$. In Section 3, we establish inclusion (1.4) for any operators $A, F \in \mathcal{B}(\mathcal{H})$ with $\operatorname{rank}(F) \leqslant r$, where $1 \leqslant r<k<\infty$. It follows that

$$
\begin{equation*}
\Lambda_{k}(A) \subseteq \bigcap\left\{\Lambda_{k-r}(A+F): F \in \mathcal{B}(\mathcal{H}) \text { has rank } \leqslant r\right\} . \tag{1.5}
\end{equation*}
$$

In particular, taking $r=k-1$, we have

$$
\begin{equation*}
\Lambda_{k}(A) \subseteq \bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { has rank }<k\} \tag{1.6}
\end{equation*}
$$

We show that the inclusions in (1.5) and (1.6) become inequalities if $\operatorname{dim} \mathcal{H}$ is finite. Examples are given to show that these are not true for infinite dimensional operators. Nevertheless, we show that equalities also hold in (1.5) and (1.6) for infinite dimensional normal operators in Section 4. The set equalities in (1.5) and (1.6) can be viewed as refinements of (1.3). Similar set equality results are given in Corollary 3.3, which can be viewed as refinements of (1.2). In Section 5, we extend the results in Sections $2-4$ to $\Lambda_{\infty}(A)$. In particular, we show in Theorem 5.1 and 5.2 that

$$
\begin{equation*}
\Lambda_{\infty}(A)=\bigcap_{k \geqslant 1} \Lambda_{k}(A)=\bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { has finite rank }\}, \tag{1.7}
\end{equation*}
$$

and $\Lambda_{\infty}(A) \neq \emptyset$ if and only if the closure of $\Lambda_{\infty}(A)$ is the essential numerical range of $A$. Then we determine the condition under which $\Lambda_{\infty}(A)$ is non-empty. The first equality in (1.7) gives an affirmative answer to a question of MartinezAvendano [14].

We close this section by listing some basic properties for the higher rank numerical range; see [4-7,9,12,13,16].
(P1) For any $a, b \in \mathbb{C}, \Lambda_{k}(a A+b I)=a \Lambda_{k}(A)+b$.
(P2) For any unitary $U \in \mathcal{B}(\mathcal{H}), \Lambda_{k}\left(U^{*} A U\right)=\Lambda_{k}(A)$.
(P3) If $A_{0}$ is a compression of $A$ on a subspace $\mathcal{H}_{0}$ of $\mathcal{H}$ such that $\operatorname{dim} \mathcal{H}_{0} \geqslant k$, then $\Lambda_{k}\left(A_{0}\right) \subseteq \Lambda_{k}(A)$.
(P4) Suppose $\operatorname{dim} \mathcal{H}<2 k$. The set $\Lambda_{k}(A)$ has at most one element.
(P5) If $\operatorname{dim} \mathcal{H} \geqslant 3 k-2$, then $\Lambda_{k}(A)$ is non-empty. Otherwise, there is $B \in \mathcal{B}(\mathcal{H})$ such that $\Lambda_{k}(B)=\emptyset$.
(P6) $\Lambda_{k}(A)$ is always convex.
(P7) If $A \in M_{n}$, then $\Lambda_{k}(A)=\Omega_{k}(A)$ with

$$
\Omega_{k}(A)=\bigcap_{\xi \in[0,2 \pi)}\left\{\mu \in \mathbb{C}: e^{i \xi} \mu+e^{-i \xi} \bar{\mu} \leqslant \lambda_{k}\left(e^{i \xi} A+e^{-i \xi} A^{*}\right)\right\},
$$

where $\lambda_{k}(H)$ denotes the $k$ th largest eigenvalue of the Hermitian matrix $H \in M_{n}$.
(P8) If $A \in M_{n}$ is a normal matrix with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$, then

$$
\Lambda_{k}(A)=\bigcap_{1 \leqslant j_{1}<\cdots<j_{n-k+1} \leqslant n} \operatorname{conv}\left\{\lambda_{j_{1}}, \ldots, \lambda_{j_{n-k+1}}\right\}
$$

## 2. The interior and closure of $\Lambda_{k}(A)$

First, we extend the definition of $\Omega_{k}(A)$ to infinite dimensional operators. For a self-adjoint operator $H$, let

$$
\lambda_{k}(H)=\sup \left\{\lambda_{k}\left(X^{*} H X\right): X \text { is an isometry from } \mathbb{C}^{k} \text { to } \mathcal{H} \text { so that } X^{*} X=I_{k}\right\} .
$$

For $A \in \mathcal{B}(\mathcal{H})$, let $\operatorname{Re}(A)=\left(A+A^{*}\right) / 2$ be the real part of $A$ and

$$
\Omega_{k}(A)=\bigcap_{\xi \in[0,2 \pi)}\left\{\mu \in \mathbb{C}: \operatorname{Re}\left(e^{i \xi} \mu\right) \leqslant \lambda_{k}\left(\operatorname{Re}\left(e^{i \xi} A\right)\right)\right\}
$$

By definition, $\Omega_{k}(A)$ is a compact convex set. It may be empty if $\operatorname{dim} \mathcal{H} \leqslant 3 k-3$; see [5, Theorem 4.7]. In the finite dimensional case, we have $\Lambda_{k}(A)=\Omega_{k}(A)$ as noted in property (P7). Let $A=I_{k} \oplus \operatorname{diag}(1,1 / 2, \ldots)$. One easily checks that $\Omega_{k}(A)=[0,1]$ and $\Lambda_{k}(A)=(0,1]$. (See also Example 3.5.) Hence, property (P7) may not hold for infinite dimensional operator $A$.

We continue to use $\mathbf{C l}(S)$ to denote the closure of a set $S$ in $\mathbb{C}$. Let $\boldsymbol{\operatorname { I n t }}(S)$ denote the relative interior of $S$. We have the following.

Theorem 2.1. Let $A \in \mathcal{B}(\mathcal{H})$ be an infinite dimensional operator, and let $k$ be a positive integer. Then

$$
\boldsymbol{\operatorname { I n t }}\left(\Omega_{k}(A)\right) \subseteq \Lambda_{k}(A) \subseteq \Omega_{k}(A)=\mathbf{C l}\left(\Lambda_{k}(A)\right)
$$

Proof. First, we establish the inclusion $\Lambda_{k}(A) \subseteq \Omega_{k}(A)$. By [12, Corollary 4], $\Lambda_{k}(A)$ is always non-empty. Suppose $\mu \in \Lambda_{k}(A)$. Then there is an isometry $X: \mathbb{C}^{k} \rightarrow \mathcal{H}$ such that $X^{*} X=I_{k}$ and $X^{*} A X=\mu I_{k}$. As a result, for any $t \in[0,2 \pi)$ we have

$$
\operatorname{Re}\left(e^{i t} \mu\right) \leqslant \lambda_{k}\left(\operatorname{Re}\left(e^{i t} A\right)\right)
$$

Thus, $\mu \in \Omega_{k}(A)$.
Next, we turn to the equality $\Omega_{k}(A)=\mathbf{C l}\left(\Lambda_{k}(A)\right)$ and the inclusion $\operatorname{Int}\left(\Omega_{k}(A)\right) \subseteq \Lambda_{k}(A)$. By Corollary 4 in [12], $\Lambda_{k}(A)$ is non-empty. We consider three cases.

Case 1. Suppose $\Omega_{k}(A)$ is a singleton. Then $\Lambda_{k}(A)=\Omega_{k}(A)$ because $\Lambda_{k}(A)$ is non-empty, and $\boldsymbol{\operatorname { I n t }}\left(\Lambda_{k}(A)\right)=\boldsymbol{\operatorname { I n t }}\left(\Omega_{k}(A)\right)=\emptyset$.
Case 2. Suppose $\Omega_{k}(A)$ has non-empty interior in $\mathbb{C}$. Let $\mu$ be an interior point of $\Omega_{k}(A)$. We may replace $A$ by $A-\mu I$ and assume that $\mu=0$, i.e., $0 \in \boldsymbol{\operatorname { I n t }}\left(\Omega_{k}(A)\right)$. Therefore, there exists $d>0$ such that

$$
\{\mu \in \mathbb{C}:|\mu| \leqslant d\} \subseteq \Omega_{k}(A)
$$

Thus, for all $t \in[0,2 \pi), \mu=d e^{i t} \in \Omega_{k}(A)$. Write $A=H+i G$ where $H$ and $G$ are self-adjoint. Then

$$
e^{-i t} A+e^{i t} A^{*}=2(\cos t H+\sin t G)
$$

Hence,

$$
\lambda_{k}\left(e^{-i t} A+e^{i t} A^{*}\right) \geqslant e^{-i t} \mu+e^{i t} \bar{\mu} \quad \Rightarrow \quad \lambda_{k}(\cos t H+\sin t G) \geqslant d
$$

Then, for each $t \in[0,2 \pi)$ there is $X_{t}: \mathbb{C}^{k} \rightarrow \mathcal{H}$ with $X_{t}^{*} X_{t}=I_{k}$ such that $\lambda_{k}\left(\cos t X_{t}^{*} H X_{t}+\sin t X_{t}^{*} G X_{t}\right)>d / 2$. Furthermore, there is $\delta_{t}>0$ such that for each $s \in\left(t-\delta_{t}, t+\delta_{t}\right)$,

$$
\left\|\left(\cos t X_{t}^{*} H X_{t}+\sin t X_{t}^{*} G X_{t}\right)-\left(\cos s X_{t}^{*} H X_{t}+\sin s X_{t}^{*} G X_{t}\right)\right\|<d / 4 .
$$

Note that $\left|\lambda_{k}(R)-\lambda_{k}(S)\right| \leqslant\|R-S\|$ for any two Hermitian matrices $R$ and $S$ by the Weyl's inequality; for example, see [2, III.2]. It follows that

$$
\left|\lambda_{k}\left(\cos t X_{t}^{*} H X_{t}+\sin t X_{t}^{*} G X_{t}\right)-\lambda_{k}\left(\cos s X_{t}^{*} H X_{t}+\sin s X_{t}^{*} G X_{t}\right)\right|<d / 4
$$

Consequently,

$$
\lambda_{k}\left(\cos s X_{t}^{*} H X_{t}+\sin s X_{t}^{*} G X_{t}\right)>\lambda_{k}\left(\cos t X_{t}^{*} H X_{t}+\sin t X_{t}^{*} G X_{t}\right)-d / 4>d / 4
$$

Since $[0,2 \pi]$ is compact, there exists a finite sequence $0 \leqslant t_{1}<\cdots<t_{m}<2 \pi$ so that

$$
[0,2 \pi] \subseteq \bigcup_{j=1}^{m}\left(t_{j}-\delta_{t_{j}}, t_{j}+\delta_{t_{j}}\right)
$$

Let $A_{0}=H_{0}+i G_{0}$ be a compression of $A$ onto a subspace spanned by the range spaces of $X_{t_{1}}, \ldots, X_{t_{m}}$. Then $\lambda_{k}\left(\cos t H_{0}+\right.$ $\left.\sin t G_{0}\right)>d / 4$ for all $t \in[0,2 \pi)$ and so $0 \in \Omega_{k}\left(A_{0}\right)$. Thus, $0 \in \Lambda_{k}\left(A_{0}\right) \subseteq \Lambda_{k}(A)$ by Theorem 2.2 in [13]. Hence, $\operatorname{Int}\left(\Omega_{k}(A)\right) \subseteq$ $\Lambda_{k}(A)$ and thus $\mathbf{C l}\left(\Lambda_{k}(A)\right)=\Omega_{k}(A)$.

Case 3. Suppose $\Omega_{k}(A)$ is not a singleton and has no interior in $\mathbb{C}$. Since $\Omega_{k}(A)$ is a compact convex set in $\mathbb{C}$, if it is not a singleton and has no interior in $\mathbb{C}$, then it is a non-degenerate line segment. We will show that $\Lambda_{k}(A)$ contains all the relative interior points of $\Omega_{k}(A)$. The result will then follow.

Assume $\gamma$ is a (relative) interior point of the line segment. By property ( P 1 ), we may assume that $[-1,1] \subseteq \Omega_{k}(A) \subseteq \mathbb{R}$ and $\gamma=0$. Write $A=H+i G$ where $H$ and $G$ are self-adjoint. Since $-1,1 \in \Omega_{k}(A)$, we have $\lambda_{k}(\cos t H+\sin t G) \geqslant|\cos t|$ for all $t \in[0,2 \pi]$. We claim that $\lambda_{k}(G)=0$. If it is not true, then there exist $\varepsilon, \delta>0$ such that $\lambda_{k}(\cos t H+\sin t G) \geqslant \varepsilon>0$ for each $t \in[\pi / 2-\delta, \pi / 2+\delta]$. By decreasing $\varepsilon$, if necessary, we may assume that $|\cos (\pi / 2+\delta)|=|\cos (\pi / 2-\delta)| \geqslant \varepsilon$. Therefore, we have $\lambda_{k}(\cos t H+\sin t G) \geqslant \varepsilon$ for all $t \in[0, \pi]$. Let $\mu=i \varepsilon$. Then, we have

$$
\operatorname{Re}\left(\mu e^{-i t}\right) \leqslant \begin{cases}\varepsilon \leqslant \lambda_{k}(\cos t H+\sin t G) & \text { if } t \in[0, \pi] \\ 0 \leqslant \lambda_{k}(\cos t H+\sin t G) & \text { if } t \in[\pi, 2 \pi]\end{cases}
$$

Therefore, $i \varepsilon \in \Omega_{k}(A)$. This contradicts that $\Omega_{k}(A)$ is a line segment in $\mathbb{R}$. Similarly, we can show that $\lambda_{k}(-G)=0$. So, we may assume that $G$ has operator matrix $D \oplus 0$ with

$$
D=\operatorname{diag}\left(d_{1}, \ldots, d_{p+q}\right)
$$

such that $d_{1}, \ldots, d_{p}>0$ and $d_{p+1}, \ldots, d_{p+q}<0$, where $p<k$ and $q<k$. Let $H_{0}$ and $A_{0}$ be the compressions of $H$ and $A$ to the kernel of $G$, respectively.

Suppose $\lambda_{k}\left(H_{0}\right)>0$ and $\lambda_{k}\left(-H_{0}\right)>0$. Then $H_{0}$ has a compression $\tilde{H}_{0} \in M_{2 k}$ such that $\tilde{H}_{0}$ has $k$ positive eigenvalues and $k$ negative eigenvalues. Clearly, $0 \in \Lambda_{k}\left(\tilde{H}_{0}\right)$ and $\tilde{H}_{0}$ is also a compression of $A$. Then $0 \in \Lambda_{k}\left(H_{0}\right) \subseteq \Lambda_{k}(A)$. So, we assume that $\lambda_{k}\left(H_{0}\right) \leqslant 0$ without loss of generality.

Suppose the kernel of $H_{0}$ has dimension at least $k$. Then again we have $0 \in \Lambda_{k}\left(H_{0}\right)=\Lambda_{k}\left(A_{0}\right) \subseteq \Lambda_{k}(A)$. Thus, we may assume that the kernel of $H_{0}$ has dimension less than $k$. Then $H_{0}$ has operator matrix of the form

$$
H_{22} \oplus H_{33}
$$

so that $H_{22} \in M_{r}$, with $r<2 k-1$ is positive semi-definite and $H_{33}$ is negative definite such that the kernel of $H_{33}$ is the zero space. Clearly, there is a negative real number in $\Lambda_{k}\left(H_{33}\right) \subseteq \Lambda_{k}\left(A_{0}\right) \subseteq \Lambda_{k}(A)$. We will show that $\Lambda_{k}(A)$ also contains a positive real number. By the convexity of $\Lambda_{k}(A)$, it will then follow that $0 \in \Lambda_{k}(A)$.

Note that 0 is an interior point, and $H_{22}$ is finite dimensional. We may find a small $\varepsilon>0$ such that $\varepsilon \in \Omega_{k}(A)$ and $H_{0}-\varepsilon I=\hat{H}_{22} \oplus \hat{H}_{33}$ so that $\hat{H}_{22}$ is positive semi-definite and $\hat{H}_{33}$ is negative definite bounded above by $-\varepsilon<0$. Thus, $\hat{H}_{33}$ is invertible, and there is an orthonormal basis of $\mathcal{B}(\mathcal{H})$ so that the operator matrices of $G$ and $\hat{H}=H-\varepsilon I$ equal

$$
D \oplus 0 \quad \text { and }\left[\begin{array}{ccc}
\hat{H}_{11} & \hat{H}_{12} & \hat{H}_{13} \\
\hat{H}_{21} & \hat{H}_{22} & 0 \\
\hat{H}_{31} & 0 & \hat{H}_{33}
\end{array}\right]
$$

for $\hat{H}_{22} \in M_{r^{\prime}}$ with $r^{\prime} \leqslant r$. For notational simplicity, we rename $r^{\prime}$ as $r$. Suppose $S \in \mathcal{B}(\mathcal{H})$ has operator matrix

$$
\left[\begin{array}{ccc}
I_{p+q} & 0 & -\hat{H}_{13} \hat{H}_{33}^{-1} \\
0 & I_{r} & 0 \\
0 & 0 & I
\end{array}\right]
$$

Then SGS* and $S \hat{H} S^{*}$ have operator matrices

$$
D \oplus 0 \text { and }\left[\begin{array}{cc}
\hat{H}_{11}-\hat{H}_{13} \hat{H}_{33}^{-1} \hat{H}_{31} & \hat{H}_{12} \\
\hat{H}_{21} & \hat{H}_{22}
\end{array}\right] \oplus \hat{H}_{33}
$$

Since $0 \in \Omega_{k}(A-\varepsilon I)$, we see that for each $t \in[0,2 \pi)$, we have $\lambda_{k}(\cos t \hat{H}+\sin t G) \geqslant 0$ and hence $\lambda_{k}\left(\cos t S \hat{H} S^{*}+\right.$ $\left.\sin t S G S^{*}\right) \geqslant 0$. Consequently, if we let $\tilde{H}_{33}$ be the leading $k \times k$ submatrix of $\hat{H}_{33}$ and let $\tilde{A}=\tilde{H}+i \tilde{G} \in M_{p+q+r+k}$ with

$$
\tilde{H}=\left[\begin{array}{cc}
\hat{H}_{11}-\hat{H}_{13} \hat{H}_{33}^{-1} \hat{H}_{31} & \hat{H}_{12} \\
\hat{H}_{21} & \hat{H}_{22}
\end{array}\right] \oplus \tilde{H}_{33} \quad \text { and } \quad \tilde{G}=D \oplus 0_{r+k},
$$

then $\lambda_{k}(\cos t \tilde{H}+\sin t \tilde{G}) \geqslant 0$ for all $t \in[0,2 \pi)$ and hence $0 \in \Omega_{k}(\tilde{A})$. By Theorem 2.2 in [13], there is a $(p+q+r+k) \times k$ matrix $X$ such that

$$
X^{*} X=I_{k} \quad \text { and } \quad X^{*} \tilde{A} X=0_{k}
$$

Consequently, as $\tilde{A}$ is a finite compression of $S(A-\varepsilon I) S^{*}$, there is a partial isometry $Y: \mathbb{C}^{k} \rightarrow \mathcal{H}$ such that $Y^{*} S(A-\varepsilon I) S^{*} Y=0_{k}$. Note that $S^{*} Y=Z T$ with $Z^{*} Z=I_{k}$ for some invertible $T \in M_{k}$. Thus, $Z^{*}(A-\varepsilon I) Z=0_{k}$, i.e., $\varepsilon \in \Lambda_{k}(A)$.

In the finite dimensional case, $\Lambda_{k}(A)$ is always closed. If $\operatorname{dim} \mathcal{H}$ is uncountable, then for any bounded convex set $S$ in $\mathbb{C}$ one can construct a normal operator $B$ using the points in $S$ as eigenvalues so that $\Lambda_{k}(A)=S$ for $A=B \otimes I_{k}$. In the following, we give examples of $A$ acting on a separable Hilbert space such that $\Lambda_{k}(A)$ has non-empty interior with no, some or all its boundary points. It is known that $\Lambda_{k}(A)$ is a singleton if $A$ is a scalar operator, and that $\Lambda_{k}(A) \subseteq \mathbb{R}$ if $A=A^{*}$. We give examples different from these trivial cases.

Example 2.2. In the following examples, let $B=\left[\begin{array}{ll}0 & 2 \\ 0 & 0\end{array}\right]$.
(a) Let $A=B \otimes I_{k-1} \oplus 0$. Then $\Omega_{k}(A)=\Lambda_{k}(A)=\{0\}$.
(b) Let $A=B \otimes I_{k-1} \oplus \operatorname{diag}(1,1 / 2,1 / 3, \ldots)$. Then $\Lambda_{k}(A)=(0,1]$. One can easily modify the example so that $\Lambda_{k}(A)=[0,1]$ or $\Lambda_{k}(A)=(0,1)$.
(c) Let $A=B \otimes I_{k-1} \oplus C$. If $C=B \oplus 0$ then $\Lambda_{k}(A)$ is the closed unit disk; if $C$ is the unilateral shift, then $\Lambda_{k}(A)$ is the open unit disk; if $C=\operatorname{diag}(-1, i,-i, 1 / 2,2 / 3,3 / 4,4 / 5, \ldots)$ then $\Omega_{k}(A)$ is the convex hull of $\{-1, i,-i, 1\}$, and $\Lambda_{k}(A)$ is the union of the interior of $\Omega_{k}(A)$ and the convex hull of $\{-1, i,-i\}$.

## 3. Low rank perturbations of general operators

For a positive integer $r$, let $\mathcal{F}_{r}$ be the set of operators in $\mathcal{B}(\mathcal{H})$ with rank at most $r$, and let $\mathcal{P}_{r}$ be the set of rank $r$ orthogonal projections in $\mathcal{B}(\mathcal{H})$.

Theorem 3.1. Let $1 \leqslant r<k<\infty$. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $F \in \mathcal{F}_{r}$. Then $\Lambda_{k}(A) \subseteq \Lambda_{k-r}(A+F)$. Consequently,

$$
\Lambda_{k}(A) \subseteq \bigcap\left\{\Lambda_{k-r}(A+F): F \in \mathcal{F}_{r}\right\} .
$$

Proof. Suppose $\lambda \in \Lambda_{k}(A)$. Let $X: \mathbb{C}^{k} \rightarrow \mathcal{H}$ be an isometry such that $X^{*} A X=\lambda I_{k}$. Then $X^{*} F X$ has rank at most $r$. There is a unitary $U \in M_{k}$ such that

$$
U^{*} X^{*} F X U=\left[\begin{array}{cc}
0_{k-r} & * \\
0 & *
\end{array}\right]
$$

Let $U_{1}$ be obtained by taking the first $k-r$ column of $U$, and $V=X U_{1}$. Then $V^{*}(A+F) V=\lambda I_{k-r}$ so that $\lambda \in$ $\Lambda_{k-r}(A+F)$.

Note that one can easily adapt the above proof to show that for $A_{1}, \ldots, A_{m} \in \mathcal{B}(\mathcal{H})$, if $X^{*} A_{j} X=\lambda_{j} I_{k}$ with $X^{*} X=I_{k}$ and if $F_{1}, \ldots, F_{m} \in \mathcal{B}(\mathcal{H})$ are such that

$$
U^{*} X^{*} F_{j} X U=\left[\begin{array}{cc}
0_{k-r} & * \\
0 & *
\end{array}\right], \quad j=1, \ldots, m,
$$

then $V^{*}\left(A_{j}+F_{j}\right) V=\lambda_{j} I_{k-r}$ for all $j=1, \ldots, m$. So, the comment about a low rank perturbation of a quantum channel in Section 1 follows.

If $1 \leqslant r<k<\infty$ and $A \in \mathcal{B}(\mathcal{H})$, then $\Omega_{k}(A)$ can be written as the intersection of $\Omega_{k-r}(A+F)$ for a collection of rank $r$ operators $F$ as shown in the following.

Theorem 3.2. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leqslant r<k<\infty$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_{r}$ containing the set $\mathcal{S}_{0}=\left\{2 e^{i \xi}\|A\| P: P \in \mathcal{P}_{r}\right.$ and $\xi \in$ $[0,2 \pi)\}$. Then

$$
\Omega_{k}(A)=\bigcap\left\{\Omega_{k-r}(A+F): F \in \mathcal{S}\right\} .
$$

Proof. The inclusion ( $\subseteq$ ) follows from Theorem 3.1 and the fact that $\Omega_{k}(A)=\mathbf{C l}\left(\Lambda_{k}(A)\right)$ by Theorem 2.1.
Suppose $\lambda \notin \Omega_{k}(A)$. Then there exists $t \in \mathbb{R}$ such that $\lambda_{k}\left(\operatorname{Re}\left(e^{i t} A\right)\right)<\operatorname{Re}\left(e^{i t} \lambda\right)$. Let $e^{i t} A=H+i G$ be with $H=H^{*}$ and $G=G^{*}$. Then $H$ has an operator matrix $\operatorname{diag}\left(\lambda_{1}, \ldots, \lambda_{m}\right) \oplus H_{2}$ with $m \leqslant k-1$ such that $\sup \sigma\left(H_{2}\right)<\operatorname{Re}\left(e^{i t} \lambda\right)$. Let $F=$ $-2 e^{-i t}\|A\|\left(I_{r} \oplus 0\right) \in \mathcal{B}(\mathcal{H})$. Then $\lambda_{k-r}\left(\operatorname{Re}\left(e^{i t}(A+F)\right)\right)<\operatorname{Re}\left(e^{i t} \lambda\right)$. Hence, $\lambda \notin \Omega_{k-r}(A+F)$.

Note that for the set $\mathcal{S}$ in the above theorem, we can take the whole $\mathcal{F}_{r}$ or the much smaller subset $\mathcal{S}_{0}$. We have the following corollary.

Corollary 3.3. Under the same setting as in Theorem 3.2. Each of the following sets is equal to $\Omega_{k}(A)$.
(a) $\bigcap\left\{\Omega_{k-1}\left(A+2 e^{i \xi}\|A\| P\right): \xi \in[0,2 \pi), P \in \mathcal{P}_{1}\right\}$.
(b) $\bigcap\left\{\Omega_{1}\left(A+2 e^{i \xi}\|A\| P\right): \xi \in[0,2 \pi), P \in \mathcal{P}_{k-1}\right\}$.

With Theorem 2.1, the above result also holds if we replace $\Omega_{m}(B)$ by $\mathbf{C l}\left(\Lambda_{m}(B)\right)$. Using the fact that $\Lambda_{k}(A)=\Omega_{k}(A)$ when $A \in M_{n}$, we have the following result.

Theorem 3.4. Suppose $A \in M_{n}$ and $1 \leqslant r<k \leqslant n$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_{r}$ containing the set $\mathcal{S}_{0}=\left\{2 e^{i \xi}\|A\| P: P \in \mathcal{P}_{r}\right.$ and $\xi \in$ $[0,2 \pi)\}$. Then
(a) $\Lambda_{k}(A)=\bigcap\left\{\Lambda_{k-r}(A+F): F \in \mathcal{S}\right\}$.
(b) $\Lambda_{k}(A)=\bigcap\left\{\Lambda_{k-1}\left(A+2 e^{i \xi}\|A\| P\right): \xi \in[0,2 \pi), P \in \mathcal{P}_{1}\right\}$.
(c) $\Lambda_{k}(A)=\bigcap\left\{W\left(A+2 e^{i \xi}\|A\| P\right): \xi \in[0,2 \pi), P \in \mathcal{P}_{k-1}\right\}$.

The following example shows that Theorem 3.4 does not hold for infinite dimensional operators.
Example 3.5. Let $A=A_{1} \oplus A_{2}$, where

$$
A_{1}=\left[\begin{array}{cc}
0 & i \\
i & 2
\end{array}\right] \quad \text { and } \quad A_{2}=\operatorname{diag}\left(b_{2}, \bar{b}_{2}, b_{3}, \bar{b}_{3}, \ldots\right) \oplus \operatorname{diag}\left(b_{2}, \bar{b}_{2}, b_{3}, \bar{b}_{3}, \ldots\right)
$$

with $b_{m}=-1+e^{i \pi / m}$ for $m=2, \ldots$. Then $0 \in \mathbf{C l}\left(\Lambda_{2}(A)\right)$ and $0 \notin \Lambda_{2}(A)$, but $0 \in \bigcap\{W(A+F): F$ is rank one $\}$.
Verification. Note that every $\mu \in \Lambda_{1}\left(A_{2}\right)$ is an element of $\Lambda_{2}\left(A_{2}\right)$, and hence $\Lambda_{1}\left(A_{2}\right)=\Lambda_{2}\left(A_{2}\right)$. Clearly, $0 \in \mathbf{C l}\left(\Lambda_{1}(A)\right)=$ $\mathbf{C l}\left(\Lambda_{2}(A)\right)$.

Next, we show that $0 \notin \Lambda_{2}(A)$. Suppose $0 \in \Lambda_{2}(A)$. Then $0 \in \Lambda_{2}(H)$ for $H=\left(A+A^{*}\right) / 2$. Let $U$ be unitary such that $U^{*} A U=\left[\begin{array}{c}0_{2} \\ * \\ *\end{array}\right]$. Then $U^{*} H U$ has the same form. Since $H$ has spectrum $\{2,0\} \cup\{-1+\cos \pi / m: m=2, \ldots\}$, we may assume that $U$ has the form [1] $\oplus U_{1}$ such that the $(1,1)$ entry of $U_{1}$ is nonzero. But then $U^{*} G U$ will have nonzero $(1,2)$ entry for $G=\left(A-A^{*}\right) /(2 i)$. This contradicts the fact that $U^{*} A U$ has zero $(1,2)$ entry. So, we see that $0 \notin \Lambda_{2}(A)$.

Now, suppose $F=\left[\begin{array}{ll}F_{11} & F_{12} \\ F_{21} & F_{22}\end{array}\right]$ is a rank one operator with $F_{11} \in M_{2}$. Let $x \in \mathbb{C}^{2}$ be a nonzero vector such that $F_{11} x=0$. If $x$ is a multiple of $e_{1}$, then the (1,1) entry of $A_{1}+F_{11}$ equals 0 and we have $0 \in W\left(A_{1}+F_{11}\right) \subseteq W(A+F)$. If $x$ is not a multiple of $e_{1}$, then $\mu_{0}=x^{*}\left(A_{1}+F_{11}\right) x=x^{*} A_{1} x \in W\left(A_{1}\right)$ has positive real part and $\mu_{0} \in W\left(A_{1}+F_{11}\right) \subseteq W(A+F)$. Since $F_{22}$ has rank at most one, by Theorem 3.1 we have

$$
W\left(A_{2}\right)=\Lambda_{2}\left(A_{2}\right) \subseteq \Lambda_{1}\left(A_{2}+F_{22}\right)=W\left(A_{2}+F_{22}\right) \subseteq W(A+F)
$$

So there exist $\mu_{1}, \mu_{2} \in W\left(A_{2}\right) \subseteq W(A+F)$ on the different sides of the line passing through $\mu_{0}$ and the origin. It follows that $0 \in \operatorname{conv}\left\{\mu_{0}, \mu_{1}, \mu_{2}\right\} \subseteq W(A+F)$ by the convexity of $W(A+F)$. Consequently, we have

$$
0 \in \bigcap\{W(A+F): F \text { has rank one }\} .
$$

## 4. Low rank perturbations of infinite dimensional normal operators

In the following, we prove that Theorem 3.4 is valid for (infinite dimensional) normal operators. We first establish some auxiliary results showing that one can refine the spectral decomposition of a normal operator using the geometrical information of its numerical range.

Let $\mathcal{P}=\{z \in \mathbb{C}: \operatorname{Im}(z)>0\}$ be the open upper half plane of $\mathbb{C}$. For $A \in \mathcal{B}(\mathcal{H})$ and $k \leqslant \operatorname{dim} \mathcal{H}$, let

$$
\mu_{k}(A, t)=\lambda_{k}\left(\left(e^{-i t} A-e^{i t} A^{*}\right) /(2 i)\right) .
$$

Notice also that

$$
\Omega_{k}(A)=\bigcap_{t \in[0,2 \pi)}\left\{\mu \in \mathbb{C}: \operatorname{Im}\left(e^{-i t} \mu\right) \leqslant \mu_{k}(A, t)\right\} .
$$

Lemma 4.1. Suppose $A \in \mathcal{B}(\mathcal{H})$ is normal. If $\mu_{m}(A, t) \leqslant 0$ for some $m \geqslant 1$ and $t \in \mathbb{R}$. Then $A$ has a decomposition $A_{1} \oplus A_{2} \oplus \hat{A}$ such that $\operatorname{dim} A_{1}<m$,

$$
W\left(A_{1}\right) \subseteq e^{i t} \mathcal{P}, \quad W\left(A_{2}\right) \subseteq-e^{i t} \mathcal{P} \quad \text { and } \quad W(\hat{A}) \subseteq e^{i t} \mathbb{R}
$$

Furthermore, if $\lambda_{\ell}\left(e^{-i t} \hat{A}+e^{i t} \hat{A}^{*}\right) / 2 \leqslant 0$ for some $\ell \geqslant 1$, then $\hat{A}$ has a decomposition $A_{3} \oplus A_{4} \oplus 0$ such that dim $A_{3}<\ell$,

$$
W\left(A_{3}\right) \subseteq e^{i t}(0, \infty), \quad \text { and } \quad W\left(A_{4}\right) \subseteq e^{i t}(-\infty, 0)
$$

Note that each of the summands $A_{1}, A_{2}, \hat{A}, A_{3}, A_{4}, 0$ may be vacuous.
Proof. Without loss of generality, we may assume that $t=0$. Let $A=H+i G$, where $H, G$ are self-adjoint. Then $G=$ $G_{1} \oplus G_{2} \oplus 0$ such that $G_{1}$ is positive definite with dimension $p<m$ and $G_{2}$ is negative definite. Let

$$
H=\left[\begin{array}{lll}
H_{11} & H_{12} & H_{13} \\
H_{12}^{*} & H_{22} & H_{23} \\
H_{13}^{*} & H_{23}^{*} & H_{33}
\end{array}\right]
$$

such that $H_{12}=[D \mid 0]$, where $D=\operatorname{diag}\left(d_{1}, \ldots, d_{p}\right)$ with $d_{1} \geqslant \cdots \geqslant d_{p} \geqslant 0$. Since $G H=H G$, it follows that $G_{1}[D \mid 0]=[D \mid$ $0] G_{2}$. Since $G_{1}$ is positive definite and $G_{2}$ is negative definite, the $(1,1)$ entry on the left side is nonnegative, and the $(1,1)$ entry on the right side is nonpositive. Thus, $d_{1}=0$ and hence $H_{12}=0$. Since $G_{1} H_{13}=0$ and $G_{2} H_{23}=0$, we have $H_{13}=0$ and $H_{23}=0$. So, $H=H_{11} \oplus H_{22} \oplus H_{33}$ and $A$ has asserted properties, with $A_{1}=H_{11}+i G_{1}, A_{2}=H_{22}+i G_{2}$, and $\hat{A}=H_{33}$.

If $\lambda_{\ell}\left(e^{-i t} \hat{A}+e^{i t} \hat{A}^{*}\right) / 2 \leqslant 0$ for some $\ell$, then we can apply the above result to $\hat{A}$ and get the desired decomposition for $\hat{A}$.

The following result [1, Lemma 2 and Corollary] will be needed in later discussion.
Lemma 4.2. Suppose $\operatorname{dim} \mathcal{H}$ is infinite. Let $T \in \mathcal{B}(\mathcal{H})$. Then the following are equivalent.
(a) $\lambda \in W_{e}(T)$.
(b) There is an orthonormal set $\left\{e_{n}\right\}$ such that $\left\langle T e_{n}, e_{n}\right\rangle \rightarrow \lambda$.
(c) There is a decomposition of $\mathcal{H}$ as $\mathcal{H}_{1} \oplus \mathcal{H}_{2}$ and a sequence $\left\{\lambda_{i}\right\}$ in $\mathbb{C}$, such that $\lambda_{i} \rightarrow \lambda$ and

$$
T=\left[\begin{array}{lll|l}
\lambda_{1} & & 0 & \\
& \lambda_{2} & & * \\
0 & & \ddots & \\
\hline & * & & *
\end{array}\right]
$$

Furthermore, if $\alpha, \beta \in W_{e}(T)$, then there exist two sequences $\left\{\alpha_{i}\right\}$ and $\left\{\beta_{i}\right\}$ in $\mathbb{C}$, such that $\alpha_{i} \rightarrow \alpha, \beta_{i} \rightarrow \beta$ and a decomposition of $\mathcal{H}$ as $\mathcal{H}_{1} \oplus \mathcal{H}_{2}$ such that

$$
T=\left[\begin{array}{lllll|l}
\alpha_{1} & & & & & \\
& \beta_{1} & & 0 & & \\
& & \alpha_{2} & & & * \\
& 0 & & \beta_{2} & & \\
& & & & \ddots & \\
\hline & & & & & *
\end{array}\right]
$$

In both cases, we may take $\mathcal{H}_{2}$ to be infinite dimensional.
Lemma 4.3. Suppose $T \in \mathcal{B}(\mathcal{H})$ is a normal operator such that for some $\pi \leqslant s_{1}<s_{2} \leqslant 2 \pi$,

$$
\sigma(T) \subseteq\left\{\rho e^{i t} \in \mathbb{C}: \rho>0, t \in\left[s_{1}, s_{2}\right]\right\}
$$

Let $k$ be a positive integer and $s_{3} \in\left(s_{1}, s_{2}\right)$,

$$
\mathcal{L}=\left\{\rho e^{i t} \in \mathbb{C}: \rho>0, t \in\left(s_{1}, s_{3}\right)\right\} \quad \text { and } \quad \mathcal{R}=\left\{\rho e^{i t} \in \mathbb{C}: \rho>0, t \in\left(s_{3}, s_{2}\right)\right\} .
$$

We have
(a) If $\mathcal{L} \cap \sigma(T)$ is infinite or contains an eigenvalue of $T$ with infinite multiplicity, then $T$ has a compression $T_{1} \in M_{k}$ such that $W\left(T_{1}\right) \subseteq \mathcal{L}$.
(b) If $\mathcal{R} \cap \sigma(T)$ is infinite or contains an eigenvalue of $T$ with infinite multiplicity, then $T$ has a compression $T_{2} \in M_{k}$ such that $W\left(T_{2}\right) \subseteq \mathcal{R}$.

If both hypotheses in (a) and (b) hold, then $T$ has a compression of the form $T_{1} \oplus T_{2}$ such that $\operatorname{dim} T_{1}=\operatorname{dim} T_{2}=k$ and

$$
W\left(T_{1}\right) \subseteq \mathcal{L} \quad \text { and } \quad W\left(T_{2}\right) \subseteq \mathcal{R}
$$

Proof. We will prove the last assertion. The proofs of (a) and (b) are similar. Suppose both $\mathcal{R} \cap \sigma(T)$ and $\mathcal{L} \cap \sigma(T)$ contain only isolated points of $\sigma(T)$. Then we can construct $T_{1}$ (respectively, $T_{2}$ ) from any $k$ (counting multiplicity) eigenvalues of $T$ in $\mathcal{L}$ (respectively, in $\mathcal{R}$ ) and the corresponding eigenvectors.

Suppose one of the sets $\mathcal{L} \cap \sigma(T)$ or $\mathcal{R} \cap \sigma(T)$, say $\mathcal{L} \cap \sigma(T)$, contains only isolated points of $\sigma(T)$, and the other set contains an accumulation point of $\sigma(T)$. Then we can construct $T_{1}$ from any $k$ eigenvalues of $T$ in $\mathcal{L}$ and the corresponding eigenvectors. Let $\mathcal{H}_{1}$ be the $k$-dimensional subspace spanned by the $k$ eigenvectors. Then with respect to the decomposition $\mathcal{H}=\mathcal{H}_{1} \oplus \mathcal{H}_{1}^{\perp}, T=T_{1} \oplus S$ for some normal $S$. Since $\mathcal{R}$ contains an accumulation point of $\sigma(S)$ and $S$ is normal, by Lemma 4.2, $S$ has a $k$-dimensional compression $T_{2}$ with $W\left(T_{2}\right) \subseteq \mathcal{R}$.

Finally, suppose both $\mathcal{L}$ and $\mathcal{R}$ contain an accumulation point of $\sigma(T)$. Then the result follows from the last statement in Lemma 4.2.

Theorem 4.4. Suppose $A \in B(\underset{\tilde{A}}{ })$ is normal. Then $\lambda \notin \Lambda_{k}(A)$ if and only if $A$ can be decomposed into $\tilde{A}_{1} \oplus \tilde{A}_{2}$ such that $\tilde{A}_{1}$ has dimension at most $k-1, W\left(\tilde{A}_{1}\right) \subseteq \lambda+S$ and $W\left(\tilde{A}_{2}\right) \subseteq \mathbb{C} \backslash(\lambda+S)$, where $S=e^{i t}(\mathcal{P} \cup \tilde{L})$ with $\mathcal{P}=\{z \in \mathbb{C}: \operatorname{Im}(z)>0\}$ and $\tilde{L}=(-\infty, 0]$ or $[0, \infty)$ for some $t \in \mathbb{R}$.

Proof. Suppose $A$ has the decomposition as stated with $\operatorname{dim} \tilde{A}_{1}=m \leqslant k-1$. Take $F \in M_{m}$ such that $W\left(\tilde{A}_{1}+F\right) \subseteq \mathbb{C} \backslash(\lambda+S)$. By Theorem 3.1,

$$
\Lambda_{k}(A) \subseteq W(A+(F \oplus 0))=W\left(\left(\tilde{A}_{1}+F\right) \oplus \tilde{A}_{2}\right) \subseteq \mathbb{C} \backslash(\lambda+S)
$$

Hence, $\lambda \notin \Lambda_{k}(A)$.
Conversely, suppose $\lambda \notin \Lambda_{k}(A)$. Without loss of generality, we may assume that $\lambda=0$.
Case 1. Suppose $\lambda=0 \notin \Omega_{k}(A)$, then $\mu_{k}(A, t)<0$ for some $t \in[0,2 \pi)$. By Lemma 4.1, $A=A_{1} \oplus A_{2} \oplus \hat{A}$ with $\operatorname{dim} A_{1}<k$, $W\left(A_{1}\right) \subseteq e^{i t} \mathcal{P}, W\left(A_{2}\right) \subseteq-e^{i t} \mathcal{P}$ and $W(\hat{A}) \subseteq e^{i t} \mathbb{R}$. Furthermore, as $\mu_{k}(A, t)<0$, we must have $\operatorname{dim} A_{1}+\operatorname{dim} \hat{A}<k$. Then $\hat{A}=A_{3} \oplus A_{4}$ so that $W\left(A_{3}\right) \subseteq e^{i t}[0, \infty)$ and $W\left(A_{4}\right) \subseteq-e^{i t}(0, \infty)$. Then the result follows with $\tilde{A}_{1}=A_{1} \oplus A_{3}$ and $\tilde{A}_{2}=$ $A_{2} \oplus A_{4}$.

Case 2. Suppose $\lambda=0 \in \Omega_{k}(A)$ and such decomposition mentioned in the theorem does not exist. Suppose ker $A$, the kernel of $A$, has dimension $p<k$. We may assume that $p=0$. Otherwise, replace $A$ by the compression of $A$ on (ker $A)^{\perp}$ and replace $k$ by $k-p$. We are going to derive a contradiction by showing that $A$ has a finite dimensional compression $B$ such that $0 \in \Omega_{k}(B)=\Lambda_{k}(B) \subseteq \Lambda_{k}(A)$.

To construct the matrix $B$, we first show that there exist $s_{1} \leqslant 0 \leqslant \pi \leqslant s_{2}$ with $s_{2}-s_{1} \leqslant 2 \pi$ such that $A=A_{1} \oplus A_{2} \oplus$ $A_{3} \oplus A_{4}$, where

$$
\begin{align*}
& \operatorname{dim} A_{1}<\infty, \quad W\left(A_{1}\right) \subseteq\left\{\rho e^{i t}: \rho>0, t \in\left(s_{1}, s_{2}\right)\right\}, \quad W\left(A_{2}\right) \subseteq\left\{\rho e^{i t}: \rho>0, t \in\left(s_{2}, s_{1}+2 \pi\right)\right\}, \\
& W\left(A_{3}\right) \subseteq e^{i s_{1}}(0, \infty), \quad \text { and } \quad W\left(A_{4}\right) \subseteq e^{i s_{2}}(0, \infty) \tag{4.1}
\end{align*}
$$

Then we show that $A_{2} \oplus A_{3} \oplus A_{4}$ has a finite dimensional compression $B_{2} \oplus B_{3} \oplus B_{4}$ such that $B=A_{1} \oplus B_{2} \oplus B_{3} \oplus B_{4}$ has $0 \in \Omega_{k}(B)$.

Since $0 \in \Omega_{k}(A)$, we have $\mu_{k}(A, t) \geqslant 0$ for all $t \in[0,2 \pi)$. If $\mu_{k}(A, t)>0$ for all $t \in[0,2 \pi)$, then 0 lies in the interior of $\Omega_{k}(A)$. Hence, $0 \in \Lambda_{k}(A)$. So, we may assume that there is $t_{0} \in[0,2 \pi)$ such that $\mu_{k}\left(A, t_{0}\right)=0$. We may further assume that $t_{0}=0$.

As $\mu_{k}\left(A, t_{0}\right)=0, A$ has at most $k-1$ eigenvalues in the open upper half plane. Suppose these eigenvalues have arguments $0<t_{1} \leqslant t_{2} \leqslant \cdots \leqslant t_{p}<\pi, p<k$. Take $t_{p+1}=\pi$. Let $g \in\{1, \ldots, p+1\}$ be the smallest integer such that $\mu_{m}\left(A, t_{g}-\pi\right)=0$ for some $m$ and $h \in\{0,1, \ldots, p\}$ be the largest integer satisfying $\mu_{m}\left(A, t_{h}\right)=0$ for some $m$. We are going to find $A_{j}$ for $j=1,2,3,4$ satisfying (4.1) with $s_{1}=t_{g}-\pi$ and $s_{2}=t_{h}+\pi$.

By Lemma 4.1 with $t=t_{h}$, we have $A=\hat{A}_{1} \oplus \hat{A}_{2} \oplus \hat{A}$ such that

$$
\operatorname{dim} \hat{A}_{1}<\infty, \quad W\left(\hat{A}_{1}\right) \subseteq e^{i t_{h}} \mathcal{P}, \quad W\left(\hat{A}_{2}\right) \subseteq-e^{i t_{h}} \mathcal{P}, \quad \text { and } \quad W(\hat{A}) \subseteq e^{i t_{h}} \mathbb{R}
$$

Let $\hat{H}=\left(e^{-i t_{h}} \hat{A}+e^{-i t_{h}} \hat{A}^{*}\right) / 2$. If both $\lambda_{k}(\hat{H})$ and $\lambda_{k}(-\hat{H})$ are nonnegative, then we have $0 \in \Lambda_{k}(\hat{H})$, which implies that $0 \in \Lambda_{k}(A)$, a contradiction. So, we have either $\lambda_{k}(\hat{H})$ or $\lambda_{k}(-\hat{H})$ is negative. By Lemma 4.1 and the assumption that $\operatorname{ker} A=0$, we have $\hat{A}=\hat{A}_{3} \oplus \hat{A}_{4}$ with

$$
W\left(\hat{A}_{3}\right) \subseteq e^{i t_{h}}(0, \infty) \quad \text { and } \quad W\left(\hat{A}_{4}\right) \subseteq-e^{i t_{h}}(0, \infty)
$$

If $t_{h}=t_{g}-\pi$, we take $A_{j}=\hat{A}_{j}$ for $j=1,2,3,4$. Then $A_{1}, A_{2}, A_{3}, A_{4}$ satisfy (4.1) with $s_{1}=t_{g}-\pi=0$ and $s_{2}=t_{h}+\pi=\pi$.

Suppose $t_{h}>t_{g}-\pi$. Then $\operatorname{dim} \hat{A}_{3}$ is finite. We further apply Lemma 4.1 to $\hat{A}_{2}$ with $t=t_{g}-\pi$, we have $\hat{A}_{2}=A_{1}^{\prime} \oplus A_{2}^{\prime} \oplus A_{3}^{\prime}$, with

$$
\begin{aligned}
& \operatorname{dim} A_{1}^{\prime}<\infty, \quad W\left(A_{1}^{\prime}\right) \subseteq\left\{\rho e^{i t}: \rho>0, t \in\left(t_{g}-\pi, t_{h}\right)\right\} \\
& W\left(A_{2}^{\prime}\right) \subseteq\left\{\rho e^{i t}: \rho>0, t \in\left(t_{h}-\pi, t_{g}-\pi\right)\right\}, \quad \text { and } \quad W\left(A_{3}^{\prime}\right) \subseteq e^{i\left(t_{g}-\pi\right)}(0, \infty)
\end{aligned}
$$

Note that $A_{4}^{\prime}$ is vacuous because $t_{h} \leqslant t_{g} \leqslant t_{h}+\pi$. Then $A_{1}=\hat{A}_{1} \oplus \hat{A}_{3} \oplus A_{1}^{\prime}, A_{2}=A_{2}^{\prime}, A_{3}=A_{3}^{\prime}$, and $A_{4}=\hat{A}_{4}$ will satisfy (4.1) with $s_{1}=t_{g}-\pi$ and $s_{2}=t_{h}+\pi$.

Now we choose a finite dimensional compression $B_{2} \oplus B_{3} \oplus B_{4}$ of $A_{2} \oplus A_{3} \oplus A_{4}$ and show that $\mu_{k}\left(A_{1} \oplus B_{2} \oplus B_{3} \oplus\right.$ $\left.B_{4}, t\right) \geqslant 0$ for all $t \in[0,2 \pi)$. Observe that

$$
\begin{equation*}
\mu_{k}\left(A_{1}, t\right) \geqslant 0 \text { for all } t_{g}-\pi<t<t_{h} \tag{4.2}
\end{equation*}
$$

Let $B_{3}$ be a $k$-dimensional compression of $A_{3}$, if $\operatorname{dim} A_{3}$ is infinite and $B_{3}=A_{3}$, otherwise. We claim that

$$
\begin{equation*}
\mu_{k}\left(A_{1} \oplus B_{3}, t\right) \geqslant 0 \quad \text { for all } t_{g-1}-\pi \leqslant t \leqslant t_{g}-\pi \tag{4.3}
\end{equation*}
$$

The claim is clear if $\operatorname{dim} A_{3}$ is infinite. Suppose $\operatorname{dim} A_{3}$ is finite and $\mu_{k}\left(A_{1} \oplus A_{3}, t\right)<0$ for some $t \in\left[t_{g-1}-\pi, t_{g}-\pi\right]$. Since $\operatorname{dim}\left(A_{1} \oplus A_{3}\right)$ is finite and $W\left(A_{1} \oplus A_{3}\right) \subseteq\left\{\rho e^{i t}: \rho>0, t \in\left[t_{g}-\pi, t_{h}+\pi\right)\right\}, A_{1} \oplus A_{3}$ has a decomposition $A_{1}^{\prime \prime} \oplus A_{3}^{\prime \prime}$, with

$$
\operatorname{dim} A_{1}^{\prime \prime}<k, \quad W\left(A_{1}^{\prime \prime}\right) \subseteq S, \quad \text { and } \quad W\left(A_{3}^{\prime \prime}\right) \subseteq \mathbb{C} \backslash S,
$$

where $S=e^{i\left(t_{g}-\pi\right)}(\mathcal{P} \cup[0, \infty))=\left\{\rho e^{i t}: \rho>0, t \in\left[t_{g}-\pi, t_{g}\right)\right\}$. Notice also that $W\left(A_{2} \oplus A_{4}\right) \subseteq \mathbb{C} \backslash S$. Then if we take $\tilde{A}_{1}=A_{1}^{\prime \prime}$ and $\tilde{A}_{2}=A_{2} \oplus A_{3}^{\prime \prime} \oplus A_{4}$, we have $W\left(\tilde{A}_{1}\right) \subseteq S$ and $W\left(\tilde{A}_{2}\right) \subseteq \mathbb{C} \backslash S$, which contradicts our assumption that such decomposition does not exist.

Next, let $B_{4}$ be a $k$-dimensional compression of $A_{4}$, if $\operatorname{dim} A_{4}$ is infinite and $B_{4}=A_{4}$, otherwise. By a similar argument as in the previous paragraph, we can show that

$$
\begin{equation*}
\mu_{k}\left(A_{1} \oplus B_{4}, t\right) \geqslant 0 \quad \text { for all } t_{h} \leqslant t \leqslant t_{h+1} . \tag{4.4}
\end{equation*}
$$

In the following, we will choose a finite dimension compression of $B_{2}$ of $A_{2}$ so that

$$
\begin{equation*}
\mu_{k}\left(B_{2} \oplus B_{3} \oplus B_{4}, t\right) \geqslant 0 \quad \text { for all } t_{h+1} \leqslant t \leqslant t_{g-1}+\pi \tag{4.5}
\end{equation*}
$$

Suppose $\operatorname{dim} A_{2}$ is finite. Then by the definition of $t_{h}$ and $t_{g}$, both $\operatorname{dim} A_{3}$ and $\operatorname{dim} A_{4}$ are infinite. Then $\mu_{k}\left(B_{3} \oplus B_{4}, t\right) \geqslant 0$ for all $t \in\left[t_{h+1}, t_{g-1}+\pi\right]$ and so (4.5) holds with vacuous $B_{2}$.

Now suppose $\operatorname{dim} A_{2}$ is infinite. We consider the following three cases.
Case 1. $t_{g}=t_{h}$. In this case, the summand $A_{2}^{\prime} \oplus A_{3}^{\prime}$ is vacuous and so as $A_{2} \oplus A_{3}$. Also $\operatorname{dim} A_{4}$ is infinite. Then (4.5) holds with vacuous $B_{2}$ and $B_{3}$.

Case 2. $t_{g}=t_{h+1}$. Let $B_{2}$ be a $k$-dimensional compression of $A_{2}$. Then $\mu_{k}\left(B_{2}, t\right) \geqslant 0$ for all $t \in\left[t_{h+1}, t_{g-1}-\pi\right]$ and so (4.5) holds.

Case 3. $t_{g}>t_{h+1}$. Because of the choice $t_{g}$ and $t_{h}$, both $\mathcal{L} \cap \sigma\left(A_{2}\right)$ and $\mathcal{R} \cap \sigma\left(A_{2}\right)$ are infinite or contains an eigenvalue of $A_{2}$ with infinite multiplicity, where

$$
\mathcal{L}=\left\{\rho e^{i t}: \rho>0, t \in\left(t_{h}+\pi, t_{h+1}+\pi\right)\right\} \quad \text { and } \quad \mathcal{R}=\left\{\rho e^{i t}: \rho>0, t \in\left(t_{g-1}+\pi, t_{g}+\pi\right)\right\} .
$$

By Lemma 4.3, we can get finite dimensional compressions $T_{1}$ and $T_{2}$ of $A_{2}$ such that $\operatorname{dim}\left(T_{1}\right)=\operatorname{dim}\left(T_{2}\right)=k, W\left(T_{1}\right) \subseteq \mathcal{L}$ and $W\left(T_{2}\right) \subseteq \mathcal{R}$. Then $\mu_{k}\left(T_{1}, t\right) \geqslant 0$ for all $t \in\left[t_{h+1}, t_{h}+\pi\right]$ and $\mu_{k}\left(T_{2}, t\right) \geqslant 0$ for all $t \in\left[t_{g}, t_{g-1}+\pi\right]$. Thus, $B_{2}=T_{1} \oplus T_{2}$ will satisfy (4.5).

Now let $B=A_{1} \oplus B_{2} \oplus B_{3} \oplus B_{4}$. By (4.2), (4.3), (4.4), and (4.5), we conclude that $\mu_{k}(B, t) \geqslant 0$ for all $t \in[0,2 \pi$ ) and hence $0 \in \Omega_{k}(B)=\Lambda_{k}(B)$.

Theorem 4.5. Suppose $A \in \mathcal{B}(\mathcal{H})$ is normal and $1 \leqslant r<k<\infty$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_{r}$ containing the set $\mathcal{S}_{0}=\left\{2 e^{i \xi}\|A\| P: P \in\right.$ $\mathcal{P}_{r}$ and $\left.\xi \in[0,2 \pi)\right\}$. Then

$$
\Lambda_{k}(A)=\bigcap\left\{\Lambda_{k-r}(A+F): F \in \mathcal{S}\right\}
$$

Proof. The inclusion ( $\subseteq$ ) follows from Theorem 3.1. Suppose $\lambda \notin \Lambda_{k}(A)$. By Theorem 4.4, $A$ has a decomposition $A_{1} \oplus A_{2}$ with $A_{1} \in M_{m}, W\left(A_{1}\right) \subseteq \lambda+S$ and $W\left(A_{2}\right) \subseteq \mathbb{C} \backslash(\lambda+S)$, where $m \leqslant k-1$ and $S$ is defined as in Theorem 4.4. Let $F=$ $-2 i e^{i t}\|A\|\left(I_{r} \oplus 0\right) \in \mathcal{S}_{0}$. Then $A+F$ has less than $k-r$ eigenvalues in $\lambda+\mathcal{S}$. Thus, $A+F$ has a decomposition $B_{1} \oplus B_{2}$ with $\operatorname{dim} B_{1}<k-r$ such that $W\left(B_{1}\right) \subseteq \lambda+S$ and $W\left(B_{2}\right) \subseteq \mathbb{C} \backslash(\lambda+S)$. By Theorem 4.4, $\lambda \notin \Lambda_{k-r}(A+F)$.

If $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint, Theorem 4.4 reduces to the following corollary.
Corollary 4.6. Suppose $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $1 \leqslant r \leqslant k$. Then $\lambda \in \Lambda_{k}(A)$ if and only if $A$ can be decomposed into $\tilde{A}_{1} \oplus \tilde{A}_{2}$ such that $\operatorname{dim} \tilde{A}_{1}<k, W\left(\tilde{A}_{1}\right) \subseteq L$ and $W\left(\tilde{A}_{2}\right) \subseteq \mathbb{R} \backslash L$, where $L=[\lambda, \infty)$ or $(-\infty, \lambda]$.

Using a similar argument as in the proof of Theorem 4.5, an analogous result can also be obtained for self-adjoint operators.

Theorem 4.7. Suppose $A \in \mathcal{B}(\mathcal{H})$ is self-adjoint and $1 \leqslant r<k<\infty$. Let $\mathcal{S}$ be a subset of $\mathcal{F}_{r}$ containing the set $\left\{ \pm 2\|A\| P: P \in \mathcal{P}_{r}\right\}$. Then

$$
\Lambda_{k}(A)=\bigcap\left\{\Lambda_{k-r}(A+F): F \in \mathcal{S}\right\} .
$$

In [14, Proposition 2.3], the author showed that $\Lambda_{k}(A) \subseteq \bigcap_{X \in \mathcal{V}_{k-1}} W\left(X^{*} A X\right)$, where $\mathcal{V}_{m}$ is the set of $X: \mathcal{H} \rightarrow \mathcal{H}$ with $X^{*} X=I_{\mathcal{H}}$ and $X(\mathcal{H})=\mathcal{H}_{1}^{\perp}$ for some subspace $\mathcal{H}_{1}$ of $\mathcal{H}$ satisfying $\operatorname{dim} \mathcal{H}_{1} \leqslant m$. In general, we have the following.

Proposition 4.8. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leqslant r<k<\infty$. Then

$$
\Lambda_{k}(A) \subseteq \bigcap\left\{\Lambda_{k-r}\left(X^{*} A X\right): X \in \mathcal{V}_{r}\right\}
$$

Proof. Let $\lambda \in \Lambda_{k}(A)$. Then there exists a rank $k$ orthogonal projection $P$ such that $P A P=\lambda P$. Suppose $X \in \mathcal{V}_{r}$. Then there exists a subspace $\mathcal{H}_{1}$ of $\mathcal{H}$ with $\operatorname{dim} \mathcal{H}_{1} \leqslant r$ satisfying $X^{*} X=I_{\mathcal{H}}$ and $X(\mathcal{H})=\mathcal{H}_{1}^{\perp}$. Therefore, $\operatorname{dim}\left(P(\mathcal{H}) \cap \mathcal{H}_{1}^{\perp}\right) \geqslant k-r$. Choose a $(k-r)$-dimensional subspace $\mathcal{H}_{2}$ of $\mathcal{H}$ such that $X\left(\mathcal{H}_{2}\right) \subseteq P(\mathcal{H}) \cap \mathcal{H}_{1}^{\perp}$. Let $\left\{y_{i}\right\}_{i=1}^{k-r}$ be an orthogonal basis of $\mathcal{H}_{2}$. Then $\left\{X\left(y_{i}\right)\right\}_{i=1}^{k-r}$ is an orthonormal subset of $P(\mathcal{H})$. So, for $1 \leqslant i, j \leqslant k-r$, we have

$$
\left\langle X^{*} A X y_{i}, y_{j}\right\rangle=\left\langle A\left(X y_{i}\right),\left(X y_{j}\right)\right\rangle=\delta_{i j} \lambda .
$$

Hence, $\lambda \in \Lambda_{k-r}\left(X^{*} A X\right)$.
Using Theorems 3.4 and 4.5 , we have

Corollary 4.9. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leqslant r<k<\infty$. If $\operatorname{dim} \mathcal{H}<\infty$ or $A$ is normal, then

$$
\Lambda_{k}(A)=\bigcap\left\{\Lambda_{k-r}\left(X^{*} A X\right): X \in \mathcal{V}_{r}\right\}
$$

Proof. For each $F \in \mathcal{F}_{r}$, there is $X \in \mathcal{V}_{r}$ such that $X^{*} F X=0$. Then

$$
\Lambda_{k}(A) \subseteq \bigcap_{X \in \mathcal{V}_{r}} \Lambda_{k-r}\left(X^{*} A X\right) \subseteq \bigcap_{F \in \mathcal{F}_{r}} \Lambda_{k-r}\left(X^{*}(A+F) X\right) \subseteq \bigcap_{F \in \mathcal{F}_{r}} \Lambda_{k-r}(A+F)
$$

By Theorems 3.4 and 4.5 , the inclusions are indeed equalities.
Similarly, using Theorems 2.1 and Corollary 3.3, we have the last corollary in this section.
Corollary 4.10. Suppose $A \in \mathcal{B}(\mathcal{H})$ and $1 \leqslant r<k<\infty$. Then

$$
\Omega_{k}(A)=\bigcap\left\{\Omega_{k-r}\left(X^{*} A X\right): X \in \mathcal{V}_{r}\right\}
$$

## 5. Results on $\boldsymbol{\Lambda}_{\infty}(A)$

Suppose $\mathcal{H}$ is infinite dimensional and $A \in \mathcal{B}(\mathcal{H})$. It is clear that $\Lambda_{\infty}(A)$ can be viewed as the set of $\lambda \in \mathbb{C}$ for which there exists an infinite orthonormal set $\left\{x_{i} \in \mathcal{H}: i \geqslant 1\right\}$ such that $\left\langle A x_{i}, x_{j}\right\rangle=\delta_{i j} \lambda$ for all $i, j \geqslant 1$. Extend the definition of $\Omega_{k}(A)$ to

$$
\Omega_{\infty}(A)=\bigcap_{\xi \in[0,2 \pi)}\left\{\mu \in \mathbb{C}: \operatorname{Re}\left(e^{i \xi} \mu\right) \leqslant \lambda_{k}\left(\operatorname{Re}\left(e^{i \xi} A\right)\right) \text { for all } k \geqslant 1\right\}
$$

We have the following result.

Theorem 5.1. Suppose $\operatorname{dim} \mathcal{H}$ is infinite and $A \in \mathcal{B}(\mathcal{H})$. Let $\mathcal{S}$ be a set of finite rank operators on $\mathcal{B}(\mathcal{H})$ containing the set $\left\{2 e^{i \xi}\|A\| P: \xi \in[0,2 \pi), P\right.$ is a finite rank orthogonal projection $\}$.
Then we have the following equalities.
(1) $\Omega_{\infty}(A)=\bigcap_{k \geqslant 1} \Omega_{k}(A)=\bigcap\{\mathbf{C l}(W(A+F)): F \in \mathcal{S}\}=W_{e}(A)$.
(2) $\Lambda_{\infty}(A)=\bigcap_{k \geqslant 1} \Lambda_{k}(A)=\bigcap\{W(A+F): F \in \mathcal{S}\}$.

Proof. (1) By the definition of $\Omega_{\infty}(A)$, we have $\Omega_{\infty}(A)=\bigcap_{k \geqslant 1} \Omega_{k}(A)$. By (1.2) and Corollary 3.3, we have

$$
\begin{aligned}
W_{e}(A) & =\bigcap_{\{\mathbf{C l}(W(A+F)): F \in \mathcal{B}(\mathcal{H}) \text { has finite rank }\}} \\
& =\bigcap_{k \geqslant 1}\{\mathbf{C l}(W(A+F)): F \in \mathcal{B}(\mathcal{H}) \text { has rank } k-1\} \\
& =\bigcap_{k \geqslant 1} \Omega_{k}(A) \\
& =\bigcap_{k \geqslant 1}\{\mathbf{C l}(W(A+F)): F \in \mathcal{S} \text { has rank } k-1\} \\
& =\bigcap\{\mathbf{C l}(W(A+F)): F \in \mathcal{S}\} .
\end{aligned}
$$

So, the second and third equalities in (1) hold.
(2) By Theorem 4 in [1], we have

$$
\begin{equation*}
\Lambda_{\infty}(A)=\bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { has finite rank }\} \tag{5.1}
\end{equation*}
$$

Clearly, we have the inclusion

$$
\Lambda_{\infty}(A) \subseteq \bigcap_{k \geqslant 1} \Lambda_{k}(A)
$$

To prove the reverse inclusion, suppose $\lambda \in \bigcap_{k \geqslant 1} \Lambda_{k}(A)$. Let $F \in \mathcal{B}(\mathcal{H})$ of rank $m$. Choose $k \geqslant m+1$. Then $\lambda \in \Lambda_{k}(A)$. By Theorem 3.1, we have

$$
\lambda \in \Lambda_{k}(A) \subseteq \Lambda_{1}(A+F)=W(A+F)
$$

Hence,

$$
\bigcap_{k \geqslant 1} \Lambda_{k}(A) \subseteq \bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { is of finite rank }\}=\Lambda_{\infty}(A) .
$$

Thus, we get the first equality in (2).
Next, we show that one only needs to use $F \in \mathcal{S}$ for the intersection on the right side of (5.1). To this end, note that
$\bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H})$ is of finite rank $\} \subseteq \bigcap\{W(A+F): F \in \mathcal{S}\}$.
To prove the reverse inclusion, assume that

$$
\lambda \notin \bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { is of finite rank }\} .
$$

If $\lambda \notin \Omega_{\infty}(A)$, then there is a finite rank $F \in \mathcal{B}(\mathcal{H})$ such that $\lambda \notin \mathbf{C l}(W(A+F))$ and hence $\lambda \notin W(A+F)$. So, assume that

$$
\begin{equation*}
\lambda \in W_{e}(A) \subseteq W(A) \quad \text { and thus } \quad|\lambda| \leqslant \sup \{|\mu|: \mu \in W(A)\} \leqslant\|A\| . \tag{5.2}
\end{equation*}
$$

Then there is $\xi \in[0,2 \pi)$ and a finite rank operator $F \in \mathcal{B}(\mathcal{H})$ such that

$$
\begin{equation*}
e^{i \xi} W(A+F-\lambda I) \subseteq\{\mu \in \mathbb{C}: \operatorname{Im}(\mu)<0\} \cup \mathcal{L}, \tag{5.3}
\end{equation*}
$$

with $\mathcal{L}=(0, \infty)$ or $\mathcal{L}=(-\infty, 0)$. We may replace $A$ by $e^{i \xi} A$ and assume that $\xi=0$. Without loss of generality, assume that $\mathcal{L}=(-\infty, 0)$.

Let $\lambda=a+i b$ with $a, b \in \mathbb{R}$ and $A=H+i G$ with $H=H^{*}$ and $G=G^{*}$. Since (5.3) holds with $\xi=0$ for a finite rank operator $F \in \mathcal{B}(\mathcal{H})$, there is $r$ not larger than the rank of $\operatorname{Im} F$ such that $G$ has an operator matrix of the form

$$
\begin{equation*}
\operatorname{diag}\left(g_{1}, \ldots, g_{r}\right) \oplus b I_{s} \oplus G_{2} \tag{5.4}
\end{equation*}
$$

with $g_{1} \geqslant \cdots \geqslant g_{r}>b, W\left(G_{2}\right) \subseteq(-\infty, b)$ and $0 \leqslant s \leqslant \infty$. By (5.2), we have

$$
g_{1}-b \leqslant\left|g_{1}\right|+|b| \leqslant 2\|G\| \leqslant 2\|A\| .
$$

We consider two cases.

Case 1. Suppose $g_{1}-b=2\|A\|$. Then $g_{1}=\|A\|=-b$. Since $\lambda \in W_{e}(A)$ and

$$
\|A\|=|b| \leqslant|a+i b|=|\lambda| \leqslant\|A\|,
$$

it follows that

$$
a=0 \quad \text { and } \quad \lambda=i b=-i\|A\|
$$

is the only element in $\mathbf{C l}(W(A)) \cap\{\mu \in \mathbb{C}$ : $\operatorname{Im}(\mu) \leqslant-\|A\|\}$. Thus, $G_{2}$ in (5.4) is vacuous, i.e., $G$ has operator matrix $\operatorname{diag}\left(g_{1}, \ldots, g_{r}\right) \oplus b I_{s}$. Using the same basis, we let $H$ have the operator matrix

$$
\left[\begin{array}{ll}
H_{11} & H_{12} \\
H_{12}^{*} & H_{22}
\end{array}\right] .
$$

Since $\left\|H_{22}+i b I\right\| \leqslant\|A\|=|b|$, we see that $H_{22}=0$. By the fact that

$$
|b|^{2}=\left\|A^{*} A\right\|=\left\|(H+i G)^{*}(H+i G)\right\|,
$$

we see that $H_{12}$ is zero as well. Thus, $A$ has operator matrix

$$
A_{1} \oplus i b I_{s} \quad \text { with } A_{1} \in M_{m} .
$$

Since (5.3) holds for a finite operator $F$ with $\xi=0$ and $\mathcal{L}=(-\infty, 0)$, we see that $s \neq \infty$. But then $\operatorname{dim} \mathcal{H}$ is finite, which is a contradiction.

Case 2. Suppose $g_{1}-b<2\|A\|$. If $s$ is finite in (5.4), then

$$
\tilde{F}=i 2\|A\|\left(I_{r+s} \oplus 0\right) \in \mathcal{S} \quad \text { and } \quad W(A-\tilde{F}) \subseteq\{\mu \in \mathbb{C}: \operatorname{Im}(\mu)<b\}
$$

Thus, $\lambda=a+i b \notin W(A-\tilde{F})$.
Next, assume that $s=\infty$. Suppose the compression of $H$ on the null space of $G-b I$ equals $H_{0}$. Then there is a positive integer $m$ such that $H_{0}$ has operator matrix $\operatorname{diag}\left(h_{1}, \ldots, h_{m}\right) \oplus H_{1}$ such that $h_{1} \geqslant \cdots \geqslant h_{m} \geqslant a$ and $W\left(H_{1}\right) \subseteq(-\infty, a)$. Otherwise, (5.3) cannot hold for a finite operator $F$ with $\xi=0$ and $\mathcal{L}=(-\infty, 0)$. Let $\tilde{F}=i 2\|A\|\left(I_{r+m} \oplus 0\right) \in \mathcal{S}$, and let $\hat{A}=A-\tilde{F}-\lambda I$. Then $\operatorname{Im}(\hat{A})=\left(\hat{A}-\hat{A}^{*}\right) / 2 i$ has an operator matrix $\hat{G}_{1} \oplus 0_{s-m} \oplus \hat{G}_{2}$ with $W\left(\hat{G}_{1} \oplus \hat{G}_{2}\right) \subseteq(-\infty, 0)$. Moreover, the compression of $\operatorname{Re}(\hat{A})=\left(\hat{A}+\hat{A}^{*}\right) / 2$ on the null space of $\operatorname{Im}(\hat{A})$ equal $H_{1}-a I$. As a result, if $\mu=\langle\hat{A} x, x\rangle \in W(\hat{A})$ has imaginary part 0 , then $x$ must lie in the null space of $\operatorname{Im}(\hat{A})$, and hence the real part of $\mu$ lies in $W\left(H_{1}-a I\right) \subseteq(-\infty, 0)$. Thus, $0 \notin W(\hat{A})$, equivalently, $\lambda \notin W(A-\tilde{F})$. Consequently,

$$
\bigcap\{W(A+F): F \in \mathcal{S}\} \subseteq \bigcap\{W(A+F): F \in \mathcal{B}(\mathcal{H}) \text { is of finite rank }\} .
$$

In [14], Martinez-Avendano asked whether $\Lambda_{\infty}(A)=\bigcap_{k \geqslant 1} \Lambda_{k}(A)$. Assertion (2) answers the question affirmatively.

## Theorem 5.2. Suppose $A \in \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is infinite dimensional. Then

$$
\operatorname{Int}\left(\Omega_{\infty}(A)\right) \subseteq \Lambda_{\infty}(A) \subseteq \Omega_{\infty}(A)
$$

Moreover, $\mathbf{C l}\left(\Lambda_{\infty}(A)\right)=\Omega_{\infty}(A)$ if and only if $\Lambda_{\infty}(A) \neq \emptyset$.
Proof. By the Corollary after Theorem 4 in [1], we see that $\operatorname{Int}\left(\Omega_{\infty}(A)\right) \subseteq \Lambda_{\infty}(A)$. The inclusion $\Lambda_{\infty}(A) \subseteq \Omega_{\infty}(A)$ is clear.
Note that $\Omega_{\infty}(A)$ is always a non-empty compact convex set. If $\Lambda_{\infty}(A)=\emptyset$, then $\mathbf{C l}\left(\Lambda_{\infty}(A)\right) \neq \Omega_{\infty}(A)$. Conversely, suppose $\Lambda_{\infty}(A) \neq \emptyset$. If $\boldsymbol{\operatorname { I n t }}\left(\Lambda_{\infty}(A)\right)=\boldsymbol{\operatorname { I n t }}\left(\Omega_{\infty}(A)\right)$ is non-empty, then $\mathbf{C l}\left(\Lambda_{\infty}(A)\right)=\Omega_{\infty}(A)$. If $\boldsymbol{\operatorname { I n t }}\left(\Omega_{\infty}(A)\right)$ is empty, then $\Omega_{\infty}(A)=\{\mu\}$ is a singleton and so is the non-empty set $\Lambda_{\infty}(A)$. Hence $\mathbf{C l}\left(\Lambda_{\infty}(A)\right)=\Lambda_{\infty}(A)=\{\mu\}$.

The next example show that $\Lambda_{\infty}(A)$ may indeed be empty so that $\mathbf{C l}\left(\Lambda_{\infty}(A)\right) \neq \Omega_{\infty}(A)$.
Example 5.3. Let $A=\bigoplus_{n \geqslant 2} \operatorname{diag}\left(e^{i \pi / n} / n,-1 / n\right) \in \mathcal{B}(\mathcal{H})$. Then $\Omega_{\infty}(A)=\{0\}$ but $0 \notin \Lambda_{1}(A)$ so that $\Lambda_{\infty}(A)=\bigcap\left\{\Lambda_{k}(A)\right.$ : $k=1,2, \ldots\}=\emptyset$. On the other hand, if $B=A \oplus 0_{\mathcal{H}}$, then $\Lambda_{\infty}(B)=\{0\}$.

From the proof of Theorem 5.2, we see that if $\Lambda_{\infty}(A)$ is a singleton, then $\Omega_{\infty}(A)$ is also a singleton, which can happen if and only if $A-\mu I$ is a compact operator for some $\mu \in \mathbb{C}$ by the corollary after Lemma 3 in [1]. In connection to this comment and Example 5.3, we have the following.

Proposition 5.4. Let $A \in \mathcal{B}(\mathcal{H})$ and $\mu \in \mathbb{C}$ be such that $A-\mu I$ is compact. Then the following are equivalent.
(a) $\Lambda_{\infty}(A)$ is non-empty.
(b) $\Lambda_{\infty}(A)=\{\mu\}$.
(c) $\mu \in \Lambda_{k}(A)$ for each $k=1,2, \ldots$.

Proof. The implications "(a) $\Leftrightarrow$ (b)" is clear. We have "(c) $\Leftrightarrow$ (b)" because $\Lambda_{\infty}(A)=\bigcap_{k} \Lambda_{k}(A)$ by Theorem 5.1.

## Acknowledgments

Li is an honorary professor of the University of Hong Kong. His research was partially supported by an USA NSF grant and a HK RGC grant. This research was done while the three authors were visiting the University of Hong Kong in the summer of 2007 supported by a HK RGC grant. They would like to thank the staff of the Mathematics Department for their hospitality.

## References

[1] J. Anderson, J.G. Stampfli, Compressions and commutators, Israel J. Math. 10 (1971) 433-441.
[2] R. Bhatia, Matrix Analysis, Springer, New York, 1996.
[3] M.D. Choi, Completely positive linear maps on complex matrices, Linear Algebra Appl. 10 (1975) 285-290.
[4] M.D. Choi, M. Giesinger, J.A. Holbrook, D.W. Kribs, Geometry of higher-rank numerical ranges, Linear Multilinear Algebra 56 (2008) 53-64.
[5] M.D. Choi, J.A. Holbrook, D.W. Kribs, K. Życzkowski, Higher-rank numerical ranges of unitary and normal matrices, Oper. Matrices 1 (2007) $409-426$.
[6] M.D. Choi, D.W. Kribs, K. Życzkowski, Higher-rank numerical ranges and compression problems, Linear Algebra Appl. 418 (2006) 828-839.
[7] M.D. Choi, D.W. Kribs, K. Życzkowski, Quantum error correcting codes from the compression formalism, Rep. Math. Phys. 58 (2006) 77-91.
[8] P.A. Fillmore, J.G. Stampfli, J.P. Williams, On the essential numerical range, the essential spectrum, and a problem of Halmos, Acta Sci. Math. (Szeged) 33 (1972) 179-192.
[9] H.L. Gau, C.K. Li, P.Y. Wu, Higher-rank numerical ranges and dilations, J. Operator Theory, in press.
[10] K.E. Gustafson, D.K.M. Rao, Numerical Ranges: The Field of Values of Linear Operators and Matrices, Springer, New York, 1997.
[11] E. Knill, R. Laflamme, Theory of quantum error correcting codes, Phys. Rev. A 55 (1997) 900-911.
[12] C.K. Li, Y.T. Poon, N.S. Sze, Condition for the higher rank numerical range to be non-empty, Linear Multilinear Algebra, in press, preprint, http://arxiv. org/abs/0706.1540.
[13] C.K. Li, N.S. Sze, Canonical forms, higher rank numerical ranges, totally isotropic subspaces, and matrix equations, Proc. Amer. Math. Soc. 136 (2008) 3013-3023.
[14] R.A. Martinez-Avendano, Higher-rank numerical range in infinite-dimensional Hilbert space, Oper. Matrices 2 (2008) 249-264.
[15] J.G. Stampfli, J.P. Williams, Growth conditions and the numerical range in a Banach algebra, Tôhuku Math. J. 20 (1968) 417-424.
[16] H. Woerdeman, The higher rank numerical range is convex, Linear Multilinear Algebra 56 (2008) 65-67.
[17] F. Wolf, On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators, Indag. Math. 21 (1959) 142-147.


[^0]:    * Corresponding author.

    E-mail addresses: ckli@math.wm.edu (C.-K. Li), ytpoon@iastate.edu (Y.-T. Poon), sze@math.uconn.edu (N.-S. Sze).

