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Differentials of Fuzzy Functions*

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In this paper the Rådström embedding theorem (*Proc. Amer. Math. Soc.* **3** (1952), 165) is generalized and is used to define the concept of the differential of a fuzzy function.

1. INTRODUCTION

The purpose of this paper is to define the concept of the differential of a fuzzy function which extends the differential of a set-valued function. The latter concept was first defined by Hukuhara [8]. Later, Banks and Jacobs [2] defined a more general concept of differential and investigated its properties.

Set-valued functions and their calculus were found useful in some of the problems of economics [1] and control theory [7]. From a probabilistic viewpoint, random sets (as a particular case of set-valued functions) have a rather well-developed theory [11].

Zadeh [16] introduced the notion of fuzzy set and later its relationships with random sets were investigated by Fortet and Kambouzia [5], Féron [4], and Goodman [6]. Fuzzy random variables [13] as a generalization of random sets can be used to represent inexactness due to both randomness and fuzziness.

On the other hand, the analysis of evidence [15] was related to the fuzzy

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analysis and to the theory of possibility by Zadeh [17] and, more recently, by Kampé de Fériet [9].

In defining the differential of a set-valued function [2] and the integral of such a function [3], a key result is an embedding theorem due to Rådström [14]. This theorem, which states that the collection of nonempty closed bounded and convex subsets of a Banach space can be embedded in a normed space, makes it possible to define the differential of a set-valued function as a differential of a function between normed spaces. To extend the definition of differential to fuzzy functions, it is natural to extend first the Rådström embedding theorem by considering an appropriate space of fuzzy subsets of a Banach space. This extension is given in Section 2 by using a suitable generalization of the Hausdorff metric. As an application of this embedding theorem, we introduce the concept of the differential of a fuzzy function in Section 3 and study some of its properties.

2. The Embedding Theorem

Let X be a reflexive Banach space, and let A, B be two nonempty bounded subsets of X. The *Hausdorff distance* between A and B is

$$d_H(A, B) = \max[\sup_{a\in A} \inf_{b\in B} \|a-b\|, \sup_{b\in B} \inf_{a\in A} \|a-b\|], \tag{2.1}$$

where $\| \ \|$ denotes the norm in X.

If $Q(X)$ denotes the collection of all nonempty compact and convex subsets of X, it is well known [10] that $(Q(X), d_H)$ is a complete metric space.

If M is a set, a *fuzzy subset* of M is a function $u: M \to [0, 1]$. The set of all fuzzy subsets of M, $\mathscr{F}(M)$ is a completely distributive lattice which includes the ordinary subsets of M (viewed as characteristic functions $M \to \{0, 1\}$).

For any fuzzy subset $u: M \to [0, 1]$, denote by $L_\alpha(u) = \{m \in M \mid u(m) \geqslant \alpha\}$, $\alpha \in [0, 1]$, the α-level set of u.

If M is a vector space, a fuzzy subset $u \in \mathscr{F}(M)$ is called a *fuzzy convex subset* (see [16]), if

$$u(\lambda m_1 + (1-\lambda) m_2) \geqslant \min[u(m_1), u(m_2)] \tag{2.2}$$

for every $m_1, m_2 \in M$, $\lambda \in [0, 1]$.

If X is a reflexive Banach space, in order to extend the Hausdorff distance, we shall consider the subset $\mathscr{F}_0(X)$ of $\mathscr{F}(X)$, containing all fuzzy sets $u: X \to [0, 1]$ with properties:

(i) u is upper semicontinuous,

(ii) u is fuzzy convex,

(iii) $L_\alpha(u)$ is compact, for every $\alpha \neq 0$.

If $u, v \in \mathscr{F}_0(X)$, define the *distance* between u and v by

$$d(u, v) = \sup_{\alpha > 0} d_H(L_\alpha(u), L_\alpha(v)), \tag{2.3}$$

where d_H denotes the Hausdorff distance.

We shall use the following result: The proof, given in [13] for the case $X = \mathbb{R}^n$, works in the general case, and will be omitted here.

THEOREM 2.1. *$(\mathscr{F}_0(X), d)$ is a complete metric space.*

The embedding theorem of Rådström will be extended to $\mathscr{F}_0(X)$. To do this, a linear structure is defined in $\mathscr{F}_0(X)$ by

$$(u + v)(x) = \sup\{\alpha \in [0, 1] \mid x \in L_\alpha(u) + L_\alpha(v)\} \tag{2.4}$$

$$\begin{aligned} (\lambda u)(x) &= u(\lambda^{-1}x), && \text{if } \lambda \neq 0, \\ &= 0, && \text{if } \lambda = 0,\ x \neq 0, \\ &= \sup_{y \in X} u(y), && \text{if } \lambda = 0,\ x = 0, \end{aligned} \tag{2.5}$$

for $u, v \in \mathscr{F}_0(X)$, $\lambda \in \mathbb{R}$.

It is clear that these definitions extend the corresponding operations in $Q(X)$ (addition of sets and multiplication of a set by a scalar). In the process of generalizing the embedding theorem, Lemma 2.1 will be useful.

LEMMA 2.1. *Let M be a set and let $\{M_\alpha \mid \alpha \in [0, 1]\}$ be a family of subsets of M such that:*

(a) $M_0 = M$,

(b) $\alpha \leqslant \beta \Rightarrow M_\beta \subseteq M_\alpha$,

(c) $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$, $\lim_{n\to\infty} \alpha_n = \alpha \Rightarrow M_\alpha = \bigcap_{n=1}^\infty M_{\alpha_n}$.

Then the fuzzy subset $\phi: M \to [0, 1]$ defined by $\phi(m) = \sup\{\alpha \in [0, 1] \mid x \in M_\alpha\}$ has the property that $L_\alpha(\phi) = M_\alpha$ for every $\alpha \in [0, 1]$.

The proof is given in [12].

Propositions 2.1–2.3 will lead to Theorem 2.2, which is the desired embedding result.

PROPOSITON 2.1. *If $u, v \in \mathscr{F}_0(X)$, then $L_\alpha(u + v) = L_\alpha(u) + L_\alpha(v)$ for every $\alpha \in [0, 1]$.*

Proof. Denote $X_\alpha = L_\alpha(u) + L_\alpha(v)$. Obviously, $X_0 = X$, and $\alpha \leqslant \beta \Rightarrow X_\beta \subseteq X_\alpha$. To apply Lemma 2.1, we check that $\alpha_1 \leqslant \alpha_2 \leqslant \cdots$, $\lim_{n\to\infty} \alpha_n = \alpha_0 \Rightarrow X_{\alpha_0} = \bigcap_{n=1}^\infty X_{\alpha_n}$.

Take $x \in \bigcap_{n=1}^{\infty} X_{\alpha_n}$; then $x = x_n + y_n$, $x_n \in L_{\alpha_n}(u)$, $y_n \in L_{\alpha_n}(v)$. Since $L_{\alpha_1}(u) \supset L_{\alpha_2}(u) \supset \cdots$, it follows that $\{x_n\}_n \subset L_{\alpha_1}(u)$; analogously, $\{y_n\}_n \subset L_{\alpha_1}(v)$. Since $L_{\alpha_1}(u)$, $L_{\alpha_1}(v)$ are compact, there are subsequences such that $x_{n_k} \to x_0$, $y_{n_k} \to y_0$. By using the upper semicontinuity of u and v, and $u(x_n) \geqslant \alpha_n$, $v(y_n) \geqslant \alpha_n$, it follows easily that $u(x_0) \geqslant \alpha_0$, $v(y_0) \geqslant \alpha_0$. From this and $x = x_n + y_n$, by taking subsequences we conclude that $x = x_0 + y_0 \in X_{\alpha_0}$.

From Lemma 2.1 it follows that $u + v$ as defined by (2.4) satisfies $L_\alpha(u + v) = L_\alpha(u) + L_\alpha(v)$ for every $\alpha \in [0, 1]$.

Proposition 2.2 shows that the cancellation law holds in $(\mathscr{F}_0(X), +)$.

PROPOSITION 2.2. *If $u, v, w \in \mathscr{F}_0(X)$ and $u + w = v + w$, it follows that $u = v$.*

Proof. This follows from Proposition 2.1 and [14, Lemma 2]. When scalar multiplication (2.5) is concerned, it is easy to see that $L_\alpha(\lambda u) = \lambda L_\alpha(u)$ for $u \in \mathscr{F}_0(X)$, $\lambda \in \mathbb{R}$, and every $\alpha \in [0, 1]$. Note, however, that $(\mathscr{F}_0(X), +, \cdot)$ is *not* a vector space.

PROPOSITION 2.3. *If $u, v, w \in \mathscr{F}_0(X)$, then $d(u + w, v + w) = d(u, v)$.*

Proof. This follows easily from the definition of d (2.3) and from [14, Lemma 3].

Theorem 2.2 gives the desired embedding result.

THEOREM 2.2. *There exists a normed space $\mathscr{X}$ such that $\mathscr{F}_0(X)$ can be embedded isometrically into $\mathscr{X}$.*

Proof. This follows from Propositions 2.2 and 2.3 and from [14, Theorem 1].

Remark. The knowledge of the structure of the normed space $\mathscr{X}$ will be necessary in the next section. It can be described as follows: Define in $\mathscr{F}_0(X) \times \mathscr{F}_0(X)$ the equivalence relation

$$(u, v) \sim (u', v') \Leftrightarrow u + v' = v + u'. \tag{2.6}$$

The equivalence class of (u, v) will be denoted by $\langle u, v \rangle$. The space $\mathscr{X}$ is the set of equivalence classes. A vector space structure is defined in $\mathscr{X}$ by

$$\langle u, v \rangle + \langle u', v' \rangle = \langle u + u', v + v' \rangle, \tag{2.7}$$

$$\begin{aligned} \lambda\langle u, v \rangle &= \langle \lambda u, \lambda v \rangle, && \text{if } \lambda \geqslant 0, \\ &= \langle (-\lambda) v, (-\lambda) u \rangle, && \text{if } \lambda < 0. \end{aligned} \tag{2.8}$$

The embedding $j: \mathscr{F}_0(X) \to \mathscr{X}$ is defined by

$$j(u) = \langle u, 0 \rangle, \tag{2.9}$$

where 0 is the fuzzy subset $0(x) = 0$.

Finally, the norm in $\mathscr{X}$ is defined by

$$\|\langle u, v \rangle\| = d(u, v). \tag{2.10}$$

3. The Differential of a Fuzzy Function

Let X be a normed space and U be an open subset of X. Let Y be a reflexive Banach space.

By *fuzzy function* we mean a function $F: U \to \mathscr{F}_0(Y)$; such a function associates to each point $x \in U$ a fuzzy subset $F(x)$ of Y (with properties (i)–(iii) described in Section 2). Clearly, such fuzzy functions generalize set-valued functions $U \to Q(Y)$.

To define the differential of a fuzzy function, we shall use the embedding Theorem 2.2 and the classical concept of a differential in normed spaces.

By Theorem 2.2, $\mathscr{F}_0(Y)$ can be embedded isometrically in a normed space $\mathscr{Y}$; let $j: \mathscr{F}_0(Y) \to \mathscr{Y}$ denote this embedding.

Definition 3.1. The fuzzy function $F: U \to \mathscr{F}_0(Y)$ is called *differentiable* at $x_0 \in U$ if the map $\hat{F} = j \circ F$ is differentiable at x_0.

More precisely, F is differentiable at $x_0 \in U$ if there exists a linear bounded operator $\hat{F}'(x_0): X \to \mathscr{Y}$, such that

$$\lim_{x \to x_0} [\|\hat{F}(x) - \hat{F}(x_0) - \hat{F}'(x_0)(x - x_0)\| / \|x - x_0\|] = 0. \tag{3.1}$$

This concept of differential generalizes the differential of a set-valued function $U \to Q(Y)$ as studied by Banks and Jacobs [2].

If X is a finite-dimensional vector space with basis $e_1, e_2, \ldots, e_n$, and if $\hat{F}'(x_0)(e_k) \in j(\mathscr{F}_0(Y)) \subseteq \mathscr{Y}$ for $k = 1, 2, \ldots, n$, we say that the fuzzy function F is *conically differentiable* at $x_0 \in U$. This concept will be related to the differential defined by Hukuhara [8].

To define the Hukuhara differential of a fuzzy function, we shall consider a more particular context: $X = \mathbb{R}$, $Y = \mathbb{R}^n$, and U is an open interval of the real line. A fuzzy function in this context is a function $F: U \to \mathscr{F}_0(\mathbb{R}^n)$.

If $u, v \in \mathscr{F}_0(\mathbb{R}^n)$, and if there exists a fuzzy subset $\xi \in \mathscr{F}_0(\mathbb{R}^n)$ such that $\xi + u = v$, then ξ is *unique* by Proposition 2.2. In this case, ξ is called the *Hukuhara difference* of v and u and is denoted by $v - u$.

DEFINITION 3.2. A fuzzy function $F: U \to \mathscr{F}_0(\mathbb{R}^n)$ is called *H-differentiable* at $x_0 \in U$ if there exists $DF(x_0) \in \mathscr{F}_0(\mathbb{R}^n)$, such that the limits $\lim_{h \to 0^+} [(F(x_0 + h) - F(x_0))/h]$ and $\lim_{h \to 0^+} [(F(x_0) - F(x_0 - h))/h]$ both exist and are equal to $DF(x_0)$.

The relationship between conical differentiability and H-differentiability of fuzzy functions is given in the following proposition which generalizes a corresponding result of Banks and Jacobs [2]:

PROPOSITION 3.1. *If the fuzzy function* $F: U \to \mathscr{F}_0(\mathbb{R}^n)$ *is H-differentiable at* $x_0 \in U$, *then F is conically differentiable at* x_0 *and*

$$\hat{F}'(x_0)(h) = h\langle DF(x_0), 0\rangle. \tag{3.2}$$

Proof. Observe that if $v - u$ exists for $u, v \in \mathscr{F}_0(\mathbb{R}^n)$, then $\langle v, 0\rangle - \langle u, 0\rangle = \langle v - u, 0\rangle$. We have

$$\left\| \frac{\hat{F}(x_0 + \Delta x) - \hat{F}(x_0)}{\Delta x} - \langle DF(x_0), 0\rangle \right\| = d\left(\frac{F(x_0 + \Delta x) - F(x_0)}{\Delta x}, DF(x_0) \right) \tag{3.3}$$

and the result follows as in [2].

The above theorem shows that differentiability as given in Definition 3.1 is a more general concept than H-differentiability.

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