Focal Boundary Value Problems for Nonlinear Difference Equations, I

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1. INTRODUCTION

For \(a \in \mathbb{R}\), let the interval \([a, +\infty) = \{a, a+1, a+2, \ldots\}\), and if \(b = a + m\), for some \(m \in \mathbb{N}\), let \([a, b] = \{a, a+1, \ldots, b\}\), and let \([a, b), (a, b], (a, b)\) denote the analogous discrete sets. With differences defined by \(\Delta u(t) = u(t+1) - u(t)\), and for \(i \geq 2\), \(\Delta^i u(t) = \Delta(\Delta^{i-1} u(t))\), we shall be concerned with uniqueness of solutions implying the existence of solutions of focal type boundary value problems for the \(n\)th-order nonlinear difference equation

\[u(t+n) = f(t, u(t), \ldots, u(t+n-1)), \tag{1}\]

where

(A) \(f: [a, +\infty) \times \mathbb{R}^n \to \mathbb{R}\) is continuous, and the equation \(u_{n+1} = f(t, u_1, \ldots, u_n)\) can be solved for \(u_1\) as a continuous function of the variables \(t, u_2, \ldots, u_{n+1}\).

We remark here that (A) implies (1) is an \(n\)th-order difference equation on any subinterval of \([a, +\infty)\), that solutions of initial value problems for (1) are unique and exist on \([a, +\infty)\), and that solutions of (1) depend continuously on initial conditions.

Following Hartman's [17] major paper, a number of recent papers have appeared which are devoted to results concerning boundary value problems for finite difference equations, and in many cases these results are analogues of those for boundary value problems for ordinary differential equations. For example, papers by Ahlbrandt and Hooker [6–8], Eloe [12, 13], Hankerson [14], Hankerson and Peterson [15, 16], Hooker et al. [21–24], Peterson [26–29], and Smith and Taylor [32] have dealt with disconjugacy or oscillation and nonoscillation of linear difference
equations, whereas the papers by Eloe [11,13] have also dealt with dis-
focality criteria for linear difference equations. For the nonlinear equation
(1), Agarwal [1-5], Eloe [9,10], Hankerson [14], Peterson [30], and
Rodriguez [31] have addressed questions concerning boundary value
problems, with consideration in [4,30] also given to certain focal type
problems.

In a recent paper, Henderson [20] proved that uniqueness of solutions
implies existence of solutions for conjugate boundary value problems
for (1), where such boundary value problems are defined as follows.

**Definition.** Given \( m_1 \in [a, +\infty) \) and \( m_2, \ldots, m_n \in \mathbb{N} \), let \( s_1, \ldots, s_n \in [a, +\infty) \) be defined by \( s_1 = m_1 \) and \( s_i = s_{i-1} + m_i \), \( 2 \leq i \leq n \). A boundary
value problem for (1) satisfying

\[
u(s_i) = y_i, \quad 1 \leq i \leq n,
\]

where \( y_i \in \mathbb{R}, \ 1 \leq i \leq n \), is called an \((m_1, \ldots, m_n)\) conjugate boundary value problem for (1).

In this work, we will address the question of uniqueness of solutions
implying the existence of solutions of boundary value problems for (1) that
are analogous to those which might be termed as "left focal problems" for
ordinary differential equations; see Henderson [18, 19].

**Definition.** Let \( 2 < k \leq n \) and let \( m_1, \ldots, m_k \) be positive integers such
that \( \sum_{i=1}^{k} m_i = n \). Let \( s_0 = 0 \) and for \( 1 \leq j \leq k \), \( s_j = \sum_{i=1}^{j} m_i \). For points
\( a \leq t_1 < t_{k-1} < \cdots < t_1 < +\infty \), where \( t_j + m_j + 1 \leq t_{j-1} \), \( 2 \leq j \leq k \), a boundary
value problem for (1) satisfying

\[
\Delta^j(u(t_j)) = y_{j+1}, \quad s_{j-1} \leq i \leq s_j - 1, \quad 1 \leq j \leq k,
\]

will be called an \((m_k, \ldots, m_1)\) focal boundary value problem for (1).

For our uniqueness assumption on \((m_k, \ldots, m_1)\) focal boundary value
problems, we will make use of Hartman’s [17] definition of a generalized
zero. For a mapping \( u: [a, +\infty) \to \mathbb{R} \), \( t_0 = a \) is a generalized zero of \( u \)
if \( u(a) = 0 \), and \( t_0 > a \) is a generalized zero of \( u \) if \( u(t_0) = 0 \) or there
is an integer \( j \geq 1 \) such that \((-1)^j u(t_0 - j) u(t_0) > 0 \) and if \( j > 1 \),
\( u(t_0 - j + 1) = \cdots = u(t_0 - 1) = 0 \).

In terms of generalized zeros, our uniqueness assumption on \((m_k, \ldots, m_1)\)
focal boundary value problems for (1) will be stated as:

**(B)** Given \( 2 \leq k \leq n \), positive integers \( m_1, \ldots, m_k \) such that
\( \sum_{i=1}^{k} m_i = n \), and points \( a \leq t_k < t_{k-1} < \cdots < t_1 < +\infty \), where \( t_j + m_j + 1 \leq t_{j-1} \), \( 2 \leq j \leq k \), if \( u(t) \) and \( v(t) \) are solutions of (1) such that \( \Delta^j(u(t) - v(t)) \),
s_{j-1} \leq i \leq s_j - 1, \quad (s_0 = 0 \text{ and } s_j = \sum_{i=1}^{j} m_i, \quad 1 \leq j \leq k), \text{ has a generalized zero at } t_j, \quad 1 \leq j \leq k, \text{ then it follows that } u(t) = v(t) \text{ on } [t_k, t_1 + m_1 - 1], \text{ (hence on } [a, +\infty)).

Remarks. (a) Condition (B) implies that, given \(2 \leq k \leq n\), each \((m_k, \ldots, m_1)\) focal boundary value problem for (1) has at most one solution on \([a, +\infty)\).

(b) Under condition (B), it follows from Henderson's [20] uniqueness implies existence result mentioned above that all conjugate boundary value problems for (1) have unique solutions on \([a, +\infty)\).

In Section 2, we first state theorems concerning continuous dependence of solutions of (1) on initial conditions and on \((m_k, \ldots, m_1)\) focal boundary conditions. Making use of this continuous dependence, we prove that (A) and (B) imply a large class of two-point boundary value problems for (1) have unique solutions on \([a, +\infty)\). It will follow as a corollary that each \((m_2, m_1)\) focal boundary value problem for (1) has a unique solution.

2. **Uniqueness Implies Existence for Two-Point Focal Problems**

At the outset of this section, we will state for reference a couple of theorems concerning the continuous dependence of solutions of (1) on initial conditions and on \((m_k, \ldots, m_1)\) focal boundary conditions. For a typical argument concerning the continuous dependence on boundary conditions, see Hankerson [14].

**Theorem 1.** Assume that condition (A) is satisfied. If there exist a sequence \(\{y_k(t)\}\) of solutions of (1), an interval \([t_0, t_0 + n - 1] \subset [a, +\infty)\), and an \(M > 0\) such that \(|y_k(t)| \leq M, \text{ for all } t \in [t_0, t_0 + n - 1], \text{ for all } k \in \mathbb{N}\), then there exists a subsequence \(\{y_{k_j}(t)\}\) which converges pointwise on \([a, +\infty)\) to a solution of (1).

**Theorem 2.** Assume that with respect to (1), conditions (A) and (B) are satisfied. Let \(2 \leq k \leq n\) and positive integers \(m_1, \ldots, m_k\), such that \(\sum_{i=1}^{k} m_i = n\), be given, and let \(s_j, \quad 0 \leq j \leq k,\) be the corresponding partial sums. Given a solution \(u(t)\) of (1) on \([a, +\infty)\), points \(a < t_k < t_{k-1} < \cdots < t_1 < +\infty,\) where \(t_j + m_j + 1 \leq t_{j-1}, \quad 2 \leq j \leq k,\) an interval \([a, b]\), where \(b \geq t_1 + m_1 - 1,\) and an \(\epsilon > 0\), there exists a \(\delta(\epsilon, [a, b]) > 0\) such that, if \(|\Delta^i u(t_j) - y_{j+1}| < \delta, \quad s_{j-1} \leq i \leq s_j - 1, \quad 1 \leq j \leq k,\) then there exists a solution \(v(t)\) of (1) satisfying \(\Delta^i v(t_j) = y_{j+1}, \quad s_{j-1} \leq i \leq s_j - 1, \quad 1 \leq j \leq k,\) and \(|\Delta^i v(t) - \Delta^i u(t)| < \epsilon, \quad 0 \leq i \leq n - 1,\) for all \(t \in [a, b]\).

Now, in terms of generalized zeros, there is a discrete version of Rolle's
theorem; see Hartman [17, Proposition 5.1]. In particular, if a mapping
\( u(t) \) has generalized zeros at points \( b < c \) that belong to the discrete interval
\( [a, +\infty) \), then \( \Delta u(t) \) has a generalized zero on \( [b, c) \). In our next theorem,
we show that conditions (A) and (B) imply the existence of solutions of a
large class of two-point boundary value problems for (1) of which the two-
point conjugate and \((m_2, m_1)\) focal problems are special subclasses.
Solutions of each two-point problem in this large class are unique by (B)
and the discrete Rolle theorem. For the existence, a shooting method is
used.

**THEOREM 3.** Assume that with respect to (1), conditions (A) and (B) are
satisfied. Then, for each \( 1 \leq k \leq n - 1 \) and \( 0 \leq m \leq n - k \), and for points
\( a \leq t_2 < t_1 < +\infty \), where \( t_2 + k + 1 \leq t_1 \), there exists a unique solution of (1)
satisfying

\[
\begin{align*}
\Delta^i u(t_2) &= y_{i+(n-k)} - m + 1, \\
\Delta^i u(t_1) &= y_{i+1},
\end{align*}
\]

\( 0 \leq i \leq n - k - 1, \tag{2.m.k} \)

on \( [a, +\infty) \), for every choice of \( y_i \in \mathbb{R}, \ 1 \leq i \leq n \).

**Proof.** We observe first that if \( m = 0 \), then the problems in the state-
ment of the theorem, i.e., the (1), (2.0.k) problems, are of the conjugate
type, and hence as pointed out in Remark (b) above, such problems have
unique solutions on \( [a, +\infty) \). Moreover, notice that if \( t_1 - t_2 = k \), the
boundary problems in the theorem are equivalent to initial value problems
for (1). Finally, as noted above, solutions of each problem (1), (2.m.k) are
unique by (B) and the discrete Rolle theorem.

The remainder of the proof makes use of a shooting method along with
an induction on \( k, m \), and the difference \( t_1 - t_2 \). Moreover, let \( y_i \in \mathbb{R},
1 \leq i \leq n \), be given throughout.

First let \( k = n - 1 \). Then we wish to show the existence of solutions of (1)
satisfying the boundary conditions (2.m.n - 1), for \( 0 \leq m \leq n - k = 1 \), and
\( t_1 - t_2 \geq k + 1 = n \). As noted above, the case \( m = 0 \) has been resolved, and
furthermore, since the spacing \( t_1 - t_2 = n - 1 \) corresponds to an initial value
problem, assume that \( m = 1 \), that \( t_1 \geq t_2 + n \), and that for each \( t_2 + n - 1 \leq \tau_1 < t_1 \), there exists a unique solution of each boundary value problem (1),
(2.1.n - 1) at the points \( t_2 \) and \( \tau_1 \). Now let \( z(t) \) be the solution of the initial
value problem for (1) satisfying

\[
\begin{align*}
\Delta^i z(t_2) &= y_{i+1}, \\
z(t_2) &= 0.
\end{align*}
\]

Define \( S_1 = \{ r \in \mathbb{R} \mid \text{there is a solution } y(t) \text{ of (1) satisfying } \Delta^i y(t_2) = \Delta^i z(t_2), \}
\]
1 ≤ i ≤ n - 1, and \( y(t_i) = r \), \( z(t_i) \in S_1 \), hence \( S_1 \) is nonempty. Further, it follows from Theorem 2 that \( S_1 \) is an open subset of \( \mathbb{R} \).

Our claim now is that \( S_1 \) is also a closed subset of \( \mathbb{R} \). We assume this claim to be false. Then there exist an \( r_0 \in \overline{S_1} \setminus S_1 \) and a strictly monotone sequence \( \{ r_i \} \subset S_1 \) such that \( \lim_{i \to \infty} r_i = r_0 \). We may assume without loss of generality that \( r_i \uparrow r_0 \). For each \( l \in \mathbb{N} \), let \( y_i(t) \) denote the corresponding solution of (1) satisfying

\[
A^i y_i(t_2) = A^i z(t_2), \quad 1 ≤ i ≤ n - 1,
\]

\[y_i(t_1) = r_i.
\]

It follows from conditions (A) and (B) that \( y_i(t) < y_{i+1}(t) \) on \([t_2 + n - 1, +\infty)\), for all \( l \in \mathbb{N} \). Furthermore, the induction hypothesis implies the existence of unique solutions of (1), (2.1.n - 1) at the points \( t_2 \) and \( t_2 + n - 1 \), which when coupled with Theorem 2 along with \( r_o \notin S_1 \) implies that \( y_i(t_2 - 1) \uparrow +\infty \), as \( l \to +\infty \). Moreover, by Theorem 1, there exists \( t_o \in (t_1, t_1 + n - 1) \) such that \( y_i(t_0) \uparrow +\infty \), as \( l \to +\infty \).

Now let \( u(t) \) be the solution of the conjugate problem (1), (2.0.n - 1) satisfying

\[
A^i u(t_2) = y_{i+1}, \quad 1 ≤ i ≤ n - 2,
\]

\[u(t_2) = 0,
\]

\[u(t_1) = r_o.
\]

Since \( y_i(t_1 - 1) \uparrow +\infty \) and \( y_i(t_0) \uparrow +\infty \), whereas \( y_i(t_1) = r_i < r_0 = u(t_1) \), for all \( l \), we have that, for some \( L \in \mathbb{N} \), \( u(t) - y_L(t) \) has a generalized zero at \( t_1 \) and also a generalized zero (or zero) at some \( \tau_o \in (t_1, t_0) \). Moreover, \( A^i(u(t_2) - y_L(t_2)) = 0, 1 ≤ i ≤ n - 2 \). It follows from repeated applications of the discrete Rolle theorem that there exist points \( t_2 ≤ \sigma_n < \sigma_{n-1} < \cdots < \sigma_1 ≤ \tau_o \) such that \( A^{i-1}(u(t) - y_L(t)) \) has a generalized zero at \( \sigma_i \), for \( 1 ≤ i ≤ n \). Hence, from (B), \( u(t) = y_L(t) \) on \([a, +\infty)\), but this is a contradiction.

Thus \( S_1 \) is closed and hence \( S_1 = \mathbb{R} \). Selecting \( y_1 \in S_1 \), we conclude that there exists a solution \( y(t) \) of (1) satisfying

\[
A^i y(t_2) = y_{i+1}, \quad 1 ≤ i ≤ n - 1,
\]

\[y(t_2) = y_1.
\]

In particular, given \( a ≤ t_2 < t_1 < +\infty \), with \( t_2 + n ≤ t_1 \), each boundary value problem (1), (2.1.n - 1) has a unique solution.

Inducting on \( k \), assume now that \( k < n - 1 \) and that for each \( k < h ≤ n - 1 \), each boundary value problem (1), (2.m.h), for \( 0 ≤ m ≤ n - h \),
has a unique solution. We wish to show the existence of unique solutions of (1), (2,m.k), 0 \leq m \leq n-k. In addition to the assumption on k, since m=0 corresponds to a conjugate problem, assume also that 1 \leq m \leq n-k and that each boundary value problem (1), (2,l.k) has a unique solution, where 0 \leq l < m. Moreover, given points a < t_2 < t_1 < +\infty, since any problem (1), (2,m,k) is an initial value problem if t_1 - t_2 = k, we assume in addition to the hypotheses on k and m that t_1 \geq t_2 + k + 1 and that for each t_2 + k \leq \tau_1 < t_1, there exists a unique solution of the boundary value problem (1), (2,m,k) at the points t_2 and \tau_1.

Now let z(t) be the solution of the boundary value problem (1), (2,m-1,k+1) satisfying

\[ A'z(t_2) = y_{i+(n-k)-m+1}, \quad m \leq i \leq k+m-1, \]
\[ A'^{-1}z(t_2) = 0, \]
\[ A'z(t_1) = y_{i+1}, \quad 0 \leq i \leq n-(k+1)-1. \]

This time define \( S_2 = \{ r \in \mathbb{R} \mid \text{there is a solution } y(t) \text{ of (1) satisfying } \]
\[ A'y(t_2) = A'z(t_2), \quad m \leq i \leq k+ m-1, \]
\[ A'y(t_1) = A'z(t_1), \quad 0 \leq i \leq n-k-2, \]
\[ A'^{-k-1}y(t_1) = r \}. \]

Now \( A'^{-k-1}z(t_1) \in S_2, \) so \( S_2 \) is nonempty, and it follows again from Theorem 2 that \( S_2 \) is an open subset of \( \mathbb{R}. \)

As before, we claim that \( S_2 \) is also closed. Assuming that \( S_2 \) is not closed, then there exist \( r_0 \in S_2 \setminus S_2 \) and a strictly monotone sequence \( \{ r_i \} \subset S_2 \) such that \( \lim_i r_i = r_0. \) We may assume again that \( r_i \uparrow r_0, \) and as before, let \( y_i(t) \) denote the corresponding solution of (1) satisfying

\[ A'y_i(t_2) = A'z(t_2), \quad m \leq i \leq k+m-1, \]
\[ A'y_i(t_1) = A'z(t_1), \quad 0 \leq i \leq n-k-2, \]
\[ A'^{-k-1}y_i(t_1) = r_i. \]

Note that from the boundary conditions, \( y_i(t_1) = y_{i+1}(t_1), \) for \( t = t_1, \ldots, t_1 + n-k-2, \) and that \( y_i(t_1 + n - k - 1) < y_{i+1}(t_1 + n - k - 1), \) for each \( i \in \mathbb{N}. \) It follows from (B) and repeated applications of the discrete Rolle theorem that \( y_i(t) < y_{i+1}(t) \) on \( [t_1 + n-k-1, +\infty), \) for each \( i \in \mathbb{N}. \) Since \( r_0 \notin S_2, \) we have from Theorem 1 that for some \( t_0 \in (t_1 + n-k-1, t_1 + n-1], \) \( y_i(t_0) \uparrow +\infty, \) as \( i \to +\infty. \)

Similarly, it also follows from (B) and repeated applications of the discrete Rolle theorem that, if \( n-k-1 \) is even, then \( y_i(t_1 - 1) < y_{i+1}(t_1 - 1), \) and if \( n-k-1 \) is odd, then \( y_i(t_1 - 1) > y_{i+1}(t_1 - 1), \) for each \( i \in \mathbb{N}. \) We will assume that \( n-k-1 \) is even so that \( y_i(t_1 - 1) < y_{i+1}(t_1 - 1), \) for each \( i \in \mathbb{N}. \) We claim that \( \{ y_i(t_1 - 1) \} \) is not bounded above. If the claim is false, then there is an \( M \) such that \( y_i(t_1 - 1) \uparrow M, \) as \( i \to +\infty. \) By
the induction hypothesis on the difference $t_1 - t_2$, there is a solution $v(t)$ of (1), (2.m.k) at the points $t_2$ and $t_1 - 1$ satisfying

$$A^i v(t_2) = A^i z(t_2) = A^i y_i(t_2), \quad m \leq i \leq k + m - 1, l \in \mathbb{N},$$

$$v(t_1 - 1) = M,$$

$$A^i v(t_1 - 1) = (-1)^i M + \sum_{j=1}^{i} (-1)^{j+1} A^{i-j} z(t_1)$$

$$= (-1)^i M + \sum_{j=1}^{i} (-1)^{j+1} A^{i-j} y_i(t_1),$$

$$1 \leq i \leq n - k - l, l \in \mathbb{N}.$$

It follows from Theorem 2 that $\{y_i(t)\}$ converges to $v(t)$ at each point of $[a, +\infty)$, which in turn implies $A^{n-k-1} v(t_1) = r_0$, hence contradicting $r \notin S_2$. Hence our claim is true that $\{y_i(t_1 - 1)\}$ is unbounded above; i.e., $y_i(t_1 - 1) \uparrow +\infty$, as $l \to +\infty$.

Now, let $u(t)$ be the solution of the boundary value problem (1), (2.m-1.k) satisfying

$$A^i u(t_2) = y_i + (n-k-1)m + 1, \quad m \leq i \leq k + m - 2,$$

$$A^{m-1} u(t_2) = 0,$$

$$A^i u(t_1) = y_i + 1, \quad 0 \leq i \leq n - k - 2,$$

$$A^{n-k-1} u(t_1) = r_0.$$

Since $y_i(t_1 - 1) \uparrow +\infty$ and $y_i(t_0) \uparrow +\infty$, where $t_0 \in (t_1 + n - k - 1, t_1 + n - 1]$ was discussed above, there exists an $L \in \mathbb{N}$ such that $y_L(t_1 - 1) > u(t_1 - 1)$ and $y_L(t_0) > u(t_0)$. Since $A^{n-k-1} u(t_1) = r_0 > r_L = A^{n-k-1} y_L(t_1)$, whereas $A^i(u(t_1) - y_L(t_1)) = 0$, $0 \leq i \leq n - k - 2$, it follows that $u(t) - y_L(t)$ has a generalized zero at $t_1 + n - k - 1$ and a generalized zero at some $\tau_0 \in (t_1 + n - k - 1, t_0]$. Moreover, $A^i(u(t_2) - y_L(t_2)) = 0$, $m \leq i \leq k + m - 2$. Applying the discrete Rolle theorem in terms of the conditions satisfied by $u(t) - y_L(t)$ at $t_2$, $t_1$, $t_1 + n - k - 1$, and $\tau_0$, we conclude that there exist points $t_2 \leq \sigma_n < \sigma_{n-1} < \cdots < \sigma_1 \leq \tau_0$ such that $A^{i-1}(u(t) - y_L(t))$ has a generalized zero at $\sigma_i$, for $1 \leq i \leq n$. This is a contradiction to (B).

Consequently, our assumption concerning $S_2$ is false, so that $S_2$ is also closed. In particular, $S_2 = \mathbb{R}$, and so choosing $y_{n-k} \in S_2$, we conclude that there exists a unique solution $v(t)$ of the boundary value problem (1), (2.m.k) satisfying
\[ A_i u(t_2) = y_{i+(n-k)-m+1}, \quad m \leq i \leq k+m-1, \]
\[ A_i u(t_1) = y_{i+1}, \quad 0 \leq i \leq n-k-1. \]

The proof of the theorem is complete.

We immediately have the following concerning the existence of solutions of \((m_2, m_1)\) focal boundary value problems for (1).

**Theorem 4.** Assume that with respect to (1), conditions (A) and (B) are satisfied. Then, given positive integers \(m_1\) and \(m_2\) such that \(m_1 + m_2 = n\), each \((m_2, m_1)\) focal boundary value problem for (1) has a unique solution on \([a, +\infty)\).

**Proof.** Given \(m_1\) and \(m_2\) such that \(m_1 + m_2 = n\), let \(m_2\) correspond to \(k\) and let \(m_1 = n - m_2\) correspond to the case \(m = n - k\) in Theorem 3.

**Remark.** We remark here that the half-line \([a, +\infty)\) is not necessary. In particular, the results can be extended to a finite interval \([a, b + n]\), where \(b\) is the rightmost point at which conditions are specified, so that our application of Theorem 1 can still be made in the arguments.

**References**