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Rationality properties of unipotent representations ☆

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Dedicated to Claudio Procesi on the occasion of his 60th birthday

Introduction

0.1. Let $\mathbf{k}$ be an algebraic closure of a finite field F_q with q elements. Let G be a connected simple algebraic group of adjoint type over $\mathbf{k}$ with a fixed F_q-rational structure; let $F : G \to G$ be the corresponding Frobenius map and let $r(G)$ be the F_q-rank of G. The fixed point set G^F is a finite group. Let W be the Weyl group of G. For $w \in W$ let R_w be the character of the virtual representation $R^1(w)$ of G^F defined in [DL, 1.5]. (The definition of R_w is in terms of l-adic cohomology but in fact R_w has integer values and is independent of l, see [DL, 3.3].) For an irreducible representation ρ of G^F over $\mathbb{C}$ we denote by tr_ρ the character of ρ. We say that ρ is *unipotent* if $(\rho : R_w)$, the multiplicity of ρ in R_w, is $\neq 0$ for some $w \in W$ (see [DL, 7.8]). Let $\mathcal{U}$ be the set of isomorphism classes of unipotent representations of G^F. Let $\widetilde{\mathcal{U}}_{\mathbb{Q}} = \{\rho \in \mathcal{U}\colon\ \mathrm{tr}_\rho(g) \in \mathbb{Q}\ \forall g \in G^F\}$. Let $\mathcal{U}_{\mathbb{Q}}$ be the set of all $\rho \in \mathcal{U}$ such that ρ is defined over $\mathbb{Q}$ (that is, it can be realized by a $\mathbb{Q}[G^F]$-module). We have $\mathcal{U}_{\mathbb{Q}} \subset \widetilde{\mathcal{U}}_{\mathbb{Q}} \subset \mathcal{U}$. Let $\mathcal{U}^0 = \{\rho \in \mathcal{U}\colon\ \rho \text{ cuspidal}\}$.

Unless otherwise specified, we assume that G *is split over* F_q. The following is one of our results.

Theorem 0.2. *We have* $\mathcal{U}_{\mathbb{Q}} = \widetilde{\mathcal{U}}_{\mathbb{Q}}$.

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0.3. We will also show (see Corollary 1.12) that if G is of type A, B, C or D, then $\mathcal{U}_{\mathbb{Q}} = \mathcal{U}$. (The analogous statement is false for exceptional types.) The rationality of certain unipotent cuspidal representations connected with Coxeter elements has been proved in [L1]. The method of [L1] has been extended in [L2, unpublished] to determine explicitly $\mathcal{U}_{\mathbb{Q}}$ in the general case (including non-split groups). The case where G is non-split of type A has been also considered by Ohmori [Oh], another extension of the method [L1].

Our study of rationality of unipotent representations is based on the statement that a given unipotent representation appears with multiplicity 1 in some (possibly virtual) representation R defined using l-adic cohomology and then using the Hasse principle. In the first method (that of [L2]), R is a particular intersection cohomology space of a variety; see Section 1. In the second method (which applies only in the cuspidal case), R will be one of the R_w above; see Section 2. In one case ($G = SO_5$ with q odd), we give an elementary approach to rationality (without using the Hasse principle); see Section 3.

1. First method

1.1. Let p be the characteristic of F_q. For any prime number $l \neq p$, we choose an imbedding of the field $\mathbb{Q}_l$, the l-adic numbers, into $\mathbb{C}$. This allows us to regard any representation of G^F over $\mathbb{Q}_l$ as one over $\mathbb{C}$. Let X be the flag manifold of G; let $F : X \to X$ be the map induced by $F : G \to G$. For $w \in W$ let O_w be the set of all $(B, B') \in X \times X$ that are in relative position w. As in [DL], for any $w \in W$, let $X_w = \{B \in X\colon\ (B, F(B)) \in O_w\}$; let $\overline{X}_w$ be the closure of X_w in X. Then $X_w, \overline{X}_w$ are stable under the conjugation action of G^F on X. Hence for any $j \in \mathbb{Z}$ there is an induced action of G^F on the l-adic cohomology with compact support $H_c^j(X_w, \mathbb{Q}_l)$ and on the l-adic intersection cohomology $\mathbf{H}^j(\overline{X}_w, \mathbb{Q}_l)$. (Note that $\overline{X}_w$ has pure dimension $l(w)$ where $l : W \to \mathbb{N}$ is the length function.) Recall that R_w is the character of the virtual representation $\sum_{j \in \mathbb{Z}} (-1)^j H_c^j(X_w, \mathbb{Q}_l)$ of G^F.

Lemma 1.2. *Let $\rho \in \mathcal{U}$. There exists $x \in W$ and $j \in [0, l(x)]$ such that $(\rho : \mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l)) = 1$.*

The proof is based on results of [L3]. For any $x \in W$ let $\mathcal{A}_x$ be the virtual representation of W defined in [L3, pp. 154, 156]. For any virtual representation E of W we set $R_E = |W|^{-1} \sum_{w \in W} \operatorname{tr}(w, E) R_w$ (a $\mathbb{Q}$-valued class function on G^F). Thus, $R_{\mathcal{A}_x}$ is defined. Let $a : W \to \mathbb{N}$ be as in [L3, p. 178]. Assume that

$$x \in W \text{ is such that } \quad \bigl(\rho : (-1)^{l(x)-a(x)} R_{\mathcal{A}_x}\bigr) = 1. \tag{1.2a}$$

Then from [L3, 6.15, 6.17(i), 5.13(i)] we deduce $(\rho : \mathbf{H}^{l(x)-a(x)}(\overline{X}_x, \mathbb{Q}_l)) = 1$. (Actually, in [L3], q is assumed to be sufficiently large; but this assumption is removed later in [L3].) Thus, to prove the lemma it is enough to show that (1.2a) holds for some $x \in W$. Now in [L3], the multiplicities of unipotent representations in $(-1)^{l(x)-a(x)} R_{\mathcal{A}_x}$ have been explicitly described for many x. (See the tables in [L3, pp. 304–306] for types E_8, F_4, and the results in [L3, Chapter 9] for classical types.) In particular, we see that (1.2a) holds for some $x \in W$.

Lemma 1.3. *Let $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$. Let l be a prime number invertible in* **k**. *Let x, j be as in Lemma* 1.2.

(a) *ρ is defined over $\mathbb{Q}_l$.*
(b) *ρ is defined over $\mathbb{R}$ if and only if j is even.*
(c) *If j is even then $\rho \in \mathcal{U}_{\mathbb{Q}}$.*

Clearly, (a) follows from Lemma 1.2. We prove (b). Let $c \in H^2(\overline{X}_w, \mathbb{Q}_l)$ be the Chern class of an ample line bundle on $\overline{X}_w$ (we ignore Tate twists); we may assume that this line bundle is the restriction of a line bundle on X. Since G^F acts trivially on $H^2(X, \mathbb{Q}_l)$ it follows that c is G^F-invariant. Hence the map $\mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l) \to \mathbf{H}^{2l(x)-j}(\overline{X}_x, \mathbb{Q}_l)$ given by $\xi \mapsto c^{l(x)-j}\xi$ is compatible with the G^F-action. This map is an isomorphism, by the Hard Lefschetz Theorem [BBD, 5.4.10]. Let $(\,,\,) : \mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l) \times \mathbf{H}^{2l(x)-j}(\overline{X}_x, \mathbb{Q}_l)$ be the Poincaré duality pairing. (We again ignore Tate twists.) Then $\xi, \xi' \mapsto (\xi, c^{l(x)-j}\xi')$ is a $(-1)^j$-symmetric, non-singular, G^F-invariant bilinear form $\mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l) \times \mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l) \to \mathbb{Q}_l$. This restricts to a $(-1)^j$-symmetric, G^F-invariant bilinear form on the ρ-isotypic part of $\mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l)$, which is non-singular, since ρ is isomorphic to its dual (recall that $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$). This ρ-isotypic part is isomorphic to ρ and (b) follows. Under the assumption of (c), we see from (a), (b), using the Hasse principle for division algebras with centre $\mathbb{Q}$ [Weil, Theorem 2, Chapter XI-2] that ρ is defined over $\mathbb{Q}$. (The Hasse principle is applicable even when information is missing at one place, in our case at p-adic numbers, see [Weil, Theorem 2, Chapter XIII-3].) The lemma is proved.

Lemma 1.4. *Let $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$. Let x, j be as in Lemma* 1.2. *Then j is even.*

It is known [L3] that the parity of an integer j such that $(\rho : \mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l)) \neq 0$ for some $x \in W$, is an invariant of ρ. Moreover, j is even except if G is of type E_7 and $\rho \in \mathcal{U}^0$, or G is of type E_8 and ρ is a component of the representation induced by a unipotent cuspidal representations of a parabolic of type E_7 (see [L3, Chapter 11]). In these exceptional cases, we have $\rho \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$, as one sees using [L3, 11.2]. The lemma is proved.

1.5. Now Theorem 0.2 follows immediately from Lemmas 1.3(c) and 1.4.

1.6. Let $\mathcal{X}$ be the set of all triples $(\mathcal{F}, y, \sigma)$ where $\mathcal{F}$ is a "family" [L3, 4.2] of irreducible representations of W (with an associated finite group $\mathcal{G}_{\mathcal{F}}$, see [L3, Chapter 4]), y is an element of $\mathcal{G}_{\mathcal{F}}$ defined up to conjugacy, and σ is an irreducible representation of the centralizer of y in $\mathcal{G}_{\mathcal{F}}$ defined up to isomorphism. For $(\mathcal{F}, y, \sigma) \in \mathcal{X}$, let $\lambda_{y,\sigma}$ be the scalar by which y acts on σ (a root of 1). Let $\mathcal{X}_1$ be the set of all $(\mathcal{F}, y, \sigma) \in \mathcal{X}$ such that $|\mathcal{F}| \neq 2$ and $\lambda_{y,\sigma} = \pm 1$. If q is a square, let $\mathcal{X}_2$ be the set of all $(\mathcal{F}, y, \sigma) \in \mathcal{X}$ such that $|\mathcal{F}| = 2$, $y = 1$. If q is not a square, let $\mathcal{X}_2 = \emptyset$. In any case, $\mathcal{X}_2$ is empty unless G is of type E_7 or E_8. Let $\mathcal{X}_{\mathbb{Q}} = \mathcal{X}_1 \cup \mathcal{X}_2$.

In [L3, 4.23], $\mathcal{X}$ is put in a bijection

$$(\mathcal{F}, y, \sigma) \leftrightarrow \rho_{\mathcal{F},y,\sigma} \tag{1.6a}$$

with $\mathcal{U}$.

Lemma 1.7. *Assume that $(\mathcal{F}, y, \sigma) \in \mathcal{X}_{\mathbb{Q}}$, $(\mathcal{F}', y', \sigma') \in \mathcal{X}$ are distinct. Let $\rho = \rho_{\mathcal{F},y,\sigma}$, $\rho' = \rho_{\mathcal{F}',y',\sigma'}$. Then there exists $x \in W$ such that*

$$\bigl(\rho : (-1)^{l(x)-a(x)} R_{\mathcal{A}_x}\bigr) \neq \bigl(\rho' : (-1)^{l(x)-a(x)} R_{\mathcal{A}_x}\bigr).$$

As mentioned in the proof of Lemma 1.2, the multiplicities of various unipotent representations in $(-1)^{l(x)-a(x)} R_{\mathcal{A}_x}$ have been explicitly computed in [L3] for many $x \in W$. From this the lemma follows easily.

Lemma 1.8. *Let $\rho = \rho_{\mathcal{F},y,\sigma}$, where $(\mathcal{F}, y, \sigma) \in \mathcal{X}_{\mathbb{Q}}$. Then $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$.*

Let $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Then $\gamma(\mathrm{tr}_\rho) = \mathrm{tr}_{\rho'}$ for some $\rho' \in \mathcal{U}$. Since the character of $(-1)^{l(x)-a(x)} R_{\mathcal{A}_x}$ is an integer valued, it is fixed by γ. (Here x is any element of W.) Hence ρ, ρ' have the same multiplicity in $(-1)^{l(x)-a(x)} R_{\mathcal{A}_x}$. From Lemma 1.7 it follows that $\rho = \rho'$. Thus, $\gamma(\mathrm{tr}_\rho) = \mathrm{tr}_\rho$ for any $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$, so that tr_ρ has rational values. The lemma is proved.

Lemma 1.9. *Let $\rho = \rho_{\mathcal{F},y,\sigma}$, where $(\mathcal{F}, y, \sigma) \notin \mathcal{X}_{\mathbb{Q}}$. Then $\rho \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$.*

Assume first that $\lambda_{y,\sigma} \neq \pm 1$. Then $\lambda_{y,\sigma} \notin \mathbb{Q}$ hence there exists $\gamma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ such that $\gamma(\lambda_{y,\sigma}) \neq \lambda_{y,\sigma}$. Using the interpretation of $\lambda_{y,\sigma}$ given in [L3, 11.2], it follows that $\gamma(\mathrm{tr}_\rho) \neq \mathrm{tr}_\rho$. Hence $\rho \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$. Next assume that $\lambda_{y,\sigma} = \pm 1$. Then $|\mathcal{F}| = 2$. Moreover, if q is a square, then $y \neq 1$. Let σ' be the character of $\mathcal{G}_{\mathcal{F}} = \mathbb{Z}/2\mathbb{Z}$ other than σ. Let $\rho' = \rho_{\mathcal{F},y,\sigma'}$. If $y \neq 1$, then by the results of [L1], tr_ρ is carried to $\mathrm{tr}_{\rho'}$ by an element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ that takes $\sqrt{-q}$ to $-\sqrt{-q}$. If $y = 1$, then by the known construction of representations of Hecke algebras in terms of W-graphs, tr_ρ is carried to $\mathrm{tr}_{\rho'}$ by an element of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ that takes $\sqrt{q}$ to $-\sqrt{q}$. Hence again $\rho \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$.

Proposition 1.10. *Under the bijection* $\mathcal{X} \leftrightarrow \mathcal{U}$ *in* (1.6a), *the subset* $\widetilde{\mathcal{U}}_{\mathbb{Q}}$ *of* $\mathcal{U}$ *corresponds to the subset* $\mathcal{X}_{\mathbb{Q}}$ *of* $\mathcal{X}$.

This follows immediately from Lemma 1.9.

Combining the proposition with Theorem 0.2, we obtain:

Corollary 1.11. *Under the bijection* $\mathcal{X} \leftrightarrow \mathcal{U}$ *in* (1.6a), *the subset* $\mathcal{U}_{\mathbb{Q}}$ *of* $\mathcal{U}$ *corresponds to the subset* $\mathcal{X}_{\mathbb{Q}}$ *of* $\mathcal{X}$.

If G is of type A, B, C or D, then for any family $\mathcal{F}$ we have $|\mathcal{F}| \neq 2$ and the group $\mathcal{G}_{\mathcal{F}}$ is an elementary abelian 2-group, hence $\lambda_{y,\sigma} = \pm 1$ for any $(\mathcal{F}, y, \sigma) \in \mathcal{X}$. Thus, we have $\mathcal{X}_{\mathbb{Q}} = \mathcal{X}$ and we obtain:

Corollary 1.12. *If G is of type A, B, C or D, then* $\mathcal{U}_{\mathbb{Q}} = \mathcal{U}$.

1.13. If G is non-split, the analogues of Lemmas 1.2 and 1.3 continue to hold but that of Lemma 1.4 does not (it does in type D but not in type A). Also, if G is non-split of type D, then $\mathcal{U}_{\mathbb{Q}} = \mathcal{U}$. If G is non-split of type A we have $\widetilde{\mathcal{U}}_{\mathbb{Q}} = \mathcal{U}$ but in general $\mathcal{U}_{\mathbb{Q}} \neq \mathcal{U}$.

2. Second method

2.1. Let $n \in \mathbb{N}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_a$ be a sequence of integers such that

$$\sum_i \lambda_i = n + \binom{a}{2}. \tag{2.1a}$$

We define a virtual representation $[\lambda_1, \lambda_2, \ldots, \lambda_a]$ of the symmetric group S_n as follows. If $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_a$, then $[\lambda_1, \lambda_2, \ldots, \lambda_a]$ is the irreducible representation of S_n corresponding to the partition $\lambda_1 \leqslant \lambda_2 - 1 \leqslant \cdots \leqslant \lambda_a - a + 1$ of n, as in [L3, p. 81]. If $\lambda_1, \lambda_2, \ldots, \lambda_a$ are in $\mathbb{N}$ and are distinct, then

$$[\lambda_1, \lambda_2, \ldots, \lambda_a] = \operatorname{sgn}(\sigma)\big[\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \ldots, \lambda_{\sigma(a)}\big]$$

where σ is the unique permutation of $1, 2, \ldots, a$ such that $\lambda_{\sigma(1)} < \lambda_{\sigma(2)} < \cdots < \lambda_{\sigma(a)}$. If $\lambda_1, \lambda_2, \ldots, \lambda_a$ are not distinct, or if at least one of them is < 0, we set $[\lambda_1, \lambda_2, \ldots, \lambda_a] = 0$. From the definition we see easily that

$$[\lambda_1, \lambda_2, \ldots, \lambda_a] = [0, \lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_a + 1]$$

for any sequence of integers $\lambda_1, \lambda_2, \ldots, \lambda_a$ such that (2.1a) holds.

Lemma 2.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_a$ be a sequence of integers such that* (2.1a) *holds. Let $w = (k)w' \in S_k \times S_{n-k} \subset S_n$ where (k) denotes a k-cycle in S_k and $w' \in S_{n-k}$. We have*

$$\operatorname{tr}\big(w, [\lambda_1, \lambda_2, \ldots, \lambda_a]\big) = \sum_{i=1}^{a} \operatorname{tr}\big(w', [\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i - k, \lambda_{i+1}, \ldots, \lambda_a]\big). \tag{2.2a}$$

If λ_i are not distinct or if at least one of them is < 0 then both sides of (2.2a) are 0. We may assume that $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_a$. In this case, (2.2a) can be seen to be equivalent to Murnaghan's rule, see [Weyl].

2.3. For $n \geqslant 0$ let W_n be the group of all permutations of $1, 2, \ldots, n, n', \ldots, 2', 1'$ which commute with the involution $i \leftrightarrow i'$ for $i = 1, \ldots, n$ (we have $W_0 = \{1\}$). Given two sequences of integers $\lambda_1, \ldots, \lambda_a$ and $\mu_1, \mu_2, \ldots, \mu_b$ such that

$$\sum_i \lambda_i + \sum_i \mu_i = n + \binom{a}{2} + \binom{b}{2}, \tag{2.3a}$$

we define a virtual representation

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_b \end{bmatrix} \tag{2.3b}$$

of W_n as follows. If λ_i are not distinct or if μ_i are not distinct or if at least one of λ_i or μ_i is < 0, we define (2.3b) to be 0. Assume now that $\lambda_i \in \mathbb{N}$ are distinct, and that $\mu_i \in \mathbb{N}$ are distinct. Then $r, \tilde{r}$ defined by

$$\sum_i \lambda_i = r + \binom{a}{2}, \qquad \sum_i \mu_i = \tilde{r} + \binom{b}{2}$$

satisfy $r, \tilde{r} \in \mathbb{N}$, $r + \tilde{r} = n$. We identify $W_r \times W_{\tilde{r}}$ with a subgroup of W_n as in [L3, p. 82]. The virtual representation $[\lambda_1, \lambda_2, \ldots, \lambda_a] \boxtimes [\mu_1, \mu_2, \ldots, \mu_b]$ of $S_r \times S_{\tilde{r}}$ may be regarded as a virtual representation of $W_r \times W_{\tilde{r}}$ via the obvious projection $W_r \times W_{\tilde{r}} \to S_r \times S_{\tilde{r}}$ (see [L3, p. 82]). We tensor this with the one-dimensional character of $W_r \times W_{\tilde{r}}$ which is the identity on the W_r-factor and is the restriction of $\chi : W_n \to \{\pm 1\}$ (see [L3, p. 82]) on the $W_{\tilde{r}}$-factor. Inducing the resulting virtual representation from $W_r \times W_{\tilde{r}}$ to W_n, we obtain the virtual representation (2.3b) of W_n. Note that if $\lambda_1 < \lambda_2 < \cdots < \lambda_a$ and $\mu_1 < \mu_2 < \cdots < \mu_b$, then this is an irreducible representation; if σ is a permutation of $1, 2, \ldots, a$ and σ' is a permutation of $1, 2, \ldots, b$, then

$$\begin{bmatrix} \lambda_{\sigma(1)} & \lambda_{\sigma(2)} & \cdots & \lambda_{\sigma(a)} \\ \mu_{\sigma'(1)} & \mu_{\sigma'(2)} & \cdots & \mu_{\sigma'(b)} \end{bmatrix} = \operatorname{sgn}(\sigma)\operatorname{sgn}(\sigma') \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \ldots & \mu_b \end{bmatrix}.$$

From the definition we see easily that

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \ldots & \mu_b \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 + 1 & \lambda_2 + 1 & \cdots & \lambda_a + 1 \\ 0 & \mu_1 + 1 & \mu_2 + 1 & \ldots & \mu_b + 1 \end{bmatrix}.$$

Lemma 2.4. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_a$ *and* $\mu_1, \mu_2, \ldots, \mu_b$ *be two sequences of integers such that* (2.3a) *holds. Let* $w = (2k) \times w' \in W_k \times W_{n-k} \subset W_n$ *where* $0 < k \leqslant n$, $(2k)$ *denotes an element of* W_k *whose image under the obvious imbedding* $W_k \subset S_{2k}$ *is a* $2k$*-cycle and* $w' \in W_{n-k}$. *We have*

$$\begin{aligned}
&\operatorname{tr}\left(w, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_b \end{bmatrix}\right) \\
&\quad = \sum_{i=1}^{a} \operatorname{tr}\left(w', \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{i-1} & \lambda_i - k & \lambda_{i+1} & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_b \end{bmatrix}\right) \\
&\qquad - \sum_{i=1}^{a} \operatorname{tr}\left(w', \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_a \\ \mu_1 & \mu_2 & \cdots & \mu_{i-1} & \mu_i - k & \mu_{i+1} & \cdots & \mu_b \end{bmatrix}\right).
\end{aligned}$$

This follows from Lemma 2.2, using the definitions.

2.5. Let $m \in \mathbb{N}$ and let $n = m^2 + m$. Let $w_m \in W_n$ be an element whose image under the imbedding $W_n \subset S_{2n}$ is a product of cycles $(4)(8)(12)\ldots(4m)$. Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{m+1} \quad \text{and} \quad \mu_1 < \mu_2 < \cdots < \mu_m \tag{2.5a}$$

be two sequences of integers such that $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}, \mu_1, \mu_2, \ldots, \mu_m$ is a permutation of $0, 1, 2, 3, \ldots, 2m$. Then (2.3a) holds (with $a = m + 1$, $b = m$, and $n = m^2 + m$). Consider the property

$$\lambda_i + \lambda_j \neq 2m \quad \text{for any } i \neq j \quad \text{and} \quad \mu_i + \mu_j \neq 2m \quad \text{for any } i \neq j. \tag{$*$}$$

Lemma 2.6. *In the setup of Section* 2.5, *if* $(*)$ *holds, then*

$$\operatorname{tr}\left(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = (-1)^{(m^2+m)/2}.$$

If $(*)$ *does not hold, then*

$$\operatorname{tr}\left(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = 0.$$

We argue by induction on m. The result is clear when $m = 0$. Assume now that $m > 0$. We can assume that $w_m = (4m)w_{m-1} \in W_{2m} \times W_{n-2m} \subset W_n$ where $w_{m-1} \in W_{n-2m}$ is defined in a similar way to w_m. We apply Lemma 2.4 with $w = w_m$, $k = 2m$, $w' = w_{m-1}$. Note that in the formula in Lemma 2.4, at most one term is non-zero, namely, the one in which $k = 2m$ is substracted from the largest of entries λ_i or μ_i (the other terms are zero since they contain some < 0 entry). We are in one of the four cases below.

Case 1 ($2m = \lambda_{m+1}$, $0 = \mu_1$). Using Lemma 2.4, we have

$$
\begin{aligned}
A &= \operatorname{tr}\left(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \\
&= \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m & 0 \\ 0 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right) \\
&= (-1)^m \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ 0 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right) \\
&= (-1)^m \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_m - 1 \\ \mu_2 - 1 & \mu_3 - 1 & \cdots & \mu_m - 1 \end{bmatrix}\right).
\end{aligned}
$$

Now the induction hypothesis is applicable to sequences

$$
\begin{aligned}
&\lambda_1 - 1 < \lambda_2 - 1 < \cdots < \lambda_m - 1 \quad \text{and} \\
&\mu_2 - 1 < \mu_3 - 1 < \cdots < \mu_m - 1
\end{aligned}
\tag{2.6a}
$$

instead of sequences (2.5a). (Clearly, (2.5a) satisfies $(*)$ if and only if (2.6a) satisfies the analogous condition.) Hence, if (2.5a) satisfies $(*)$, then

$$
A = (-1)^m (-1)^{(m^2-m)/2} = (-1)^{(m^2+m)/2}
$$

as required. If (2.5a) does not satisfy $(*)$, then $A = (-1)^m 0 = 0$, as required.

Case 2 ($2m = \lambda_{m+1}$, $0 = \lambda_1$). Using Lemma 2.4, we have

$$
\operatorname{tr}\left(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_m & 0 \\ \mu_1 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right)
$$

and this equals 0 since 0 appears twice in the top row.

Case 3 ($2m = \mu_m$, $0 = \lambda_1$). Using Lemma 2.4, we have

$$
\begin{aligned}
A &= \operatorname{tr}\left(w_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \\
&= -\operatorname{tr}\left(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_m & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_{m-1} & 0 \end{bmatrix}\right) \\
&= (-1)^m \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_m & \lambda_{m+1} \\ 0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} \end{bmatrix}\right) \\
&= (-1)^m \operatorname{tr}\left(w_{m-1}, \begin{bmatrix} \lambda_2 - 1 & \cdots & \lambda_m - 1 & \lambda_{m+1} - 1 \\ \mu_1 - 1 & \mu_2 - 1 & \cdots & \mu_{m-1} - 1 \end{bmatrix}\right).
\end{aligned}
$$

Now the induction hypothesis is applicable to sequences

$$
\begin{aligned}
&\lambda_2 - 1 < \cdots < \lambda_m - 1 < \lambda_{m+1} - 1 \quad \text{and} \\
&\mu_1 - 1 < \mu_2 - 1 < \cdots < \mu_{m-1} - 1
\end{aligned}
\tag{2.6b}
$$

instead of sequences (2.5a). (Clearly, (2.5a) satisfies $(*)$ if and only if (2.6b) satisfies the analogous condition.) Hence, if (2.5a) satisfies $(*)$, then

$$A=(-1)^m(-1)^{(m^2-m)/2}=(-1)^{(m^2+m)/2}$$

as required. If (2.5a) does not satisfy $(*)$, then $A=(-1)^m 0=0$, as required.

Case 4 ($2m=\mu_m$, $0=\mu_1$). Using Lemma 2.4, we have

$$\begin{aligned}&\operatorname{tr}\left(w_m,\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_{m+1}\\ \mu_1 & \mu_2 & \cdots & \mu_m\end{bmatrix}\right)\\ &\quad=-\operatorname{tr}\left(w_{m-1},\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_m & \lambda_{m+1}\\ 0 & \mu_2 & \cdots & 0 & \end{bmatrix}\right)\end{aligned}$$

and this is 0 since 0 appears twice in the bottom row. The lemma is proved.

Lemma 2.7. *In the setup of Lemma* 2.6*, if* $(*)$ *holds, then*

$$\sharp\bigl(k\in\{1,2,\dots,m\}\colon\ \mu_k=\mathit{even}\bigr)=\bigl(m^2+m\bigr)/2 \quad \text{mod } 2.$$

Since $(*)$ holds, the left-hand side is equal to the number of pairs

$$(0,2m),\quad (1,2m-1),\quad (2,2m-2),\quad \dots,\quad (m-1,m+1),$$

in which both components are even. This equals $m/2$ if m is even and $(m+1)/2$ if m is odd. Hence it has the same parity as $m(m+1)/2$. The lemma is proved.

2.8. Let m,n be as in Section 2.5. Let $z\in W_n$ be an element which has no eigenvalue 1 in the reflection representation of W_n. Then $z=z_1z_2\dots z_k$ with $k\geqslant m$, $z_j\in W_n$ for all $j\in[1,k]$, and the image of z_j under the imbedding $W_n\subset S_{2n}$ is a $(2a_j)$-cycle where $a_1\geqslant a_2\geqslant\cdots\geqslant a_k$. According to [GP, 3.4],

$$\begin{aligned}&\textit{If}\quad z \textit{ has minimal length in its conjugacy class},\\ &\textit{then}\quad l(z)=a_1+3a_2+5a_3+\cdots+(2k-1)a_k.\end{aligned}\tag{2.8a}$$

Let $a_j^0=2m-2j+2$ for $j\in[1,m]$, and $a_j^0=0$ for $j\in[m+1,k]$.

Lemma 2.9. *In the setup of Sections* 2.5 *and* 2.8*, assume that*

$$\operatorname{tr}\left(z,\begin{bmatrix}\lambda_1 & \lambda_2 & \cdots & \lambda_{m+1}\\ \mu_1 & \mu_2 & \cdots & \mu_m\end{bmatrix}\right)\neq 0.$$

(a) *For any* $j\in[1,k]$ *we have* $a_1+a_2+\cdots+a_j\leqslant a_1^0+a_2^0+\cdots+a_j^0$.
(b) *We have* $a_1+3a_2+5a_3+\cdots+(2k-1)a_k\geqslant a_1^0+3a_2^0+5a_3^0+\cdots+(2k-1)a_k^0$ *with strict inequality if* $(a_1,a_2,\dots,a_k)\neq(a_1^0,a_2^0,\dots,a_k^0)$.

Using repeatedly Lemma 2.4, we see that there exists $f:[1,k]\to[1,m]$ such that the multiset $\{0,1,2,\dots,m-1,m,m-1,\dots,2,1,0\}$ coincides with the multiset

$$\Big\{0,1,2,\dots,m-1,m,m+1-\sum_{h\in f^{-1}(1)}a_h,m+2-\sum_{h\in f^{-1}(2)}a_h,\dots,$$
$$2m-\sum_{h\in f^{-1}(m)}a_h\Big\}.$$

It follows that there exists a permutation σ of $1,2,\dots,m$ such that

$$r-\sum_{h\in f^{-1}(r)}a_h=-\sigma(r)$$

for all $r\in[1,m]$. For $j\in[1,k]$ we have

$$a_1+a_2+\dots+a_j\leqslant\sum_{r\in f[1,j]}\sum_{h\in f^{-1}(r)}a_h$$

hence

$$a_1+a_2+\dots+a_j\leqslant\sum_{r\in f[1,j]}\big(r+\sigma(r)\big).$$

Since $f[1,j]$ consists of at most j elements in $[1,m]$, we have

$$\sum_{r\in f[1,j]}r\leqslant m+(m-1)+\dots+(m-j+1),$$
$$\sum_{r\in f[1,j]}\sigma(r)\leqslant m+(m-1)+\dots+(m-j+1),$$

if $j\in[1,m]$, and

$$\sum_{r\in f[1,j]}r\leqslant m+(m-1)+\dots+1,$$
$$\sum_{r\in f[1,j]}\sigma(r)\leqslant m+(m-1)+\dots+1,$$

if $j\in[m+1,k]$. Thus,

$$a_1+a_2+\dots+a_j\leqslant 2\big(m+(m-1)+\dots+(m-j+1)\big)\quad\text{if } j\in[1,m],$$

and

$$a_1+a_2+\dots+a_j\leqslant 2\big(m+(m-1)+\dots+1\big)\quad\text{if } j\in[m+1,k].$$

This proves (a).

We prove (b). From (a) we see that

$$(a_1+a_2+\dots+a_k)+2(a_1+a_2+\dots+a_{k-1})+\dots+2a_1$$
$$\leqslant\big(a_1^0+a_2^0+\dots+a_k^0\big)+2\big(a_1^0+a_2^0+\dots+a_{k-1}^0\big)+\dots+2a_1^0,$$

hence

$$(2k-1)a_1 + (2k-3)a_2 + \cdots + a_k \leqslant (2k-1)a_1^0 + (2k-3)a_2^0 + \cdots + a_k^0.$$

Since $a_1 + a_2 + \cdots + a_k = a_1^0 + a_2^0 + \cdots + a_k^0$, it follows that

$$a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k \geqslant a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0.$$

If this is an equality, we must have $a_1 = a_1^0$, $a_1 + a_2 = a_1^0 + a_2^0, \ldots$, hence $a_1 = a_1^0$, $a_2 = a_2^0, \ldots$. The lemma is proved.

2.10. Let $m \in \mathbb{N}$, $m \geqslant 1$, and let $n = m^2$. Let $w'_m \in W_n$ be an element whose image under the imbedding $W_n \subset S_{2n}$ is a product of cycles $(2)(6)(10)\ldots(4m-2)$. Let

$$\lambda_1 < \lambda_2 < \cdots < \lambda_m \quad \text{and} \quad \mu_1 < \mu_2 < \cdots < \mu_m \tag{2.10a}$$

be two sequences of integers such that $\lambda_1, \lambda_2, \ldots, \lambda_m, \mu_1, \mu_2, \ldots, \mu_m$ is a permutation of $0, 1, 2, \ldots, 2m-1$. Then (2.3a) holds (with $a = b = m$ and $n = m^2$). Let

$$N = \sharp\bigl(k \in \{1, 2, \ldots, m\}\colon\ \mu_k \geqslant m\bigr). \tag{2.10b}$$

Consider the property

$$\begin{aligned} &\lambda_i + \lambda_j \neq 2m-1 \quad \text{for any } i \neq j \quad \text{and} \\ &\mu_i + \mu_j \neq 2m-1 \quad \text{for any } i \neq j. \end{aligned} \tag{$**$}$$

Lemma 2.11. *In the setup of Section* 2.10, *if* $(**)$ *holds, then*

$$\operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = (-1)^{N+m(m-1)/2}.$$

If $(**)$ *does not hold, then*

$$\operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = 0.$$

We argue by induction on m. The result is clear when $m = 1$. Assume now that $m > 1$. We can assume that $w'_m = (4m-2)w'_{m-1} \in W_{2m-1} \times W_{n-2m+1} \subset W_n$ where $w'_{m-1} \in W_{n-2m+1}$ is defined in a similar way to w'_m. We apply Lemma 2.4 with $w = w'_m$, $k = 2m-1$, $w' = w'_{m-1}$. Note that in the formula in Lemma 2.4 at most one term is non-zero, namely, the one in which $k = 2m-1$ is substracted from the largest of entries λ_i or μ_i (the other terms are zero since they contain some < 0 entry). We are in one of the four cases below.

Case 1 $(2m-1 = \lambda_m,\ 0 = \mu_1)$. Using Lemma 2.4, we have

$$\begin{aligned} A &= \operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \\ &= \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m-1} & 0 \\ 0 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right) \\ &= (-1)^{m-1} \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} 0 & \lambda_1 & \lambda_2 & \cdots & \lambda_{m-1} \\ 0 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right) \\ &= (-1)^{m-1} \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} \lambda_1 - 1 & \lambda_2 - 1 & \cdots & \lambda_{m-1} - 1 \\ \mu_2 - 1 & \mu_3 - 1 & \cdots & \mu_m - 1 \end{bmatrix}\right). \end{aligned}$$

Now the induction hypothesis is applicable to sequences

$$\begin{aligned} &\lambda_1 - 1 < \lambda_2 - 1 < \cdots < \lambda_{m-1} - 1 \quad \text{and} \\ &\mu_2 - 1 < \mu_3 - 1 < \cdots < \mu_m - 1 \end{aligned} \tag{2.11a}$$

instead of sequences (2.10a). (Clearly, (2.10a) satisfies $(**)$ if and only if (2.11a) satisfies the analogous condition.) Let N' be defined as N in (2.10b), in terms of (2.11a). Then $N' = N$. If (2.10a) satisfies $(**)$, then

$$A = (-1)^{m-1}(-1)^{(m-1)(m-2)/2}(-1)^{N'} = (-1)^{m(m-1)/2}(-1)^{N},$$

as required. If (2.10a) does not satisfy $(**)$, then $A = (-1)^{m-1}0 = 0$, as required.

Case 2 ($2m - 1 = \lambda_m$, $0 = \lambda_1$). Using Lemma 2.4, we have

$$\operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_{m-1} & 0 \\ \mu_1 & \mu_2 & \cdots & \mu_m & \end{bmatrix}\right)$$

and this is 0 since 0 appears twice in the top row.

Case 3 ($2m - 1 = \mu_m$, $0 = \lambda_1$). Using Lemma 2.4, we have

$$\begin{aligned} A &= \operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \\ &= -\operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_m & \\ \mu_1 & \mu_2 & \cdots & \mu_{m-1} & 0 \end{bmatrix}\right) \\ &= (-1)^{m} \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} 0 & \lambda_2 & \cdots & \lambda_m & \\ 0 & \mu_1 & \mu_2 & \cdots & \mu_{m-1} \end{bmatrix}\right) \\ &= (-1)^{m} \operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} \lambda_2 - 1 & \cdots & \lambda_m - 1 & \\ \mu_1 - 1 & \mu_2 - 1 & \cdots & \mu_{m-1} - 1 \end{bmatrix}\right). \end{aligned}$$

Now the induction hypothesis is applicable to sequences

$$\begin{aligned} &\lambda_2 - 1 < \lambda_3 - 1 < \cdots < \lambda_m - 1 \quad \text{and} \\ &\mu_1 - 1 < \mu_2 - 1 < \cdots < \mu_{m-1} - 1 \end{aligned} \tag{2.11b}$$

instead of sequences (2.10a). (Clearly, (2.10a) satisfies $(**)$ if and only if (2.11b) satisfies the analogous condition.) Let N' be defined as N in (2.10b), in terms of (2.11b). Then $N' = N - 1$. If (2.10a) satisfies $(**)$, then

$$A = (-1)^m(-1)^{(m-2)(m-1)/2}(-1)^{N'} = (-1)^{m(m-1)/2}(-1)^N$$

as required. If (2.10a) does not satisfy $(**)$, then $A = (-1)^m 0 = 0$, as required.

Case 4 ($2m-1 = \mu_m$, $0 = \mu_1$). Using Lemma 2.4, we have

$$\operatorname{tr}\left(w'_m, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) = -\operatorname{tr}\left(w'_{m-1}, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ 0 & \mu_2 & \cdots & 0 \end{bmatrix}\right)$$

and this is 0 since 0 appears twice in the bottom row. The lemma is proved.

Lemma 2.12. *Assume that we are in the setup of Lemma* 2.11, *that* $(**)$ *holds, and that* $m = 2m'$ *for some integer* $m' > 0$. *Then*

$$\sharp(k \in \{1,2,\ldots,m\}\colon\ \mu_k \geqslant m) - \sharp(k \in \{1,2,\ldots,m\}\colon\ \mu_k \textit{ even}) = m' \mod 2, \tag{2.12a}$$

$$\sharp(k \in \{1,2,\ldots,m\}\colon\ \mu_k \textit{ even}) = N + m(m-1)/2 \mod 2. \tag{2.12b}$$

Among the m' pairs $(0, 4m'-1)$, $(2, 4m'-3)$, $\ldots$, $(2m'-2, 2m'+1)$ there are, say, α pairs with the first component of form λ_i and second component of form μ_j and β pairs with the first component of form μ_j, and second component of form λ_i. Clearly, $\alpha + \beta = m'$. Among the m' pairs

$$(1, 4m'-2),\quad (3, 4m'-4),\quad \ldots,\quad (2m'-1, 2m')$$

there are, say, γ pairs with the first component of form λ_i and second component of form μ_j, and δ pairs with the first component of form μ_j and second component of form λ_i. Clearly, $\gamma + \delta = m'$. From the definitions we have

$$\sharp\big(k \in \{1,2,\ldots,2m'\}\colon\ \mu_k \geqslant 2m'\big) = \alpha + \gamma,$$
$$\sharp\big(k \in \{1,2,\ldots,2m'\}\colon\ \mu_k \text{ even}\big) = \beta + \gamma.$$

Hence the left-hand side of (2.12a) is equal to $\alpha + \gamma - (\beta + \gamma) = \alpha - \beta$, which has the same parity as $\alpha + \beta = m'$. This proves (2.12a). Now (2.12b) follows from (2.12a) since $m' = 2m'(2m'-1)/2 \bmod 2$. The lemma is proved.

Proposition 2.13. *Assume that* G *in Section* 0.1 *is of type* B_n *or* C_n *where* $n = m^2 + m$, $m \in \mathbb{N}$, $m \geqslant 1$. *We identify the Weyl group* W *of* G *with* W_n (*see Section* 2.3) *in the standard way.* (*The simple reflections of* W *become the permutations* $s_i = (i, i+1)(i', (i+1)')$, $i \in [1, n-1]$, *and* $s_n = (n, n')$ *in* W_n.) *Let* $w = w_m$, *see Section* 2.5. *Let* $\rho \in \mathcal{U}^0$. *Then* $(\rho : R_w) = 1$.

For any subset J of cardinal m of $I = \{0, 1, 2, \ldots, 2m\}$ let

$$E_J = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}$$

(an irreducible representation of W), where $\mu_1 < \mu_2 < \cdots < \mu_m$ are the elements of J in increasing order, and $\lambda_1 < \lambda_2 < \cdots < \lambda_{m+1}$ are the elements of $I - J$ in increasing order; let $f(J) = \sharp(j \in J\colon\ j \text{ even})$. By [L3, 4.23],

$$(\rho, R_w) = 2^{-m} \sum_J (-1)^{f(J)} \operatorname{tr}(w_m, E_J) \tag{2.13a}$$

where J runs over all subsets of I of cardinal m. Using Lemmas 2.6 and 2.7 we see that (2.13a) equals $2^{-m}\sharp(J\colon\ J \cap (2m - J) = \emptyset) = 1$. The proposition is proved.

Proposition 2.14. *Assume that G in Section* 0.1 *is of type D_n where $n = m^2$, $m = 2m'$, $m' \in \mathbb{N}$, $m' \geqslant 1$. We identify the Weyl group W of G with the subgroup of W_n generated by $s_1, s_2, \ldots, s_{n-1}, s_n s_{n-1} s_n$ (a Coxeter subgroup on these generators). Let $w = w'_m$, see Section* 2.10. (*We have $w'_m \in W$.*) *Let $\rho \in \mathcal{U}^0$. Then $(\rho : R_w) = 1$.*

For any subset J of cardinal m of $I = \{0, 1, 2, \ldots, 2m - 1\}$ let E_J be the restriction of

$$\begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}$$

from W_n to W (an irreducible representation of W), where $\mu_1 < \mu_2 < \cdots < \mu_m$ are the elements of J in increasing order, and $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ are the elements of $I - J$ in increasing order; let $f(J) = \sharp(j \in J\colon\ j \text{ even})$. Note that $E_J = E_{I-J}$ and $f(J) = f(I - J)$. By [L3, 4.23],

$$(\rho : R_w) = 2^{-m} \sum_J (-1)^{f(J)} \operatorname{tr}(w_m, E_J), \tag{2.14a}$$

where J runs over all subsets of I of cardinal m. Using Lemmas 2.11 and 2.12 we see that (2.14a) equals $2^{-m}\sharp(J\colon\ J \cap (2m - J) = \emptyset) = 1$. The proposition is proved.

2.15. Let m, n, W be as in Proposition 2.14. Let $z \in W_n$ be an element which has no eigenvalue 1 in the reflection representation of W_n. Then $z = z_1 z_2 \ldots z_k$ with $k \geqslant m$, $z_j \in W_n$ for all $j \in [1, k]$, and the image of z_j under the imbedding $W_n \subset S_{2n}$ is a $(2a_j)$-cycle where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k$. Assume also that $M = \sharp(j \in [1, k]\colon\ a_j > 0)$ is even. Then $z \in W$. According to [GP, 3.4],

$$\textit{If}\quad z \textit{ has minimal length in its conjugacy class in } W,$$
$$\textit{then}\quad l(z) = a_1 + 3a_2 + 5a_3 + \cdots + (2k - 1)a_k - M. \tag{2.15a}$$

Let $a_j^0 = 2m - 2j + 1$ for $j \in [1, m]$ and $a_j^0 = 0$ for $j \in [m+1, k]$.

Lemma 2.16. *In the setup of Sections* 2.10 *and* 2.15, *assume that*

$$\operatorname{tr}\left(z, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \neq 0.$$

(a) *For any* $j \in [1, k]$ *we have* $a_1 + a_2 + \cdots + a_j \leqslant a_1^0 + a_2^0 + \cdots + a_j^0$. *If* $m \leqslant j < M$, *we have* $a_1 + a_2 + \cdots + a_j \leqslant a_1^0 + a_2^0 + \cdots + a_j^0 - 1$.

(b) *We have* $a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k - M \geqslant a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0 - m$ *with strict inequality if* $(a_1, a_2, \ldots, a_k) \neq (a_1^0, a_2^0, \ldots, a_k^0)$.

Using repeatedly Lemma 2.4, we see that there exists $f : [1, k] \to [1, m]$ such that the multiset $\{0, 1, 2, \ldots, m-1, m-1, \ldots, 2, 1, 0\}$ coincides with the multiset

$$\left\{0, 1, 2, \ldots, m-1, m - \sum_{h \in f^{-1}(1)} a_h, m+1 - \sum_{h \in f^{-1}(2)} a_h, \ldots, 2m-1 - \sum_{h \in f^{-1}(m)} a_h\right\}.$$

It follows that there exists a permutation σ of $1, 2, \ldots, m$ such that

$$r - 1 - \sum_{h \in f^{-1}(r)} a_h = -\sigma(r)$$

for all $r \in [1, m]$. For $j \in [1, k]$ we have

$$a_1 + a_2 + \cdots + a_j \leqslant \sum_{r \in f[1,j]} \sum_{h \in f^{-1}(r)} a_h,$$

hence

$$a_1 + a_2 + \cdots + a_j \leqslant \sum_{r \in f[1,j]} \bigl(r + \sigma(r) - 1\bigr).$$

Since $\sharp(f[1, j]) = j'$ where $j' \leqslant \min(j, m)$, we have

$$\sum_{r \in f[1,j]} r \leqslant m + (m-1) + \cdots + (m - j' + 1),$$

$$\sum_{r \in f[1,j]} \sigma(r) \leqslant m + (m-1) + \cdots + (m - j' + 1),$$

$$a_1 + a_2 + \cdots + a_j \leqslant (2m-1) + (2m-3) + \cdots + (2m - 2j' + 1).$$

Thus,

$$a_1 + a_2 + \cdots + a_j \leqslant (2m-1) + (2m-3) + \cdots + (2m-2j+1) \quad \text{if } j \in [1,m]$$

and

$$a_1 + a_2 + \cdots + a_j \leqslant (2m-1) + (2m-3) + \cdots + 1 \quad \text{if } j \in [m+1,k].$$

If $M > m$, then for $j \in [m, M-1]$ we have $a_{j+1} \geqslant 1$ hence $a_{j+1} + a_{j+2} + \cdots + a_k \geqslant 1$. Since $a_1 + a_2 + \cdots + a_k = a_1^0 + a_2^0 + \cdots + a_k^0$, we have, for any $j \in [m, M-1]$:

$$\begin{aligned} a_1 + a_2 + \cdots + a_j &= a_1^0 + a_2^0 + \cdots + a_k^0 - (a_{j+1} + a_{j+2} + \cdots + a_m) \\ &= a_1^0 + a_2^0 + \cdots + a_j^0 - (a_{j+1} + a_{j+2} + \cdots + a_k) \\ &\leqslant a_1^0 + a_2^0 + \cdots + a_j^0 - 1. \end{aligned}$$

This proves (a).

We shall prove (b). Assume first that $M \leqslant m$. From (a) we see that

$$\begin{aligned} &(a_1 + a_2 + \cdots + a_k) + 2(a_1 + a_2 + \cdots + a_{k-1}) + \cdots + 2a_1 \\ &\quad \leqslant \bigl(a_1^0 + a_2^0 + \cdots + a_k^0\bigr) + 2\bigl(a_1^0 + a_2^0 + \cdots + a_{k-1}^0\bigr) + \cdots + 2a_1^0 \end{aligned}$$

hence

$$(2k-1)a_1 + (2k-3)a_2 + \cdots + a_k \leqslant (2k-1)a_1^0 + (2k-3)a_2^0 + \cdots + a_k^0.$$

Since $a_1 + a_2 + \cdots + a_k = a_1^0 + a_2^0 + \cdots + a_k^0$, it follows that

$$a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k \geqslant a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0.$$

Hence

$$\begin{aligned} &a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k - M \\ &\quad \geqslant a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0 - m. \end{aligned}$$

If this is an equality, we must have $a_1 = a_1^0, a_1 + a_2 = a_1^0 + a_2^0, \ldots$, hence $a_1 = a_1^0$, $a_2 = a_2^0, \ldots$.

Assume next that $M > m$. From (a) we see that

$$\begin{aligned} &(a_1 + a_2 + \cdots + a_k) + 2(a_1 + a_2 + \cdots + a_{k-1}) + \cdots + 2a_1 \\ &\quad \leqslant \bigl(a_1^0 + a_2^0 + \cdots + a_k^0\bigr) + 2\bigl(a_1^0 + a_2^0 + \cdots + a_{k-1}^0\bigr) + \cdots + 2a_1^0 \\ &\qquad - \sharp[m, M-1] \end{aligned}$$

hence

$$\begin{aligned} &(2k-1)a_1 + (2k-3)a_2 + \cdots + a_k \\ &\quad \leqslant (2k-1)a_1^0 + (2k-3)a_2^0 + \cdots + a_k^0 - 2(M-m). \end{aligned}$$

Since $a_1 + a_2 + \cdots + a_k = a_1^0 + a_2^0 + \cdots + a_k^0$, it follows that

$$a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k \geqslant a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0 + 2(M-m),$$

hence

$$a_1 + 3a_2 + 5a_3 + \cdots + (2k-1)a_k - M > a_1^0 + 3a_2^0 + 5a_3^0 + \cdots + (2k-1)a_k^0 - m.$$

The lemma is proved.

We return to the general case.

2.17. Let $\rho \in \mathcal{U}^0$. We attach to ρ a conjugacy class C_ρ in W as follows. If G is of type B_n or C_n, $n = m^2 + m$, let C_ρ be the conjugacy class of w_m (see Section 2.5). If G is of type D_n, $n = m^2$, m even, let C_ρ be the conjugacy class of w'_m (see Section 2.10). If G is of exceptional type, C_ρ is the conjugacy class of w whose characteristic polynomial $|w|$ in the reflection representation of W (a product of cyclotomic polynomials Φ_d) is described in Table 1 where we specify also the minimum length $l(w)$ for $w \in C_\rho$.

Here ρ is specified by the notation $\rho = \rho_{\mathcal{F},y,\sigma} = \rho_{y,\sigma}$ (we omit writing $\mathcal{F}$), where for the pairs (y, σ) we use the notation of [L3, 4.3]. The information on $l(w)$ is taken from [GP, Appendix].

Table 1

Type	$\|w\|$	ρ	$l(w)$
E_6	$\Phi_{12}\Phi_3$	$\rho_{g_3,\theta^{\pm 1}}$	6
E_7	$\Phi_{18}\Phi_2$	$\rho_{g_2,1}$, $\rho_{g_2,\epsilon}$	7
E_8	Φ_{30}	ρ_{g_5,ζ^j}, $j = 1, 2, 3, 4$; $\rho_{g_6,\theta^{\pm 1}}$	8
	Φ_{24}	$\rho_{g_4,i^{\pm 1}}$	10
	$\Phi_{18}\Phi_6$	$\rho_{g_3,\epsilon\theta^{\pm 1}}$	14
	Φ_{12}^2	$\rho_{g'_2,\epsilon}$	20
	$\Phi_{12}\Phi_6^2$	$\rho_{g_2,-\epsilon}$	22
	Φ_6^4	ρ_{1,λ^4}	40
F_4	Φ_{12}	$\rho_{g_3,\theta^{\pm 1}}$, $\rho_{g_4,i^{\pm 1}}$	4
	Φ_8	$\rho_{g_2,\epsilon}$	6
	Φ_6^2	$\rho_{g'_2,\epsilon}$	8
	Φ_4^2	ρ_{1,λ^3}	12
G_2	Φ_6	$\rho_{g_3,\theta^{\pm 1}}$, $\rho_{g_2,\epsilon}$	2
	Φ_3	ρ_{1,λ^2}	4

Remark. The cases E_8, F_4, G_2, $C = C_\rho$ for ρ equal to ρ_{1,λ^4}, ρ_{1,λ^3}, ρ_{1,λ^2}, respectively, have the following properties:

- the length function is constant on C;
- the cardinal of C is equal to the number of left cells in the two-sided cell $\mathbf{c}$ of W attached to ρ.

This suggests that $C \subset \mathbf{c}$ and that any left cell in $\mathbf{c}$ contains a unique element of C.

Theorem 2.18. *Let $\rho \in \mathcal{U}^0$. Let w be an element of minimal length in C_ρ.*

(a) *We have $(\rho : R_w) = (-1)^{r(G)}$.*
(b) *If $z \in W$ satisfies $l(z) < l(w)$, then $(\rho : R_z) = 0$.*
(c) *If $z \in W$ satisfies $l(z) = l(w)$ and $z \notin C_\rho$, then $(\rho : R_z) = 0$.*

Let W^0 be the set of all $z \in W$ such z has no eigenvalue 1 in the reflection representation of W.

We prove (a). In view of Propositions 2.13 and 2.14, we may assume that G is of exceptional type. In each case, one can compute $(\rho_{y,\sigma} : R_w)$ using [L3, 4.23]; for the computation we need the character table of W and the explicit entries of the non-abelian Fourier transform [L3, pp. 110–113]. The result in each case is $(-1)^{r(G)}$. This proves (a).

We prove (b), (c) assuming that G is of type B_n or C_n, $n = m^2 + m$. Let $z \in W$ be such that $(\rho : R_z) \neq 0$. Since $\rho \in \mathcal{U}^0$, it follows that $z \in W^0$. Define $a_1 \geqslant a_2 \geqslant \ldots \geqslant a_k$ in terms of z as in Section 2.8. Let z_0 be an element of minimal length in the conjugacy class of z. From our assumption it follows that

$$\operatorname{tr}\left(z_0, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_{m+1} \\ \mu_1 & \mu_2 & \cdots & \mu_m \end{bmatrix}\right) \neq 0$$

for some $\lambda_1, \lambda_2, \ldots, \lambda_{m+1}, \mu_1, \mu_2, \ldots, \mu_m$ as in Section 2.5. By Lemma 2.9(b) we have (using (2.8a)) that either z_0 is conjugate to w and $l(z_0) = l(w)$, or that $l(z_0) > l(w)$. Since $l(z) \geqslant l(z_0)$, we have that either z is conjugate to w and $l(z) \geqslant l(w)$ or that $l(z) > l(w)$. Hence (b), (c) are proved in our case.

The proof of (b), (c) in the case where G is of type D_n, $n = m^2$, m even, is entirely similar; it uses (2.15a) and Lemma 2.16(b) instead of (2.8a) and Lemma 2.9(b).

Assume now that G is of an exceptional type. We can make an explicit list of the conjugacy classes z in W other than C_ρ, such that $z \in W^0$ and such that the minimum length of an element in this conjugacy class is $\leqslant l(w)$ (to do this we use [GP, Appendix]). For each z in this list, we compute $(\rho : R_z)$ using again [L3, 4.23] (we again need the character table of W and the explicit entries of the non-abelian Fourier transform). The result in each case is 0. This proves (b) and (c).

Remark. For $w \in W$ let $K_w = K_w^{\mathcal{L}}$ be as in [L4, 2.4] with $\mathcal{L} = \overline{\mathbb{Q}}_l$ (a constructible complex of l-adic sheaves on G). For any character sheaf A on G let $(A : K_w)$

be the alternating sum of multiplicities of A in the various perverse cohomology sheaves of K_w. According to [L5], if p is not too small, the cuspidal character sheaves A of G such that $(A : K_w) \neq 0$ for some w are in natural bijection $A \leftrightarrow \rho$ with $\mathcal{U}^0$ so that $(A : K_w) = (\rho : R_w)$. Hence the theorem has the following consequence for cuspidal character sheaves.

Corollary. *Let A be as above. Then $\{w \in W\colon\ (A : K_w) \neq 0,\ l(w)$ is minimum possible$\}$ is contained in a single conjugacy class of W (namely C_ρ, where ρ corresponds to A as above); moreover, for $w \in C_\rho$ we have $(A : K_w) = (-1)^{r(G)}$.*

2.19. In this subsection we assume that G is non-split over F_q. The action of Frobenius on W is now given by an automorphism $\gamma : W \to W$ of order $c > 1$ and we may form the corresponding semidirect product $\widetilde{W}$ of W with $\mathbb{Z}/c\mathbb{Z}$ (whose generator is again denoted by γ). The reflection representation of W extends naturally to a representation of $\widetilde{W}$; hence for $w \in W$, the characteristic polynomial $|w\gamma|$ of $w\gamma$ is defined. To any $\rho \in \mathcal{U}^0$ we associate a subset C_ρ of W such that $C_\rho\gamma$ is a single orbit for the conjugation action of W on $\widetilde{W}$, as follows.

If G is of type A_{n-1} where n is a triangular number and w_0 is the longest element of $W = S_n$, C_ρ consists of all w such that ww_0 is a product of cycles $(1)(5)(9)\ldots$ or $(3)(7)(11)\ldots$.

If G is of type D_n where $n = m^2$, m odd, and $c = 2$, we identify $W\gamma = W_n - W$ (see Proposition 2.14) in the standard way; C_ρ consists of the elements of w such that the image of $w\gamma$ under the imbedding $W_n \subset S_{2n}$ is a product of cycles $(2)(6)(10)\ldots(4m-2)$.

If G is of type D_4 and $c = 3$, we have $\mathcal{U}^0 = \{\rho_1, \rho_2\}$. Then C_{ρ_1} consists of all w such that $|w\gamma| = \Phi_{12}$ and C_{ρ_2} consists of all w such that $|w\gamma| = \Phi_6^2$. We arrange the notation so that $(\rho_1, R_w) = 1$ for $w \in C_{\rho_1}$.

If G is of type E_6, we have $\mathcal{U}^0 = \{\rho_1, \rho_2, \rho_3\}$ where $\rho_1 \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$, $\rho_2 \notin \widetilde{\mathcal{U}}_{\mathbb{Q}}$, $\rho_3 \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$. Then $C_{\rho_1} = C_{\rho_2}$ consists of all w such that $|w\gamma| = \Phi_{18}$ and C_{ρ_3} consists of all w such that $|w\gamma| = \Phi_6^3$.

The statement of Theorem 2.18 continues to hold in the present case. The proof is along similar lines as in the split case (but we use [GKP] instead of [GP]). (The equality $(\rho : R_w) = (-1)^{r(G)}$ for G non-split of type A with $\rho \in \mathcal{U}^0$, $w \in C_\rho$, appeared in [Oh].)

2.20. A statement like Theorem 2.18(a) was made without proof in [L3, p. 356] (for not necessarily split G). In that statement, the assumption that ρ is cuspidal was missing. That assumption is in fact necessary, as Lemma 2.21(ii) (for G of type C_4) shows.

Lemma 2.21. (i) *Let $\epsilon : W_2 \times W_2 \to \{\pm 1\}$ be a character. Then*

$$\operatorname{tr}\Big(w, \operatorname{ind}_{W_2 \times W_2}^{W_4}(\epsilon)\Big) \in 2\mathbb{Z} \tag{2.21a}$$

for all $w \in W_4$.

(ii) *Let* $E = \left[\begin{smallmatrix}1 & & 2\\ 2 & & \end{smallmatrix}\right]$ *(an irreducible representation of* W_4*). Then* R_E *is of the form* tr_ρ *for some* $\rho \in \mathcal{U}$ *and* $(\rho : R_w)$ *is even for any* $w \in W = W_4$.

The residue class mod 2 of the left-hand side of (2.21a) is clearly independent of the choice of ϵ. Hence to prove (2.21a) we may assume that $\epsilon = 1$. Let $\pi : W_4 \to S_4$ be the canonical homomorphism. We have

$$\mathrm{tr}\Big(w, \mathrm{ind}_{W_2\times W_2}^{W_4}(1)\Big) = \mathrm{tr}\Big(\pi(w), \mathrm{ind}_{S_2\times S_2}^{S_4}(1)\Big).$$

But if $y \in S_4$, then $\mathrm{tr}(y, \mathrm{ind}_{S_2\times S_2}^{S_4}(1))$ is 6 if $y = 1$, is 2 if y has order 2, and is 0 otherwise; in particular, it is even for any y. This proves (i).

In (ii), the multiplicity of ρ in R_w is $\mathrm{tr}(w, E)$, that is, the left-hand side of (2.21a) for a suitable ϵ. Hence it is even by (i). The lemma is proved.

We now shall prove anew the following special case of Theorem 0.2 and Corollary 1.12.

Theorem 2.22. (a) *Assume that* $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}} \cap \mathcal{U}^0$*. Then* $\rho \in \mathcal{U}_{\mathbb{Q}}$*.*
(b) *If* G *is of type* B, C *or* D *and* $\rho \in \mathcal{U}^0$*, then* $\rho \in \mathcal{U}_{\mathbb{Q}}$*.*

Before starting the proof, note that for any $x \in W$:

(c) $(\rho : \sum_j (-1)^j \mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l)) = (\rho : \sum_j (-1)^j H_c^j(X_x, \mathbb{Q}_l)) +$ *an integer linear combination of* $(\rho : \sum_j (-1)^j H_c^j(X_{x'}, \mathbb{Q}_l))$ *for various* x' *with* $l(x') < l(x)$*.*
(d) *the* G^F*-modules* $\mathbf{H}^j(\overline{X}_x, \mathbb{Q}_l), \mathbf{H}^{2l(x)-j}(\overline{X}_x, \mathbb{Q}_l)$ *are mutually dual, hence they contain* ρ *with the same multiplicity* (recall that ρ is self-dual).

We prove (a). Let $w \in C_\rho$, so that $(\rho : R_w) = (-1)^{r(G)}$ (see Theorem 2.18(a)). We assume that w has minimal length in C_ρ. In our case, $(-1)^{r(G)} = 1$. Since R_w is the character of a virtual representation defined over $\mathbb{Q}_l$, it follows that ρ is defined over $\mathbb{Q}_l$ (here l is any prime $\neq p$). Using the Hasse principle it is then enough to show that ρ is defined over $\mathbb{R}$. Using Theorem 2.18(b) and (c) for $x = w$, it follows that $(\rho : \sum_j (-1)^j \mathbf{H}^j(\overline{X}_w, \mathbb{Q}_l)) = 1$. Using (d) and the first sentence in the proof of Lemma 1.4, we deduce that $(\rho : \mathbf{H}_c^{l(w)}(X_w, \mathbb{Q}_l)) = 1$ and that $l(w)$ is even. Since $\mathbf{H}_c^{l(w)}(X_w, \mathbb{Q}_l)$ admits a symmetric, non-degenerate, G^F-invariant $\mathbb{Q}_l$-bilinear form with values in $\mathbb{Q}_l$ with ρ self-dual and $(\rho : \mathbf{H}^{l(w)}(\bar{X}_w, \mathbb{Q}_l)) = 1$, it follows that ρ itself (regarded as a $\mathbb{Q}_l[G^F]$-module) admits a symmetric, non-degenerate, G^F-invariant $\mathbb{Q}_l$-bilinear form with values in $\mathbb{Q}_l$. Hence ρ is defined over $\mathbb{R}$.

In the setup of (b), ρ is unique up to isomorphism, hence it is automatically in $\widetilde{\mathcal{U}}_{\mathbb{Q}}$. Thus, (a) is applicable and $\rho \in \mathcal{U}_{\mathbb{Q}}$. The theorem is proved.

2.23. In this subsection we assume that G is of type F_4, that $p = 2$ and that q is an odd power of 2. Then $F : G \to G$ admits a square root $F' : G \to G$ so that

$G^{F'}$ is a Ree group. The concepts in Section 0.1 are well defined for $G^{F'}$ (instead of G^F). It is known [L3] that there is a unique $\rho \in \mathcal{U}^0$ such that $(\rho : R_w)$ is even for all $w \in W$. We necessarily have $\rho \in \widetilde{\mathcal{U}}_{\mathbb{Q}}$ but the methods of this paper do not allow to decide whether ρ is defined over $\mathbb{Q}_l$ or $\mathbb{R}$.

3. An example in SO_5

3.1. In this section we assume that $p \neq 2$ and that $G = SO(V)$ where V is a 5-dimensional **k**-vector space with a fixed F_q-rational structure and a fixed non-degenerate symmetric bilinear form $(,)$ defined over F_q. Let C be the set of all $g \in G$ such that $g = su = us$ where $-s \in O(V)$ is a reflection and $u \in SO(V)$ has Jordan blocks of sizes $2, 2, 1$. Then C is a conjugacy class in G and $F(C) = C$. A line L in $V(F_q)$ is said to be of type 1 if $(x, x) \in F_q^2 - 0$ for any $x \in L - \{0\}$ and of type -1 if $(x, x) \in F_q - F_q^2$ for any $x \in L - \{0\}$. Let $\mathcal{L}_1$ (respectively $\mathcal{L}_{-1}$) be the set of lines of type 1 (respectively -1) in $V(F_q)$. For $\epsilon, \delta \in \{1, -1\}$, let $C^{\epsilon,\delta}$ be the set of all $g \in C^F$ such that the line L in $V(F_q)$ such that $g|_L = 1$ is in $\mathcal{L}_\epsilon$ and any line L in $V(F_q)$ such that $g|_L = -1$ and $(L, L) \neq 0$ is in $\mathcal{L}_\delta$. Then $C^{\epsilon,\delta}$ is a conjugacy class of G^F and C^F is union of the four conjugacy classes $C^{1,1}$, $C^{1,-1}$, $C^{-1,1}$, $C^{-1,-1}$. We define a class function $\phi : G^F \to \mathbb{Z}$ by $\phi(g) = 2\delta q$ if $g \in C^{\epsilon,\delta}$ and $\phi(g) = 0$ if $g \in G - C^F$. (This is the characteristic function of a cuspidal character sheaf on G.)

Let O_+ (respectively O_-) be the stabilizer in G of a 4-dimensional subspace of V defined over F_q on which $(,)$ is non-degenerate and split (respectively non-split). Let $\det : O_+ \to \{\pm 1\}$ (respectively $\det : O_- \to \{\pm 1\}$) be the unique non-trivial homomorphism of algebraic groups. The restriction of det to O_+^F or O_-^F is denoted again by det. Consider the virtual representation

$$\Phi = \mathrm{ind}_{O_+^F}^{G^F}(1) - \mathrm{ind}_{O_+^F}^{G^F}(\det) - \mathrm{ind}_{O_-^F}^{G^F}(1) + \mathrm{ind}_{O_-^F}^{G^F}(\det)$$

of G^F. For $g \in G^F$ we have

$$\mathrm{tr}(g, \Phi) = 2\sharp(L \in \mathcal{L}_1 \colon g|_L = -1) - 2\sharp(L \in \mathcal{L}_{-1} \colon g|_L = -1).$$

It follows easily that $\mathrm{tr}(g, \Phi) = \phi(g)$.

Let θ be the unique unipotent cuspidal representation of G^F. Then $(\theta : \phi) = 1$. It follows that $(\theta : \Phi) = 1$. Since tr_θ is $\mathbb{Q}$-valued and Φ is a difference of two representations defined over $\mathbb{Q}$, it follows that θ *is defined over* $\mathbb{Q}$. Thus we have proved the rationality of θ without using the Hasse principle.

3.2. Assume now that $q = 3$. Then $SO(V)$ is isomorphic to a Weyl group W of type E_6 while O_+^F is isomorphic to a Weyl group of type F_4 and O_-^F is isomorphic to a Weyl group of type $A_5 \times A_1$ (imbedded in the standard way in the W). Now θ corresponds to the 6-dimensional reflection representation of W (Kneser).

Its restriction to the Weyl group of type F_4 contains no one-dimensional invariant subspace while its restriction to the Weyl group of type $A_5 \times A_1$ splits into a 5-dimensional irreducible representation and a non-trivial 1-dimensional representation. Since $(\theta : \Phi) = 1$ (see Section 3.1) it follows that $(\theta : \mathrm{ind}_{O_-^F}^{G^F}(\det)) = 1$.

References

[BBD] A.A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1982).

[DL] P. Deligne, G. Lusztig, Representations of reductive groups over finite fields, Ann. of Math. 103 (1976) 103–161.

[GP] M. Geck, G. Pfeiffer, Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras, Clarendon Press, Oxford, 2000.

[GKP] M. Geck, S. Kim, G. Pfeiffer, Minimal length elements in twisted conjugacy classes of finite Coxeter groups, J. Algebra 229 (2000) 570–600.

[L1] G. Lusztig, Coxeter orbits and eigenspaces of Frobenius, Invent. Math. 38 (1976) 101–159.

[L2] G. Lusztig, Lecture at the US–France Conference on Representation Theory, Paris, 1982, unpublished.

[L3] G. Lusztig, Characters of Reductive Groups Over a Finite Field, in: Ann. of Math. Stud., Vol. 107, Princeton University Press, 1984.

[L4] G. Lusztig, Character sheaves, I, Adv. Math. 56 (1985) 193–237.

[L5] G. Lusztig, Character sheaves, V, Adv. Math. 61 (1986) 103–155.

[Oh] Z. Ohmori, The Schur indices of the cuspidal unipotent characters of the finite unitary groups, Proc. Japan Acad. Ser. A Math. Sci. 72 (1996) 111–113.

[Weil] A. Weil, Basic Number Theory, Springer-Verlag, 1967.

[Weyl] H. Weyl, The Classical Groups, Princeton University Press, 1939.