NONLINEAR FILTERING OF REFLECTING DIFFUSION PROCESSES

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In the nonlinear filtering model
\[ y_t = h_t(X_t) + e_t, \quad 0 \leq t \leq T, \]
where \((e_t)\) is a finitely additive white noise, the problem of finding the conditional density \(u(t, x)\) of \(X_t\) given observations \(\{y_u: 0 \leq u \leq t\}\) is considered when \((X_t)\) is a reflecting diffusion process. It is shown that \(u(t, x)\) can be obtained as the unique classical solution of an initial-boundary value problem for a parabolic PDE.

1. Introduction

The aim of this paper is to show how the conditional density in a nonlinear filtering problem can be obtained as the classical solution of an initial-boundary value problem for a parabolic partial differential equation, when the signal process is a multidimensional diffusion process with reflection. We are utilizing the (finitely additive) white noise approach to the filtering problem, developed by Kallianpur and Karandikar (1985) to obtain a 'pathwise' solution to the problem. Using stochastic calculus, Pardoux (1977, 1978) derived the conditional filtering density as the unique weak solution of a linear stochastic PDE. The white noise calculus, producing a pathwise solution, has thus the advantage of being more operational in that the observed sample path appears only as a 'parameter' in a classical PDE. We will only briefly sketch the finitely additive filtering model that will serve as our basis. For details we refer to the papers by Kallianpur and Karandikar (1983, 1985).

Let \(T := [0, T]\) and \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. Then the signal process we are considering is assumed to be an unobservable Markov process \((X_t)_{t \in \mathbb{T}}\), taking values in a complete, separable metric space \((S, \mathcal{S})\). We assume that the map \(h : T \times S \to \mathbb{R}^m\) is measurable and such that
\[
E \int_0^T |h_t(X_t)|^2 \, dt < \infty
\]
thus, setting
\[
\xi(\omega) := (\xi_t(\omega))_{t \in \mathbb{T}} = (h_t(X_t(\omega)))_{t \in \mathbb{T}}
\]
we have defined a random element in the space
\[ L^2 := \left\{ g : T \to \mathbb{R}^m \mid g \text{ is measurable and } \int_0^T |g(t)|^2 \, dt < \infty \right\}. \]

On \( L^2 \) we consider the canonical gaussian measure defined e.g. through its characteristic functional
\[ C(g) := \exp\left\{ -\frac{1}{2} \|g\|^2 \right\}, \quad g \in L^2. \]
This measure is only finitely additive, which follows e.g. from the Minlos-Sazonov theorem. The map \( e : L^2 \to L^2 \) defined by
\[ e(g) := g, \quad g \in L^2, \]
will thus be taken as a random element in \( L^2 \), having the canonical gaussian measure for its distribution. \( e \) will play the role of the observation error in the nonlinear filtering model and is commonly referred to as 'white noise'. The filtering model in this setup can now be written as
\[ y(\omega, g) = \xi(\omega) + e(g) \quad \text{for all } (\omega, g) \in \Omega \times L^2. \]

Using the above model the filtering problem consists of giving an estimator of \( f(X_t) \) given observations \( \{y_u: 0 \leq u \leq t\} \) of the element \( y \) for fixed \( t \in T \) and all measurable functions \( f \) such that \( E[|f(X_t)|] < \infty \). The best MSE-estimator in this case is the conditional expectation \( E[f(X_t) \mid (\xi, y)] \), which we will write as \( E[f(X_t) \mid Q, y] \), with \( Q \) denoting the orthogonal projection onto the subspace
\[ H_t := \left\{ g \in L^2 \mid \int_0^T |g(u)|^2 \, du = 0 \right\}. \]

For details of the definition and computation of expectations in the finitely additive white noise model (1.6) we refer the reader to Kallianpur and Karandikar (1985).

The following Bayes formula is fundamental for the further study of the filtering problem:
\[
E[f(X_t) \mid Q, y] = \frac{\sigma_r(f(X_t), Q, y)}{\sigma_r(1, Q, y)} = \frac{E[f(X_t) \cdot \exp\{\int_0^T (y_u h_u(X_u)) - \frac{1}{2} |h_u(X_u)|^2 \, du\}]}{E[\exp\{\int_0^T (y_u h_u(X_u)) - \frac{1}{2} |h_u(X_u)|^2 \, du\}]}, \tag{1.8}
\]
\( \sigma_r(f(X_t), Q, y) \) is the so called 'unnormalized' conditional expectation. The Bayes formula allows the derivation of recursion formulas for the conditional expectation and facilitates the study of the filtering problem for special types of signal processes \( (X_t) \), since the expectations on the right-hand side of (1.8) are only taken with respect to the probability measure \( \Pi \). One class of signal processes will be studied in the remaining part of the paper—the reflecting multidimensional diffusion processes.
2. The signal process

We will assume that the signal process \((X_t)\) is a \(d\) dimensional diffusion process which is confined to a bounded, open subset \(D\) of \(\mathbb{R}^d\). Although there are various types of boundary behaviour possible for \((X_t)\) (cf. Freidlin, 1969; Venttsel, 1959) we restrict our attention to such processes which are instantaneously reflected into the interior of \(D\) in the conormal direction. The reason for this restriction lies in the fact that we are in this case able to establish the existence of marginal densities under suitable conditions on the characteristics of the process.

We first state our assumption about the region \(D\).

**Definition 2.1.** Let \(D \subset \mathbb{R}^d\) be an open and bounded set and assume that there exists a function \(\phi \in \mathcal{C}_b^2(\mathbb{R}^d)\) such that we can write
\[
D = \{x \in \mathbb{R}^d : \phi(x) > 0\} \quad \text{and} \quad \partial D = \{x \in \mathbb{R}^d : \phi(x) = 0\}.
\]

\((\mathcal{C}_b^2(\mathbb{R}^d)\) denotes the set of all twice continuously differentiable functions which are bounded along with their derivatives.) The closure of \(D\) is denoted by \(\bar{D}\). Denoting by \(\nabla \phi(x)\) the column vector of partial derivatives of \(\phi\) we will w.u.l.g. choose the defining function \(\phi\) so that for all \(x \in \partial D\), \(\nabla \phi(x)\) agrees with the unit normal vector \(n\) at \(x\), directed into the interior of \(D\).

We now want to define the signal process. From the various methods of defining a reflecting diffusion process the so-called sub-martingale problem (cf. Stroock and Varadhan, 1971) is most convenient for our purposes.

**Definition 2.2.** Let \(S_d^+\) be the space of all real, symmetric, positive definite \(d\) by \(d\) matrices and suppose that the maps
\[
a = (a_{ij}) : T \times D \to S_d^+, \quad b = (b_i) : T \times D \to \mathbb{R}^d \quad \text{and} \quad \delta = (\delta_i) : T \times \partial D \to \mathbb{R}^d
\]
are bounded and continuous.

Furthermore suppose that \(\delta\) is Lipschitz continuous and satisfies the condition
\[
\langle \delta(t, x), \nabla \phi(x) \rangle > 0 \quad \text{for all } t \in T, \ x \in \partial D. \tag{2.1}
\]

Then we define the two differential operators
\[
L_t := \sum_{i,j=1}^{d} a_{ij}(t, x) \cdot \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \cdot \frac{\partial}{\partial x_i} \tag{2.2}
\]
and
\[
J_t := \sum_{i=1}^{d} \delta_i(t, x) \cdot \frac{\partial}{\partial x_i}. \tag{2.3}
\]
The submartingale problem for a reflecting diffusion process is then defined in the following way.

**Definition 2.3.** A probability measure \( P \) on the space \( \mathcal{C}(T; \mathbb{R}^d) \) is called the solution of the sub-martingale problem for \( a, b \) and \( \delta \) if

(i) \( P(x(t) \in \bar{D}) = 1 \) for all \( t \in T \);

(ii) the process

\[
X_f(t) := f(t, x(t)) - \int_0^t I_D(x(u)) \left( L_u + \frac{\partial}{\partial u} \right) f(u, x(u)) \, du
\]

(2.4)

is a \( P \)-sub-martingale for every function \( f \in \mathcal{C}^{1,2}_{b}(T \times \mathbb{R}^d) \) which satisfies

\[
J_f \geq 0 \quad \text{on} \quad T \times \partial D.
\]

(2.5)

Here \( L_r \) and \( J_r \) are defined as in (2.2) and (2.3). \( \mathcal{C}(T; \mathbb{R}^d) = \{ f: T \rightarrow \mathbb{R}^d : f \text{ is continuous} \} \); \( \mathcal{C}^{1,2}_{b}(T \times \mathbb{R}^d) \) is the space of real valued functions that are twice continuously differentiable w.r.t. \( x \in \mathbb{R}^d \) and once w.r.t. \( t \in T \) and are bounded along with their derivatives.

The following result, which we state as a lemma, was proved in Stroock and Varadhan (1971).

**Lemma 2.1.** Under the conditions stated in Definition 2.2 the submartingale problem of Definition 2.3 is well posed, i.e. has a unique solution \( P \). This solution possesses the strong Markov property. □

We will thus define our signal process \( (X_t) \) to be a Markov process with sample paths in \( \mathcal{C}(T; \mathbb{R}^d) \), having \( P \) for its distribution.

Before moving to the filtering problem we state two auxiliary results (proved in Stroock and Varadhan, 1971) that will be useful in the proof of the Feynman-Kac formula.

**Lemma 2.2.** Let \( P \) and \( f \) be as in Definition 2.3, but assume now that

\[
J_f = 0 \quad \text{on} \quad T \times \partial D.
\]

(2.6)

Then the process \( (X_f(t)) \) is a \( P \)-martingale. □

**Lemma 2.3.** If \( P \) is the solution of the sub-martingale problem, then

\[
P \left\{ \int_0^T I_{\partial D}(x(u)) \, du > 0 \right\} = 0
\]

(2.7)

(i.e. the process spends no time on the boundary a.s.). □

Lemma 2.1 implies that all functions \( f \in \mathcal{C}^{1,2}_{b}(\mathbb{R}^d) \) satisfying (2.6) are in the common domain of the family of infinitesimal generators of the Markov process and that the generator agrees with the differential operator \( L_r \) on this set.
3. The second initial-boundary value problem

We will now study a pair of partial differential equations (PDE’s) which are associated with the signal process. These PDE’s will provide the key to the solution of the filtering problem. Conditions under which these PDE’s have unique classical solutions will be stated in terms of the coefficients of the operators $L_t$ and $J_t$.

In addition to the assumptions of Definition 2.2 we state the following set of hypotheses:

(A1) Let $c: T \times \bar{D} \to \mathbb{R}$ be a bounded, Lipschitz-continuous function.

(A2) For all $i, j \in \{1, \ldots, d\}$ the functions

$$a_{ij}, \quad b_i, \quad \frac{\partial}{\partial x_i} b_i, \quad \frac{\partial}{\partial x_i} a_{ij} \quad \text{and} \quad \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}$$

are bounded and Lipschitz continuous.

(A3) For all $(t, x) \in T \times \bar{D}$ the matrix $(a_{ij})$ is strictly positive definite.

(A4) The probability distribution of $X_0$ admits a Lebesgue density $p_0$.

Furthermore we define the boundary function $\delta$ in the following way:

$$\delta_i(t, x) := \sum_{j=1}^{d} a_{ij}(t, x) \cdot \frac{\partial}{\partial x_j} \phi(x) \quad \forall (t, x) \in T \times \partial D.$$  \hspace{1cm} (3.1)

Under conditions (A1)–(A4) the sub-martingale problem for $a, b$ and $\delta$ is well posed and the associated Markov process $(X_t)_{t \in T}$ is a $d$-dimensional diffusion process with reflection on the boundary. This is easily seen to be true by observing that (A1)–(A3) imply the conditions of Definition 2.2.

Next we define two boundary operators.

Definition 3.1. (i) Let $u: T \times \bar{D} \to \mathbb{R}$ be differentiable in $\bar{D}$ then we define the derivative of $u$ in the direction of the inward pointing normal $n_s$ as

$$\frac{\partial}{\partial n} u(t, x) := \sum_{i,j=1}^{d} a_{ij}(t, x) \cdot \frac{\partial}{\partial x_i} u(t, x) \cdot \frac{\partial}{\partial x_j} \phi(x) \quad \text{for all} \quad t \in T \quad \text{and} \quad x \in \partial D.$$  \hspace{1cm} (3.2)

(This definition is valid, since $\nabla \phi(x) = n_s$.)

(ii) Define the function $\beta: T \times \partial D \to \mathbb{R}$ by

$$\beta(t, x) := \sum_{i=1}^{d} \left[ \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_{ij}(t, x) \cdot \frac{\partial}{\partial x_i} \phi(x) \right] - b_i(t, x) \cdot \frac{\partial}{\partial x_i} \phi(x).$$  \hspace{1cm} (3.3)

The formal adjoint of the operator $L_t$ is denoted by $L_t^*$ and can under the assumption (A2) be written as

$$L_t^* = \sum_{i,j=1}^{d} a_{ij}(t, x) \cdot \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b'_i(t, x) \cdot \frac{\partial}{\partial x_i} + c'(t, x)$$  \hspace{1cm} (3.4)

with

$$b'_i(t, x) = b_i(t, x) + 2 \cdot \sum_{j=1}^{d} \frac{\partial}{\partial x_j} a_{ij}(t, x)$$  \hspace{1cm} (3.5)
and

\[ c'(t, x) = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} b_i(t, x) + \sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j} a_{ij}(t, x). \]  

(3.6)

In the sequel we will prove that the unnormalized filtering density \( p(t, x) \) can be obtained as the unique classical solution of the following initial-boundary value problem:

\[ L^* u(t, x) + c(t, x) \cdot u(t, x) - \frac{\partial}{\partial t} u(t, x) = 0, \quad 0 < t \leq T, \ x \in D, \]

(3.7)

\[ \frac{\partial}{\partial n} u(t, x) + \beta(t, x) \cdot u(t, x) = 0, \quad 0 < t \leq T, \ x \in \partial D, \]

(3.8)

\[ u(0, x) = p_0(x), \]

with a suitable choice of \( c(t, x) \).

To establish that the solution of (3.7) is actually the filtering density we also need to consider the adjoint problem

\[ L_v v(t, x) + c(t, x) \cdot v(t, x) + \frac{\partial}{\partial t} v(t, x) = 0, \quad 0 \leq t < T, \ x \in D, \]

(3.9)

\[ \frac{\partial}{\partial n} v(t, x) = 0, \quad 0 \leq t < T, \ x \in \partial D, \]

(3.10)

\[ v(T, x) = f(x), \quad x \in \bar{D}, \]

where \( f \) denotes a measurable function.

First we give conditions for both problems to have unique solutions.

**Lemma 3.1.** Suppose that assumptions (A1)-(A2) are satisfied. Also assume that the functions \( p_0, f \) and \( a_0(0, x) \) are defined and continuously differentiable in a neighborhood of \( \partial D \). Then problem (3.7) has a unique solution \( u \in \mathcal{C}^{1,2}(T \times \bar{D}) \). The adjoint problem (3.8) also has a unique solution \( v \in \mathcal{C}^{1,2}(T \times \bar{D}) \).

**Proof.** Under the stated conditions the coefficients \( a, b', c' \) and \( c \) of (3.7) are all bounded and Lipschitz continuous. This property also holds for the function \( \beta \). Since furthermore \( a(t, x) \) is uniformly positive definite and \( p_0 \) is continuously differentiable in a neighborhood of \( \partial D \), we can apply Corollary 2, Chapter 5 of Friedman (1964), yielding existence and uniqueness of the solution in the stated class of functions. The proof of the second assertion follows along the same lines by a time change. \( \square \)
Next we exploit the duality of problems (3.7) and (3.8).

**Lemma 3.2.** If $u$ and $v$ are the solutions of (3.7) and (3.8) then

$$
\int_D u(t, x) \cdot v(t, x) \, dx
$$

is constant over $[0, T]$.

**Proof.** We observe that

$$
\frac{\partial}{\partial t} \int_D u(t, x) \cdot v(t, x) \, dx
$$

$$
= \int_D \left[ u(t, x) \cdot \frac{\partial}{\partial t} v(t, x) + v(t, x) \cdot \frac{\partial}{\partial t} u(t, x) \right] \, dx
$$

$$
= \int_D \left[ v(t, x) \cdot (L^t + c(t, x)) u(t, x) - u(t, x) \cdot (L_t + c(t, x)) v(t, x) \right] \, dx
$$

$$
= -\int_{\partial D} \sum_{i=1}^d \frac{\partial}{\partial x_i} \phi(x) \cdot \beta_i[u, v] \, dS_x
$$

by Green's identity (cf. Friedman, 1964). Here $dS_x$ denotes integration w.r.t. the surface measure on $\partial D$ and the bilinear forms $\beta_i[u, v]$ are defined as:

$$
\beta_i[u, v] = \sum_{i=1}^d \left[ v \cdot a_{ij} \cdot \frac{\partial}{\partial x_j} u - u \cdot a_{ij} \cdot \frac{\partial}{\partial x_j} v + u \cdot c \cdot \frac{\partial}{\partial x_j} a_{ij} \right] - u \cdot v \cdot b_i. \quad (3.9)
$$

But now

$$
\sum_{i=1}^d \frac{\partial}{\partial x_i} \phi(x) \cdot \beta_i[u, v] = v \cdot \left[ \frac{\partial}{\partial n} u + \beta \cdot v \right] - u \cdot \frac{\partial}{\partial n} v
$$

which vanishes on $T \times \partial D$, since $u$ and $v$ satisfy the boundary conditions. Thus

$$
\frac{\partial}{\partial t} \int_{\partial D} v(t, x) \cdot u(t, x) \, dx = 0
$$

which completes the proof. □

**4. A Feynman–Kac formula**

To establish the link between the statistical nature of the filtering problem and the solutions of the partial differential equations of the previous section, we now derive a so called Feynman–Kac formula for the solution $v$ of (3.8), thereby providing a probabilistic interpretation of this solution in terms of a conditional expectation.
Theorem 4.1. If \( v \) is the unique solution of (3.8) then
\[
\int_{\mathcal{D}} v(0, x) \cdot p_0(x) \, dx = E \left[ f(X_\tau) \cdot \exp \left\{ \int_0^T c(s, X_s) \, ds \right\} \right]
\]
where \((X_t)\) denotes the signal process defined in Section 2.

Proof. Let \( v \) be the solution to (3.8) and define
\[
X_v(t) := v(t, X_t) - \int_0^t I_D(X_s) \cdot \left( L_s + \frac{\partial}{\partial s} \right) v(s, X_s) \, ds.
\]
(4.1)

Now since \((\partial/\partial n)v = 0\) we can apply Lemma 2.2 with
\[
J_t = \sum_{i=1}^d \delta_i(t, x) \cdot \frac{\partial}{\partial x_i}
\]
which implies that \((X_v(t))\) is a \( P \)-martingale. If we set
\[
A(t) := \exp \left\{ \int_0^t c(s, X_s) \, ds \right\}
\]
(4.2)
then the process \((N(t))\) defined by
\[
N(t) = X_v(t) \cdot A(t) - \int_0^t X_v(s) \, dA(s)
\]
(4.3)

is also a \( P \)-martingale. This follows from the 'integration by parts' formula for martingales (cf. Stroock and Varadham, 1979, Section 1.2). Using the particular forms of (4.1) and (4.2), (4.3) can be expressed as
\[
N(t) = v(t, X_t) \cdot A(t) - \int_0^t I_D(X_s) \cdot \left( L_s + c(s, X_s) + \frac{\partial}{\partial s} \right) v(s, X_s) \cdot A(s) \, ds
\]
\[
- \int_0^t I_{aD}(X_s) \cdot v(s, X_s) \cdot c(s, X_s) \cdot A(s) \, ds
\]
\[
= v(t, X_t) \cdot A(t) - \int_0^t I_{aD}(X_s) \cdot v(s, X_s) \cdot c(s, X_s) \cdot A(s) \, ds
\]
since \( v \) solves (3.8). Due to the martingale property of \((N(t))\) we have \( E[N(0)] = E[N(T)] \). Now
\[
E[N(T)] = E[v(T, X_\tau) \cdot A(T)]
\]
\[
- E \left[ \int_0^T I_{aD}(X_s) \cdot v(s, X_s) \cdot c(s, X_s) \cdot A(s) \, ds \right]
\]
(4.4)
but by Lemma 2.3 the integral term is a.s. equal to zero, so that the last expectation vanishes. Since according to (3.8), \( v(T, x) = f(x) \), we obtain:
\[
E[v(T, X_\tau) \cdot A(T)] = E \left[ f(X_\tau) \cdot \exp \left\{ \int_0^T c(s, X_s) \, ds \right\} \right].
\]
(4.5)
For the computation of $E[v(0, X_0) \cdot A(0)]$ we can utilize the fact that $X_0$ admits the density $p_0$ and $A(0) = 1$:

$$E[v(0, X_0) \cdot A(0)] = \int_\mathcal{D} v(0, x) \cdot p_0(x) \, dx. \quad (4.6)$$

Combining (4.5), (4.6) and using the martingale property we arrive at the desired result. $\square$

**Remark.** It is now easy to deduce a probabilistic representation for the solution $v(t, x)$ of (3.8). In fact, if we define the process $A(t)$ in (4.2) as

$$A(t) = \exp \left\{ \int_s^t c(u, X_u) \, du \right\}$$

for some fixed $s \in [0, T]$ and $t \in [s, T]$ then by analogous arguments we can deduce

$$E[v(s, X_s)] = E \left[ f(X_T) \cdot \exp \left\{ \int_s^T c(u, X_u) \, du \right\} \right].$$

This equality implies after conditioning on $X_s = x$ that

$$v(s, x) = E \left[ f(X_T) \cdot \exp \left\{ \int_s^T c(u, X_u) \, du \right\} \bigg| X_s = x \right]. \quad (4.7)$$

Such a relationship, expressing the solution of a PDE through a conditional expectation, is commonly referred to as a Feynman-Kac type formula.

Although formula (4.7) is sufficient for our purposes, we remark here that Freidlin gives a more general version of the Feynman-Kac formula, corresponding to a different boundary behaviour of the signal process (cf. Freidlin, 1985, Chapter II).

The next theorem combines the results of Lemma 3.2 and Theorem 4.1.

**Theorem 4.2.** Let $u$ be the unique solution to (3.7), then for every $f \in \mathcal{C}_b^2(\mathcal{D})$ we have

$$\int_\mathcal{D} f(x) \cdot u(T, x) \, dx = E \left[ f(X_T) \cdot \exp \left\{ \int_0^T c(s, X_s) \, ds \right\} \right]. \quad (4.8)$$

**Proof.** From Lemma 3.2 we obtain

$$\int_\mathcal{D} f(x) \cdot u(T, x) \, dx = \int_\mathcal{D} v(0, x) \cdot p_0 \, dx.$$

Now if $f$ satisfies the conditions of Lemma 3.1, the Feynman-Kac formula implies

$$\int_\mathcal{D} f(x) \cdot u(T, x) \, dx = E \left[ f(X_T) \cdot \exp \left\{ \int_0^T c(s, X_s) \, ds \right\} \right]$$

and the extension of this equality to all $f \in \mathcal{C}_b^2(\mathcal{D})$ is straightforward. $\square$
Theorem (4.2) has an important consequence.

**Corollary 4.1.** If assumptions (A1)-(A4) are satisfied, then the distribution of $X_{t_0}$ admits a Lebesgue density for all $t_0 \in T$. This density is the unique solution of (3.8) with $c = 0$.

**Proof.** If $c \equiv 0$ the existence and uniqueness results of the previous section still hold, but (4.8) now reads
\[ \int f(x) \cdot u(T, x) \, dx = E[ f(X_T) ] \]
for all $f \in \mathcal{C}_b^2(\bar{D})$. This shows that $u(T, x)$ is the Lebesgue density of the distribution of $X_T$. Since we can choose $T$ arbitrarily the existence of a density for all positive times follows. \( \square \)

Next we will apply the results of this section to solve the filtering problem under the assumptions of definition (3.1).

5. Solution of the filtering problem

From the Bayes formula (1.8) we know that the unnormalized conditional expectation of $f(X_T)$ given $Q_{T}\gamma$ is given by
\[ \sigma_T(f(X_T), Q_{T}\gamma) = E \left[ f(X_T) \cdot \exp \left\{ \int_0^T \langle y_u, h_u(X_u) \rangle - \frac{1}{2} |h_u(X_u)|^2 \, du \right\} \right] \]
If we define
\[ c_\gamma(t, x) = \langle y_t, h_t(x) \rangle - \frac{1}{2} |h_t(x)|^2 \quad (5.1) \]
for all $t \in T, x \in \bar{D}$, then to show that an unnormalized conditional density $p_T(x, Q_{T}\gamma)$ exists we have to show that
\[ \int f(x) \cdot p_T(x, Q_{T}\gamma) \, dx = E \left[ f(X_T) \cdot \exp \left\{ \int_0^T c_\gamma(u, X_u) \, du \right\} \right] \]
Hence it is obvious that the solution of (3.7) with $c$ defined as in (5.1) is the prime candidate for the conditional filtering density.

To prove that the solution of (3.7) is indeed the unnormalized conditional density and can be obtained by solving the PDE, we have to impose conditions on $h$ and $y$ that allow the application of the results from Sections 3 and 4.

**Theorem 5.1.** Suppose that assumptions (A2)-(A4) hold. Further assume that the distribution of $X_0$ has a Lebesgue-density $p_0(x) \in \mathcal{C}_b^1(D)$. If we also assume that the signal function $h$ is bounded and Lipschitz continuous and that $y \in H_T$ is Lipschitz continuous, then the unnormalized conditional filtering density $p_T(x, Q_{T}\gamma)$ is the unique classical solution of the PDE (3.7) when the function $c$ is defined as in (5.1).
**Proof.** The stated conditions on $h$ and $y$ imply that the function $c_r(t, x)$ satisfies the assumption (A1). Hence, according to Lemma 3.1, the problem (3.7) has a unique classical solution. That this solution is indeed the unnormalized filtering density follows from the duality result (Lemma 3.2) and the Feynman–Kac formula (4.8). □

**Remark.** Since the space of all Lipschitz-continuous functions is dense in $H_T$ and because the map $y \to p_T(x, Q_{fy})$ is continuous (cf. Kallianpur and Karandikar, 1985; Hucke, 1987) the required Lipschitz continuity of $y$ poses no restriction. For an arbitrary $y \in H_T$ the density $p_T(x, Q_{fy})$ can be approximated by a sequence $p_T(x, Q_{fy_n})$ of solutions to (3.7), where $(y_n)_{n \in \mathbb{N}}$ is a sequence of Lipschitz continuous functions converging to $y \in H_T$.

The value of the above result lies in the fact that in the finitely additive white noise model we are able to show that for a signal process of the type considered in Section 2 the filtering density can be obtained by solving a well known boundary value problem for PDE’s. This is in contrast to the results of Pardoux (1977, 1978) who studied the same filtering problem in the stochastic calculus model. His results establish the filtering density as the unique weak solution to a stochastic partial differential equation. The advantage of the finitely additive model is thus again, as also pointed out by Kallianpur and Karandikar (1985), to provide us with a solution which is ‘pathwise’, i.e. in which the observations appear as a ‘parameter’.

Furthermore smoothness properties of the filtering density (as a function of $x$) are easily obtained.

6. Concluding remarks

Using results from martingale theory and partial differential equations we have reduced the nonlinear filtering problem for reflecting diffusion process signals to the solution of a classical boundary value problem for partial differential equations. Having these results at our disposal the next question should be concerned with the numerical solution—implementation of a solution algorithm—of this problem. This question brings up problems of numerical analysis, as well as problems as to how an appropriate discrete approximation to the continuous observed function $y$ should look. We hope to investigate this problem in the future.

References
