Relativistic transformations at variable velocities

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Relativistic transformations at constant velocities \( V \) and \( v \) considered in [A. Einstein, Zur Elektrodynamik der bewegter Körper, Ann. Phys., 17 (1905) 891–921] are modified for the general case of variable speed \( V(.) \) of synchronization signal and variable relative velocity \( v(t) \) between two frames (\( K \)) and (\( k \)) linked through that signal. A differential approach to time synchronization is considered which makes clear that with variable velocities the observed time in the moving frame is represented by a nonlinear functional of those velocities \( V(.) \) and \( v(t) \) over the segments of trajectory covered by the synchronization (observation) signal, and the difference with respect to Einstein’s PDE at constant velocities is in the range of several decimal orders. Moreover, the application of linear transformations at constant velocities to the case of variable velocities may produce illusory effects of a relativistic mirage of expansion. For practical applications, and to preserve the simplicity of relativistic transformations, the averaging approach is developed, and a new form of relativistic transformations is obtained for discretized trajectories with on-line identification of average values for variable relative velocity \( v(t) \) and in consideration of a nominal velocity \( V \) with possible variations \( \delta V(.) \). An estimation of errors is made through the \( \gamma \)-representation based on direct distance measurements. Natural time delays due to the finite speed of information transmittal are included into relativistic transformations at variable velocities, and equations for software development are presented to support computation of real time trajectories and error estimation. The results open new avenues for theoretical and experimental studies of special relativity in media moving with variable speed and for control of processes by means of actually available signals.

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1. Introduction

The consideration of constant velocities \( V \) and \( v \) in special relativity theory developed by Albert Einstein [1] prevents the application of Einstein’s relativistic transformations to signals and motions which for some reason may include time intervals of non-constant velocities. In this paper, we consider the case of non-constant velocities \( V(.) \) and \( v(t) \), in view of obtaining the special relativity transformations for these practically important kinds of signals and processes.

The paper is organized as follows. In Sections 2 and 3, Einstein’s definition of simultaneity and his original derivation of the time transformation are reproduced in quotations from his basic paper [1, Sections 1–3]. Section 4 presents a differential approach to time synchronization at variable velocities. In Section 5, the averaging approach to time synchronization is developed for variable velocities. Section 6 contains the calculation of Einstein’s calibrating factor and its use in relativistic transformations obtained by the averaging approach. In Section 7, distance measurements are used to compute the approximations to average velocities of a moving frame (\( k \)) as observed from (\( K \)), and the equation for velocity variations is obtained from the \( \gamma \)-representation. Section 8 presents modification of Einstein’s transformations for the case of variable velocities. In Section 9, the abstract and real time are briefly discussed and the generalized relativistic transformations for

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variable velocities in real time are presented with evaluation of errors that may be introduced by the use of rays of light for distance measurements. Section 10 presents some concluding remarks followed by references immediately relative to the problems considered.

2. Definition of simultaneity [1, Sections 1, 2]

This is the title of the first section from which we reproduce the original Einstein’s description of time and simultaneity in an English translation from the Russian edition [2, pp. 8-10]. For a coordinate system “in which are valid the equations of mechanics of Newton”, called a “still system”, or system at rest, the following is written.

“When required to describe a motion of a material point, we specify the values of its coordinates as functions of time. Thereby it should be noted that such mathematical description has a physical sense only if it is first understood what is meant by “time”. We should pay attention to the fact that all our considerations in which time plays a role are always the considerations about simultaneous events”. Then we read on page 9 of [2]:

“If at point A of a space there is a clock, then an observer at A can establish the time of events in an immediate proximity of A by observing the positions of the hands of the clock simultaneous to those events. If at another point of the space, called B, there is also a clock (we add “identical to the one at A”), then in an immediate proximity of B it is also possible to make a time estimate of events by an observer at B. However, it is impossible without further hypotheses to compare the timing of an event at A with an event at B; we have yet defined only “A-time” and “B-time” but not the common for A and B “time”. The latter can be established by introducing a definition that the “time” necessary for passing of a ray of light from A to B is equal to the “time” necessary for passing of a ray of light from B to A. Consider that at a moment \( t_A \) of “A-time” a ray of light leaves from A to B and is reflected at a moment \( t_B \) of “B-time” from B to A returning back at A at a moment \( t_A' \) of “A-time”.

The clocks at A and B will be, by definition, synchronized, if

\[
t_B - t_A = t_A' - t_B. \tag{1}
\]

We assume that this definition of synchronization can be made in a non-contradictory manner, and furthermore, for as many points as desired, thus, the following statements are valid:

1. if the clock at B is synchronized with the clock at A, then the clock at A is synchronized with the clock at B;
2. if the clock at A is synchronized with the clock at B and with the clock at C, then the clocks at B and C are also synchronized with respect to each other.

Thus, using certain (thoughtful) physical experiments, we have established what should be understood as synchronized located in different places still clocks, and thereby we evidently achieved definitions of the concepts: “simultaneity” and “time”. “Time” of an event means simultaneous with the event indication of a still clock which is located at the place of the event and which is synchronized with certain still clock, thereby with one and the same clock under all definitions of time.

According to experiments, we also assume that the value

\[
2AB/(t_A' - t_A) = V \quad \text{(AB is the length of a segment)} \tag{2}
\]

is a universal constant (the speed of light in vacuum).

It is essential that we have defined time with the help of still clocks in a system at rest; we shall call this time that belongs to a system at rest, “the time of the still system”.

Further considerations are based on the principle of relativity and on the principle of constancy of the speed of light. We formulate both principles as follows.

1. Laws which govern the changes of state of physical systems do not depend on which of the two coordinate systems, moving with respect to each other with a constant speed along a right line, these changes relate.
2. Every ray of light propagates in a “still” system of coordinates with certain speed \( V \) irrespective of whether the ray of light is issued by a resting or moving source.

Thereby, formula (2) applies, and the “segment of time” should be understood in the sense of the above definition.

3. Derivation of Einstein’s time transformation [1, Section 3]

We now quote passages from [2, pp. 13–14] related to the theory of time transformation. “Consider in a “still” space two 3D Cartesian frames with a common origin and parallel axes, each equipped with scales and clocks which are identical in both frames. Now, let the origin of one of those frames \((k)\) be in motion with a constant speed \( v \) in direction of increasing \( x \) of the other frame \((K)\) which is at rest. Then, to each moment \( t \) of a still frame, \((K)\) corresponds certain position of the axes of moving frame \((k)\) whose axes can be assumed parallel to the axes of still frame \((K)\).

Let the space in the still frame \((K)\) be graduated with its scale at rest, and same for the space in the moving frame \((k)\) graduated with its scale, at rest with respect to \((k)\), yielding coordinates \( x, y, z \) in \((K)\) and \( \xi, \eta, \zeta \) in \((k)\). Using light signals as described in [1, Section 1], see above, let us define time \( t \) in \((K)\) and \( \tau \) in \((k)\) with the clocks at rest in each frame.

In this way, to the values \( x, y, z, t \) which define the place and time of an event in the still frame \((K)\), there will correspond the values \( \xi, \eta, \zeta, \tau \) that define the same event in the moving frame \((k)\), and we have to find the system of equations that link those values of coordinates and times.
First of all, it is clear that those equations must be linear according to the property of homogeneity which we ascribe to the space and time.

If we denote \( x' = x - vt \), then it is clear that to a point at rest in the system \((k)\) will correspond certain, independent of time values \( x', y, z \). Let us determine \( \tau \) as a function of \( x', y, z, t \), which would mean that \( \tau \) corresponds to the readings of clocks at rest in the moving frame \((k)\), synchronized with the clocks in the still frame \((K)\) by the rule \((1)\). Choosing in \((1)\) the point \( A \) as the origin of the moving frame \((k)\) and sending at the moment \( t_0 = t_4 \) a ray of light along the \( X \)-axis to the point \( x' \) (point \( B \)) which ray is reflected back at the moment \( t_1 = t_2 \) to the origin where it comes at the moment \( t_2 = t_3 \), we have from \((1)\) the following equation: \( t_1 - t_0 = t_2 - t_1 \) which is written in [1, Section 3], quote from [2, p. 14, the first equation], in the form:

\[
0.5(t_0 + t_2) = t_1,
\]

or, specifying the arguments of the function \( \tau \) and using the principle of constancy of the speed of light in the system at rest \((K)\), we have

\[
0.5[\tau_0(0, 0, 0, t) + \tau_2(0, 0, 0, t + x'/(V - v) + x'/(V + v))] = \tau_1[x', 0, 0, t + x'/(V - v)].
\]

If \( x' \) is taken infinitesimally small, then it follows

\[
0.5[1/(V - v) + 1/(V + v)]\partial \tau /\partial t = \partial \tau /\partial x' + [1/(V - v)]\partial \tau /\partial t,
\]

or

\[
\partial \tau /\partial x' + [v/(V^2 - v^2)]\partial \tau /\partial t = 0.
\]

It must be noted that we could take, instead of the origin, any other point to send a ray of light, therefore, the last equation is valid for all values \( x', y, z \).

Since the light along the axes \( Y \) and \( Z \), if observed from the system at rest, always propagates with the velocity \((V^2 - v^2)^{0.5}\), so the similar argument applied to these axes yields \( \partial \tau /\partial y = 0, \partial \tau /\partial z = 0 \). Since \( \tau \) is a linear function, so from these equations it follows

\[
\tau = a(t - vx)/(V^2 - v^2),
\]

where \( a = \phi(v) \) is yet unknown function, and for brevity it is taken that at the origin of the moving frame \((k)\) if \( \tau = 0 \), so also \( t = 0 \).” (Einstein’s notations, see [2, pp. 14-15].)

Alternatively, making use of a linear function with undetermined coefficients,

\[
\tau(x', y, z, t) = at + bx',
\]

and trying to satisfy equation \((4)\) identically with respect to \( t \) and \( x' \), without other assumptions, the reader can find the unique linear relation between \( a \) and \( b \) resulting in the formula \((7)\) for all three axes, see [3]. Substituting \( x' = x - vt \) into \((7)\) yields

\[
\tau = a[t - v(x - vt)/(V^2 - v^2)] = a\alpha\tau/(V^2 - v^2), \quad \alpha^2 = V^2/(V^2 - v^2),
\]

and the observed time \( \tau \) is really homogeneous in \( t, x' \) of \((7)\) and in \( t, x \) of \((9)\).

The factor \( a = a(v, V) \) is determined by additional requirements, see Eqs. \((28)-(36)\) in Section 6, yielding \( a = [1 - (v/V)^2]^{0.5}, \quad \alpha^2 = [1 - (v/V)^2]^{-0.5} = \beta, \) thus, \( a = \beta^{-1} \leq 1 \) and \( \alpha^2 = \beta^2. \) With these values, the relativistic transformations well known in the literature are obtained by Einstein [1]:

\[
\tau = \beta(t - vx/V^2), \quad \xi = \beta(x - vt), \quad \eta = y, \quad \zeta = z, \quad \beta = [1 - (v/V)^2]^{-0.5} \geq 1.
\]

where \( \beta \) is Einstein’s calibration factor, see Section 6. Furthermore, Einstein writes in [2, p. 16]: “If no suppositions are made about the initial position of a moving system and a zero point of the variable \( \tau \), then to each right-hand side of these equations one has to append one additive constant.” It means that Eqs. \((10)\) with specified initial conditions provide a complete description of the observed motion in relativistic space–time coordinates \( \xi, \eta, \zeta, \tau \) for the case \( V = \text{const}, \ v = \text{const}. \)

It is tempting to make use of relativistic transformations \((10)\) in the general case with relative velocity \( v(t) \neq \text{const} \) and/or signal velocity \( V \neq \text{const} \) by simple consideration of variable \( v(t), V(t) \) in Eqs. \((10)\). Although acceptable (by continuity) as an approximation for a small segment of trajectory close to initial data, this “solution” is mathematically unfounded and does not provide any reliable approximation to the actually observed motion in relativistic space–time coordinates. Moreover, such unconventional use of relativistic transformations \((10)\) may produce unreal, illusory effects, as can be seen in the following example.

**Example 3.1. Relativistic mirage of expansion.** Suppose that observations are being made from a still point on Earth or from a satellite still with respect to Earth. This point is designated as being a “still” point \( x \) on the \( Ox \) axis of a still frame \((K)\), and it is indicated in Einstein’s transformations \((10)\). Being “still” in this sense is only approximation since the Earth is rotating around the Sun and around its own axis of rotation (360° in 24 h), and also a satellite is moving in some abstract frame \((K^*)\), the “still space” in which transformations \((10)\) are valid. This frame \((K^*)\) postulated as being absolutely still is, unfortunately,
unknown, so the real frame (K) which is not absolutely still is used instead of (K*). We see that a realistic observation point x postulated to be still in (K) is, in fact, not still in (K*), which amounts to its unaccounted movement in (K*). If it moves with a constant speed with respect to (K*), say, x = x' + vt, w = const, x' ∈ (K*) is still, there is no problem since the independent of the time value with respect to (K*) is x' = x - vt in (4) to (8), hence, in Einstein’s transformations (10) actually related to (K*) one should consider the values v* = v + w, β* = β(v*, V), x' = x - w, where v + w is the true relative velocity of (k) with respect to (K*). We see that Einstein’s transformations (10) stay intact, yielding a neighboring relativistic trajectory ξ* = β*(x* - vt) = β*[x - (v + w)t], a little different from the one defined by (10) if the known frame (K) were absolutely still.

In orbital motions, however, velocities are not constant, so d(x - x*)/dt = w(t) ≠ const. Now, using linear relativistic transformations in this non-stationary case (for which they are invalid) and measuring distance ξ*(t) from a point x to an object ξ*(t) in the universe, we obtain dξ*/dt = -β*v* - β*tdv*/dt = -β*(v + w) - β*tdw/dt → +∞ as t → ∞, if dw/dt ≤ ρ < 0, ∀v. Thus, if the velocity dx/dt = w(t), is decreasing, we observe a relativistic mirage of expansion ξ*(t) → +∞.

4. Differential approach to time synchronization at variable velocities

For simplicity, and to conform to notations and arguments used in classical literature, let us follow the consideration and notation of Einstein as close as possible. We assume that vectors v, V, 0ξ are collinear with velocity v in direction of increasing x and |v| < V, the same as in [1, 2]. However, we consider cases when v(.) and V(.) may not be constant, thus the principles 1, 2 of Section 2 do not apply. Since V(.) is not necessarily the speed of light [3], the experimentally confirmed universal constancy of V in (2) is considered only as a special case in our study. It is worth noting that the synchronization Eq. (1) and formula (2) are postulated for arbitrary finite differences of time corresponding to specific information transmitting signals—the rays of light. In reality, other signals may serve as information transmitters, and this, together with possible non-constancy of velocities, requires the modification of the basic assumptions (1) and (2) imposed on the synchronized time in the whole space.

In the framework of Einstein, see Sections 2 and 3, let us consider different pairs of points A ∈ (k), B ∈ (K), where (k) is moving with a speed v(t) ≠ const and (K) is still (v = 0), such that distances AB = Δl < e where e is arbitrarily small. Consider the topology produced by a set of neighborhoods N = ∪Nε, a compact E in some finite volume of the space of points that is covered by a finite number of those neighborhoods. Now, consider pairs of points A, B that belong to one and the same Nε within which the velocities v, V can be considered constant due to arbitrarily small e. For all such points in (1), (2), synchronization equation (1) and the calibration equation (2) can be preserved: t₀ - tₐ = t₀ - tₐ, 2AB/(t₀ - tₐ) = V, where V = V(x) is some common value of V(.) in Nε to both which A and B belong. It means that principles 1, 2 and constancy v = const postulated by Einstein are considered locally, within Nε, where Einstein’s derivation is justified and his transformations (7), (10) stay intact with different constants a, v, V for different neighborhoods Nε. For a small e(ρ), the union of such piecewise linear transformations joined at boundaries of adjacent Nε presents an approximation to the unknown continuous transformation over the compact E up to any precision ρ specified in advance. To obtain a single continuous transformation for variable velocities, we have to consider that at the moment t₀' point A is located in adjacent neighborhood Nε', so that small changes in velocities V + dV, v + dv corresponding to that adjacent neighborhood Nε' have to be taken into account. For a variable speed v(t) ≠ const, the independent of time value x' = x - vt corresponding to a point at rest in a moving frame (k), as considered by Einstein in (4)–(7), should be replaced by the value

\[ x' = x - \int_0^t v(s)ds, \quad t ≥ 0, \]  

and, in accordance with Einstein’s initial conditions, we set t = 0 if τ = 0. Now, we have, to the first order with respect to the variables t, x'

\[ t₁ - t₀ = t₀ - tₐ = τ[x', 0, 0, 0, t + x'/(V - v)] = \tau(0, 0, 0, t) = x'[\partial τ/\partial x' + (\partial τ/\partial t)/(V - v)]_{V,v}, \]  

\[ t₂ - t₁ = tₐ - tₐ = \tau[0, 0, 0, t + x'/(V - v)] = \tau(x', 0, 0, t + x'/(V - v)]_{V,v} \]

\[ = -x'[\partial τ/\partial x' + (\partial τ/\partial t)(x'(t + dt))/(V + dV - v - dv) - x'(t)/(V - v)] + x'(t + dt)/(V + dV + v + dv)], \quad (V, v) ∈ Nε, (V + dV, v + dv) ∈ Nε', \]

where τ + dt corresponds to Nε' with different velocities V + dV, v + dv in (13)–(13').

Denoting dt' = x'(t + dt)/(V + dV - v - dv) - x'(t)/(V - v) and equating (12) and (13'), we obtain:

\[ x'[2\partial τ/\partial x' + (\partial τ/\partial t)/(V - v)] - (\partial τ/\partial t)(dt' + x'(t + dt))/(V + dV + v + dv)] = 0. \]  

If relation (14) corresponds to velocities in just one neighborhood Nε, then dV, dv, dr, dt' are all zeros, and (14) is identical to (6) yielding Einstein’s transformation (7). If (12), (13)–(13') correspond to adjacent neighborhoods, then those values are nonzero, and we have to evaluate the contribution of those increments in (14) corresponding to different velocities V + dV,
\( v + dv \) in \( N' \). Denoting the second bracket in (14) by \( F_\cdot \), we can write
\[
2\beta^2\langle \partial \tau / \partial x \rangle [ (v(V^2 - v^2)] \partial \tau / \partial t \rangle - (\partial \tau / \partial t) [ F_\cdot - x'(t)/(V + v)] = 0, \tag{15}
\]
\( F_\cdot - x'(t)/(V + v) = \partial x'/\partial t - (\partial x'/\partial t) [ F_\cdot - x'(t)/(V + v)] = \partial x'(t) + dt/\partial x' \rangle - \partial x'(t)/(V + v)
\]
\( = 2x'(t) + v(t) dt/\partial x' \rangle [(V + dv)/(V - v)] - \partial x'(t)/(V + v) = 2x'(t) + v(t) dt/\partial x' \rangle [(V + dv)/(V^2 - v^2)] - 2v(\partial x'/\partial t)/(\partial x'/\partial t) \rangle = 2\beta^2 v dt/\partial x', \tag{16}
\]
where we dropped the increments \( dv, \) which are small with respect to \( v, \) leading to an approximation in (16) with \( \beta \geq 1 \) from (10). Now, if \( F_\cdot - x'(t)/(V + v) = 0 \) in (15), we would get Einstein’s equation (6), so the second term in (15), which for \( v \) in (16), presents a difference due to a change in velocities for \( N' \). To evaluate the order of this difference, we can use (6) with its solution (7) as an approximation to (15) and make an estimation in (16) with respect to this solution. For velocities \( \theta(t) \rightarrow V \), the factor \( \beta^2 \rightarrow \infty \) in (16) and if \( dt > 0, \) \( \partial \tau / \partial t \rangle \neq 0 \), we obtain that the brace \( \{ \ldots \} \rightarrow \pm \infty \) in (15), so the linear transformations (7), (9) are invalid for variable velocities \( \theta(t) \) close to \( V \). Considering lower velocities \( \theta(t) \in [0.03, 3000] \text{ m/s (up to 9 Mach)}, \) the speed of light \( V = 3 \times 10^8 \text{ m/s,} \) and classical transformation (7) with \( a = \beta^{-1} = (1 - \theta/V)^2 \rangle \sigma = [1 - 10^{-10} \text{ th order}], \) we have \( \partial \tau / \partial t = \theta \pm 1 \). Now, with \( \beta^2 \geq 1, \) it remains to compare the values of \( \partial \tau / \partial x' \rangle \) defined by (6) and by (15). We have
\[
\partial \tau / \partial x' = -v/(V^2 - v^2) \partial \tau / \partial t < -3000/(9 \times 10^{16} - 9 \times 10^{9}) \geq 0.3 \times 10^{-13} \text{ from (6),}
\]
whereas for the brace in (15) which is zero in (6) we have from (15) and (16) with \( \partial \tau / \partial t = 1: \)
\[
\{ \ldots \} = [F_\cdot - x'(t)/(V + v)]/2x' \rangle \beta^2 v dt/\partial x' \rangle < 0.001 \text{ (17)}
\]
which means that values of \( \partial \tau / \partial x' \rangle \) defined by (6) and (15) differ by several decimal orders already for \( dt/\partial x' \rangle > 0.001 \text{ in (17).} \)
Hence, the classical PDE (6) represented by the brace in (15) is invalid for variable velocities \( \theta(t) \) in the range (0.03, 3000) m/s too.

This invalidity can be seen also from another argument. For constant velocities, we have from the first equation in (10):
\[
dt/\partial t = \beta \geq 1, \text{ if } \theta(t) \neq \text{ const, } V = \text{ const, then we have } \partial t/\partial t = \beta dt/\partial t = \beta dt/\partial x v/\partial x \rangle \text{ dt/\partial x} \rangle \geq 1 \text{ and the solution of the transformation (15) is}
\]
\[
\text{the same as in the case of variable velocities.}
\]

An attempt to derive a relativistic time transformation using PDEs to the 2nd or higher order requires the use of relation (15) with exact (to the order desired) transformation of \( F_\cdot \), its integration for the specified velocities \( \theta(t), \) and determining the factor \( d_\cdot \) as a functional of those velocities \( \theta(t), \) \( \theta(v). \) Such an approach is not only prohibitively complex analytically, but it is also unrealistically computationally because the functions \( \theta(t), \) \( \theta(v) \) are unknown for real life motions. Hence, it is required that \( \theta(t), \) \( \theta(v) \) be somehow measured, which leads to another approach for relativistic time synchronization at variable velocities.

5. Averaging approach to time synchronization at variable velocities

The variable speed of propagation of a synchronization signal does not depend on time, but rather it depends on a path along which the signal propagates. In the air, the speed of light \( V \) is variable due to variable density of the air, cf. the Huygens law of 1690 for the relative index of refraction \( n_{21} = c_1/c_2 \) (which ratio is equal to \( \sin \alpha_1/\sin \alpha_2 \) by Snell’s law of 1615 for two isotropic media separated by transparent boundary), and the absolute index of refraction \( n = cV > 1 \) where \( c \) is the speed of light in vacuum. Hence, for a right line segment \( BA > \varepsilon \) of signal propagation through adjacent neighborhoods, we have to consider the average speed
\[
V = V(x') = \int_0^T V(x, v(x)) dx/r, \quad r = |x'|, \tag{18}
\]
where \( v(\cdot) \) may affect \( V(x') \) if transmitting signals are produced by flows of fluid. For the same right line segment of motion, we have to consider the average velocity:
\[
v_1 = \int_t^T v(s) ds/(T - t), \quad T > t \geq 0, \tag{19}
\]
which can be split in two values: \( v_1 \) for \( T = t_\cdot > t = t_\cdot \), and \( v_2 \) for \( T = t_\cdot > t = t_\cdot \). With these notations, the Einstein equation (4) takes the form:
\[
0.5[t_0(0, 0, 0, t) \pm t_\cdot (0, 0, 0, 0, t + x'/(V - v_1) + x'/(V - v_2))] = t_\cdot [x', 0, 0, 0, t + x'/(V - v_1)]. \tag{20}
\]
Note that in the parentheses of (20) stand the resulting relative average velocities born by the observation signal and the motion combined. Of course, one could consider average values \( V_1(x') \) over \( (AB) \) and \( V_2(x') \) over \( (BA) \), \( V_1(x') \neq V_2(x') \), yielding in (20) the denominators \( V_1 - v_1 \) and \( V_2 + v_2 \). However, a difference \( \Delta V = V_2 - V_1 \) in the signal velocities averaged over large intervals \( (AB) \) and \( (BA) \) with a relatively small increment \( (AA') \) in system’s trajectory due to a much smaller
velocity \( v(t) \ll V(x') \) is far smaller than \( \Delta v = v_2 - v_1 \), so for this reason and for the sake of simplicity, we keep in (20) one and the same average velocity \( V \) from (18). Now, we can use the common notation \( t_{0,1,2} = \tau \) and representation (8), yielding, instead of (20), the identity:

\[
0.5[\alpha t + a(x'(v_1 - v_2)) = b(x'(v_1 - v_2)], \quad \forall t, \forall x'.
\]

(21)

Multiplying (21) by 2 and canceling the terms with \( \alpha t \) on both sides, we get:

\[
ax'[1/(V - v_1) + 1/(V + v_2)] = 2bx' + 2ax'(V - v_1), \quad \forall t, \forall x'.
\]

(22)

Simplifying (22), we have

\[
x'[2b + a(1/(V - v_1) - 1/(V + v_2)) = x'[2b + a(v_1 + v_2)/(V - v_1)(V + v_2)] = 0, \quad \forall t, \forall x'.
\]

(23)

so that identity (23) holds if and only if \( a, b \) are chosen from the equation

\[
2b + a(v_1 + v_2)/(V - v_1)(V + v_2) = 0, \quad \forall t,
\]

(24)

which yields the time transformation:

\[
\tau(x', y, z, t) = \alpha t + b x' = a(x - 0.5x'(v_1 + v_2)/(V - v_1)(V + v_2)),
\]

(25)

where \( V, v_1, v_2 \) should be expressed through (18), (19) and the factor \( a(\cdot) \) is to be determined by additional requirements. Clearly, if \( v_1 = v_2 = v = \text{const} \) and \( V = \text{const} \) are average velocities corresponding to the entire time segment \( \Delta t' = t' - t, \) then (25) coincides with the linear function in (7).

From (25) with (18) and (19), we see that the observed time \( \tau \) is a nonlinear function depending on the variable velocity of motion and on the path of the signal propagation along a right line of the relative motion of the two systems. If velocities are constant and known, then we return to Einstein’s transformation (7) or (9). Otherwise, the use of (25) with (18) and (19) requires prior identification of velocities \( v_1, v_2 \) for every starting moment \( t = t_a \) in (19) and corresponding adjustment of the limits for \( x' \) in the integral of (11). Also, the factor \( a(\cdot) \) should be computed which is not equal to \( (v/V)^2 \) as in Section 3. However, the very existence of transformation (25) for variable \( V(\cdot), v(\cdot) \) proves that all motions and processes are interacting under relativistic links depending on the relative velocities of bodies, media and information transmitting signals involved.

Similarly to Einstein’s derivation in Section 3, it is clear that “if \( x' \) is taken infinitesimally small, then it follows” (4), (5), (6) from (20) since the average velocities in (20) tend to constant velocities in (4) as \( x' \to 0 \). In this case, keeping the average velocities for better approximation, we obtain a generalization of (6) in the form:

\[
2x' \partial \tau / \partial x' + x'(v_1 + v_2)/(V - v_1)(V + v_2)) \partial \tau / \partial t = 0, \quad V = V(x'),
\]

(26)

which is the first approximation of the Taylor series for (20) with the common notation \( t_{0,1,2} = \tau \). If \( \tau(\cdot) \) is a linear function of \( t, x' \) with \( \partial \tau / \partial y = 0, \partial \tau / \partial z = 0, \) see Section 3, then transformation (25) follows from (26) for all \( x' \) and all \( v_1, v_2, V \).

A noteworthy difference between (6), (7) and (26), (25) is that (7) is valid globally, for the entire trajectory of motion assuming that \( V, v \) are constant and known, whereas (26), (25) are valid only on a finite segment of a trajectory corresponding to the moments \( t_a, t_b, t' \) for which the average velocities \( V, v_1, v_2 \) are measured, but this, — for any variable velocities \( V(x), v(t) \) which may be unknown outside the segment \( [t_a, t'_b] \).

Example 5.1. For the illustration, consider the flight of a spacecraft along a right line away from Earth with the same weight conditions as on Earth, that is, with a velocity \( v(t) = gt \), where \( g = 9.8 \text{ m/s}^2 \). Starting at some point in \( t_a = 100 \text{ s} \) the flight, the speed is \( v(t_a) = v(100) = gt_a = 980 \text{ m/s} \), and in the 60 s thereafter, we have the speeds \( v(t_b) = v(130) = 1274 \text{ m/s} \), \( v(t_a) = v(160) = 1568 \text{ m/s} \), so that \( v_1 = (980 + 1274)/2 = 1127 \text{ m/s} \), \( v_2 = (1274 + 1568)/2 = 1421 \text{ m/s} \).

For such velocities, \( a(\cdot) \cong \beta^{-1} = [1 - (v/V)^2]^{0.5} \cong 1 \), see below, and we have from (25), (11) with the constant speed of light \( V = 3 \times 10^8 \text{ m/s} \)

\[
\tau = t - 1274 \left[ x - \int_{100}^{t} v(s)ds \right] / (V - v_1)(V + v_2) = t - 1274[x - 0.5g(t^2 - 10^4)]/(V - v_1)(V + v_2) \cong t - 1274x + 0.5g(t^2 - 10^4)V^{-2} = t - 1274x + 0.54(t^2 - 10^4) \times 10^{-16}, \quad t \in [100, 160] \text{ s},
\]

so that the time transformation, indeed, contains a small nonlinearity if \( v(t) \neq \text{const} \). However, if \( v(t) \to V \), which will be the case in \( V/g = 3 \times 10^7 \text{ s} \), \( 1.05 \) year of the flight, then \( \tau \to \infty \), and the flight becomes unobservable. If the observation (synchronization) signal has the speed of sound or less, then dependence on time and on the path of the signal becomes really important, and Einstein’s transformations should be upgraded to take into account variable velocities of real life processes.
6. Calibrating factor for average velocities

Recall that for constant velocities \( V, v \), and given, according to experiments, that light in a moving frame \((k)\) propagates with the same speed \( V \), Einstein writes [2, p. 15]: “For a ray of light issued at the moment \( \tau = 0 \) in the direction of an increasing \( \xi \), we have \( \xi = V \tau \), or \( \xi = aV(t - x/v/(V^2 - v^2)) \). However, with respect to the origin of system \((k)\), the ray of light, if observed in the still system \((K)\), propagates with the speed \( V - v \), so it follows that

\[
\frac{x'}{(V - v)} = t. \tag{27}
\]

Substituting this \( t \) into equation for \( \xi \), we get \( \xi = axV^2/(V^2 - v^2) \).” Now, with \( x' = x - vt \) in the expression for \( \xi \), and for \( \tau \) in (7), it yields

\[
\tau = a[t - uV/(V^2 - v^2)] = aV^2/(V^2 - v^2), \quad \alpha^2 = V^2/(V^2 - v^2), \tag{28}
\]

\[
\xi = axV^2/(V^2 - v^2) = aV^2/(V^2 - v^2). \tag{29}
\]

Further, Einstein writes [2, p. 15]: “Considering rays propagating along two other axes, we find

\[
\eta = V \tau = aV[t - uV/(V^2 - v^2)], \quad \text{whereby } t = y/(V^2 - v^2)^{0.5}, x' = 0; \tag{30}
\]

hence (with our notation in (28) for \( \alpha^2 \))

\[
\eta = aV^2/(V^2 - v^2)^{0.5} = axy, \quad \xi = aV^2/(V^2 - v^2)^{0.5} = axz. \tag{31}
\]

To determine the function \( a(u, V) \) in (28)–(31), Einstein writes in [2, pp. 16–17]: “For this purpose, we introduce one more, the third coordinate system \((K')\), which with respect to system \((k)\) is in translational motion parallel to the \( \xi \)-axis in such a way that its origin moves with a velocity \( -v \) along \( \xi \)-axis. Suppose that at the moment \( t = 0 \) all three axes coincide, and for \( t = x = y = z = 0 \) the time \( t' \) in \((K')\) is 0. Suppose that \( x', y', z' \) are coordinates measured in system \((K')\). After applying twice our transformation formulae (28), (29), (31), we obtain” (our derivation in (32)–(35)):

\[
t' = a\alpha^2\tau'(\tau + v\xi/V^2) = a^2\alpha^4[t - uV/(V^2 + v(x - vt)/(V^2)] = a^2\alpha^2 t, \tag{32}
\]

\[
x' = a\alpha^2(\xi + vt) = a^2\alpha^4(x - vt + vt - v^2x/V^2) = a^2\alpha^2 x, \quad a > 0, \tag{33}
\]

\[
y' = a\alpha^2(\eta - \alpha^2 r), \quad z' = a\alpha^2 \zeta = a^2\alpha^2 z. \tag{34}
\]

“Since relations between \( x', y', z' \) and \( x, y, z \) do not contain time, the systems \((K)\) and \((K')\) are at rest with respect to each other, so it is clear that the transformation from \((K)\) into \((K')\) must be the identity transformation [2, p. 17].”

Hence, \( a^2\alpha^2 = 1 \) and also \( \alpha \alpha = 1 \) since the axes \( \eta, \gamma, \xi, \zeta \) have the same directions. Using the value \( \alpha^2 \) from (28), we get, with Einstein’s notation for \( \beta > 0 \):

\[
a^2\alpha^2 = a^2V^2/(V^2 - v^2) = 1, \quad a = [1 - (u/V^2)]^{0.5}, \quad \alpha^2 = [1 - (u/V^2)^2]^{0.5} = \beta. \tag{35}
\]

Substituting the values of \( \alpha^2 = \beta \) from (35) into (28) and (29) and \( \alpha \alpha = 1 \) into (31) yields relativistic transformations [1, 2] well known in the literature:

\[
\tau = \beta(t - uV/V^2), \quad \xi = \beta(x - vt), \quad \eta = x, \quad \xi = z, \quad \beta = [1 - (u/V^2)^2]^{0.5} \geq 1. \tag{36}
\]

To consider variable velocities over time segment \([t_A, t_A']\) in the above derivation of the calibrating factor \( a(\cdot) \), we can use the same argument with \( u, V \) denoting the average velocities over \([t_A, t_A']\). This allows us to bypass the complex derivation of a new formula for calibration factor \( \beta' \) \((v_1, v_2, V)\) based on three averages by using \( \beta \) of (36) as its approximation with only two averages \( V, v \in (v_1, v_2) \). Since \( a = \beta^{-1} \), see (35), so from (25) with (18) and (19) we obtain, instead of (28)–(30):

\[
\tau = \beta^{-1}[t - 0.5x'(v_1 + v_2)/(V - v_1)(V + v_2)], \quad x' = x - \int_0^t v(s) \, ds, \quad V = \int_0^T V(s) \, ds/r, \quad r = |x'|, \tag{37}
\]

\[
v_1 = \int_t^{tA} v(s) \, ds/(t_A - t) - t, \quad v_2 = \int_t^{tA} v(s) \, ds/(t_A' - t), \quad t = t_A \geq 0. \tag{38}
\]

\[
\xi = \beta^{-1}V[t - 0.5x'(v_1 + v_2)/(V - v_1)(V + v_2)]; \quad \beta^{-1} = [1 - (v/V^2)^2]^{0.5}, \quad v \in (v_1, v_2), \tag{39}
\]

\[
\eta = \beta^{-1}V[t - 0.5x'(v_1 + v_2)/(V - v_1)(V + v_2)] = \beta^{-1}VT, \quad \text{since } x' = 0. \tag{40}
\]

In the case of constant velocities, Einstein writes [2, p. 14, bottom] that “…light along the axes \( Y, Z \), if observed from the system at rest, always propagates with the speed \( V' = (V^2 - v^2)^{0.5} \), yielding for those axes \( \partial \tau / \partial y = 0, \partial \tau / \partial z = 0 \).” In our case of variable velocities, we have in (39) and (40) the product \((V - v_1)(V + v_2)\), instead of \((V^2 - v^2)\) in denominators of (28)–(31), since average velocities \( v_1, v_2 \) of (20) affect differently the observed time \( \tau \) and coordinates \( \xi, \eta, \zeta \) in a distributed manner which yields the Einstein value \( V' = (V^2 - v^2)^{0.5} \) for constant velocities and for small time of observation in (29)–(31). For this reason, we have to consider the speed of rays of light along the axes \( Y, Z \), if observed from the system at
rest, being equal to \( V' = [(V - v_1)(V + v_2)]^{0.5} \) with \( V = V(x') \), in agreement with the speed for constant velocities. This yields in (40) the value \( t = y/V' \) for \( Y \)-axis, and \( t = z/V' \) for \( Z \)-axis, which implies, instead of (40):

\[
\eta = \beta^{-1}Vt = \beta^{-1}yV/V', \quad \zeta = \beta^{-1}Vt = \beta^{-1}zV/V', \quad V = V(x'), \quad V' = [(V - v_1)(V + v_2)]^{0.5} \tag{41}
\]

For small \( \Delta t' = t'_n - t_n \), we can use the average velocities \( v_1 = v_2 = v = \text{const} \), \( V = \text{const} \), for which relations (37)–(41) coincide with (28)–(31) and (36). Clearly, many different models of averaging can be employed with a different precision and complexity to obtain relativistic transformations over a small segment of trajectory with unavoidable problem of determining the involved average velocities by the measurement and computation.

7. Identification of velocities through measurement and computation

Suppose that point \( \xi(,) \) represents a rocket, asteroid or spacecraft for which initial conditions of the motion are unknown, and also velocity \( v \) of the frame \((k)\) moving with respect to the still frame \((K)\) is not known. In such cases, accurate observation of that body \( \xi(,) \) is possible only after the velocity \( v \) and actual position of the body at some moment in time are identified and the average speed \( V \) in (37), (39) of the signal (carrier of information) is somehow evaluated. For the general case of variable velocities \( v(t) \neq \text{const} \), \( V(x') \neq \text{const} \), the Einstein transformations (36) can be used if average velocities are introduced on a discretized trajectory, which velocities are identified over the pieces where the observation of the moving body needs to be supported.

7.1. Design of experiments

Consider a still point \( x_0 \) on the \( X \)-axis of a still frame \((K)\) at which point a source of light is fixed beaming along the \( X \)-axis with short pulses of light. The reader can imagine the origin of \((K)\) at the center of Earth, the point \( x_0 \) at the top of a hill at a place with clear air and good weather, the axis \( 0x \) pointing to the outer space where an asteroid at a distance \( \xi(t, x_0) \) is observed at \( x_0 \) moving along the right line \( 0x \) with \( y = z = 0 \). Short pulses can be extracted from a continuous beam of light with a thin evenly perforated disc with windows (openings, gaps) of 1 mm wide and closures of the same or different width rotating with a high speed in a vacuum enclosure. To control the pulses, the vertical shaft of the disc can be turned at small angles to the vertical and the speed of rotation can be varied. The stand is similar to the setup of Fizeau [4] and Cornu [5], see also [6, pp. 1276–1277] for details and calculations.

Consider the time moments \( t_n = n\Delta t, n = 0, 1, \ldots \) at which pulses are sent to the asteroid and the later moments \( t'_n = t_n + \Delta t_n \), at which reflected light of those pulses is received at the same point \( x_0 \) where the source of light is located. Here, the increments \( \Delta t \) and \( \Delta t_n \) are small finite time differences such that the ray of light (pulse) sent at \( t_n \) is reflected and received back at the moment \( t'_n, n = 0, 1, \ldots \). The length of discretization interval \( \Delta t \) can be varied at will through the disc control [5].

7.2. Computation of the average velocities as observed in \((K)\)

We shall use the scheme of Einstein, with a difference that, instead of sending a ray \( \xi \rightarrow x' \rightarrow \xi \) (there are no people on asteroid, point \( \xi \), who could send that ray to the point \( x' \)), in order to synchronize the timing of events at \( \xi(t, x_0) \), on asteroid, and at \( x' \in (K) \), see Eqs. (7), (9) and (36), the rays are sent in opposite directions \( x_0 \rightarrow \xi \rightarrow x_0 \), to measure the actual distances to the points of reflection of the rays from the moving asteroid, whatever its velocity \( w(t) \) may be. We assume that \( w(t) > 0 \) corresponds to direction of increasing \( x \), so that the asteroid moves away from Earth.

At a moment \( t_n \) when a pulse is sent, the body (asteroid) is at some unknown distance from \( x_0 \). When the pulse is reflected, the body is at a greater distance \( x_n \), which can be computed, upon reception of reflected ray, by the formula:

\[
x_n = 0.5V_n\Delta t_n, \quad \text{although at the moment } t'_n = t_n + \Delta t_n \text{ of reception, the body will be at a still greater (unknown) distance from } x_0. \quad \text{Sending the next pulse at the moment } t_{n+1}, \text{ we can compute in the same way } x_{n+1} = 0.5V_{n+1}\Delta t_{n+1}, \text{yielding }
\]

\[
\Delta x_n = x_{n+1} - x_n = 0.5(V_{n+1}\Delta t_{n+1} - V_n\Delta t_n) \tag{42}
\]

\[
= w_n(t_{n+1} + 0.5\Delta t_{n+1} - t_n - 0.5\Delta t_n) = w_n(\Delta t + 0.5\Delta t_{n+1} - 0.5\Delta t_n). \tag{43}
\]

Here \( w(t) \) is the unknown velocity of the moving body \( \xi(t) \) with respect to the time \( t \) as observed from the still frame \((K)\) on Earth. This \( w(t) \) is not equal to any of the velocities \( v, v_1, v_2 \) in (37) to (41) which represent the constant or average velocities of the moving frame \((k)\) with respect to the still frame \((K)\). Similarly, we used notation \( x_n \) in (42), (44), (46) and \( x(t) \) in (47) for the distance \( x_0\xi, \) not for the value of coordinate \( \xi(t) \) in (36), (39). Thus, we consider here direct measurements, without the use of Einstein’s relativistic coordinates or transformations. In (42) we have used the first mean value theorem for integrals with \( w_n \) as a notation for yet unknown average velocity on the interval \((a, b)\) specified in (42). Comparing the
entries in (42) and (43), we find
\[ w_n = (V_{n+1} \Delta t_{n+1} - V_n \Delta t_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n) = 2(x_{n+1} - x_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n). \]  

(44)

If \( V_n \) were known, this would allow us to compute \( w_n \) through measurements of the time increments in (44). Since \( \Delta t_{n+1} > \Delta t_n, x_{n+1} > x_n, w_n > 0 \), so we have
\[ 2\Delta t + \Delta t_{n+1} - \Delta t_n = 2\Delta t + \varepsilon, \quad \varepsilon > 0, \]
and if \( \Delta t \to 0 \), then \( \varepsilon = \Delta t_{n+1} - \Delta t_n \to 0 \), since the whole sequence of pulses contracts into one single pulse. In this case, from (44) it follows: \( w_n = \Delta x_n/(\Delta t + 0.5\varepsilon) \), yielding
\[ \Delta x_n/\Delta t = w_n(\Delta t + 0.5\varepsilon)/\Delta t > w_n, \quad n = 0, 1, \ldots. \]

(46)

and as \( \Delta t \to 0 \) we get, in the limit: \( dx/dt = w(t)[1 + 0.5 \lim(\varepsilon/\Delta t)] = w(t) \), since \( \varepsilon/\Delta t \) is positive, so its limit must be zero according to definition of the mean value \( w_n > 0 \) in (42). Noting the second equality in (44), it proves the following result:
\[ dx/dt = w(t) = \lim W_n(\Delta t) = \lim[(V_{n+1} \Delta t_{n+1} - V_n \Delta t_n)/(2\Delta t + \varepsilon)], \quad \text{as } \Delta t \to 0, \forall n. \]

(47)

Note that observed \( w(t) \) depends on the variable signal velocity \( V(.) \). If \( dx/dt = w(t) = p = \text{const} \), then \( w_n = p \), and we return to the model of Einstein with constant velocities \( V, v \) for which transformations (10) hold, cf. Lemma 9.1 in [3].

7.3. Rods, clocks and signals in measurement and computation

A rod with a fixed unity of length provides uniform graduation of a coordinate frame. A clock in uniform motion with a fixed unity of the angle in rotation of its hands provides uniform graduation of time. Graduated length and time constitute the basis of kinematics in special relativity theory, cf. the first three paragraphs of quotations from [1] in Section 3. However, large distances cannot be conveniently measured by a rod or a measuring tape that are available in stores. To measure large distances, certain signals are used, such as radar or rays of light in Einstein’s special relativity [1,2]. Signals are not directly characterized by a length or time, — but they propagate at certain velocities which are defined by the length of propagation per unit of time. The velocity of propagation of a signal does not have to be constant, but it has to be known in order to use that signal for measurement, calibration or synchronization.

If unity of length or time is unknown, there are no coordinates in space, or in time. Similarly, if the velocity of a signal is unknown, we cannot synchronize clocks or measure large distances and other physical parameters at large distances with that signal, even if units of length and time are known. It does not mean that physical processes cease to exist — they evolve as usual, being unobservable since we cannot measure or compute their characteristics or parameters. Thus, units of length and time and signal velocities must be known to measure physical processes or compute their parameters. This knowledge is obtained prior to measurement or synchronization, by comparison of the unit measuring rod with the standard one (scale), the unit interval of time with the standard clock, and calibrated velocity of a measuring or synchronization signal by special methods.

If that signal is a ray of light, its nominal speed is already measured, see, e.g., [6] and references therein. For other synchronization signals (sound, flows of fluid), their nominal speed of propagation can be measured using the known distance between two points \( x_1, x_2 \) and the time \( t_2 - t_1 \) of travel of the signal between those two points, whereby the speed of propagation is \( V^* = (x_2 - x_1)/(t_2 - t_1) \). The distance itself \( x_2 - x_1 \) can be measured by the unit rod or, if large, by radar or a ray of light issued from \( x_1 \) and reflected from \( x_2 \) at measured interval \( \Delta t \) so that \( x_2 - x_1 = 2V\Delta t \), yielding \( V^* = 2V/\Delta t/(t_2 - t_1) \). Of course, this knowledge is approximate, and its precision defines the accuracy of experiment and computations in relativistic applications. In contrast, velocities of moving frames need not be known in advance since they can be measured from a still frame (K) using clocks and signals calibrated by appropriate standard scales for time and length.

Denoting by \( V \) some nominal average value of the approximately known velocity (measured or postulated) of a signal actually employed in observation over a piece of trajectory, we can express the (possibly variable) real value of the signal velocity as \( V_n = V + \delta V_n \) and rewrite the basic equation (44) in the form of the three equations as follows
\[ W_n = V(\Delta t_{n+1} - \Delta t_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n), \]
\[ \delta W_n = (\delta V_{n+1} \Delta t_{n+1} - \delta V_n \Delta t_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n), \]
\[ w_n = W_n + \delta W_n = 2(x_{n+1} - x_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n). \]

(48)

(49)

(50)

The nominal value \( W_n \) in (48) is computable since the nominal value of \( V \) is known and time differences are measured, thereby without time delays due to information transmittal which delays are cancelled in time differences of the same nature. The real velocities of the body \( w_n \) in (50) can be approximated by \( W_n \) for \( n = 0, 1, \ldots \), up to the increments \( \delta W_n \), with relative errors
\[ \delta W_n/W_n = (\delta V_{n+1} \Delta t_{n+1} - \delta V_n \Delta t_n)/V(\Delta t_{n+1} - \Delta t_n) \equiv \delta V_n/V, \quad n = 0, 1, \ldots. \]

(51)

which are of the same order as relative variations in the signal velocity. For the rays of light or radar, or even sound signals, this is a small imprecision which justifies the computation of velocity \( w_n \equiv W_n \) by (48) and of distances \( x_n = 0.5V_n \Delta t_n \equiv 0.5V \Delta t_n \), on the basis of measured successive time differences for the average signal velocity \( V \) corresponding to successive time intervals of observation.
7.4. Computation of average velocities \( v_n \) from the \( \gamma \)-representation

Using Lemma 9.1, see [3], for the case of constant average velocities over pieces of discretized trajectory, noting that in (47) \( x(t) \equiv \xi(t) \) of (36) and denoting \( p = d\xi/dt \), we get from (36)

\[
dx/dt \equiv d\xi/dt = -\beta v = -v[1 - (v/V)^2]^{-0.5} = p, \quad \text{if } v = \text{const}, \quad p = \text{const}.
\]

Solving (52) for \( v \), we get

\[
v = -p[1 + (p/V)^2]^{-0.5} = -p\gamma^{-1}(p), \quad \beta(v) = \gamma(p) = [1 + (p/V)^2]^{0.5},
\]

which yields, after the substitution of \( v(p), \beta(v) \) into (36)

\[
\tau = \beta(t - vx_0/V^2) \equiv \gamma(p)t + px_0/V^2, \quad \gamma(p) = [1 + (p/V)^2]^{0.5},
\]

\[
\xi = \beta(x_0 - vt) \equiv \gamma(p)x_0 + pt, \quad x_0 = \text{const}, \quad p = dx/dt \equiv d\xi/dt = \text{const}.
\]

It follows from (52) that \( v = 0 \) if \( p = 0 \), and if \( p \neq 0 \), then \( v^2 < p^2 \) and \( v^2 < V^2 \), thus the physical condition \( |v| < V \) assumed in [1, Section 4], cf. Section 4, is automatically satisfied. The identities in (54) and (55) on the right provide the \( \gamma \)-representation for motions with constant velocities which is based on directly measured derivative in (55).

If we consider the discretization of motion with varying average velocities \( w_n \) between adjacent pulses, it is clear that over each interval \((a, b) = (t_n + 0.5\Delta t_n, t_{n+1} + 0.5\Delta t_{n+1})\) in (42) and (43) the motion with variable speed \( w(t) \) is represented by the uniform motion with constant average velocity \( w_n \) and relativistic transformations (54) and (55) with constant parameters \( v_n, p_n = w_n \) \((n = 0, 1, \ldots)\) of (44) are valid over those intervals. Computed by (44) values of \( w_n \) can be substituted for \( p \) in (52) to compute \( v_n, \beta_n \) whereupon transformations (36) can be used. However, it is much simpler to use \( p_n = w_n / \Delta x_n / (b - a) > 0 \) of (42) and (43), then compute \( v_n = \gamma(p_n) \) from (54), and estimate the trajectory making use of the expressions in (54) and (55) on the right for the \( \gamma \)-representation. We see that relativistic transformations (36), (54) and (55) derived for the relative velocity \( v = \text{const} \) can be used with discretization and on-line observation of the actual motion in appropriate segments along its trajectory.

The \( \gamma \)-representation provides a means to estimate distortion produced by relativistic identification of average velocities \( w_n \) based on direct measurements of distances \( \xi_n = x_n = 0.5V\Delta t_n \geq 0.5V\Delta t_n \) used in (42), (50). Indeed, for a fine discretization of trajectory into sufficiently small pieces \( \Delta x_n = x_{n+1} - x_n \) of (50), the nominal velocities \( w_n = W_n + \delta W_n \geq w_n = p_n \) represent the measured piecewise constant values of the unknown \( p(t) = d\xi/dt \). Since the definition of variables \( p, \gamma(p) \) in (52) and (53), considered as exact invertible mathematical transformation \( v \leftrightarrow p \) (a mapping), does not depend on the existence of a motion nor on relativistic transformations thereof, so the measurements \( p_n = w_n \) can be considered as point-wise values of some unknown smooth function \( p(t) \) without any connection to relativistic considerations nor to formulas (54) and (55). Because of this fact, we can differentiate (53) to obtain deviations of unknown variable velocity

\[
dv/dp = -\gamma^{-3}(p).
\]

However, the values \( p(t) = d\xi/dt \geq \Delta \xi_n / \Delta t = [\xi(t_{n+1}) - \xi(t_n)]/\Delta t \) are measured along the trajectory, so by (56) they determine the actual deviations \( \Delta v \equiv -\gamma^{-3}(p)\Delta p \) of the unknown velocity \( v(t) \) in (54) and (55) by discrete measurements along a discretized trajectory, without recourse to transformations (54) and (55) nor to the exact unknown relativistic transformations for variable velocities considered in Section 4.

8. Modification of Einstein’s transformations for variable velocities

Considerations in Sections 7.2 and 7.4 present discrete approximations for the generalized relativistic transformations in two forms according to the identities (53)–(55) where \( v, p \) should be indexed by \( n = 0, 1, \ldots \) in the process of discretized observation. In practice, it is not necessary to have formula-like transformations for variable velocities which would be much more complicated than those in (36) and, if computed from measurements, would be discretized in a digital computer anyway with constant average velocities of a motion given in the image \( \xi(x, x_0) \). Since the velocity identification process can be performed on a chosen piece of trajectory without prior consideration of preceding motion from a starting point, with the intervals \( \Delta t > 0 \) of (44)–(46) chosen as small as need be, the proposed averaging method can be effectively used in case \( v(t) \neq \text{const} \) with transformations based on computed \( w_n, v_n = v(p_n) = v(w_n), n = 0, 1, \ldots \). The resulting transformations follow from (53)–(55) with \( x_0 = x = \text{const} \):

\[
\tau_n = \beta_n(t - vx_n/V^2) \equiv \gamma_n t + wx_n/V^2, \quad v_n = -w_n\gamma_n^{-1}, \quad \beta_n = \gamma_n = [1 + (w_n/V)^2]^{0.5},
\]

\[
\xi_n = \beta_n(x - vt) \equiv \gamma_n x + wnt, \quad t \in \left[t_n + 0.5\Delta t_n, t_{n+1} + 0.5\Delta t_{n+1}\right], n = 0, 1, \ldots ,
\]

\[
w_n = V(\Delta t_{n+1} - \Delta t_n)/(2\Delta t + \Delta t_{n+1} - \Delta t_n), \quad \tau_n = \tau(t_n + 0.5\Delta t_n), \quad \xi_n = \xi(t_n + 0.5\Delta t_n).
\]

In (57), (58), the values \( \tau(t), \xi(t) \) are the image time and distance provided by the transformations over time intervals in (58), whereas the values \( \tau_n, \xi_n \) in (59) are the observed time and distance of reflection of a ray of light from a body in space computed for the known values \( x, V \) and measured time intervals in (58) and (59). The functions \( \tau(t), \xi(t) \) in (57) and (58) present continuous piecewise linear trajectories that tend to some continuous transformations when \( \Delta t \to 0 \) which are not identical to linear transformations (54) and (55) in the limit for \( \Delta t \to 0 \), as demonstrated by the example below.
Remark 8.1. If system \((k)\), moving along a right line by inertia with velocities \(v = \text{const}, V = \text{const}\), starts to accelerate or makes a turn, then those velocities vary and all the processes that were unfolding in \((k)\) at constant velocities are modified. For this reason, within the intervals \(t \in (t_0 + 0.5 \Delta t_0, t_{n+1} + 0.5 \Delta t_{n+1})\) of constant velocities \(w_n\) in (59), the processes observed in \((k)\) from a still frame \((K)\) in coordinates \(\tau(\cdot), \xi(\cdot)\) of (57) and (58), with approximation of variable velocities by their averages over those intervals, present approximations to those modified processes at variable velocities which are physically different from the processes that would be observed in a system moving with a constant velocity. These approximations contain distortions and contractions produced by relativistic transformations as demonstrated in [3]. To retrieve the processes that would have been unfolding during the motion by inertia, an additional transformation is necessary which would take into account the dynamics of the non-uniform motion of system \((k)\) with respect to the still system \((K)\). Such questions are beyond the scope of this paper.

Example 8.1. For variable speeds less than 3000 m/s (9 mach), we have \([p(t)/V]^2 < 10^{-10}\), so that, due to (53), one can set \(\beta(v(t)) = \gamma(p(t)) \cong 1\), although we still have the motion with variable speed \(-v(t) = d\xi/dt \neq \text{const}\). For this case, we have according to (55): \(\xi = x + t d\xi/dt, x = \text{const}\). Writing this equation in the form: \(d(\xi/t) + xdt/t^2 = 0\), we obtain the integral \(\xi(t) = x + (\xi_0 - x)t/t_0, \xi(t_0) = \xi_0, t_0 > 0\), thus, \(d\xi/dt = \text{const}\). This example shows that the exact constancy of the speed \(v\), and \(p\), is not only sufficient for derivation of transformations (36), but it is also necessary for their very existence in this form, no matter how small may be deviations of the speed from a constant value. Hence, continuous relativistic transformations at variable velocities \(v, V\), valid for the entire trajectory (globally), must have a form different from (36), (54), (55). This proves the expediency of the piecewise linear approximations (57), (58) as a way to bypass difficulties related to possible variations of velocities \(v, V\) along the trajectory of motion and to the application of relativistic considerations in a moving space, i.e., with respect to a designated "still" (actually orbiting) frame \((K)\), avoiding illusory effects of expansion and/or contraction, see Example 3.1.

9. Relativistic transformations at variable velocities in real time

Observers at \(A\) and \(B\) clearly do not physically coincide with the points \(A\) and \(B\), thus, to be observed (received, registered), the time estimates of the moments of arrival at \(A\) and \(B\) in (1) must be transmitted to the observers near \(A\) and \(B\) visually or otherwise, by a physical process which takes some time \(\delta > 0\). Thus, if we want to consider in (1) the time estimates of the moments registered by a sensor (observer), we have to agree that those estimates of the moments of arrival of the ray of light at \(A\) and \(B\) will not be received by the observers, or registered by the sensors, at the very same instants as the light arrives at those points, but a little later. It means that reception, or registration, of time estimates of arrivals is not simultaneous with the actual arrival time of the ray at \(A\) and \(B\) but relates, in fact, to past moments, due to a finite speed of information transmitted to the sensors (observers). Hence, if we want to consider the real time estimate registered by a sensor, not some arrival that actually occurred but is not yet detected (received), we have to replace the moments in (1) by the instants of actual reception of past arrivals, and add to \(t_0\) certain time interval \(\tau^o \geq 0\) of reflection in the mirror at \(B\) which time interval is contained in time differences of (1) if reflection in a mirror is not instantaneous. This renders the equation for experimentally observed time estimates that correspond to the genuine moments of arrival already past:

\[
(t_0 + \tau^o + \delta_B) - (t_0 + \delta_A) = (t^*_A + \delta_A) - (t^*_B + \tau^o + \delta_B), \quad \delta_A, \delta_B \in (0, \delta).
\]  

The time estimates in parentheses we call real time, which is the instants registered by the sensor as times of arrival, with delays due to information transmission. The moments indicated in (1) we call abstract time. The real and abstract times do not coincide, except for an unlikely event when \(\tau^o + \delta_B = \delta_A = 0\) throughout the whole time interval of observation.

Abstract time in not a fictitious moment, — it has really occurred but cannot be known at the very moment of arrival. It can only be estimated up to some precision and with a delay equal to duration of information transmitted by an available physical process. Classical relativity theory operates with abstract time, thus, ignoring delays due to information transmission. Of course, this simplifies the analysis, but makes its results subject to additional imprecision which in some cases may be quite large and comparable with purely relativistic effects. For this reason, it is important to include real time of (60) into relativistic transformations (57)–(59).

Omitting formal derivation which can be found in [3], Sections 7 and 10, and using the real time \(\tau^*\) instead of abstract time \(\tau\) in (57)–(59), (30) and (31) with the values \(a_n = \beta_n^{-1}, n = 0, 1, \ldots\), where \(\beta_n\) are the Einstein calibrating factors for average velocities in (57), the complete set of relativistic transformations in real time is as follows:

\[
\tau^* = \tau - a_n \delta = \tau - \beta_n^{-1} \delta, \quad \beta_n^{-1} = [1 - (v_n/V)^2]^{0.5} \leq 1, n = 0, 1, \ldots, \quad \delta = \tau^o + \delta_B - \delta_A, \quad (61)
\]

\[
\xi^* = \xi - V a_n \delta = \xi - V \beta_n^{-1} \delta, \quad \eta^* = \eta - V a_n \delta = y - V \beta_n^{-1} \delta, \quad \xi^* = z - V \beta_n^{-1} \delta.
\]  

The affine transformation formulae (61) and (62) reflect the fact that real time relativistic transformations present an affinely connected time-space structure with affinors being conditioned on the actually interacting physical processes.

Measuring distances by the rays of light explains the large factor \(V = 3 \times 10^{10} \text{ cm/s} = 300,000 \text{ km/s}\) in (62) that magnifies the effect of small time delays in (61) between actual arrivals of the rays and their reception (detection) by sensors, which affects the measurement, computation and control at large distances. This effect is not critical when low speed signals transmit the information between processes interacting in real time.
If \( \delta = 0 \), transformations (61) and (62) coincide with transformations (36) or (57)–(59). If \( \delta = 0 \) and \( v_n = 0 \), then (61) and (62) become trivial identities. However, if \( \delta \neq 0 \), \( v_n = 0 \), then (61) and (62) present transformations at rest relative to the information transmitting signals alone. In case of classical relativity, those signals are rays of light or radar in moving media, \( v \neq 0 \). In general, those signals may be any signals propagating with some velocity \( V \neq 0 \) in media at motion \( (v \neq 0) \) or at rest \( (v = 0) \) between sensors (observers) at a distance. In this general sense, relativities are all around us, synchronizing physical, chemical, and other life processes in their co-existence and interaction.

Since \( \delta \) cannot be known exactly, it is important to evaluate its influence on the real values of time and coordinates in (61) and (62) of which (62) contain common deduced values corresponding to time delay in reception of information:

\[
\Delta_n = Vn^{-\delta} \delta, \quad n = 0, 1, \ldots
\]  

(63)

For the case \( \delta_B = \delta_A \) we have \( \delta = \tau^0 \sim 10^{-10} \) s, thus, with \( a_n = \beta_n^{-1} \leq 1 \) distortion of time is negligible. On the contrary, distortion of distances (coordinates) may be quite large. Indeed, for velocities \( v_n \sim 300 \) m/s with the speed of light \( V = 3 \times 10^8 \) m/s, we have \( \beta_n^{-1} = (1 - 10^{-12})^{0.5} \approx 1 \), thus, \( \Delta_n = Vn^{-\delta} \tau^0 = 0.03 \) m = 3 cm. However for \( \delta = 0.1 \) s, we have \( \Delta_n = Vn^{-1} \delta = 3 \times 10^7 \) m = 30 000 km (equatorial diameter of Earth is 12 756 km). Imprecision (63) is present in measurements of all three distance coordinates in (62), even at rest if \( v_n = 0 \). Thus, real time measurements delivered by a ray of light or radar may include substantial errors in measurements of location. We see that all motions and processes, in deterministic or stochastic consideration, are observed in real time and their images are distorted by relativistic effects, time uncertainty, and imprecision of instruments.

In practice, the value of \( \delta \) in (61)–(63) is included in time measurements through additive constants mentioned by Einstein [2, p. 16] in relation to his transformations (10) of which Eqs. (57) and (58) and (61) and (62) are local representations over small segments of a discretized trajectory. It means that, if those equations are used with real time measurements, then these additive constants are already included in the real time transformations in accordance with actually realized value of \( \delta \). However, the time differences in (42)–(44) and (48)–(50), (59) do not contain time delays due to information transmittal which delays are cancelled out. Same relates to time differences \( \Delta t = t_{n+1} - t_n \) and \( \Delta t_m = t_{m} - t_{n} \), so the only imprecision in the average velocities \( w_n \) of (44), (48), (50), (59), and measured distances in (42), (44), (50), is the imprecision of the measured \( [6] \) speed of light \( V \). This is an important advantage of the identification method, Section 7, based on measured differences, which excludes time delays due to finite speed of information transmittal.

If \( v_n \to V \), then \( \beta_n \to \infty, \beta_n^{-1} \to 0 \), so in (61) and (62) we have \( \beta_n^{-1} \delta \to 0 \). However, from (36), (61) and (62) we see that \( \tau \to \infty, \xi \to \infty, \tau^* \to \infty, \xi^* \to \infty \), thus, physical processes in \( (k) \) become undetected since their images in \( (K) \) cannot be obtained in finite images. The situations when \( v_n \) is close to \( V \) are of much practical interest. If \( V \) is the speed of sound, or a lower speed, the ratio \( v_n/V \) becomes of paramount importance, and if it is close to 1, the experiments would produce deceptive images and wrong results. It implies that experiments and computations cannot give us more than the nature allows us to obtain through the signals employed in those experiments and computations.

10. Conclusions

Classical transformations of special relativity considered by Albert Einstein [1] are modified and upgraded for the general case of variable speed of synchronization signal and variable relative velocity \( v(t) \neq const \) between two frames \((K)\) and \((k)\) linked through that signal. It is demonstrated that classical transformations with constant velocities \( V, v \), if formally applied to frames at variable relative velocity, may produce an illusory effect (mirage) of expansion which does not exist in reality. This is due to unconventional use of such transformations for systems with variable relative velocity, the case for which classical transformations are invalid. Differential and averaging approaches are considered, and a new form of relativistic transformations is obtained for discretized trajectories with on-line identification of variable relative velocity \( v(t) \) and estimation of errors through the \( \gamma \)-representation based on direct distance measurements. Natural time delays due to finite speed of information transmittal are included into relativistic transformations at variable velocities, and equations for software development are presented to support computation of real time trajectories and for error estimation.

References