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Some notes on the threshold graphs

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ABSTRACT

In this paper we consider threshold graphs (also called nested split graphs) and investigate some invariants of these graphs which can be of interest in bounding the largest eigenvalue of some graph spectra.

1. Introduction

Threshold graphs represent a well-studied class of graphs motivated from numerous directions. They were first introduced by Chvátal and Hammer in 1977 [4] (with motivation in integer linear programming) as graphs for which there exists a linear threshold function separating independent subsets from non-independent (vertex) subsets. Since then, depending on pedigree, many different definitions and/or characterizations have been found, including constructive ones, those based on forbidden configurations, etc. We mention in passing that threshold graphs appear in studying graphical degree sequences, simplicial complexes, etc. The importance of threshold graphs can be also seen through numerous applications (not to be listed here).

The detailed treatment of threshold graphs first appeared in the book by Golumbic [10]; the most complete reference on the topic is the book by Mahadev and Peled [11] (which includes nine different characterizations). Needless to say there are many different generalizations of threshold graphs. They can be viewed as special cases of some wider classes of graphs like cographs, split graphs, interval graphs, etc.

Our motivation for considering threshold graphs comes from the spectral graph theory. These graphs arise (within the graphs with fixed order and/or size) as graphs with the largest eigenvalue of the adjacency matrix. Brualdi and Hoffman [3] observed that they admit the stepwise form of the adjacency matrix, while later Hansen (see, for example, [2]) observed that they are split graphs distinguished by a nesting property imposed on vertices in the maximal co-clique, and hence called them the nested split graphs. As far as we know, it was first observed in [12], they are $\{2K_2, P_4, C_4\}$-free graphs, and thus the threshold graphs. In [5,6] it was observed that they appear in the same role with respect to the signless Laplacian spectrum.

Recall, a *split graph* is a graph which admits a partition (or colouring) of its vertex set into two parts (say white and black) so that the vertices of the white part (say U) are independent (induce a co-clique), while the vertices of the black part (say V) are non-independent (induce a clique). All other edges, the *cross edges*, join a vertex in U to a vertex in V. To get a *nested split*

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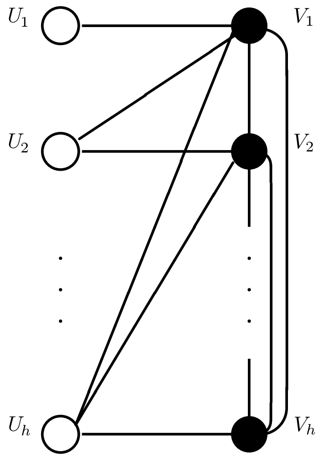

Fig. 1. The structure of a nested split graph.

graph (or NSG for short) we add cross edges in accordance to partitions of U and V into h cells (namely, $U = U_1 \cup U_2 \cup \cdots \cup U_h$ and $V = V_1 \cup V_2 \cup \cdots \cup V_h$) in the following way: each vertex $u \in U_i$ is adjacent to all vertices $v \in V_1 \cup V_2 \cup \cdots \cup V_i$ (see Fig. 1; for more details see [12]). The vertices $U_i \cup V_i$ form the ith level of some NSG (h is the number of levels). The NSG as described can be denoted by $\mathrm{NSG}(m_1, m_2, \ldots, m_h; n_1, n_2, \ldots, n_h)$, where $m_i = |U_i|$ and $n_i = |V_i|$ $(i = 1, 2, \ldots, h)$.

We now fix some notation and terminology. Given a (simple) graph $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$ is its vertex set, while $E = \{e_1, e_2, \ldots, e_m\}$ its edge set (n is its order, while m its size). As usual, if $v \in V$, then $\deg(v)$ (or d_v for short) is the degree of v. The average degree of G $(=1/n \sum_{v \in V} d_v)$ is denoted by $\bar{d}$, while the average degree of the neighbors of v $(=1/d_v \sum_{u \sim v} d_u)$ is denoted by $\bar{d}_v$ (here $\sim$ denotes that the vertices in question are adjacent; we write $e = uv$ if $u \sim v$). If $e \in E$, then $\deg^*(e)$ (or d_e^* for short) is the edge degree of e — it is the number of edges adjacent to e, or alternatively, the degree of e in $L(G)$, the line graph of G; clearly, $d_e^* = d_u + d_v - 2$. Recall, the *line graph* of a graph G, denoted by $L(G)$, has as the vertex set the edge set of G, with two vertices in $L(G)$ adjacent if the corresponding edges in G are adjacent. The average edge degree of G $(=1/m \sum_{e \in E} d_e^*)$ is denoted by $\bar{d}^*$, while the average edge degree of the neighbors of e $(=1/d_e^* \sum_{f \sim e} d_f^*)$ is denoted by $\bar{d}_e^*$ (note, here $\sim$ denotes that the edges in question are adjacent).

The following (derived) parameters of the NSGs will be useful in the next section:

- $a_i = \sum_{s=1}^{i-1} m_s$ the number of white vertices contained in the first $i - 1$ levels (we also set $a_{ij} = a_j - a_i$);
- $b_i = \sum_{t=1}^{i} n_t$ the number of black vertices contained in the first i levels (we also set $b_{ij} = b_j - b_i$);
- $A_i = \sum_{s=1}^{i-1} m_s b_s$ the number of cross edges emerging from the vertex set $U_1 \cup \cdots \cup U_{i-1}$ (we also set $A_{ij} = A_j - A_i$);
- $B_i = \sum_{t=1}^{i} n_t(n - 1 - a_t)$ the number of cross non-edges (or edges in $\bar{G}$) emerging from the vertex set $V_1 \cup \cdots \cup V_i$ (we also set $B_{ij} = B_j - B_i$).

Note $1 \leqslant i \leqslant j \leqslant h$; we also assume that $a_1 = A_1 = 0$. We will also need the following parameters (defined for each $1 \leqslant i \leqslant h$):

- d_i $(=b_i)$, the degree of any (white) vertex in U_i;
- D_i $(=n - 1 - a_i)$, the degree of any (black) vertex in V_i;
- $\bar{d}_i$ $(=\bar{d}_{u_i})$, the average degree of any (white) vertex in U_i;
- $\bar{D}_i$ $(=\bar{d}_{v_i})$, the average degree of any (black) vertex in V_i.

The rest of the paper is organized as follows: in Section 2 (main results) we prove some inequalities for the quantities based on vertex or edge degrees, and in Section 3 give some considerations related to the spectral graph theory.

2. Main results

Let $G = (V, E)$ be a connected NSG of order $n \geqslant 5$. In this section we focus our attention on some questions related to average vertex (resp. edge) degrees of G.

We first consider these quantities with respect to vertices. Recall first that for (distinct) vertex degrees the following holds:

$$d_1 < d_2 < \cdots < d_h \leqslant D_h < D_{h-1} < \cdots < D_1 = n - 1. \tag{2.1}$$

In contrast, for average vertex degrees of neighbors in NSGs, we have another type of monotonicity (see Proposition 2.1). To prove this we first invoke the following fact:

$(*)$ for any strictly monotone sequence, say (s_n), the sequence of weighted arithmetic means (S_n) (here $S_n = \frac{\sum_{i=1}^{n} w_i s_i}{\sum_{i=1}^{n} w_i}$, where (w_n) is a positive sequence of weights) is also a strictly monotone sequence.

Proposition 2.1. *If G is an NSG then:*

$$n-1=\bar{d}_1>\bar{d}_2>\cdots>\bar{d}_h\geqslant\bar{D}_h>\bar{D}_{h-1}>\cdots>\bar{D}_1. \tag{2.2}$$

In addition, $\bar{d}_h=\bar{D}_h$ only if U_h is a singleton.

Proof. Note first that $\bar{d}_1=n-1$. The monotonicity of $\bar{d}_i$'s easily follows from $(*)$ (and (2.1)). Note next that $\bar{d}_h$ is an average degree of all black vertices (i.e. vertices from $V_1\cup V_2\cup\cdots\cup V_h$ of G). On the other hand $\bar{D}_h$ has a similar interpretation on the almost the same vertex set but with one black vertex from V_h removed while all white vertices from U_h added. So $\bar{d}_h\geqslant\bar{D}_h$, and it follows at once that equality holds if U_h is a singleton (i.e. if $m_h=1$). Finally, we prove that

$$\bar{D}_i<\bar{D}_{i+1} \tag{2.3}$$

for $1\le i\le h-1$. Let $Q_i=\sum_{s=i}^{h}m_sd_s+\sum_{t=1}^{h}n_tD_t$, while $q_i=\sum_{s=i}^{h}m_s+\sum_{t=1}^{h}n_t-1$. Then

$$\bar{D}_i=\frac{Q_i-D_i}{q_i},\qquad \bar{D}_{i+1}=\frac{Q_i-m_id_i-D_{i+1}}{q_i-m_i},$$

and (2.3) is equivalent to $Q_i+\frac{q_i(D_i-D_{i+1})}{m_i}>q_id_i+D_i$. Since $Q_i\geqslant q_id_i+D_i$ (by (2.1)), we are done.

This completes the proof. □

The next example (constructed ad hoc) shows that other graphs (like line graphs of NSGs) do not have such a nice property.

Example 2.1. Let $G=\mathrm{NSG}(1,2,1;1,1,1)$. Let the vertices of G be labelled so that $U_1=\{1\}$, $U_2=\{2,3\}$, $U_3=\{4\}$, $V_3=\{5\}$, $V_2=\{6\}$ and $V_1=\{7\}$. Consider the line graph of G, i.e. the graph $H=L(G)$, with $V(H)=\{45,17,26,36,27,37,46,56,47,57,67\}$. The degrees of the vertices of H (edges of G) are: $\{4,5,5,5,5,6,6,6,7,7,9\}$, respectively. It is now easy to see that $d_u=4$, $\bar{d}_u=6.5$ $(u=45)$ $d_v=5$, $\bar{d}_v=6.8$ $(v=17)$ and $d_w=6$, $\bar{d}_w=6.0$ $(w=46)$. So $d_u<d_v<d_w$, but nothing analogous holds for $\bar{d}_u$, $\bar{d}_v$ and $\bar{d}_w$. □

We now consider the invariant $d_v+\bar{d}_v$ (v is a vertex of G, where G is not necessary an NSG). It can be easily shown (as expected from (2.1) and (2.2)) that this invariant is not monotonic for NSGs in the sense of Proposition 2.1. On the other hand, this invariant was considered by [7], where it was shown that $\max\{d_v+\bar{d}_v : v\in V(G)\}\leqslant\frac{2m}{n-1}+n-2$, for any (connected) graph of order n and size m. Here, to make a paper more self-contained (in view of spectral context) we give a short proof of this result, but only for NSGs.

Proposition 2.2. *If G is an NSG, then*

(i) $\max_{1\le i\le h}\{d_i+\bar{d}_i\}\leqslant\frac{2m}{n-1}+n-2$;
(ii) $\max_{1\le i\le h}\{D_i+\bar{D}_i\}\leqslant\frac{2m}{n-1}+n-2$;

The equality in (i) *holds only for $G=K_n$, while in* (ii) *only for $i=1$.*

Proof. To prove (i) we have to show that

$$d_i+\frac{\sum_{t=1}^{i}n_tD_t}{d_i}\leqslant\frac{\sum_{t=1}^{i}n_tD_t+S_i}{n-1}+n-2$$

holds for $i=1,2,\ldots,h$ (here $S_i=\sum_{t=i+1}^{h}n_tD_t+\sum_{s=1}^{h}m_sd_s$). The latter is equivalent to

$$d_i+(D_1-d_i)\sum_{t=1}^{i}\frac{n_tD_t}{d_iD_1}\leqslant n-2+\frac{S_i}{D_1};$$

note $D_1=n-1$. Since $\frac{D_t}{D_1}\leqslant 1$ and $\sum_{t=1}^{i}n_t=d_i$, the left hand side is $\leqslant n-1$. On the other hand $S_i\geqslant\sum_{s=1}^{h}m_sd_s\geqslant n-1$; note, $\sum_{s=1}^{h}m_sd_s$ is the number of cross edges in G, and it is $\geqslant n-1$ since the corresponding bipartite graph (on n vertices formed by the cross edges) is connected. So the right hand side is $\geqslant n-1$, and we are done.

For (ii) we have to show that

$$D_i+\frac{\sum_{s=i}^{h}m_sd_s+\sum_{t=1}^{h}n_tD_t-D_i}{D_i}\leqslant\frac{\sum_{s=1}^{h}m_sd_s+\sum_{t=1}^{h}n_tD_t}{D_1}+n-2,$$

holds for $i = 1, 2, \ldots, h$, or equivalently, that

$$D_i + \left(\frac{D_1 - D_i}{D_1 D_i}\right)(D_i\bar{D}_i + D_i) \leqslant \frac{\sum_{s=1}^{i-1} m_s d_s}{D_1} + n - 1;$$

note, $\sum_{s=i}^{h} m_s d_s + \sum_{t=1}^{h} n_t D_t = D_i\bar{D}_i + D_i$. Since $\frac{\bar{D}_i}{D_1} \leqslant 1$, and since $D_1 - D_i = \sum_{s=1}^{i-1} m_s$, the left hand side is $\leq \frac{\sum_{s=1}^{i-1} m_s}{D_1} + n - 1$, and this is clearly $\leqslant \frac{\sum_{s=1}^{i-1} m_s d_s}{D_1} + n - 1$. So, we are again done.

This completes the proof. □

We now switch to the analogous quantities related to edges. We first note that the analogy of Proposition 2.1 now does not hold (see again Example 2.1). In sequel we will consider NSGs G for which the quantity $\max_{e\in E}\{\bar{d}_e^*\}$ (or equivalently, $\max\{\bar{d}_v : v \in V(L(G))\}$) is exceeding the value equal to $n - 3 + \bar{d}$. (Note, the quantity $\max_{v\in V}\{\bar{d}_v\}$ is always equal to $n - 1$ in NSGs, the maximal possible value by Proposition 2.1, and therefore is not interesting to be studied.)

Let $e = uv$ be an edge of any graph G (not necessarily an NSG). Then

$$\bar{d}_e^* = \frac{\sum_{f\sim u, f\neq e} \deg^*(f) + \sum_{f\sim v, f\neq e} \deg^*(f)}{\deg(u) + \deg(v) - 2}.$$

Putting $p = \deg(u)$ and $q = \deg(v)$ we get

$$\bar{d}_e^* = \frac{\sum_{w\sim u, w\neq v} [p + d_w - 2] + \sum_{w\sim v, w\neq u} [q + d_w - 2]}{p + q - 2},$$

which yields

$$\bar{d}_e^* = \frac{p^2 + q^2 - 3p - 3q + 4}{p + q - 2} + \frac{\sum_{w\sim u, w\neq v} d_w + \sum_{w\sim v, w\neq u} d_w}{p + q - 2}.$$

Therefore, we get

$$\bar{d}_e^* = f(p, q) + \frac{S(u) + S(v)}{p + q - 2},$$

where

$$S(u) = \sum_{w\sim u, w\neq v} d_w, \qquad S(v) = \sum_{w\sim v, w\neq u} d_w,$$

and

$$f(p, q) = p + q - 1 - 2\frac{pq - 1}{p + q - 2}.$$

In what follows we assume that G is an NSG as depicted in Fig. 1, other than a complete graph.

Lemma 2.1. *If $e = uv$, where $u \in V_i$ and $v \in V_j$ $(1 \leqslant i \leqslant j \leqslant h)$, then*

$$\bar{d}_e^* < n - 3 + \bar{d}.$$

Proof. We have: $p = n - 1 - a_i$, $q = n - 1 - a_j$ $(=p - a_{ij})$, $S(u) = 2m - A_i - p - q$ and $S(v) = 2m - A_j - p - q$ $(=S(u) - A_{ij})$. So it follows that

$$f(p, q) = p - 2 - \frac{a_{ij}}{2} + \frac{(\frac{a_{ij}}{2})^2}{p - 1 - \frac{a_{ij}}{2}},$$

$$\frac{S(u) + S(v)}{p + q - 2} = -2 + \frac{2m - 2 - A_i - \frac{1}{2}A_{ij}}{p - 1 - \frac{a_{ij}}{2}}$$

and therefore

$$\bar{d}_e^* = p - 4 - \frac{a_{ij}}{2} + \frac{2m - 2 - A_i - \frac{1}{2}A_{ij} + (\frac{a_{ij}}{2})^2}{p - 1 - \frac{a_{ij}}{2}}.$$

So we will consider the following inequality:

$$n - a_i - 5 - \frac{a_{ij}}{2} + \frac{2m - A_i - 2 - \frac{1}{2}A_{ij} + (\frac{a_{ij}}{2})^2}{n - a_i - 2 - \frac{a_{ij}}{2}} \leqslant n - 3 + \frac{2m}{n},$$

which is equivalent to

$$\frac{2m}{n-a_i-2-\frac{a_{ij}}{2}}-\frac{2m}{n}\leqslant 2+a_i+\frac{a_{ij}}{2}+\frac{A_i+2+\frac{1}{2}A_{ij}-(\frac{a_{ij}}{2})^2}{n-a_i-2-\frac{a_{ij}}{2}}$$

and also to

$$2m\leqslant n\left(n-2-a_i-\frac{a_{ij}}{2}\right)+\frac{n\left(2+A_i+\frac{1}{2}A_{ij}-(\frac{a_{ij}}{2})^2\right)}{2+a_i+\frac{a_{ij}}{2}}.$$

For $i=j=1$ the latter inequality reduces to $m\leqslant\binom{n}{2}$, and we are done. To prove it for $j>1$, we first estimate the upper bound for m, in the case that G' (in the role of G) is an NSG of order n having the first $j-1$ levels the same as G (namely, $U_1, V_1, \ldots, U_{j-1}, V_{j-1}$), and the remaining levels chosen so that the size of G' is maximal. It is next easy to see that the maximum (denoted by m') is attained when G' is *"the closest"* to the complete graph, i.e. when G' has exactly j levels, and the clique induced by the black vertices is the largest possible. This happens when U'_j has only one element, and $V_1\cup\cdots\cup V_{j-1}\cup V'_j$ has $n-a_j-1$ elements. Then $2m'=(n-a_j)(n-a_j-1)+2A_j$. To complete the proof, it suffices to verify that

$$2m'<n\left(n-2-a_i-\frac{a_{ij}}{2}\right)+\frac{n[2+A_i+\frac{1}{2}A_{ij}-(\frac{a_{ij}}{2})^2]}{2+a_i+\frac{a_{ij}}{2}}.$$

We first observe that for every $s\in\{1,\ldots,i-1\}$

$$n-a_j-1+\frac{n(b_s-1)}{2+a_i+\frac{a_{ij}}{2}}\geqslant b_j+m_j-1+\frac{n(b_s-1)}{2+a_i+\frac{a_{ij}}{2}}\geqslant 2b_s,$$

since $n-a_j\geqslant b_j+m_j$ and $n\geqslant 2+a_i+\frac{a_{ij}}{2}$. Next, for every $s\in\{i,\ldots,j-1\}$ we have

$$n(b_s+a_i+1)\geqslant(a_i+a_{ij}+b_s+2)(b_s+a_i+1)\geqslant b_s(2a_i+a_{ij}+4),$$

since $n-a_j-1\geqslant b_s+1$. Therefore we get

$$n-1-a_j+\frac{1}{2}\frac{n(b_s+a_i+1)}{2+a_i+\frac{a_{ij}}{2}}\geqslant b_s+1+b_s>2b_s.$$

Using the above inequalities, we obtain

$$\begin{aligned}
n\left(n-2-a_i-\frac{a_{ij}}{2}\right)+\frac{n[2+A_i+\frac{1}{2}A_{ij}-(\frac{a_{ij}}{2})^2]}{2+a_i+\frac{a_{ij}}{2}} &= n(n-a_j-1)+n\left(\frac{a_{ij}}{2}-1\right)+\frac{n\left[2+A_i+\frac{1}{2}A_{ij}-(\frac{a_{ij}}{2})^2\right]}{2+a_i+\frac{a_{ij}}{2}}\\
&=(n-a_j)(n-a_j-1)+(a_i+a_{ij})(n-a_j-1)+\frac{n}{2+a_i+\frac{a_{ij}}{2}}\left[A_i-a_i+\frac{1}{2}(A_{ij}+a_{ij}(a_i+1))\right]\\
&=(n-a_j)(n-a_j-1)+\sum_{s=1}^{i-1}m_s\left[n-a_j-1+\frac{n(b_s-1)}{2+a_i+\frac{a_{ij}}{2}}\right]+\sum_{s=i}^{j-1}m_s\left[n-a_j-1+\frac{1}{2}\frac{n(b_s+a_i+1)}{2+a_i+\frac{a_{ij}}{2}}\right]\\
&>(n-a_j)(n-a_j-1)+2A_j\\
&=2m'.
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *If $e=uv$, $u\in U_j$ and $v\in V_i$ $(1\leqslant i\leqslant j\leqslant h)$, then*

$$\bar{d}^*_e\leqslant n-3+\bar{d}$$

holds, unless $i=j=1$, $|V_1|=1$ and $\bar{d}>\frac{n}{2}$.

Proof. We now have: $p=b_j=b_i+b_{ij}$, $q=n-1-a_i$, $S(u)=B_j-q=B_i+B_{ij}-q$ and $S(v)=2m-A_i-p-q$. So $S(u)+S(v)=2m-p-2q+R$, where $R=B_i+B_{ij}-A_i$. Let $C_{ij}=\sum_{s=i+1}^{j}n_sa_{is}$. It is a matter of routine calculations to show that $R=pq-C_{ij}$. But then $S(u)+S(v)=2m+pq-p-2q-C_{ij}$, and consequently we have that

$$\bar{d}^*_e=p+q-2+\frac{2m-pq-q-C_{ij}}{p+q-2},$$

or equivalently

$$\bar{d}_e^* = n - 3 - a_i + b_j + \frac{2m - (n-1-a_i)(b_j+1) - C_{ij}}{n-3-a_i+b_j}.$$

So we have to prove the following inequality:

$$2m(a_i - b_j + 3) \leqslant n[(a_i - b_j)(n - 3 + b_j - a_i) + (n - 1 - a_i)(b_j + 1) + C_{ij}].$$

We next consider the following two cases depending on the sign of $a_i - b_j + 3$.

Case 1: $a_i - b_j + 3 \geqslant 0$. If $a_i - b_j + 3 = 0$ then the above inequality reduces to $n(b_j - 2) + C_{ij} \geqslant (a_i + 1)(b_j + 1)$. Since $b_j - 2 = a_i + 1$ the latter inequality becomes $(n - b_j - 1)(b_j - 2) + C_{ij} \geqslant 0$, which holds since $n \geqslant a_i + b_j + 1$ and $b_j = a_i + 3 > 2$. So we next assume that $a_i - b_j + 3 > 0$. If $i = 1$ then $a_1 = 0$ and $b_j \leqslant 2$. If $b_j = 1$ (i.e. if $|V_1| = 1$) then we easily get that the above inequality reduces to $m \leqslant \frac{n^2}{4}$. So, if $|V_1| = 1$ and $\bar{d} > \frac{n}{2}$ we get that the inequality in question does not hold (an exceptional case from the lemma). Otherwise, if $b_j = 2$ then the above inequality reduces to $2m \leqslant n(n-1+C_{1j})$ which clearly holds. So we next assume that $i \geqslant 1$ and consequently we have to prove that

$$2m \leqslant \frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) + (b_j + 1)(n - a_i - 1) + C_{ij}].$$

Assume now that G' is the graph with maximal number of edges obtained in the same way as in the proof of Lemma 2.1 Then $2m' = (n - a_j)(n - a_j - 1) + 2A_j$. Therefore we have to prove that

$$2m' \leqslant \frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) + (b_j + 1)(n - a_i - 1) + C_{ij}].$$

This can be done as follows:

$$\begin{aligned}
&\frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) + (b_j + 1)(n - 1 - a_i) + C_{ij}] \\
&\quad = (n - a_j)(n - a_j - 1) + \frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) \\
&\qquad + (b_j + 1)(n - 1 - a_i) - (n - 2a_j - 1)(a_i - b_j + 3) + C_{ij}] - a_j(a_j + 1) \\
&\quad \geqslant (n - a_j)(n - a_j - 1) + a_j(n - a_j - 1) + na_{ij} + \frac{n}{a_i - b_j + 3}[a_j(b_j - 2) + C_{ij}] \\
&\quad \geqslant (n - a_j)(n - a_j - 1) + a_jb_j + a_j(b_j - 2) \\
&\quad \geqslant (n - a_j)(n - a_j - 1) + 2a_jb_{j-1} \\
&\quad \geqslant (n - a_j)(n - a_j - 1) + 2A_j \\
&\quad = 2m'.
\end{aligned}$$

Note that the first inequality in the chain follows since $n - a_j - b_j - 1 \geqslant 0$, while the second one is based on the following facts: $n > a_i - b_j + 3$, $n - a_j - 1 \geqslant b_j$, $a_{ij} \geqslant 0$ and $C_{ij} \geqslant 0$. Finally, since the third one is self-evident, we are done in this case.

Case 2: $a_i - b_j + 3 < 0$. Now we have to verify that

$$2m > \frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) + (b_j + 1)(n - a_i - 1) + C_{ij}].$$

In contrast to the previous case we will now have to minimize the number of edges in G'. For this aim we take that the first $j - 1$ levels are the same as in G, while the remaining vertices are in the jth level distributed so that V_j' is the same as V_j. Then $2m' = b_j(b_j - 1) + 2(n - a_j - b_j)b_j + 2A_j$. Therefore we have to prove that

$$2m' \geqslant \frac{n}{a_i - b_j + 3}[(a_i - b_j)(n - 3 + b_j - a_i) + (b_j + 1)(n - a_i - 1) + C_{ij}].$$

This can be done as follows:

$$\begin{aligned}
&\frac{n}{b_j - a_i - 3}[(b_j - a_i)(n - 3 + b_j - a_i) - (b_j + 1)(n - a_i - 1) - C_{ij}] \\
&\quad = b_j(b_j - 1) + 2(n - a_j - b_j)b_j + (2a_j + b_j + 1 - n)b_j + \frac{n}{b_j - a_i - 3}(a_i^2 - na_i + 4a_i + b_j + 1 - n - C_{ij}) \\
&\quad \leqslant b_j(b_j - 1) + 2(n - a_j - b_j)b_j + a_jb_j - na_i - \frac{n}{b_j - a_i - 3}[(n - b_j - 1)(a_i + 1) + C_{ij}] \\
&\quad \leqslant b_j(b_j - 1) + 2(n - a_j - b_j)b_j + a_jb_j - na_i - a_j(a_i + 1) - C_{ij}
\end{aligned}$$

$$\leqslant b_j(b_j-1)+2(n-a_j-b_j)b_j+a_{ij}b_j-C_{ij}$$
$$\leqslant b_j(b_j-1)+2(n-a_j-b_j)b_j+2A_j$$
$$=2m'.$$

Note that the first three inequalities in the chain follow since $n-a_j-b_j \geqslant 1$. On the other hand, the fourth one easily follows by observing that $A_j = A_i + A_{ij}$ and $A_{ij} + C_{ij} = a_{ij}b_j$. So we are again done.

This completes the proof. □

Collecting the above results we get that $\max_{e\in E}\{\bar{d}_e^*\} \leqslant n-3+\bar{d}$ for all NSGs G for which $n_1 > 1$, or $n_1 = 1$ and $\bar{d} \leqslant \frac{n}{2}$. In other words, if G has at least two vertices of degree $n-1$, or one vertex of degree $n-1$ and the average vertex degree $\bar{d} \leqslant \frac{n}{2}$ then $\max_{e\in E}\{\bar{d}_e^*\} \leqslant n-3+\bar{d}$ holds. We will say that these graphs are of type-I. In contrast, the graphs for which $\max_{e\in E}\{\bar{d}_e^*\} > n-3+\bar{d}$ are of type-II.

We will now consider in more details the graphs of type-II. Any such graph has a unique vertex of degree $n-1$ (thus $m_1 = k$ and $n_1 = 1$) and big average vertex degree ($>\frac{n}{2}$). The latter fact also implies that k cannot be too big. By a simple calculation we can get that $k < (1-\frac{\sqrt{2}}{2})n < 0.3n$ (note, k is the largest if $G = \mathrm{NSG}(k, 1; 1, n-k-2)$). Next, it is also reasonable to ask how large the quantity $\max_{e\in E}\{\bar{d}_e^*\}$ can be for a fixed k. But then, due to Lemmas 2.1 and 2.2, we have to take that $G = \mathrm{NSG}(k, m_2, \ldots, m_h; 1, n_2, \ldots, n_h)$ for some choice its parameters, and to consider an edge $e = uv$ with $u \in U_1$ and $v \in V_1$. By adding some edges if necessary (note, then the observed quantity cannot decrease) we arrive at the graph $G' = \mathrm{NSG}(k, 1; 1, n-k-2)$. But then, by simple calculations, we get that

$$\bar{d}_e^* \leqslant \frac{k^2-(2n-3)k+(n-2)(2n-3)}{n-2},$$

with the largest possible value equal to $2n-5$ (attained for $G = \mathrm{NSG}(1, 1; 1, n-3)$, if $k = 1$). Note, for the latter graph we have that

$$\max_{e\in E}\{\bar{d}^*\} = \max\left\{2n-6+\frac{1}{2n-5}, 2n-5, 2n-6+\frac{2}{2n-5}, 2n-6+\frac{5}{2n-5}\right\},$$

and so, as expected, for $n \geq 5$ the second value is the right one (note, the cases with $n < 5$ are excluded from considerations). Collecting the above results, we arrive at:

Proposition 2.3. *If G is an NSG then, depending on the type of G (I or II, respectively), we have:*

(i) $\max_{e\in E}\{\bar{d}_e^*\} \leqslant n-3+\bar{d}$, *or*
(ii) $n-3+\bar{d} < \max_{e\in E}\{\bar{d}_e^*\} \leqslant \frac{k^2-(2n-3)k+(n-2)(2n-3)}{n-2}$, *where k is the number of vertices of degree one.*

3. Concluding remarks

We will now use the above results to give some comments related to spectral graph theory. More precisely, we will highlight some phenomena related to Conjecture 7 from [6], the conjecture generated by the computer program AutoGraphiX (AGX). Let $\kappa(G)$ the largest eigenvalue of the signless Laplacian of a graph G (not necessarily an NSG). Recall, $Q(G) = D(G) + A(G)$, where $A(G)$ is the adjacency matrix of G, and $D(G)$ the diagonal matrix of its vertex degrees, is the signless Laplacian of G. According to [6], Conjecture 7 reads:

If G is a connected graph of order $n \geqslant 5$ and average vertex degree $\bar{d}(G)$, then $\kappa(G) \leqslant n-1+\bar{d}(G)$ with equality if and only if G is complete.

The next theorem covers some cases for which the above conjecture is true.[1]

Theorem 3.1. *Let G be a connected graph of order n and size m, and average vertex degree $\bar{d}(G) \leqslant \frac{n}{2}$. Then $\kappa(G) < n-1+\bar{d}(G)$.*

Proof. Based on Theorem 5.4 from [6], it suffices to verify the conjecture only for NSGs. Since $\kappa(G) = \rho(L(G)) + 2$, where $\rho(G)$ is the largest eigenvalue of the adjacency matrix of a graph G (see, for example, Eq. (2) in [6]), we in fact have to prove that $\rho(L(G)) < n-3+\bar{d}(G)$. Due to Favaron et al. (cf. [8]), $\rho(L(G)) \leqslant \max_{e\in E}\{\bar{d}_e^*\}$. The final conclusion (for graphs in question) now follows by using Proposition 2.3. □

The following remark is worth mentioning:

Remark 3.1. In particular, we immediately have that Conjecture 7 from [6] holds for all bipartite graphs G (including some non-bipartite graphs). On the other hand, it is true in general, as we recently learnt from D. Cvetković, who pointed us the Reference [9], where the authors have made a short proof of the conjecture in question by proving first that $\kappa(G) \leqslant \max\{d_v + \bar{d}_v\}$ (see also [1]), and by using the sophisticated bound from [7] (see also Proposition 2.2). □

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