Dynamics of certain class of critically bounded entire transcendental functions

M. Guru Prem Prasad *, Tarakanta Nayak ¹

Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781 039, Assam, India

Received 18 April 2006
Available online 21 August 2006
Submitted by William F. Ames

Abstract

Let E denote the class of all transcendental entire functions \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \mathbb{C} \) and \( a_n \geq 0 \) for all \( n \geq 0 \) such that \( f(x) > 0 \) for \( x < 0 \) and the set of all (finite) singular values of \( f \) forms a bounded subset of \( \mathbb{R} \). For each \( f \in E \), one parameter family \( S = \{ f_{\lambda} \equiv \lambda f: \lambda > 0 \} \) is considered. In this paper, we mainly study the dynamics of functions in the one parameter family \( S \). If \( f(0) \neq 0 \), we show that there exists a positive real number \( \lambda^* \) (depending on \( f \)) such that the bifurcation and the chaotic burst occur in the dynamics of functions in the one parameter family \( S \) at the parameter value \( \lambda = \lambda^* \). If \( f(0) = 0 \), it is proved that the Julia set of \( f_{\lambda} \) is equal to the complement of the basin of attraction of the super attracting fixed point 0 for all \( \lambda > 0 \). It is also shown that the Fatou set \( F(f_{\lambda}) \) of \( f_{\lambda} \) is connected whenever it is an attracting basin and the immediate basin contains all the finite singular values of \( f_{\lambda} \). Finally, a number of interesting examples of entire transcendental functions from the class \( E \) are discussed.

Keywords: Bifurcation; Chaotic burst; Julia sets; Fatou sets

1. Introduction

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be an entire transcendental function and \( f^n \) denote \( n \)-times composition of \( f \). The set of all points \( z \) in \( \mathbb{C} \) at which the sequence of iterates \( \{ f^n(z) \}_{n=0}^{\infty} \) forms a normal family

* Corresponding author.

E-mail address: mgpp@iitg.ernet.in (M. Guru Prem Prasad).

1 The research of Tarakanta Nayak is supported by CSIR Senior Research Fellowship, Grant No. 9/731(31)/2004-EMR-I.

0022-247X/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2006.06.095
in the sense of Montel is called the Fatou set of \( f \) and is denoted by \( \mathcal{F}(f) \). The Julia set of \( f \), denoted by \( \mathcal{J}(f) \), is the complement of the Fatou set of \( f \) in the extended complex plane \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \). The Fatou set and the Julia set of a function are the primary objects of investigation in Complex Dynamics. It is well known that the Fatou set is open and the Julia set is a no where dense subset of \( \hat{\mathbb{C}} \) unless it is equal to \( \hat{\mathbb{C}} \). A detailed treatment of the dynamics of entire functions and survey of results can be found in [1,5,7,13,14].

A point \( \tilde{z} \) is said to be a critical point of \( f \) if \( f'(\tilde{z}) = 0 \). If \( \tilde{z} \) is a critical point then the value \( f(\tilde{z}) \) is called a critical value of \( f \). A point \( \tilde{a} \) is called an asymptotic value of \( f \) if there exists a path \( \gamma(t) \) (a continuous function from \( (0, \infty) \) to \( \mathbb{C} \)) satisfying \( \lim_{t \to \infty} \gamma(t) = \infty \) and \( \lim_{t \to \infty} f(\gamma(t)) = \tilde{a} \). All the critical values and finite asymptotic values of a function \( f \) are known as the singular values of \( f \). Let \( S_f \) denote the set of all singular values of \( f \) and \( O^+(S_f) = \{ f^n(z) : z \in S_f, n = 0, 1, \ldots \} \) denote the forward orbits of all singular values of \( f \). Let \( \tilde{a} \) be an asymptotic value of \( f \) and \( D_r(\tilde{a}) \) be a disk of radius \( r \) with center \( \tilde{a} \). For every \( r > 0 \), choose a component \( U(r) \) of the pre-image \( f^{-1}(D_r(\tilde{a})) \) such that \( r_1 < r_2 \) implies \( U(r_1) \subset U(r_2) \). In this case, \( \bigcap_{r > 0} U(r) = \emptyset \). The choice \( r \to U(r) \) defines a transcendental singularity of \( f^{-1} \). It is called direct if there exists \( r > 0 \) such that \( f(z) \neq \tilde{a} \) for \( z \in U(r) \) and indirect otherwise (refer to [2] for more details). A function \( f \) is said to be critically bounded (or critically finite) if \( S_f \) is bounded (or finite). The entire function \( f \) is said to be of finite order (growth) if there is a positive number \( A \) such that, as \( |z| = r \to \infty \), \( |f(z)| < Ke^{rA} \) for some constant \( K \). The lower bound \( \rho = \rho(f) \) of all such \( A \) is called the order (growth) of the function \( f \) and it is given by

\[
\frac{1}{\rho} = \lim \inf_{n \to \infty} \frac{\log(1/|a_n|)}{n \log n},
\]

where \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) for \( z \in \mathbb{C} \) [10].

A bifurcation is said to occur at the parameter value \( \lambda^* \) in the dynamics of one parameter family \( \{f_\lambda(z) : \lambda \in \text{a real parameter}\} \) if there exists \( \epsilon > 0 \) such that whenever \( a \) and \( b \) satisfy \( \lambda^* - \epsilon < a < \lambda^* \) and \( \lambda^* < b < \lambda^* + \epsilon \), the dynamics of \( f_a(z) \) is different from the dynamics of \( f_b(z) \). In other words, the dynamics of the function changes when the parameter value crosses through the point \( \lambda^* \). Devaney and Durkin [6] showed that a bifurcation occurs at the parameter value \( \lambda^* = \frac{1}{e} \) in the dynamics of one parameter family \( \mathcal{E} = \{E_\lambda(z) = e^{\lambda z^2} : \lambda > 0\} \).

They also proved that the Julia set of critically finite entire function \( \lambda e^z \) for \( 0 < \lambda \leq \frac{1}{e} \) is a nowhere dense subset of \( \hat{\mathbb{C}} \) and as soon as the parameter \( \lambda \) exceeds the value \( \frac{1}{e} \), the Julia set \( \mathcal{J}(\lambda e^z) \) becomes the whole of extended complex plane. This phenomena is referred to as explosion in the Julia sets or chaotic burst in the dynamics of functions in the one parameter family. Similar explosion in the Julia sets occurs in one parameter family \( \{\lambda i \cos z : \lambda > 0\} \) at the parameter value \( \lambda \approx 0.66274 \). Kapoor and Prasad [11,12] and Prasad [9] proved the occurrence of bifurcation and chaotic burst in the dynamics of non-critically finite (having infinitely many singular values), but critically bounded entire transcendental functions in one parameter families \( \{\lambda e^{z-1} : \lambda > 0\}, \{\lambda I_0(z) : \lambda \in \mathbb{R} \setminus \{0\}\} \), where \( I_0(z) \) is the modified Bessel function of zero order and \( \{\lambda \sinh z : \lambda \in \mathbb{R} \setminus \{0\}\} \).

Let \( E \) be the class of entire transcendental functions defined by

\[
E = \left\{ f : \begin{array}{l}
\text{(i) } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ for } z \in \mathbb{C}, \text{ where } a_n \geq 0 \text{ for all } n \geq 0 \\
\text{(ii) } f(x) > 0 \text{ for all } x < 0 \\
\text{(iii) the set } S_f \text{ is a bounded subset of } \mathbb{R}
\end{array} \right\}.
\]

Define

\[
E_0 = \{ f \in E : f(0) = 0 \} \quad \text{and} \quad E_1 = \{ f \in E : f(0) \neq 0 \}.
\]
For each $f \in E$, consider the one parameter family $S = \{ f_{\lambda} \equiv \lambda f : \lambda > 0 \}$. It is worth noting that the class $E$ contains the interesting examples of functions such as $I_{2n}(z)$ and $z^{-n}I_0(z)$ for $n \in \mathbb{N}$, where $I_n(z)$ is the modified Bessel function of first kind and order $n$. The class $E_1$ includes the functions $\frac{\sinh z}{z}$, $I_0(z)$ and $e^z$ whose dynamics have already been studied.

The change in the dynamics of functions in the one parameter family $S = \{ f_{\lambda} \equiv \lambda f : \lambda > 0 \}$, where $f \in E$ as the parameter $\lambda$ varies over positive real numbers is the main subject of investigation of this paper. In Section 2, we show that the class $E$ is closed under composition of functions. We also show that the compositions of certain kind of polynomials with the functions in $E$ yield functions belonging to $E$. In Section 3, the dynamics of $f_{\lambda} \in S$ on the real line is investigated. It is shown for $f \in E_1$ that there exists a positive real number $\lambda^*$ (depending on $f$) such that the bifurcation in the dynamics of functions in the one parameter family $S$ occurs at $\lambda = \lambda^*$. A study of dynamics of one parameter family $S = \{ f_{\lambda}(z) = \lambda f(z) : z \in \mathbb{C} : \lambda > 0 \}$ is carried out individually for $f \in E_0$ and $f \in E_1$ in Section 4. In this section, it is shown for $f \in E_1$ that the chaotic burst in the dynamics of function in the one parameter family $S$ occurs at $\lambda = \lambda^*$. Under certain circumstances, we show that the Fatou set of $f_{\lambda} \equiv \lambda f$, where $f \in E$ is connected. In the last section, a number of interesting examples of functions in $E$ are discussed.

2. Properties of $E$

It is easy to observe that, if $\lambda > 0$ then $\lambda + f \in E_1$ whenever $f \in E$ and $f_{\lambda} \equiv \lambda f \in E_j$ for $f \in E_j$, $j = 0, 1$. Besides this, certain compositions of functions also yield functions in the class $E$ and is shown in Proposition 3. A relation between the set $S_f$ of all singular values of a composition function $f \equiv g \circ h$ and the sets of all singular values of individual functions $g$ and $h$ is obtained in the following proposition.

Proposition 1. Let $g$ and $h$ be two non-constant entire functions and $f(z) = g(h(z))$ be their composition. Then, $S_f \subseteq S_g \cup \{g(z) : z \in S_h\}$.

Proof. Let $\tilde{w}$ be a singular value of $f$. We show that if $\tilde{w}$ is a finite asymptotic (or critical) value of $f$ then it is either a finite asymptotic (or critical) value of $g$ or a $g$-image of a finite asymptotic (or critical) value of $h$.

Suppose that $\tilde{w}$ is a finite asymptotic value of $f$. Let $\gamma : [0, \infty) \rightarrow \mathbb{C}$ be an asymptotic path corresponding to the asymptotic value $\tilde{w}$ of the function $f(z)$. Let $M = \{ t :$ there is a sequence $\{t_k\}$ of positive real numbers such that $\lim_{k \to \infty} t_k = \infty$ and $\lim_{k \to \infty} h(\gamma(t_k)) = t \}$. Observe that $g(t) = \tilde{w}$ for every $t \in M$. Since $g$ is a non-constant entire function, the set $M$ cannot have any limit point in $\mathbb{C}$. Therefore $M \cap \mathbb{C}$ is a discrete set. Now we claim that $M$ contains only one element in $\mathbb{C}$. If possible, let the set $M$ contains more than one element in $\mathbb{C}$. Suppose that $m_1$ and $m_2$ are in $M$ with $m_1 \neq m_2$. Then, there exist open disks $D_1(m_1)$ and $D_2(m_2)$ such that $D_1(m_1) \cap M = \{m_1\}$ and $D_2(m_2) \cap M = \{m_2\}$. The curve $h(\gamma(t))$ intersects the disks $D_1(m_1)$ and $D_2(m_2)$ infinitely many times and also the boundaries $B_1 = \partial D_1(m_1)$ and $B_2 = \partial D_2(m_2)$ of these disks infinitely many times. Note that, if $\{h(\gamma(t)) : t \geq 0\} \cap B_i, i = 1, 2$, is a finite set $S$ (say), then $S \cap M \neq \emptyset$ which is a contradiction to $D_1(m_1) \cap M = \{m_1\}$ for $i = 1, 2$. Suppose that $\{h(\gamma(t)) : t \geq 0\} \cap B_i$ contains infinitely many complex numbers. Then, the intersecting points $\{h(\gamma(t)) : t \geq 0\} \cap B_i$ will have a limit point $l_i$ (say), since $B_1$ and $B_2$ are compact. It implies that $l_i \in M$ which is a contradiction to $D_1(m_1) \cap M = \{m_1\}$ for $i = 1, 2$. So $M$ is a singleton set. If $M = \{b\}$ where $b \in \mathbb{C}$ then $\tilde{w} = g(b)$ and $b$ is an asymptotic value of $h(z)$.
If $M = \{\infty\}$ then $\tilde{w}$ is an asymptotic value of $g(z)$. Therefore, $\tilde{w}$ is either an asymptotic value of $g$ or equal to $g(b)$ where $b$ is a finite asymptotic value of $h$.

Now, suppose that $\tilde{w}$ is a critical value of $f$. If $\tilde{z}$ is a critical point such that $f(\tilde{z}) = \tilde{w}$ then $g'(h(\tilde{z}))h'(\tilde{z}) = 0$ and hence, either $h(\tilde{z})$ is a critical point of $g$ or $\tilde{z}$ is a critical point of $h$. Consequently, $\tilde{w} = g(h(\tilde{z}))$ is a critical value of $g$ or a $g$-image of a critical value of $h$. □

Remark 2. For two entire functions $g$ and $h$, if $S_g$ and $S_h$ are bounded subsets of $\mathbb{R}$ and $g$ is an entire function preserving the real axis then $g(S_h) = \{g(z): z \in S_h\}$ is a bounded subset of $\mathbb{R}$ and consequently, $S_{g \circ h} \subseteq S_g \cup \{g(z): z \in S_h\}$ is a bounded subset of $\mathbb{R}$.

**Proposition 3.** Let $f \in E$, $g \in E_0$ and $h \in E_1$. Let $P(z) = (z + a_1)^{m_1}(z + a_2)^{m_2} \cdots (z + a_n)^{m_n}$ be a non-constant polynomial where $a_1, a_2, \ldots, a_n$ are positive real numbers and $m_1, m_2, \ldots, m_n$ are non-negative integers. Then,

1. $\phi = h \circ f \in E_1$ and $\psi = g \circ h \in E_1$. In particular, the class $E_1$ is closed under composition.
2. The class $E_0$ is closed under composition.
3. $\Phi = P \circ f \in E_1$ and $\Psi = h \circ P \in E_1$.

**Proof.**

1. Let $\phi(z) = h(f(z))$ for $z \in \mathbb{C}$, where $h \in E_1$ and $f \in E$. If $h(z) = \sum_{n=0}^{\infty} a_n z^n$ for $z \in \mathbb{C}$ then $h(f(z)) = \sum_{n=0}^{\infty} a_n (f(z))^n = \sum_{n=0}^{\infty} b_n z^n$ for $z \in \mathbb{C}$ (say). All the coefficients in the Taylor series of $(f(z))^n$ about the origin are non-negative, so all $b_n$’s are non-negative. It is obvious that $\phi(x) = h(f(x)) > 0$ for $x < 0$ and $\phi(0) = h(f(0)) > 0$. As $f$ and $h$ are in $E$, $S_f$ and $S_h$ are bounded subsets of $\mathbb{R}$ and $h$ is an entire function that preserves the real axis. The set $S_{h \circ f}$ is a bounded subset of $\mathbb{R}$ by Remark 2. Thus, $\phi = h \circ f \in E_1$ for $h \in E_1$ and $f \in E$. Taking $f$ in $E_1$, it is seen that the class $E_1$ is closed under composition.

It can be shown similarly that all the coefficients of Taylor series of $\psi = g \circ h$ about the origin are non-negative and $S_{g \circ h}$ is a bounded subset of $\mathbb{R}$ for all $g \in E_0$. Since $g(h(x)) > 0$ for all $x \leq 0$, it follows that $\psi = g \circ h \in E_1$.

2. Let $g$ and $\tilde{g}$ be in $E_0$. It follows by similar arguments used in the first paragraph of this proposition that, all the coefficients of the Taylor series of $g \circ \tilde{g}$ about the origin are non-negative and $S_{g \circ \tilde{g}}$ is a bounded subset of $\mathbb{R}$. Clearly, $g(\tilde{g}(0)) = 0$. Since $\tilde{g}(x) > 0$ for $x < 0$, $g(\tilde{g}(x)) > 0$ for all $x < 0$. Therefore, $g \circ \tilde{g}$ belongs to $E_0$.

3. Observe that all the coefficients in the Taylor series of $\Phi = P \circ f$ and $\Psi = h \circ P$ about the origin are non-negative. Since all zeros of $P(z)$ are real, the zeros of $P'(z)$ are real by Lucas theorem. Further, $P(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ which gives that critical values of $P$ are real. As $P$ has no finite asymptotic value, $S_P$ is a finite subset of $\mathbb{R}$. For any function $f$ in $E$, the set of all singular values $S_f$ is a bounded subset of $\mathbb{R}$ and, $P$ and $f$ preserve the real axis. So $S_P$ and $S_\Psi$ are bounded subsets of $\mathbb{R}$ by Remark 2. Clearly, $P(f(x)) > 0$ for $x \leq 0$ and $h(P(x)) > 0$ for $x \leq 0$. Thus, $\Phi = P \circ f$ and $\Psi = h \circ P$ belong to $E_1$ for all $f \in E$ and $h \in E_1$. □

3. Dynamics of $f_\lambda(x)$ on $\mathbb{R}$

In this section, the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$, where $f \in S$ is studied. For the functions $f$ in the class $E_0$, the dynamics of $f_\lambda \equiv \lambda f$ on $\mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$ is described in Theo-
Theorem 6. Let \( f_\lambda \equiv \lambda f \) where \( f \in E_0 \) and \( \lambda > 0 \). Then, \( f_\lambda \) has only two real periodic points \( 0 \) and \( r_\lambda \) with \( 0 < r_\lambda \), where \( 0 \) is a super attracting fixed point and \( r_\lambda \) is a repelling fixed point. Further, \( \lim_{n \to \infty} f_\lambda^n(x) = 0 \) for \( 0 \leq x < r_\lambda \) and \( \lim_{n \to \infty} f_\lambda^n(x) = \infty \) for \( x > r_\lambda \).

Proof. Let \( f_\lambda(x) = \lambda \sum_{n=0}^{\infty} a_n x^n \) for \( x \in \mathbb{R} \) where \( a_n \geq 0 \) for all \( n \geq 0 \). Observe that \( f_\lambda(x) > 0 \) for \( x \in \mathbb{R} \) with \( x \neq 0 \) and \( f_\lambda'(x) > 0 \) for \( x > 0 \). Therefore, any non-zero real periodic point of \( f_\lambda \) lies only in \( \mathbb{R}^+ = \{ x \in \mathbb{R} : x > 0 \} \) and it must be a fixed point. Since \( f \in E_0 \), \( f_\lambda(0) = \lambda f(0) = 0 \).

Further, the point \( x = 0 \) is a fixed point of \( f_\lambda \). Since \( f_\lambda(x) > 0 \) for \( x < 0 \), \( f_\lambda(x) = 0 \) and \( f_\lambda(x) > 0 \) for \( x > 0 \), it follows that \( f_\lambda'(0) = 0 \) and hence the point \( x = 0 \) is a super attracting fixed point of \( f_\lambda \). Thus, \( f_\lambda(x) = \lambda \sum_{n=0}^{\infty} a_n x^n \) for \( x \in \mathbb{R} \) and \( a_n \geq 0 \) for all \( n \geq 0 \).

Let \( g_\lambda(x) = f_\lambda(x) - x \) for \( x \in \mathbb{R} \). Then, \( g_\lambda'(x) = f_\lambda'(x) - 1 = \lambda (\sum_{n=2}^{\infty} n a_n x^{n-1}) - 1 \) and \( g_\lambda''(x) = f_\lambda''(x) = \lambda \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \). It shows that \( g_\lambda''(x) > 0 \) for \( x > 0 \) and \( g_\lambda'(x) \to +\infty \) as \( x \to +\infty \). Since \( g_\lambda(0) = -1 \) and \( g_\lambda'(x) \) is increasing in \( \mathbb{R}^+ \), there is a unique \( x_\lambda > 0 \) such that \( g_\lambda'(x) < 0 \) for \( x \in (0, x_\lambda) \), \( g_\lambda'(x_\lambda) = 0 \) and \( g_\lambda'(x) > 0 \) for \( x > x_\lambda \). It shows that \( g_\lambda \) decreases in \( (0, x_\lambda) \) and increases thereafter. Since \( g_\lambda(0) = 0 \) and \( \lim_{x \to +\infty} g_\lambda(x) = +\infty \), there exists a unique point \( r_\lambda \) in \( (x_\lambda, \infty) \) such that \( g_\lambda(x) < 0 \) for \( x \in (0, r_\lambda) \), \( g_\lambda(r_\lambda) = 0 \) and \( g_\lambda(x) > 0 \) for \( x \in (r_\lambda, \infty) \). Therefore, the point \( r_\lambda \) is a unique real fixed point of \( f_\lambda \) and is repelling as \( f_\lambda'(x) > 1 \) in \( (x_\lambda, \infty) \).

For \( 0 < x < r_\lambda \), \( f_\lambda(x) < x \). It gives that the sequence \( \{ f_\lambda^n(x) \}_{n=0}^{\infty} \) is decreasing and bounded below by \( 0 \). Therefore, \( \lim_{n \to \infty} f_\lambda^n(x) = 0 \) for \( 0 < x < r_\lambda \). The sequence \( \{ f_\lambda^n(x) \}_{n=0}^{\infty} \) is increasing and unbounded for \( x > r_\lambda \). So \( \lim_{n \to \infty} f_\lambda^n(x) = +\infty \) when \( x > r_\lambda \). \( \square \)

Remark 5. Let \( A = \{ x \in \mathbb{R} : x < 0 \) and \( f_\lambda(x) \in [0, r_\lambda) \} \) and \( B = \{ x \in \mathbb{R} : x < 0 \) and \( f_\lambda(x) \in (r_\lambda, \infty) \} \), where \( f \in E_0 \) and \( \lambda > 0 \). It follows by Theorem 4 that \( f_\lambda^n(x) \to 0 \) as \( n \to \infty \) for \( x \in A \) and \( f_\lambda^n(x) \to +\infty \) as \( n \to \infty \) for \( x \in B \).

Theorem 6. Let \( f_\lambda \equiv \lambda f \) where \( f \in E_1 \) and \( \lambda > 0 \). There exists a unique positive real number \( \lambda^* \) such that

1. For \( 0 < \lambda < \lambda^* \), \( f_\lambda \) has only two real fixed points \( a_\lambda \) and \( r_\lambda \) with \( a_\lambda < r_\lambda \), where \( a_\lambda \) is attracting and \( r_\lambda \) is repelling. Further, \( \lim_{n \to \infty} f_\lambda^n(x) = a_\lambda \) for \( 0 \leq x < r_\lambda \) and \( \lim_{n \to \infty} f_\lambda^n(x) = \infty \) for \( x > r_\lambda \).

2. For \( \lambda = \lambda^* \), \( f_\lambda \) has only one real fixed point \( x^* \), where \( x^* \) is the unique real solution of \( f(x) - x f'(x) = 0 \) and \( x^* \) is rationally indifferent. Further, \( \lim_{n \to \infty} f_\lambda^n(x) = x^* \) for \( 0 \leq x < x^* \) and \( \lim_{n \to \infty} f_\lambda^n(x) = x^* \) for \( x > x^* \).

3. For \( \lambda > \lambda^* \), \( f_\lambda \) has no real fixed point. Further, \( \lim_{n \to \infty} f_\lambda^n(x) = \infty \) for all \( x \geq 0 \).

Proof. As \( f \in E_1 \), \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) for \( x \in \mathbb{R} \), where \( a_n \geq 0 \) for all \( n \geq 0 \). It is easy to see that \( f(x) > 0 \) for all \( x \in \mathbb{R} \). Observe that \( f'(x) > 0 \) and \( f''(x) > 0 \) for \( x > 0 \). Therefore, \( f'(x) \) is increasing and tends to \( +\infty \) as \( x \to +\infty \). Further, we conclude that \( \lim_{x \to +\infty} f(x) = +\infty \) and \( \lim_{x \to +\infty} f(x) - x f'(x) = -\infty \).

For \( \lambda > 0 \), \( f_\lambda(x) \) is positive for any real number \( x \) and \( f_\lambda'(x) \) is positive for \( x > 0 \). Therefore, any real periodic point of \( f_\lambda \) must be a fixed point and positive.
Let $g_2(x) = f_2(x) - x$ for $x \in \mathbb{R}$. Observe that $g_2''(x) = f_2''(x) > 0$ for $x > 0$. Therefore, $g_2'(x)$ is increasing in $\mathbb{R}^+$ and $g_2'(x) \to +\infty$ as $x \to +\infty$. Suppose that $f'(0) \neq 0$. Then, $f'(0) > 0$ since $f'(x) > 0$ for $x > 0$. If $\lambda \geq \frac{1}{f'(0)}$, then $g_2'(0) = f_2'(0) - 1 \geq 0$. It gives that $g_2'(x) > g_2'(0) \geq 0$ for all $x > 0$. Therefore, $g_2$ is strictly increasing in $\mathbb{R}^+$. As $g_2(0) > 0$, $g_2(x)$ has no zero in $\mathbb{R}^+$. In other words, $f_2(x)$ has no fixed point in $\mathbb{R}$ when $\lambda \geq \frac{1}{f'(0)}$. If $0 < \lambda < \frac{1}{f'(0)}$, then $g_2'(0) < 0$. Also, if $f'(0) = 0$, then $g_2'(0) < 0$ for all $\lambda$. As $g_2'(x)$ is increasing and tends to $+\infty$ as $x \to +\infty$, there exists a real number $x_\lambda > 0$ such that $g_2'(x_\lambda) < 0$ for $x \in (0, x_\lambda)$, $g_2'(x) = 0$ for $x = x_\lambda$ and $g_2'(x) > 0$ for $x > x_\lambda$. It shows that $g_2(x)$ decreases in $(0, x_\lambda)$ and attains a minimum value at $x = x_\lambda$, and then increases to $+\infty$ in $(x_\lambda, \infty)$. It gives that $\lambda = \frac{1}{f'(x_\lambda)}$.

Consider $\phi(x) = f(x) - xf'(x)$ for $x \in \mathbb{R}^+$. Since $\phi'(x) = -xf''(x) < 0$ for all $x > 0$, the function $\phi(x)$ is decreasing in $(0, \infty)$. Using the facts that $\phi(0) = f'(0) > 0$ and $\lim_{x \to +\infty} \phi(x) = -\infty$, by the continuity of $\phi$, it follows that there exists unique $x^* > 0$ such that

$$\phi(x) = \begin{cases} > 0 & \text{for } x < x^*, \\ = 0 & \text{for } x = x^*, \\ < 0 & \text{for } x > x^*. \end{cases}$$

In the sequel, let

$$\lambda^* = \frac{1}{f'(x^*)},$$

where $x^*$ is the positive real root of $\phi(x) = f(x) - xf'(x)$ for $f \in E_1$. Since $\frac{1}{f'(x)}$ is decreasing in $\mathbb{R}^+$ and $x^* > 0$, it follows that $\lambda^* < \frac{1}{f'(0)}$ if $f'(0) \neq 0$.

1. If $0 < \lambda < \lambda^*$, then $\frac{1}{f'(x_\lambda)} < \frac{1}{f'(x^*)}$ and $x_\lambda > x^*$. It follows from Eq. (1) that $\phi(x_\lambda) < 0$. Consequently, $g_2(x_\lambda) = f_2(x_\lambda) - x_\lambda < 0$. Therefore, there exists two real numbers $a_\lambda$ and $r_\lambda$ (say) with $0 < a_\lambda < x_\lambda < r_\lambda$ such that $g_2(a_\lambda) = 0 = g_2(r_\lambda)$. Therefore, $f_2$ has exactly two real fixed points $a_\lambda$ and $r_\lambda$. Observe that $0 < f_2'(a_\lambda) = f_2'(r_\lambda) = 1 < f_2'(x_\lambda)$, since $0 < a_\lambda < x_\lambda < r_\lambda$. Therefore, $a_\lambda$ is attracting and $r_\lambda$ is repelling fixed points of $f_2$. Note that $f_2(x) > x$ for $0 < x < a_\lambda$ and $f_2(x) < x$ for $a_\lambda < x < r_\lambda$. Since $f_2(x)$ is increasing in $\mathbb{R}^+$, the sequence $\{f_2^n(x)\}_{n=0}^{\infty}$ is increasing and bounded above by $a_\lambda$ for $0 \leq x < a_\lambda$. Similarly, the sequence $\{f_2^n(x)\}_{n=0}^{\infty}$ is decreasing and bounded below by $a_\lambda$ for $a_\lambda < x < r_\lambda$. Hence, $\lim_{n \to \infty} f_2^n(x) = a_\lambda$ for $0 \leq x < r_\lambda$ by monotone convergence theorem. Now, $f_2(x) > x$ and $f_2'(x) > 1$ for $x > r_\lambda$. It implies that the sequence $\{f_2^n(x)\}_{n=0}^{\infty}$ is increasing and not bounded above. Consequently, $\lim_{n \to \infty} f_2^n(x) = \infty$.

2. When $\lambda = \lambda^*$, it can be shown that $g_2(x_\lambda) = 0$ and $x_\lambda = x^*$ by similar arguments in the case $0 < \lambda < \lambda^*$. As $g_2^*(x^*)$ is the minimum value of $g_2^*(x)$, the point $x^*$ is the only zero of $g_2^*(x)$. Hence $f_2^*(x)$ has only one fixed point $x^*$ and it is rationally indifferent. The sequence $\{f_2^*(x)\}_{n=0}^{\infty}$ is increasing and bounded above by $x^*$ for $0 \leq x < x^*$. By monotone convergence theorem, it follows that $\lim_{n \to \infty} f_2^*(x) = x^*$. For $x > x^*$, the sequence $\{f_2^*(x)\}_{n=0}^{\infty}$ is increasing and not bounded above. Therefore, $f_2^*(x)$ tends to $+\infty$ as $n \to \infty$.

3. Let $\lambda > \lambda^*$. If $f'(0) = 0$ then there exists a $x_\lambda$ such that $f_2'(x_\lambda) = 1$ and $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$. Similarly, if $f'(0) \neq 0$ and $\lambda^* < \lambda < \frac{1}{f'(0)}$ then there exists a $x_\lambda$ such that $f_2'(x_\lambda) = 1$ and $\frac{1}{f'(x_\lambda)} > \frac{1}{f'(x^*)}$. This implies that $x_\lambda < x^*$ and $\phi(x_\lambda) > 0$. Therefore, $g_2(x) > g_2(x^*) = 0$ for all $x > 0$ showing that $f_2(x)$ has no fixed point for $\lambda > \lambda^*$. If $f'(0) \neq 0$ and $\lambda \geq \frac{1}{f'(0)}$, then...
it is already shown in the beginning that $f_\lambda$ has no fixed point. In all the cases, observe that $f_\lambda(x) > x$ for all $x \geq 0$. Therefore, the sequence $\{f^n_\lambda(x)\}_{n \geq 0}$ is strictly increasing and not bounded above for all $x \geq 0$ and hence $\lim_{n \to \infty} f^n_\lambda(x) = \infty$ for all $x \geq 0$. \qed

Remark 7.

(1) Let $A = \{x \in \mathbb{R}: x < 0$ and $\lambda f(x) \in [0, r_\lambda]\}$ and $B = \{x \in \mathbb{R}: x < 0$ and $\lambda f(x) \in (r_\lambda, \infty)\}$, where $f \in E_1$ and $0 < \lambda < \lambda^*$. Then, it follows by Theorem 6(1) that $f^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in A$ and $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x \in B$.

(2) Let $A = \{x \in \mathbb{R}: x < 0$ and $\lambda^* f(x) \in [0, x^*]\}$ and $B = \{x \in \mathbb{R}: x < 0$ and $\lambda^* f(x) \in (x^*, \infty)\}$, where $f \in E_1$. Then, it follows by Theorem 6(2) that $f^n_\lambda(x) \to x^*$ as $n \to \infty$ for $x \in A$ and $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x \in B$.

(3) If $f \in E_1$ and $\lambda > \lambda^*$ then $f_\lambda(x) > 0$ for $x < 0$. Since $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x \geq 0$ by Theorem 6(3), it also happens that $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for $x < 0$. Therefore, $f^n_\lambda(x) \to +\infty$ as $n \to \infty$ for all $x \in \mathbb{R}$, if $f \in E_1$ and $\lambda > \lambda^*$.

4. Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$

In this section, it is shown that rotational domains, wandering domains and Baker domains are non-existent in the Fatou sets of functions in $E$. Then an investigation of dynamics of functions $f_\lambda$ is made separately for $f \in E_0$ and $f \in E_1$. It is well known that the Fatou set of an entire transcendental function does not contain Herman ring [1]. If $f \in E$, we prove that the Siegel disks, Baker domains and wandering domains also do not exist in the Fatou set of $f$ in Theorems 8 and 9. The dynamics of $f_\lambda(z) = \lambda f(z)$ for $z \in \mathbb{C}$ and $\lambda > 0$ is described in Theorem 10 for $f \in E_0$ and in Theorem 12 for $f \in E_1$.

Theorem 8. Let $f \in E$. Then, $\mathcal{F}(f)$ does not contain a Siegel disk.

Proof. Suppose that the Fatou set of $f$ contains a Siegel disk $D$ (say). For $f \in E$, it follows from the facts $f(x) > x$ for $x \in \mathbb{R}$ and $S_f \subset \mathbb{R}$ is bounded that the closure of the forward orbits of all singular values of $f$ is properly contained in $\mathbb{R}$. It is known that the set of forward orbits of all singular values of $f$ is dense in the boundary $\partial D$ of the Siegel disk $D$. Therefore, $\partial D \subset \mathbb{R}$ and the set $(\partial D)^c = (\mathbb{R} \setminus \partial D)$ is path connected and $D \subseteq (\partial D)^c$. Now, we claim that the complement of the closure of $D$, denoted by $(\bar{D})^c$, is empty. If possible, let there is a point $z^*$ in $(\bar{D})^c = (D \cup \partial D)^c = D^c \cap (\partial D)^c$. Then, $\{z^*\} \cup D$ is a subset of $(\partial D)^c$ and a path $\gamma$ can be found in $\partial D$ joining the point $z^*$ and a point of $D$. Since $z^* \in D^c$, the path $\gamma$ must intersect $\partial D$ which is not possible. Therefore, $(\bar{D})^c$ is an empty set. As any component of the Fatou set other than $D$ must be in $(\bar{D})^c$, the Fatou set of $f$ cannot contain any component other than $D$. Since the Fatou set of $f$ is completely invariant, it follows that $D$ is completely invariant. By Picard’s theorem, all points of $D$, except at most one have infinitely many pre-images. Since $D$ is completely invariant, $f$ is not one-to-one on $D$ which is a contradiction to the definition of Siegel disk. Therefore, the Fatou set of $f$ does not contain a Siegel disk. \qed

Theorem 9. Let $f \in E$. Then, the Fatou set of $f$ contains neither a Baker domain nor a wandering domain.

Proof. Since $f$ is a critically bounded entire function, $f^k$ is critically bounded for each $k \in \mathbb{N}$. It is shown in [7] that the sequence $\{f^{kn}(z)\}_{n \geq 0}$ does not converge to $\infty$ for any $k \in \mathbb{N}$ and for $z \in \mathcal{F}(f)$. Therefore, the Fatou set of $f$ cannot contain a Baker domain.
Suppose that $W$ is a wandering domain in the Fatou set of $f$. Let $z_0$ be a point in $W$. Then the sequence $\{f^n(z)\}_{n>0}$ is normal at $z = z_0$. Let $w_0$ be a finite limit point of the set $A = \{z_n = f^n(z_0): n = 0, 1, 2, 3, \ldots\}$. Then, there is a subsequence $\{z_{n_k} = f^{nk}(z_0)\}_{k>0}$ of $\{f^n(z_0)\}_{n>0}$ converging to $w_0$. Without loss of generality, it is assumed that $\zeta_k = n_k+1 - n_k$ is an increasing and unbounded sequence. It is known that if $W$ is a wandering domain in the Fatou set of a transcendental entire function $f$, then each limit function of $\{f^n(z)\}_{n>0}$ for $z \in W$ is in the intersection of the Julia set and the derived set of $O^+(S_f)$ [3]. Therefore, it follows that $\lim_{k \to \infty} f^{nk}(z_0) = w_0$ is in the derived set of $O^+(S_f)$ which is a subset of $\mathbb{R}$. Since $z_0$ is in the wandering domain $W$, $w_0$ cannot be a periodic point. From Theorems 4 and 6 and Remarks 5 and 7, it is clear that any real number which is not a periodic point of $f$ either tends to a non-repelling (i.e., super attracting, attracting or rationally indifferent) fixed point or to $\infty$ under iteration of $f$. Since $w_0$ is in the Julia set of $f$, further iterations of the point $w_0$ (that is, $\{f^m(w_0)\}_{m>0}$) cannot tend to a non-repelling periodic point, but tends to infinity (as $m \to \infty$). For $M > |w_0|$, we can find a $\tilde{m} \in \mathbb{N}$ such that $|f^m(w_0)| > M$ for all $m > \tilde{m}$. Since $w_0 = \lim_{k \to \infty} f^{nk}(z_0)$ and, $f^m$ and the modulus function are continuous in $\mathbb{C}$, we have $\lim_{k \to \infty} |f^m(f^{nk}(z_0))| > M$ for all $m > \tilde{m}$. It means that there is a $\tilde{k} \in \mathbb{N}$ such that $|f^{\tilde{m}}(f^{nk}(z_0))| > M$ for $m > \tilde{m}$ and $k > \tilde{k}$. Since $\{\zeta_k\}_{k>0}$ is increasing and unbounded, there is a $k_0 \in \mathbb{N}$ such that $\zeta_k > \tilde{m}$ for all $k > k_0$. Choose $m = \zeta_{k_0+1}, \zeta_{k_0+2}, \zeta_{k_0+3}, \ldots$ and observe that $|f^{\tilde{m}}(f^{nk}(z_0))| > M$ for all $k > \tilde{k}$ and $k > k_0$. Consequently, $|(f^{nk+1}(z_0))| > M > |w_0|$ for all $k > k'$, where $k' = \max\{\tilde{k}, k_0\}$ which is impossible since $\lim_{k \to \infty} f^{nk}(z_0) = w_0$. Therefore, $w_0$ is infinite and $\lim_{k \to \infty} f^{nk}(z_0) = \infty$. Thus, $w_0 = \infty$ is only the limit point of $A$ and it implies that $f^n(z) \to \infty$ as $n \to \infty$ which is not possible as $f$ is a critically bounded entire function [7]. Therefore, the Fatou set of $f$ cannot contain a wandering domain. □

Theorem 10. Let $f_\lambda (z) = \lambda f(z)$ for $z \in \mathbb{C}$, where $f \in E_0$ and $\lambda > 0$. Then, the Fatou set of $f_\lambda$ is the basin of attraction of the super attracting fixed point 0 of $f_\lambda$.

Proof. If $f \in E_0$ and $\lambda > 0$, then $f_\lambda \equiv \lambda f \in E_0$. By Theorems 8 and 9, it follows that the Fatou set of $f_\lambda$ does not contain the rotational domains, Baker domains and wandering domains. Therefore, the Fatou set of $f_\lambda$ contains only the basin of attractions or the parabolic domains. Suppose that $U$ is a basin of attraction (or a parabolic domain) associated with a non-real attracting (or rationally indifferent) periodic point $c_\lambda$ (say) of period $p$ of $f_\lambda$. Then, there is at least one singular value $\tilde{w}$ (say) of $f_\lambda$ such that $f^{np}_\lambda(\tilde{w})$ converges to $c_\lambda$ as $n \to \infty$. It shows that there exists a natural number $n_0$ such that $f^{np}(\tilde{w})$ are non-real for all $n > n_0$ which is a contradiction to the fact the forward orbits of all singular values are subset of the real line. Therefore, it is concluded that the basin of attractions or parabolic domains corresponding to the real periodic points are only possible components in the Fatou set of $f_\lambda$. It is proved in Theorem 4 that the super attracting fixed point $x = 0$ is the only such real periodic point of $f_\lambda$ for $\lambda > 0$. Therefore, the Fatou set $F(f_\lambda)$ is the basin of attraction of the super attracting fixed point 0 of $f_\lambda$. □

The following computationally useful characterization of the Julia set of $f_\lambda \equiv \lambda f$, where $f \in E_0$ is an immediate consequence of Theorem 10.

Corollary 11. Let $f_\lambda (z) = \lambda f(z)$ for $z \in \mathbb{C}$, where $f \in E_0$ and $\lambda > 0$. Then, the Julia set of $f_\lambda$ is the complement of the basin of attraction of the super attracting fixed point 0 of $f_\lambda$.

Theorem 12. Let $f_\lambda (z) = \lambda f(z)$ for $z \in \mathbb{C}$, where $f \in E_1$ and $\lambda > 0$. 
(1) For $0 < \lambda < \lambda^*$, the Fatou set of $f_{\lambda}$ is the basin of attraction of the real attracting fixed point $a_{\lambda}$.
(2) For $\lambda = \lambda^*$, the Fatou set of $f_{\lambda}$ is the parabolic domain corresponding to the real rationally indifferent fixed point $x^*$.
(3) For $\lambda > \lambda^*$, the Fatou set of $f_{\lambda}$ is empty.

Proof. If $f \in E_1$ and $\lambda > 0$, then $f_{\lambda} \equiv \lambda f \in E_1$. The Fatou set of $f_{\lambda}$ has no rotational domains, Baker domains and wandering domains by Theorems 8 and 9. Suppose that $U$ is a basin of attraction or a parabolic domain of $f_{\lambda}$ associated with a non-real attracting or rationally indifferent periodic point of period $p$. Then, there is at least one singular value $\tilde{w}$ of $f_{\lambda}$ and a natural number $n_0$ such that $f_{\lambda}^{n_0}(\tilde{w})$ is non-real for all $n > n_0$ which is not possible since $O^+(S_{f_{\lambda}})$ is a subset of the real line. Therefore, if $U$ is a basin of attraction or a parabolic domain of $f_{\lambda}$ then the periodic point of $f_{\lambda}$ associated with $U$ is real. Let $\lambda^*$ be given in Eq. (2).

(1) For $0 < \lambda < \lambda^*$, it follows from Theorem 6(1) that $f_{\lambda}$ has only one real attracting fixed point $a_{\lambda}$. Therefore, the Fatou set $\mathcal{F}(f_{\lambda})$ is equal to the basin of attraction of $a_{\lambda}$ for $0 < \lambda < \lambda^*$.
(2) If $\lambda = \lambda^*$, then $f_{\lambda}$ has only one real fixed point $x^*$, where $x^*$ is the unique real solution of $f(x) - xf'(x) = 0$ and $x^*$ is the real rationally indifferent fixed point by Theorem 6(2). Therefore, it follows that the Fatou set of $f_{\lambda}$ is the parabolic domain corresponding to the real rationally indifferent fixed point $x^*$.
(3) For $\lambda > \lambda^*$, the function $f_{\lambda}$ has no real fixed point by Theorem 6(3) and $f_{\lambda}^n(x) \to \infty$ for all $x \in \mathbb{R}$ as $n \to \infty$ by Remark 7(3). Therefore, it is concluded that the Fatou set $\mathcal{F}(f_{\lambda})$ is an empty set for $\lambda > \lambda^*$.

Corollary 13. Let $f_{\lambda}(z) = \lambda f(z)$ for $z \in \mathbb{C}$, where $f \in E_1$ and $\lambda > 0$. Then,

(1) For $0 < \lambda < \lambda^*$, the Julia set of $f_{\lambda}$ is the complement of the basin of attraction of the real attracting fixed point $a_{\lambda}$.
(2) For $\lambda = \lambda^*$, the Julia set of $f_{\lambda}$ is the complement of parabolic domain corresponding to the real rationally indifferent fixed point $x^*$.
(3) For $\lambda > \lambda^*$, the Julia set of $f_{\lambda}$ is equal to $\hat{\mathbb{C}}$.

Remark 14. It follows from Corollary 13 that the Julia set of $f_{\lambda}$ is a nowhere dense subset of $\hat{\mathbb{C}}$ for $0 < \lambda \leq \lambda^*$. If the parameter $\lambda$ exceeds the value $\lambda^*$, then the Julia set of $f_{\lambda}$ explodes and equals the extended complex plane. Thus, the chaotic burst occurs in the dynamics of functions in the one parameter family $S = \{f_{\lambda} \equiv \lambda f: \lambda > 0\}$ for $f \in E_1$ at the parameter value $\lambda = \lambda^*$.

4.1. Connected Fatou set

If $f$ is an entire transcendental function, then each preperiodic component of the Fatou set of $f$ is simply connected. For the functions $f$ in $E$, we show in the following that the Fatou set of $f$ is (simply) connected in certain cases.

Lemma 15. Let $f$ be a transcendental entire function and the set of all singular values of $f$ be contained in a bounded Jordan domain $D$ containing 0 and with smooth boundary. Then each
component of $f^{-1}(D^c)$ is a simply connected domain whose boundary is a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to $\infty$.

**Proof.** It is known that, if $D_r^2(0) = \{z: |z| < r^2\}$ contains all the singular values of a transcendental entire function $f$ then every component of $f^{-1}(\mathbb{C} \setminus D_r(0))$ is a simply connected domain bounded by a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to $\infty$ [8]. The proof follows since $D$ is homeomorphic to $D_r(0)$. □

**Theorem 16.** Let $f_\lambda \equiv \lambda f$, where $f \in \mathbb{E}$ and $\lambda > 0$. Suppose that the Fatou set of $f_\lambda$ is a basin of attraction of an attracting fixed point $a_\lambda$ (say). If all the singular values of $f_\lambda$ are in the immediate basin of attraction of $a_\lambda$ then the Fatou set of $f_\lambda$ is connected and each maximally connected subset of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ is an unbounded simple curve.

**Proof.** Let $I(a_\lambda)$ be the immediate basin of attraction of the attracting fixed point $a_\lambda$ and $D \subset I(a_\lambda)$ be a Jordan domain with smooth boundary containing all the singular values of $f_\lambda$ and $0$ such that $f_\lambda(D) \subset D$. Then, $f_\lambda^{-1}(D^c)$ does not intersect $D$ and $f_\lambda^{-1}(D) = \mathbb{C} \setminus f_\lambda^{-1}(D^c)$. Therefore, $D \subset f_\lambda^{-1}(D)$. By Lemma 15, each component of $f_\lambda^{-1}(D^c)$ is a simply connected domain whose boundary is a single non-closed analytic curve in $\mathbb{C}$ both ends of which tend to $\infty$. In other words, $f_\lambda^{-1}(D^c)$ is connected and its boundary has no self intersections in $\mathbb{C}$ which means that $f_\lambda^{-1}(D)$ is (simply) connected. Since each component of $f_\lambda^{-1}(I(a_\lambda))$ intersects $f_\lambda^{-1}(D)$, $f_\lambda^{-1}(I(a_\lambda))$ is connected. Further, $a_\lambda \in f_\lambda^{-1}(I(a_\lambda)) \cap I(a_\lambda)$ implies that $I(a_\lambda)$ is backward invariant. Therefore, the Fatou set of $\mathcal{F}(f_\lambda)$ is equal to $I(a_\lambda)$ and is connected. It gives that each maximally connected subset of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ is a simple curve. Since the Fatou set of $f_\lambda$ is simply connected, each maximally connected subset of $\mathcal{J}(f_\lambda) \setminus \{\infty\}$ is an unbounded curve. □

5. Examples

In the present section, the dynamics of some interesting functions $f$ in the class $\mathbb{E}$ are described. First, we prove a proposition on the number of (finite) asymptotic values of $f$ that is needed for our discussion.

**Proposition 17.** Let $f$ be an entire transcendental function of order (growth) one and $\tilde{w}$ be a finite asymptotic value of $f$. If all the critical values of $f$ has only one limit point in $\mathbb{C}$ and that is equal to $\tilde{w}$ then $\tilde{w}$ is the only finite asymptotic value of $f$.

**Proof.** Suppose that $w^*$ is a finite asymptotic value of $f$ with $w^* \neq \tilde{w}$. It is well known that an entire function of finite order (growth) $\rho$ has at most $2\rho$ finite asymptotic values. Since $f$ has order (growth) one, the function $f$ has exactly two finite asymptotic values. If an entire function of finite order (growth) $\rho$ has $2\rho$ finite asymptotic values then none is direct [15, p. 307]. Therefore, the asymptotic values $\tilde{w}$ and $w^*$ are indirect singularities of $f^{-1}$. Now, by Theorem 1 in [2], it follows that any indirect singularity of $f^{-1}$ must be a limit point of critical values of $f$. By hypothesis, all the critical values of $f$ has only one finite limit point $\tilde{w}$ and hence $\tilde{w} = w^*$ which is a contradiction. Hence, it is concluded that $\tilde{w}$ is the only finite asymptotic value of $f$. □
For \( n = 0, 1, 2, \ldots \), define
\[
J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} \left( \frac{z}{2} \right)^{2k+n}
\]
for \( z \in \mathbb{C} \).

The entire transcendental function \( J_n(z) \) is known as the Bessel function of first kind and order \( n \).

Let
\[
I_n(z) = \frac{J_n(iz)}{i^n}
\]
for \( z \in \mathbb{C} \).

Then, the function \( I_n(z) \) is called the modified Bessel function of first kind and order \( n \). The orders (growth) of \( J_n(z) \), \( I_n(z) \) and \( z^{-n}I_n(z) \) are computed for each non-negative integer \( n \) in the following proposition.

**Proposition 18.** For each non-negative integer \( n \), the orders (growth) of \( J_n(z) \), \( I_n(z) \) and \( z^{-n}I_n(z) \) are equal to one.

**Proof.** The Bessel functions \( J_n \) satisfies the recurrence relation \( J_{n+1}(z) = \frac{n}{z}J_n(z) - J'_n(z) \) for \( n = 0, 1, 2, \ldots \) \([4, p. 93]\). Since for any two entire functions \( g \) and \( h \), \( \rho(g \pm h) = \max\{\rho(g), \rho(h)\} \) provided \( g \neq h \) and the order (growth) of \( g \) is equal to the order (growth) of derivative of \( g \), the order (growth) of \( J_{n+1} \) is equal to the order (growth) of \( J_n \) for \( n = 0, 1, 2, \ldots \). Further, it is observed that the order (growth) of \( J_n(z) \), \( I_n(z) \) and \( z^{-n}I_{2n}(z) \) are equal for \( n = 0, 1, 2, \ldots \). Therefore, it is enough to show that the order of \( J_0 \) is 1.

The Taylor series of \( J_0(z) \) about the point \( z = 0 \) is given by \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{z}{2} \right)^{2k} \) for \( z \in \mathbb{C} \). Using the characterization of order of entire functions in terms of coefficients of Taylor series \([10]\) and Stirling formula \( k! = k^ke^{-k}\sqrt{2\pi k} \) for large enough \( k \), it is easy to see that the order of \( J_0 \) is 1. \( \square \)

5.1. **Example I:** \( B_n(z) = z^{-n}I_n(z), n \geq 0 \)

For \( n = 0, 1, 2, 3, \ldots \), let
\[
I_n(z) = \frac{J_n(iz)}{i^n}
\]
(3)

for \( z \in \mathbb{C} \). The following proposition locates all the singular values of \( z^{-n}I_n(z) \).

**Proposition 19.** For any non-negative integer \( n \), the function \( B_n(z) = z^{-n}I_n(z) \) has infinitely many singular values and all of them lie in \([-\frac{1}{2^n!}, \frac{1}{2^n!}]\).

**Proof.** Observe that \( B'_n(z) = \frac{dz}{dz} \left( \frac{J_n(iz)}{(iz)^n} \right) = \frac{-iJ_{n+1}(iz)}{(iz)^n} \). Since the set of all zeros of \( J_{n+1}(z) \) is an unbounded subset of \( \mathbb{R} \), the set of all critical points of \( B_n(z) \) (that is the zeros of \( J_{n+1}(iz) \)) is an unbounded subset of the imaginary axis. Let all the critical points whose imaginary part is positive be arranged in an increasing sequence, say \( \{z_k = ix_k\}_{k>0} \). Then \( B_n(z_k) = B_n(ix_k) = \frac{J_n(-x_k)}{(-x_k)^n} \neq 0 \) for each \( k \) because the roots of \( J_n \) and \( J_{n+1} \) interlace and \( J_{n+1}(-x_k) = \pm J_{n+1}(x_k) = 0 \) for each \( k \). As \( B_n(iy) = \frac{J_n(-y)}{(-y)^n} \to 0 \) as \( y \to +\infty \), \( \lim_{k \to \infty} B_n(z_k) = 0 \). Since \( B_n \) is an even function, \(-z_k\) is also a critical point of \( B_n \) for each \( k \) and \( \lim_{k \to \infty} B_n(-z_k) = 0 \). It shows that \( B_n(z_k) \) are distinct for infinitely many values of \( k \). Therefore \( B_n(z) \) has infinitely many critical
values. Since $B_n(iy) \rightarrow 0$ as $|y| \rightarrow \infty$, the point $z = 0$ is an asymptotic value of $B_n(z)$. It also follows that 0 is the only limit point of all the critical values of $B_n(z)$. By Proposition 17, the point $z = 0$ is the only finite asymptotic value of $B_n$.

If $z_k = ix_k$ is a critical point of $B_n(z)$ then
\[
\frac{n}{x_k^n} J_n(x_k) = \frac{J_{n+1}(x_k) + J'_n(x_k)}{x_k^{n-1}} = \frac{J'_n(x_k)}{x_k^{n-1}}
\]
by using the recurrence relation $\frac{n}{x} J_n(x) = J_{n+1}(x) + J'_n(x)$ [4, p. 93] and the fact that $J_{n+1}(x_k) = 0$. Again by recurrence relation $\frac{n}{x} J_n(x) = J_{n-1}(x) - J'_n(x)$, we get
\[
\frac{n}{x_k^n} J_n(x_k) = \frac{J_{n-1}(x_k) - \frac{n}{x_k} J_n(x_k)}{x_k^{n-1}} = \frac{J_{n-1}(x_k)}{x_k^{n-1}} - \frac{n}{x_k^n} J_n(x_k),
\]
which gives that
\[
\frac{J_n(x_k)}{x_k^n} = \frac{1}{2n} \frac{J_{n-1}(x_k)}{x_k^{n-1}}
\]
for each $n = 1, 2, 3, \ldots$, and hence
\[
\frac{J_n(x_k)}{x_k^n} = \frac{1}{2n(2n-1) \cdots 1} \frac{J_0(x_k)}{x_k^0} = \frac{1}{2^n n!} J_0(x_k).
\]
Since $|J_0(x)| \leq 1$ for $x \in \mathbb{R}$, $\left|\frac{J_n(x_k)}{x_k^n}\right| \leq \frac{1}{2^n n!}$. Thus, all the critical values lies in $[\frac{1}{2^n n!}, \frac{1}{2^n n!}]$.

Therefore all the singular values of $B_n(z)$ are in $[\frac{1}{2^n n!}, \frac{1}{2^n n!}]$. □

From Eq. (3), it is obvious that all the coefficients in the Taylor series of $B_n(z) = z^{-n} I_n(z)$ about origin are positive, $B_n(x) > 0$ for all $x < 0$ and $B_n(0) = \frac{1}{2^n n!}$. This fact along with Proposition 19 shows that $z^{-n} I_n(z) \in E_1$ for each non-negative integer $n$. For a fixed non-negative integer $n$, the dynamics of functions in the one parameter family $\{\lambda B_n(z) = \lambda z^{-n} I_n(z): \lambda > 0\}$ follows from Theorem 12. Thus, there is a critical parameter $\lambda^*_n$ such that $\mathcal{F}(\lambda B_n)$ is a nowhere dense subset of complex plane for $0 < \lambda \leq \lambda^*_n$ and is equal to entire complex plane for $\lambda > \lambda^*_n$. The chaotic burst in the dynamics of $\{\lambda B_n(z) = \lambda z^{-n} I_n(z): \lambda > 0\}$ occurs at the parameter value $\lambda = \lambda^*_n$. If $a_\lambda$ denotes the attracting fixed point of $\lambda B_n$ for $0 < \lambda < \lambda^*_n$ then $[0, a_\lambda]$ is in the immediate basin of attraction of $a_\lambda$ by Theorem 6. As $B_n(z)$ is an even function, it follows that $[-a_\lambda, 0]$ is in the immediate basin of attraction of $a_\lambda$. Since $B_\lambda$ is increasing in $\mathbb{R}^+$ and $0 < a_\lambda$, it follows that $B_\lambda(0) = \frac{1}{2^n n!} < a_\lambda$. By Proposition 19, all the singular values of $B_\lambda$ are in $[\frac{1}{2^n n!}, \frac{1}{2^n n!}]$ and hence, in the immediate basin of attraction of $a_\lambda$. By Theorem 16, $\mathcal{F}(\lambda B_n)$ is connected for $0 < \lambda < \lambda^*_n$.

For $n = 0$, the dynamics of functions in the family $\{\lambda I_0(z): \lambda > 0\}$ is studied in [12]. For $n = 1$, it is numerically found that the critical parameter $\lambda^*_1$ for the family $\{\lambda z^{-1} I_1(z): \lambda > 0\}$ is approximately equal to 2.529. The pictures of the Julia sets of $2.5z^{-1} I_1(z)$ and $2.5291z^{-1} I_1(z)$ are computationally generated using an algorithm (refer [12]) based on Corollary 13 and are given in Figs. 1 and 2.

5.2. Example II: $I_{2n}(z)$, $n > 0$

For $n = 1, 2, 3, \ldots$, $I_{2n}(z) = \sum_{k=0}^{\infty} \frac{1}{k!(k+2n)!} (\xi)^{2k+2n}$ for $z \in \mathbb{C}$. All the coefficients in the Taylor series of $I_{2n}$ about origin are positive. Further, $I_{2n}(x) > 0$ for all $x < 0$ and, $I_{2n}(0) = 0$. 


1457
Note that, \( I'_{2n}(z) = i^{2n+1} J'_{2n}(iz) \). Observe that the order of \( J_{2n}(z) \) is one. The function \( J_{2n}(z) \) is real for real \( z \) and it has only real zeros. By Laguerre theorem, all the roots of \( J'_{2n}(z) \) are real and separated from each other by zeros of \( J_{2n}(z) \). Thus, all the critical points of \( I_{2n}(z) \) are purely imaginary and are separated by the zeros of \( I_{2n}(z) \). Observe that, the number of critical points is infinite and they form an unbounded subset of the imaginary axis. Further, 0 is not a critical value and \( I_{2n}(z) \) → 0 when \( z \rightarrow \infty \) along positive (or negative) imaginary axis. Therefore, 0 is an asymptotic value. By the same argument used in Proposition 19, it follows that 0 is the only limit point of all critical values of \( I_{2n}(z) \). By Proposition 17, the point \( z = 0 \) becomes the only finite asymptotic value of \( I_{2n}(z) \). Thus, \( I_{2n}(z) \in E_0 \) for \( n > 0 \) and the Fatou set \( \mathcal{F}(\lambda I_{2n}(z)) \) is the basin of attraction of super attracting fixed point 0 for all \( \lambda > 0 \) by Theorem 10.

5.3. Example III: \( S_{m,n}(z) = \frac{\sinh^m z}{z^n}, \ m \geq n > 0 \)

The Taylor series of \( \sinh z \) about origin is given by \( \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!} \). For \( m, n \in \mathbb{N} \) and \( m \geq n \), all the coefficients of Taylor series of \( S_{m,n}(z) = \frac{\sinh^m z}{z^n} \) are positive. Let both \( m \) and \( n \) be even or both of them be odd. Then \( S_{m,n}(x) > 0 \) for \( x < 0 \).

Note that \( S'_{m,n}(z) = \frac{\sinh^{m-1} z (mz \cosh z - n \sinh z)}{z^{n+1}} \). Since all the solutions of \( \alpha \tan z = z \) are real for \( 0 < \alpha \leq 1 \), it follows that \( m \tanh z = z \) has only purely imaginary solutions when \( \frac{m}{n} \leq 1 \) and these solutions form an unbounded set. Further, the solutions of \( \frac{\sinh^{m-1} z}{z^n} = 0 \) are purely imaginary. Therefore, all the critical points of \( S_{m,n}(z) \) are purely imaginary. Note that, \( S_{m,n} \) takes the imaginary axis to a bounded interval in the real axis. So all the critical values of \( S_{m,n} \) are in a bounded interval of the real axis. The function \( S_{m,n}(z) \) tends to 0 as \( z \rightarrow \infty \) along either positive or negative imaginary axis. So the point \( z = 0 \) is an asymptotic value of \( S_{m,n}(z) \) and, same argument used in Proposition 19 gives that it is the only limit point of all the critical values of \( S_{m,n} \). By Proposition 17, the point \( z = 0 \) is the only finite asymptotic value of \( S_{m,n} \). Note that \( S_{m,n}(0) = 1 \) for \( n = m \) and \( S_{m,n}(0) = 0 \) for \( m > n \). Thus, \( \frac{\sinh^m z}{z^n} \in E_0 \) for \( m > n \) and \( \frac{\sinh^m z}{z^n} \in E_1 \) for \( m = n \) when \( m, n \in \mathbb{N}, m \geq n \) and both \( m \) and \( n \) are even or both of them are odd. The dynamics of the functions in \( \{ \lambda S_{m,n}: \lambda > 0 \} \) follows from Theorem 10 when \( m > n \) and from Theorem 12 when \( m = n \). Thus, there is a critical parameter \( \lambda_m^* \) such that \( \mathcal{J}(\lambda S_{m,m}) \) is a no where dense subset of \( \hat{\mathbb{C}} \) for \( 0 < \lambda \leq \lambda_m^* \) and is equal to \( \hat{\mathbb{C}} \) for \( \lambda > \lambda_m^* \).
The case \( m = n = 1 \) is studied in [9]. For \( m = n = 2 \), the critical parameter \( \lambda_2^* \) is found to be approximately equal to 0.7618. The computationally generated pictures of the Julia sets of \( 0.75 \sinh^2 \frac{z}{z^2} \) and \( 0.7619 \sinh^2 \frac{z}{z^2} \) are given in Figs. 3 and 4.

References