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Boundedness of cohomology

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ABSTRACT

Let $d \in \mathbb{N}$ and let $\mathcal{D}^d$ denote the class of all pairs (R, M) in which $R = \bigoplus_{n \in \mathbb{N}_0} R_n$ is a Noetherian homogeneous ring with Artinian base ring R_0 and such that M is a finitely generated graded R-module of dimension $\leqslant d$. For such a pair (R, M) let $d_M^i(n)$ denote the (finite) R_0-length of the n-th graded component of the i-th R_+-transform module $D_{R+}^i(M)$.

The cohomology table of a pair $(R, M) \in \mathcal{D}^d$ is defined as the family of non-negative integers $d_M := (d_M^i(n))_{(i,n) \in \mathbb{N} \times \mathbb{Z}}$. We say that a subclass $\mathcal{C}$ of $\mathcal{D}^d$ is of finite cohomology if the set $\{d_M \mid (R, M) \in \mathcal{C}\}$ is finite. A set $\mathbb{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$ is said to bound cohomology, if for each family $(h^\sigma)_{\sigma \in \mathbb{S}}$ of non-negative integers, the class $\{(R, M) \in \mathcal{D}^d \mid d_M^i(n) \leqslant h^{(i,n)}$ for all $(i, n) \in \mathbb{S}\}$ is of finite cohomology. Our main result says that this is the case if and only if $\mathbb{S}$ contains a quasi diagonal, that is a set of the form $\{(i, n_i) \mid i = 0, \ldots, d-1\}$ with integers $n_0 > n_1 > \cdots > n_{d-1}$.

We draw a number of conclusions of this boundedness criterion.

1. Introduction

This paper continues our investigation [6], which was driven by the question *“What bounds cohomology of a projective scheme?”*

A considerable number of contributions has been given to this theme, mainly under the aspect of bounding some cohomological invariants in terms of other invariants (see [1–4,7–9,11–13,15–19,21, 22] for example).

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Our aim is to start from a different point of view, focusing on the notion of cohomological pattern (see [5]). So, our main result characterizes those sets $\mathbb{S} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$ "which bound cohomology of projective schemes of dimension $< d$".

To make this precise, fix a positive integer d and let $\mathcal{D}^d$ be the class of all pairs (R, M) in which $R = \bigoplus_{n \geqslant 0} R_n$ is a Noetherian homogeneous ring with Artinian base ring R_0 and M is a finitely generated graded R-module with $\dim(M) \leqslant d$. In this situation let $R_+ = \bigoplus_{n>0} R_n$ denote the irrelevant ideal of R.

For each $i \in \mathbb{N}_0$ consider the graded R-module $D^i_{R_+}(M)$, where $D^i_{R_+}$ denotes the i-th right derived functor of the R_+-transform functor $D_{R_+}(\bullet) := \varinjlim_n \operatorname{Hom}_R((R_+)^n, \bullet)$. In addition, for each $n \in \mathbb{Z}$ let $d^i_M(n)$ denote the (finite) R_0-length of the n-th graded component $D^i_{R_+}(M)_n$ of $D^i_{R_+}(M)$.

Finally, for $(R, M) \in \mathcal{D}^d$ let us consider the so-called cohomology table of (R, M), that is the family of non-negative integers

$$d_M := \left(d^i_M(n)\right)_{(i,n) \in \mathbb{N}_0 \times \mathbb{Z}}.$$

A subclass $\mathcal{C} \subseteq \mathcal{D}^d$ is said to be of finite cohomology if the set $\{d_M \mid (R, M) \in \mathcal{C}\}$ is finite. The class $\mathcal{C}$ is said to be of bounded cohomology if the set $\{d^i_M(n) \mid (R, M) \in \mathcal{C}\}$ is finite for all pairs $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$. It turns out that these two conditions are both equivalent to the condition that the class $\mathcal{C}$ is of finite cohomology "along some diagonal", e.g. there is some $n_0 \in \mathbb{Z}$ such that the set $\triangle_{\mathcal{C},n_0} := \{d^i_M(n_0 - i) \mid (R, M) \in \mathcal{C},\ 0 \leqslant i < d\}$ is finite (see Theorem 3.5).

So, if one bounds the values of $d^i_M(n)$ along a "diagonal subset"

$$\left\{(j, n_0 - j) \mid j = 0, \dots, d-1\right\} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$$

for an arbitrary integer n_0 one cuts out a subclass $\mathcal{C} \subseteq \mathcal{D}^d$ of finite cohomology. Motivated by this observation we say that the subset $\mathbb{S} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$ bounds cohomology in the class $\mathcal{C} \subseteq \mathcal{D}^d$ if for each family $(h^\sigma)_{\sigma \in \mathbb{S}}$ of non-negative integers $h^\sigma \in \mathbb{N}_0$ the class

$$\left\{(R, M) \in \mathcal{C} \;\middle|\; \forall (i, n) \in \mathbb{S}\colon\ d^i_M(n) \leqslant h^{(i,n)}\right\}$$

is of finite cohomology. Now, we may reformulate our previous result by saying that for arbitrary n_0 the diagonal set $\{(j, n_0 - j) \mid j = 0, \dots, d-1\}$ bounds cohomology in $\mathcal{D}^d$. It seems rather natural to ask, whether one can characterize the shape of those subsets $\mathbb{S} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$ which bound cohomology in $\mathcal{D}^d$. This is indeed done by our main result (see Corollary 4.10):

A subset $\mathbb{S} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$ bounds cohomology in $\mathcal{D}^d$ if and only if it contains a quasi-diagonal, that is a set of the form $\{(i, n_i) \mid i = 0, \dots, d-1\}$ with

$$n_0 > n_1 > \dots > n_{d-1}.$$

Our next aim is to apply our main result in order to cut out classes $\mathcal{C} \subseteq \mathcal{D}^d$ of finite cohomology by fixing some numerical invariants which are defined on the class $\mathcal{C}$. A finite family $(\mu_i)_{i=1}^r$ of numerical invariants μ_i on $\mathcal{C}$ is said to bound cohomology in $\mathcal{C}$ if for all $n_1, \dots, n_r \in \mathbb{Z} \cup \{\pm\infty\}$ the class $\{(R, M) \in \mathcal{C} \mid \mu_i(M) = n_i \text{ for } i = 1, \dots, r\}$ is of finite cohomology.

We define a numerical invariant $\varrho : \mathcal{D}^d \to \mathbb{N}_0$ by setting $\varrho(M) := d^0_M(\operatorname{reg}^2(M))$, where $\operatorname{reg}^2(M)$ denotes the Castelnuovo–Mumford regularity of M at and above level 2. Then, we show (see Theorem 5.8):

The pair of invariants $(\operatorname{reg}^2, \varrho)$ bounds cohomology in $\mathcal{D}^d$.

As an application of this we prove (see Theorem 5.9 and Corollary 5.10)

Fix a polynomial $p \in \mathbb{Q}[t]$ and an integer r. Let $\mathcal{C} \subseteq \mathcal{D}^d$ be the class of all pairs (R, M) such that M is a graded submodule of a finitely generated graded R-module N with Hilbert polynomial $p_N = p$ and $\operatorname{reg}^2(N) \leqslant r$. Then reg^2 bounds cohomology in $\mathcal{C}$.

An immediate consequence of this is (see Corollary 5.11):

Let $(R, N) \in \mathcal{D}^d$, let $r \in \mathbb{Z}$ and let M run through all graded submodules $M \subseteq N$ with $\operatorname{reg}^2(M) \leqslant r$. Then only finitely many cohomology tables d_M occur.

As applications of this, we generalize two finiteness results of Hoa and Hyry [17] for local cohomology modules of graded ideals in a polynomial ring over a field to graded submodules $M \subseteq N$ for a given pair $(R, N) \in \mathcal{D}^d$ (see Corollaries 5.13 and 5.14).

In order to translate our results to sheaf cohomology of projective schemes observe that for all $(i, n) \in \mathbb{N}_0 \times \mathbb{Z}$ and all pairs $(R, M) \in \mathcal{D}^d$ we have $H^i(X, \mathcal{F}(n)) \cong D^i_{R_+}(M)_n$, where $X := \operatorname{Proj}(R)$ and $\mathcal{F} := \widetilde{M}$ is the coherent sheaf of $\mathcal{O}_X$-modules induced by M (see [10, Chapter 20] for example).

2. Preliminaries

In this section we recall a few basic facts which shall be used later in our paper.

Notation 2.1. Let $R = \bigoplus_{n \geqslant 0} R_n$ be a homogeneous Noetherian ring, so that R is positively graded, R_0 is Noetherian and $R = R_0[l_0, \ldots, l_r]$ with finitely many elements $l_0, \ldots, l_r \in R_1$. Let R_+ denote the irrelevant ideal $\bigoplus_{n>0} R_n$ of R.

Reminder 2.2 *(Local cohomology and Castelnuovo–Mumford regularity).* (A) Let $i \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. By $H^i_{R_+}(\bullet)$ we denote the i-th local cohomology functor with respect to R_+. Moreover by $D^i_{R_+}(\bullet)$ we denote the i-th right derived functor of the ideal transform functor $D_{R_+}(\bullet) = \varinjlim_n \operatorname{Hom}_R((R_+)^n, \bullet)$ with respect to R_+.

(B) Let $M := \bigoplus_{n \in \mathbb{Z}} M_n$ be a graded R-module. Keep in mind that in this situation the R-modules $H^i_{R_+}(M)$ and $D^i_{R_+}(M)$ carry natural gradings. Moreover we then have a natural exact sequence of graded R-modules

(i) $0 \to H^0_{R_+}(M) \to M \to D^0_{R_+}(M) \to H^1_{R_+}(M) \to 0$

and natural isomorphisms of graded R-modules

(ii) $D^i_{R_+}(M) \cong H^{i+1}_{R_+}(M)$ for all $i > 0$.

(C) If T is a graded R-module and $n \in \mathbb{Z}$, we use T_n to denote the n-th graded component of T. In particular, we define the *beginning* and the *end* of T respectively by

(i) $\operatorname{beg}(T) := \inf\{n \in \mathbb{Z} \mid T_n \neq 0\}$,
(ii) $\operatorname{end}(T) := \sup\{n \in \mathbb{Z} \mid T_n \neq 0\}$,

with the standard convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

(D) If the graded R-module M is finitely generated, the R_0-modules $H^i_{R_+}(M)_n$ are all finitely generated and vanish as well for all $n \gg 0$ as for all $i > \dim(M)$. So, we have

$$-\infty \leqslant a_i(M) := \operatorname{end}\big(H^i_{R_+}(M)\big) < \infty \quad \text{for all } i \geqslant 0,$$

with $a_i(M) := -\infty$ for all $i > \dim(M)$.

If $k \in \mathbb{N}_0$, the *Castelnuovo–Mumford regularity of M at and above level k* is defined by

$$\operatorname{reg}^k(M) := \sup\{a_i(M) + i \mid i \geqslant k\} \quad (< \infty).$$

The *Castelnuovo–Mumford regularity of M* is defined by $\operatorname{reg}(M) := \operatorname{reg}^0(M)$.

(E) We also shall use the *generating degree* of M, which is defined by

$$\operatorname{gendeg}(M) = \inf\Big\{n \in \mathbb{Z} \;\Big|\; M = \sum_{m \leqslant n} RM_m\Big\}.$$

If the graded R-module M is finitely generated, we have $\operatorname{gendeg}(M) \leqslant \operatorname{reg}(M)$.

Reminder 2.3 *(Cohomological Hilbert functions).* (A) Let $i \in \mathbb{N}_0$ and assume that the base ring R_0 is Artinian. Let M be a finitely generated graded R-module. Then, the graded R-modules $H^i_{R_+}(M)$ are Artinian. In particular for all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$ we may define the non-negative integers

(i) $h^i_M(n) := \operatorname{length}_{R_0}(H^i_{R_+}(M)_n)$,
(ii) $d^i_M(n) := \operatorname{length}_{R_0}(D^i_{R_+}(M)_n)$.

Fix $i \in \mathbb{N}_0$. Then the functions

(iii) $h^i_M : \mathbb{Z} \to \mathbb{N}_0$, $n \mapsto h^i_M(n)$,
(iv) $d^i_M : \mathbb{Z} \to \mathbb{N}_0$, $n \mapsto d^i_M(n)$

are called the i-th *Cohomological Hilbert functions* of the *first* respectively the *second kind* of M.

(B) Let M be a finitely generated graded R-module and let $x \in R_1$. We also write $\Gamma_{R_+}(M)$ for the R_+-torsion submodule of M which we identify with $H^0_{R_+}(M)$. By $\operatorname{NZD}_R(M)$ and $\operatorname{ZD}_R(M)$ we respectively denote the set of non-zerodivisors or of zero divisors of R with respect to M. The linear form $x \in R_1$ is said to be $(R_+$-*)filter regular with respect to M if* $x \in \operatorname{NZD}_R(M/\Gamma_{R_+}(M))$.

Reminder 2.4. (Cf. [6, Definition 5.2].) For $d \in \mathbb{N}$ let $\mathcal{D}^d$ denote the class of all pairs (R, M) in which $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian homogeneous ring with Artinian base ring R_0 and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a finitely generated graded R-module with $\dim(M) \leqslant d$.

3. Finiteness and boundedness of cohomology

We keep the notations and hypotheses introduced in Section 2.

Definition 3.1. The *cohomology table* of the pair $(R, M) \in \mathcal{D}^d$ is the family of non-negative integers

$$d_M := \big(d^i_M(n)\big)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}.$$

Reminder 3.2. (A) According to [5] the *cohomological pattern* $\mathcal{P}_M$ of the pair $(R, M) \in \mathcal{D}^d$ is defined as the set of places at which the cohomology table of (R, M) has a non-zero entry:

$$\mathcal{P}_M := \{(i, n) \in \mathbb{N}_0 \times \mathbb{Z} \mid d^i_M(n) \neq 0\}.$$

(B) A set $P \subseteq \mathbb{N}_0 \times \mathbb{Z}$ is called a *tame combinatorial pattern of width* $w \in \mathbb{N}_0$ if the following conditions are satisfied:

(π_1) $\exists m, n \in \mathbb{Z}$: $(0, m), (w, n) \in P$;
(π_2) $(i, n) \in P \Rightarrow i \leqslant w$;
(π_3) $(i, n) \in P \Rightarrow \exists j \leqslant i$: $(j, n + i - j + 1) \in P$;
(π_4) $(i, n) \in P \Rightarrow \exists k \geqslant i$: $(k, n + i - k - 1) \in P$;
(π_5) $i > 0 \Rightarrow \forall n \gg 0$: $(i, n) \notin P$;
(π_6) $\forall i \in \mathbb{N}$: ($\forall n \ll 0$: $(i, n) \in P$) *or else* ($\forall n \ll 0$: $(i, n) \notin P$).

By [5] we know:

(a) If $(R, M) \in \mathcal{D}^d$ with $\dim(M) = s > 0$, then $\mathcal{P}_M$ is a tame combinatorial pattern of width $w = s - 1$.
(b) If P is a tame combinatorial pattern of width $w \leqslant d - 1$, then there is a pair $(R, M) \in \mathcal{D}^d$ such that the base ring R_0 is a field and $P = \mathcal{P}_M$.

By the previous observation, the set of patterns $\{\mathcal{P}_M \mid (R, M) \in \mathcal{D}^d\}$ is quite large, and hence so is the set of cohomology tables $\{d_M \mid (R, M) \in \mathcal{D}^d\}$. Therefore, one seeks for decompositions $\bigcup_{i \in \mathbb{I}} \mathcal{C}_i = \mathcal{D}^d$ of $\mathcal{D}^d$ into "simpler" subclasses $\mathcal{C}_i$ such that for each $i \in \mathbb{I}$ the set $\{d_M \mid (R, M) \in \mathcal{C}_i\}$ is finite. Bearing in mind this goal, we define the following concepts:

Definitions 3.3. (A) Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a subclass. We say that $\mathcal{C}$ *is a subclass of finite cohomology* if

$$\sharp\{d_M \mid (R, M) \in \mathcal{C}\} < \infty.$$

(B) We say that $\mathcal{C} \subseteq \mathcal{D}^d$ is a *subclass of bounded cohomology* if

$$\forall (i, n) \in \mathbb{N}_0 \times \mathbb{Z}: \quad \sharp\{d_M^i(n) \mid (R, M) \in \mathcal{C}\} < \infty.$$

Remark 3.4. (A) Let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{D}^d$ be subclasses of $\mathcal{D}^d$. Then clearly

(a) If $\mathcal{C} \subseteq \mathcal{D}$ and $\mathcal{D}$ is of finite cohomology or of bounded cohomology, then so is $\mathcal{C}$ respectively.

(B) If $r \in \mathbb{Z}$, we have a bijection

$$\{d_M \mid (R, M) \in \mathcal{C}\} \twoheadrightarrow \{d_{M(r)} \mid (R, M) \in \mathcal{C}\} \quad \text{given by } d_M \mapsto d_{M(r)}.$$

Now, we show how the finiteness and boundedness conditions defined above are related.

Theorem 3.5. *For a subclass $\mathcal{C} \subseteq \mathcal{D}^d$ the following statements are equivalent*:

(i) *$\mathcal{C}$ is a class of finite cohomology.*
(ii) *$\mathcal{C}$ is a class of bounded cohomology.*
(iii) *For each $n_0 \in \mathbb{Z}$ the set $\triangle_{\mathcal{C}, n_0} := \{d_M^i(n_0 - i) \mid (R, M) \in \mathcal{C},\ 0 \leqslant i < d\}$ is finite.*
(iv) *There is some $n_0 \in \mathbb{Z}$ such that the set $\triangle_{\mathcal{C}, n_0}$ of statement* (iii) *is finite.*

Proof. The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear from the definitions. To prove the implication (iv) $\Rightarrow$ (i) fix $n_0 \in \mathbb{Z}$ and assume that the set $\triangle_{\mathcal{C}, n_0}$ is finite. Then there is some non-negative integer h such that $d^i_{M(n_0)}(-i) \leqslant h$ for all pairs $(R, M) \in \mathcal{C}$ and all $i \in \{0, \ldots, d-1\}$. By [6, Theorem 5.4] it thus follows that the set of functions

$$\{d^i_{M(n_0)} \mid (R, M) \in \mathcal{C},\ i \in \mathbb{N}_0\}$$

is finite. By Remark 3.4(B) we now may conclude that the class $\mathcal{C}$ is of finite cohomology. $\square$

So, by Theorem 3.5 boundedness and finiteness of cohomology are the same for a given class $\mathcal{C} \subseteq \mathcal{D}^d$.

Definition 3.6. Let $d \in \mathbb{N}_0$, let $\mathcal{C} \subseteq \mathcal{D}^d$ and let $\mathbb{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$ be a subset. We say that the set $\mathbb{S}$ *bounds cohomology in* $\mathcal{C}$ if for each family $(h^\sigma)_{\sigma\in\mathbb{S}}$ of non-negative integers h^σ the class

$$\big\{(R, M) \in \mathcal{C} \,\big|\, \forall (i, n) \in \mathbb{S}\colon\ d^i_M(n) \leqslant h^{(i,n)}\big\}$$

is of finite cohomology.

Remark 3.7. (A) Let $d \in \mathbb{N}_0$, let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{D}^d$ and $\mathbb{S}, \mathbb{T} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$. Then obviously we can say

If $\mathbb{S} \subseteq \mathbb{T}$ and $\mathbb{S}$ bounds cohomology in $\mathcal{C}$, then so does $\mathbb{T}$.

(B) If $r \in \mathbb{Z}$, we can form the set $\mathbb{S}(r) := \{(i, n + r) \mid (i, n) \in \mathbb{S}\}$. In view of the bijection of Remark 3.4(B) we have

$\mathbb{S}(r)$ bounds cohomology in $\mathcal{C}(r) := \{(R, M(r)) \mid (R, M) \in \mathcal{C}\}$ if and only if $\mathbb{S}$ does in $\mathcal{C}$.

(C) For all $s \in \{0, \ldots, d\}$ we set

$$\mathbb{S}^{<s} := \mathbb{S} \cap \big(\{0, \ldots, s-1\} \times \mathbb{Z}\big)$$

as $\mathcal{D}^s \subseteq \mathcal{D}^d$ it follows easily:

If $\mathbb{S}$ bounds cohomology in $\mathcal{C}$, then $\mathbb{S}^{<s}$ bounds cohomology in $\mathcal{D}^s \cap \mathcal{C}$.

Corollary 3.8. *Let $\mathcal{C} \subseteq \mathcal{D}^d$ and $n \in \mathbb{Z}$. Then, the "n-th diagonal"*

$$\big\{(i, n-i) \,\big|\, i = 0, \ldots, d-1\big\}$$

bounds cohomology in $\mathcal{C}$.

Proof. This is immediate by Theorem 3.5. □

4. Quasi-diagonals

Our first aim is to generalize Corollary 3.8 by showing that not only the diagonals bound cohomology on $\mathcal{C}$, but rather all "quasi-diagonals". We shall define below, what such a quasi-diagonal is.

Lemma 4.1. *Let $t \in \{1, \ldots, d\}$, let $(n_i)_{i=d-t}^{d-1}$ be a sequence of integers such that $n_{d-1} < \cdots < n_{d-t}$ and let $\mathcal{C} \subseteq \mathcal{D}^d$ be a class such that the set $\{d^i_M(n_i) \mid (R, M) \in \mathcal{C}\}$ is finite for all $i \in \{d-t, \ldots, d-1\}$. Then the set $\{d^i_M(n) \mid (R, M) \in \mathcal{C}\}$ is finite whenever $n_i \leqslant n$ and $d - t \leqslant i \leqslant d-1$.*

Proof. By our hypothesis there is some $h \in \mathbb{N}_0$ with $d^i_M(n_i) \leqslant h$ for all $i \in \{d-t, \ldots, d-1\}$ and all pairs $(R, M) \in \mathcal{C}$.

Let $(R, M) \in \mathcal{C}$. On use of standard reduction arguments we can restrict ourselves to the case where the Artinian base ring R_0 is local with infinite residue field. Replacing M by $M/\Gamma_{R_+}(M)$ we may assume that M is R_+-torsion free. Therefore, there exists $x \in R_1 \cap \mathrm{NZD}(M)$. For each $i \in \mathbb{N}_0$ and $m \in \mathbb{Z}$, the short exact sequence $0 \to M(-1) \to M \to M/xM \to 0$ induces a long exact sequence.

$$(*_{i,m}) \qquad D^i_{R_+}(M)_{m-1} \to D^i_{R_+}(M)_m \to D^i_{R_+}(M/xM)_m \to D^{i+1}_{R_+}(M)_{m-1}.$$

As $\dim(M/xM) < d$, the sequences $(*_{d-1,m})$ imply that $d_M^{d-1}(m) \leqslant d_M^{d-1}(m-1)$ for all $m \in \mathbb{Z}$. This proves our claim if $t = 1$. So, let $t > 1$.

Assume inductively that the set $\{d_M^i(n_i) \mid (R, M) \in \mathcal{C}\}$ is finite whenever $n_i \leqslant n$ and $d - t + 1 \leqslant i \leqslant d - 1$. It remains to find a family of non-negative integers $(h_n)_{n \geqslant n_{d-t}}$ such that $d_M^{d-t}(n) \leqslant h_n$ for all $n \geqslant n_{d-t}$ and all pairs $(R, M) \in \mathcal{C}$. Let $\mathcal{E}$ denote the class of all pairs $(R, M/xM) = (R, \overline{M})$ in which $(R, M) \in \mathcal{C}$ and $x \in R_1 \cap \mathrm{NZD}(M)$. As $n_i - 1 \geqslant n_{i+1}$ for all $i \in \{d-t, \ldots, d-2\}$, the sequences $(*_{i,n_i})$ show that

$$d_{M/xM}^i(n_i) \leqslant d_M^{i+1}(n_i - 1) + h \quad \text{for } i \in \{d-t, \ldots, d-2\}.$$

This means that the set $\{d_{\overline{M}}^i(n_i) \mid (R, \overline{M}) \in \mathcal{E}\}$ is finite whenever $(d-1) - (t-1) \leqslant i \leqslant d-2$. So, by induction the set $\{d_{\overline{M}}^i(n_i) \mid (R, \overline{M}) \in \mathcal{E}\}$ is finite whenever $n_i \leqslant n$ and $(d-1) - (t-1) \leqslant i \leqslant d-2$.

In particular there is a family of non-negative integers $(k_m)_{m \geqslant n_{d-t}}$ such that $d_{M/xM}^{d-t}(m) \leqslant k_m$ for all $m \geqslant n_{d-t}$ and all (R, M) and x as above. Now, for each $n \geqslant n_{d-t}$ set $h_n := h + \sum_{n_{d-t} < m \leqslant n} k_m$. If we choose $(R, M) \in \mathcal{C}$, the sequences $(*_{d-t,n})$ imply that $d_M^{d-t}(n) \leqslant h_n$ for all $n \geqslant n_{d-t}$. □

Proposition 4.2. *Let $(n_i)_{i=0}^{d-1}$ be a sequence of integers such that $n_{d-1} < \cdots < n_0$ and let $\mathcal{C} \subseteq \mathcal{D}^d$. Then the set $\{(i, n_i) \mid i = 0, \ldots, d-1\}$ bounds cohomology in $\mathcal{C}$.*

Proof. Let $(h^i)_{i=0}^{d-1}$ be a family of non-negative integers and let $\mathcal{C}'$ be the class of all pairs $(R, M) \in \mathcal{C}$ such that $d_M^i(n_i) \leqslant h^i$ for $i = 0, \ldots, d-1$. Then, by Lemma 4.1 the set $\{d_M^i(n) \mid (R, M) \in \mathcal{C}'\}$ is finite, whenever $n \geqslant n_i$ and $0 \leqslant i \leqslant d-1$. Therefore the set $\triangle_{\mathcal{C}', n_0} := \{d_M^i(n_0 - i) \mid (R, M) \in \mathcal{C}',\ 0 \leqslant i < d\}$ is finite. So, by Theorem 3.5 the class $\mathcal{C}'$ is of finite cohomology. It follows that $\{(i, n_i) \mid i = 0, \ldots, d-1\}$ bounds cohomology in $\mathcal{C}$. □

Definition 4.3. A set $\mathbb{T} \subseteq \{0, 1, \ldots, d-1\} \times \mathbb{Z}$ is called a *quasi-diagonal* if there is a sequence of integers $(n_i)_{i=0}^{d-1}$ such that $n_{d-1} < n_{d-2} < \cdots < n_0$ and

$$\mathbb{T} = \big\{(i, n_i) \bigm| i = 0, \ldots, d-1\big\}.$$

Observe, that diagonals in $\{0, \ldots, d-1\} \times \mathbb{Z}$ are quasi-diagonals. So, the next result generalizes Corollary 3.8.

Corollary 4.4. *Let $\mathbb{S} \subseteq \{0, 1, \ldots, d-1\} \times \mathbb{Z}$ be a set which contains a quasi-diagonal. Then $\mathbb{S}$ bounds cohomology in each subclass $\mathcal{C} \subseteq \mathcal{D}^d$.*

Proof. Clear by Proposition 4.2. □

Our next goal is to show that the converse of Corollary 4.4 holds, namely: if a set $\mathbb{S} \subseteq \{0, 1, \ldots, d-1\} \times \mathbb{Z}$ bounds cohomology in $\mathcal{D}^d$, then $\mathbb{S}$ contains a quasi-diagonal.

Reminder 4.5. Let K be a field, let $R = K \oplus R_1 \oplus \cdots$ and $R' = K \oplus R'_1 \oplus \cdots$ be two Noetherian homogeneous K-algebras. Let $R \boxtimes_K R' := K \oplus (R_1 \otimes R'_1) \oplus (R_2 \otimes R'_2) \oplus \cdots \subseteq R \otimes_K R'$ be the *Segre product ring of R and R'*, a Noetherian homogeneous K-algebra. For a graded R-module $M = \bigoplus_{n \in \mathbb{Z}} M_n$ and a graded R'-module $M' = \bigoplus_{n \in \mathbb{Z}} M'_n$ let $M \boxtimes_K M' := \bigoplus_{n \in \mathbb{Z}} M_n \otimes_K M'_n \subseteq M \otimes_K M'$ the Segre product module of M and M', a graded $R \boxtimes_K R'$-module. Keep in mind, that the *Künneth relations* (for Segre products) yield isomorphism of graded $R \boxtimes_K R'$-modules

$$D^i_{(R \boxtimes_K R')_+}(M \boxtimes_K M') \cong \bigoplus_{j=0}^{i} D^j_{R_+}(M) \boxtimes_K D^{i-j}_{R'_+}(M')$$

for all $i \in \mathbb{N}_0$ (cf. [23,14,20]).

Lemma 4.6. *Let $d > 1$ and let $R := K[x_1, \dots, x_d]$ be a polynomial ring over some infinite field K. Let $\mathbb{S} \subseteq \{0, 1, \dots, d-1\} \times \mathbb{Z}$ such that*

(1) *$\mathbb{S}$ contains no quasi-diagonal,*
(2) *$\mathbb{S} \cap (\{0, \dots, d-2\} \times \mathbb{Z})$ contains a quasi-diagonal $\{(i, n_i) \mid i = 0, \dots, d-2\}$ and*
(3) *$\mathbb{S} \cap (\{d-1\} \times \mathbb{Z}) \neq \emptyset$.*

Then

(a) *$(d-1, n) \notin \mathbb{S}$ for all $n \ll 0$.*
(b) *There is a family $(M_k)_{k\in\mathbb{N}}$ of finitely generated graded R-modules, locally free of rank $\leqslant ((d-1)!)^2$ on $\operatorname{Proj}(R)$ such that the set $\{d^i_{M_k}(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in \mathbb{S}$ and*

$$\lim_{k\to\infty} d^{d-1}_{M_k}(r) = \infty, \quad \text{where } r := \inf\bigl\{n \in \mathbb{Z} \bigm| (d-1, n) \in \mathbb{S}\bigr\} - 1.$$

Proof. For all $i \in \{1, \dots, d\}$ we write $R^i := K[x_1, \dots, x_i]$ and $\mathbb{S}^i := \mathbb{S} \cap (\{i\} \times \mathbb{Z})$. Statement (a) follows immediately from our hypotheses on the set $\mathbb{S}$. So, it remains to prove statement (b). After shifting appropriately we may assume that $r = -1$.

By our hypotheses on $\mathbb{S}$ it is clear that $\mathbb{S}^i \neq \emptyset$ for all $i \in \{0, \dots, d-1\}$. Let

$$\alpha_i := \sup\bigl\{n \in \mathbb{Z} \bigm| (i, n) \in \mathbb{S}^i\bigr\} \quad \text{for all } i \in \{0, \dots, d-1\}.$$

Then by our hypothesis on $\mathbb{S}$ we have $\alpha_i < \infty$ for some $i \in \{1, \dots, d-2\}$. Let

$$s := \min\bigl\{i \in \{0, \dots, d-2\} \bigm| \alpha_i < \infty\bigr\}$$

and

$$n_s := \alpha_s = \max\bigl\{n \in \mathbb{Z} \bigm| (s, n) \in \mathbb{S}^s\bigr\}.$$

Now, we may find a quasi-diagonal $\{(i, n_i) \mid i = 0, \dots, d-2\}$ in $\mathbb{S} \cap (\{0, \dots, d-2\} \times \mathbb{Z})$ such that for all $i \in \{s+1, \dots, d-2\}$ we have

$$n_i = \max\bigl\{n < n_{i-1} \bigm| (i, n) \in \mathbb{S}\bigr\}.$$

As $\mathbb{S}$ contains no quasi-diagonal, we must have $n_{d-2} \leqslant 0$. For all $m, n \in \mathbb{Z} \cup \{\pm\infty\}$ we write $]m, n[:= \{t \in \mathbb{Z} \mid m < t < n\}$. Using this notation we set

$$t_{-1} := \infty; \qquad t_{d-s-1} := -\infty; \qquad t_i := \max\{d-s-i-2, n_{i+s}\}, \quad \forall i \in \{0, \dots, d-s-2\},$$

and write

$$P := \bigcup_{i=0}^{d-s-1} \bigl(\{i\} \times]t_i, t_{i-1}[\bigr).$$

Observe, that by our choice of the pairs (i, n_i) we have

$$(*) \qquad \text{if } s \leqslant i \leqslant d-1 \text{ and } (i, n) \in \mathbb{S}, \quad \text{then } (i-s, n) \notin P.$$

Moreover by [5, 2.7] the set $P \subseteq \{0, \dots, d-s-1\} \times \mathbb{Z}$ is a minimal combinatorial pattern of width $d-s-1$. So, by [5, Proposition 4.5], there exists a finitely generated R^{d-s}-module N, locally free of rank $\leqslant (d-s-1)!$ on $\mathrm{Proj}(R^{d-s})$ such that $\mathcal{P}_N = P$.

Now, consider the Segre product ring $S := R^{s+1} \boxtimes_K R^{d-s}$ and for each $k \in \mathbb{N}$ let M_k be the finitely generated graded S-module $R^{s+1}(-k) \boxtimes_K N$, which is locally free of rank $\leqslant (d-1)!/s!$ on $\mathrm{Proj}(S)$. Observe that

$$d^j_{R^{s+1}} \equiv 0 \quad \text{for all } j \neq 0, s \quad \text{and} \quad d^l_N \equiv 0 \quad \text{for all } l > d-s-1.$$

Now, we get from the Künneth relations (cf. Reminder 4.5) for all $i \in \{0, \dots, d-1\}$ and all $n \in \mathbb{Z}$

$$d^i_{M_k}(n) = \begin{cases} d^0_{R^{s+1}}(-k+n) d^i_N(n) & \text{for } 0 \leqslant i < s, \\ d^0_{R^{s+1}}(-k+n) d^i_N(n) + d^s_{R^{s+1}}(-k+n) d^{i-s}_N(n) & \text{for } s \leqslant i \leqslant d-s-1, \\ d^s_{R^{s+1}}(-k+n) d^{i-s}_N(n) & \text{for } d-s-1 < i \leqslant d-1. \end{cases}$$

As $P = \mathcal{P}_N$ and in view of $(*)$ we have $d^{i-s}_N(n) = 0$ for all $(i, n) \in \mathbb{S}$ with $s \leqslant i \leqslant d-1$. Moreover, for all $n \in \mathbb{Z}$ and all $k \in \mathbb{N}$ we have $d^0_{R^{s+1}}(-k+n) \leqslant d^0_{R^{s+1}}(n-1)$. So for all $k \in \mathbb{N}$ and all $(i, n) \in \mathbb{S}$ we get

$$d^i_{M_k}(n) \begin{cases} \leqslant d^0_{R^{s+1}}(n-1) d^i_N(n) & \text{for } 0 \leqslant i \leqslant d-s-1, \\ = 0 & \text{if } d-s-1 < i \leqslant d-1. \end{cases}$$

Therefore the set $\{d^i_{M_k}(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in \mathcal{S}$.

Moreover $d^{d-1}_{M_k}(-1) = d^s_{R^{s+1}}(-k-1) d^{d-s-1}_N(-1)$. As $(d-s-1, -1) \in P$ we have $d^{d-s-1}_N(-1) > 0$ and hence $d^s_{R^{s+1}}(-k-1) = \binom{k}{s}$ implies that

$$\lim_{k \to \infty} d^{d-1}_{M_k}(-1) = \infty.$$

As $\dim(S) = d$, there is a finite injective morphism $R \to S$ of graded rings, which turns S in an R-module of rank $(d-1)!/s!(d-s-1)!$. So M_k becomes an R-module locally free of rank $\leqslant [(d-1)!/s!(d-s-1)!][(d-1)!/s!] \leqslant ((d-1)!)^2$ on $\mathrm{Proj}(R)$. Moreover, by Graded Base Ring Independence of Local Cohomology, we get isomorphisms of graded R-modules $D^j_{S_+}(M_k) \cong D^j_{R_+}(M_k)$ for all $j \in \mathbb{N}_0$. Now, our claim follows easily. □

Definition 4.7. A class $\mathcal{D} \subseteq \mathcal{D}^d$ is said to be *big*, if for each $t \in \{1, \dots, d\}$ there is an infinite field K such that $\mathcal{D}$ contains all pairs (R, M) in which R is the polynomial ring $K[x_1, \dots, x_t]$.

Proposition 4.8. *Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a big class and let $\mathbb{S} \subseteq \{0, \dots, d-1\} \times \mathbb{Z}$ be a set which bounds cohomology in $\mathcal{C}$. Then $\mathbb{S}$ contains a quasi-diagonal.*

Proof. There is an infinite field K such that with $R := K[x_1, \dots, x_d]$ we have $(R, R(-k)) \in \mathcal{C}$ for all $k \in \mathbb{N}$. The set $\{d^i_{R(-k)}(n) \mid k \in \mathbb{N}\}$ is finite for all $(i, n) \in \{0, \dots, d-2\} \times \mathbb{Z}$ and $\lim_{k \to \infty} d^{d-1}_{R(-k)}(0) = \infty$. It follows that $\mathbb{S}^{d-1} := \mathbb{S} \cap (\{d-1\} \times \mathbb{Z}) \neq \emptyset$. This proves our claim if $d = 1$.

So, let $d > 1$. Clearly $\mathcal{D}^{d-1} \cap \mathcal{C} \subseteq \mathcal{D}^{d-1}$ is a big class and $\mathbb{S}^{<(d-1)} = \mathbb{S} \cap (\{0, \dots, d-2\} \times \mathbb{Z})$ bounds cohomology in $\mathcal{D}^{d-1} \cap \mathcal{C}$ (see Remark 3.7(C)). So, by induction the set $\mathbb{S}^{<(d-1)}$ contains a quasi-diagonal. If $\mathbb{S}$ would contain no quasi-diagonal, Lemma 4.6 would imply that for our polynomial ring R there is a class $\mathcal{D}$ of pairs $(R, M) \in \mathcal{D}^d$ which is not of bounded cohomology but such that the set $\{d^i_M(n) \mid (R, M) \in \mathcal{D}\}$ is finite for all $(i, n) \in \mathbb{S}$. As $\mathcal{C}$ is a big class, we have $\mathcal{D} \subseteq \mathcal{C}$, and this would imply the contradiction that $\mathbb{S}$ does not bound cohomology in $\mathcal{C}$. □

Theorem 4.9. *Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a big class and let $\mathbb{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$. Then $\mathbb{S}$ bounds cohomology in $\mathcal{C}$ if and only if $\mathbb{S}$ contains a quasi-diagonal.*

Proof. Clear by Corollary 4.4 and Proposition 4.8. □

Corollary 4.10. *The set $\mathbb{S} \subseteq \{0, \ldots, d-1\} \times \mathbb{Z}$ bounds cohomology in $\mathcal{D}^d$ if and only if $\mathbb{S}$ contains a quasi-diagonal.*

Proof. Clear by Theorem 4.9. □

5. Bounding invariants

In this section we investigate numerical invariants which bound cohomology.

Definitions 5.1. (A) (See [2,8,9].) Let $\mathcal{C} \subseteq \mathcal{D}^d$ be a subclass. A *numerical invariant* on the class $\mathcal{C}$ is a map

$$\mu : \mathcal{C} \to \mathbb{Z} \cup \{\pm\infty\}$$

such that for any two pairs $(R, M), (R, N) \in \mathcal{C}$ with $M \cong N$ we have $\mu(R, M) = \mu(R, N)$. We shall write $\mu(M)$ instead of $\mu(R, M)$.

(B) Let $(\mu_i)_{i=1}^r$ be a family of numerical invariants on the subclass $\mathcal{C} \subseteq \mathcal{D}^d$. We say that the family $(\mu_i)_{i=1}^r$ *bounds cohomology on the class* $\mathcal{C}$, if for each $(n_1, \ldots, n_r) \in (\mathbb{Z} \cup \{\pm\infty\})^r$ the class

$$\{(R, M) \in \mathcal{C} \mid \mu_i(M) = n_i \text{ for all } i \in \{1, \ldots, r\}\}$$

is of bounded cohomology.

(C) A numerical invariant μ on the class $\mathcal{C} \subseteq \mathcal{D}^d$ is said to be *finite* if $\mu(M) \in \mathbb{Z}$ for all $(R, M) \in \mathcal{C}$.

(D) A numerical invariant μ on the class $\mathcal{C} \subseteq \mathcal{D}^d$ is said to be *positive* if $\mu(M) \geqslant 0$ for all $(R, M) \in \mathcal{C}$.

Remark 5.2. (A) If $\mu : \mathcal{C} \to \mathbb{Z} \cup \{\pm\infty\}$ is a numerical invariant on the class $\mathcal{C} \subseteq \mathcal{D}^d$ and if $\mathcal{D} \subseteq \mathcal{C}$, then the restriction $\mu \restriction_{\mathcal{D}} : \mathcal{D} \to \mathbb{Z} \cup \{\pm\infty\}$ is a numerical invariant on the class $\mathcal{D}$. Clearly, if μ is finite (resp. positive) then so is $\mu \restriction_{\mathcal{D}}$.

(B) If $(\mu_i)_{i=1}^r$ bounds cohomology on the class $\mathcal{C} \subseteq \mathcal{D}^d$ and if $\mathcal{D} \subseteq \mathcal{C}$, then $(\mu_i \restriction_{\mathcal{D}})_{i=1}^r$ bounds cohomology in $\mathcal{D}$.

(C) A family $(\mu_i)_{i=1}^r$ of positive numerical invariants bounds cohomology in $\mathcal{C}$ if and only if for all $(n_1, \ldots, n_r) \in (\mathbb{N}_0 \cup \{\infty\})^r$ the class

$$\{(R, M) \in \mathcal{C} \mid \mu_i(M) \leqslant n_i \text{ for all } i \in \{1, \ldots, r\}\}$$

is of bounded cohomology.

(D) A family $(\mu_i)_{i=1}^r$ of finite positive invariants bounds cohomology on $\mathcal{C}$ if and only if the sum invariant $\sum_{i=1}^r \mu_i : \mathcal{C} \to \mathbb{N}_0$ bounds cohomology in $\mathcal{C}$.

Remark 5.3. Let $i \in \mathbb{N}_0$ and $n \in \mathbb{Z}$. Then, the map

$$d^i_\bullet(n) : \mathcal{D}^d \to \mathbb{N}_0 \quad ((R, M) \mapsto d^i_M(n))$$

is a finite positive numerical invariant on $\mathcal{D}^d$.

Theorem 5.4. *Let $(n_i)_{i=0}^{d-1}$ be a sequence of integers such that $n_0 > n_1 > n_2 > \cdots > n_{d-1}$. Then the family of numerical invariants $(d^i_{\bullet}(n_i))_{i=0}^{d-1}$ bounds cohomology in $\mathcal{D}^d$.*

Proof. Clear by Proposition 4.2. □

Reminder 5.5. For each $k \in \mathbb{N}_0$ we may define the numerical invariant

$$\mathrm{reg}^k : \mathcal{D}^d \to \mathbb{Z} \cup \{-\infty\} \quad \big((R, M) \mapsto \mathrm{reg}^k(M)\big).$$

Notation 5.6. For $(R, M) \in \mathcal{D}^d$ we set

$$\varrho(M) := \begin{cases} d^0_M(\mathrm{reg}^2(M)) & \text{if } \dim(M) > 1, \\ d^0_M(0) & \text{if } \dim(M) \leqslant 1. \end{cases}$$

Remark 5.7. (A) If $(R, M) \in \mathcal{D}^d$ with $\dim(M) \leqslant 1$, the cohomological Hilbert function d^0_M of M is constant, and this constant is strictly positive if and only if $\dim(M) = 1$.

(B) The function

$$\varrho : \mathcal{D}^d \to \mathbb{N}_0 \quad \big((R, M) \mapsto \varrho(M)\big)$$

is a finite positive numerical invariant on $\mathcal{D}^d$.

Theorem 5.8. *The pair of invariants* $(\mathrm{reg}^2, \varrho)$ *bounds cohomology in* $\mathcal{D}^d$.

Proof. Fix $u, v \in \mathbb{Z}$ and set

$$\mathcal{C} := \big\{(R, M) \in \mathcal{D}^d \;\big|\; \mathrm{reg}^2(M) = u,\ \varrho(M) = v\big\}.$$

If $(R, M) \in \mathcal{C}$ we have $d^0_M(u) = d^0_M(\mathrm{reg}^2(M)) = v$.

Let $i \in \mathbb{N}$. Then $u - i = \mathrm{reg}^2(M) - i > a_{i+1}(M)$ and hence $d^i_M(u - i) = h^{i+1}_M(u - i) = 0$. Therefore (R, M) belongs to the class

$$\mathcal{D} := \big\{(R, M) \in \mathcal{D}^d \;\big|\; d^0_M(u) = v \text{ and } d^i_M(u - i) = 0 \text{ for all } i \in \{1, \ldots, d-1\}\big\}.$$

But according to Theorem 5.4 the class $\mathcal{D}$ is of bounded cohomology. □

Lemma 5.9. *Let $(R, M) \in \mathcal{D}^d$ be such that $\dim(R/\mathfrak{p}) \neq 1$ for all $\mathfrak{p} \in \mathrm{Ass}_R(M)$. Then*

$$d^0_M(n - 1) \leqslant \max\big\{0, d^0_M(n) - 1\big\} \quad \textit{for all } n \in \mathbb{Z}.$$

Proof. For an arbitrary finitely generated graded R-module N let

$$\lambda(N) := \inf\big\{\mathrm{depth}(N_{\mathfrak{p}}) + \mathrm{height}\big((\mathfrak{p} + R_+)/\mathfrak{p}\big) \;\big|\; \mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Var}(R_+)\big\}.$$

Clearly, for all $n \in \mathbb{Z}$ we have $\lambda(N(n)) = \lambda(N)$. So, for all $n \in \mathbb{Z}$, we get by our hypotheses that $\lambda(M(n)) = \lambda(M) > 1$. Now, according to [8, Proposition 4.6] we obtain

$$d^0_M(n - 1) = d^0_{M(n)}(-1) \leqslant \max\big\{0, d^0_{M(n)}(0) - 1\big\} = \max\big\{0, d^0_M(n) - 1\big\}. \qquad \square$$

Theorem 5.10. *Let $r, s \in \mathbb{Z}$ and let $p \in \mathbb{Q}[t]$ be a polynomial. Let $\mathcal{C} \subseteq \mathcal{D}^d$ be the class of all pairs $(R, M) \in \mathcal{D}^d$ satisfying the following conditions*:

(α) *There is a finitely generated graded R-module N with Hilbert polynomial $p_N = p$ and $\operatorname{reg}^2(N) \leqslant r$ such that $M \subseteq N$.*
(β) $\operatorname{reg}^2(M) \leqslant s$.

Then, $\mathcal{C}$ is a class of finite cohomology.

Proof. Let $v := \max\{r, s\}$. We first show that for each pair $(R, M) \in \mathcal{C}$ we have

$$(*) \qquad \varrho(M) \leqslant p(v)$$

and

$$(**) \qquad \dim(M) \leqslant 1 \quad \text{or} \quad \operatorname{reg}^2(M) \geqslant -v - p(v).$$

So, let $(R, M) \in \mathcal{C}$. Then, there is a monomorphism of finitely generated graded R-modules $M \overset{\epsilon}{\rightarrowtail} N$ such that $p_N = p$ and $\operatorname{reg}^2(N) \leqslant r \leqslant v$.

Assume first that $\dim(M) > 1$. As $\operatorname{reg}^2(M) \leqslant v$ we then get

$$\varrho(M) = d_M^0\big(\operatorname{reg}^2(M)\big) \leqslant d_M^0(v) \leqslant d_N^0(v) = p_N(v) = p(v).$$

If $\dim(M) \leqslant 1$, the function d_M^0 is constant and therefore

$$\varrho(M) = d_M^0(0) = d_M^0(v) \leqslant d_N^0(v) = p_N(v) = p(v).$$

Thus we have proved statement $(*)$.

To prove statement $(**)$ we assume that $\dim(M) > 1$. Then there is a short exact sequence of finitely generated graded R-modules

$$0 \to H \to M \to \overline{M} \to 0$$

such that $\dim(H) \leqslant 1$ and $\operatorname{Ass}_R(\overline{M})$ does not contain any prime $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) \leqslant 1$. As $\dim(H) \leqslant 1$, we have $H^i_{R_+}(H) = 0$ for all $i > 1$. Therefore $H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M})$ for all $i > 1$ and hence $\operatorname{reg}^2(M) = \operatorname{reg}^2(\overline{M})$. Moreover by the observation made on $\operatorname{Ass}_R(\overline{M})$, we have (see Lemma 5.9)

$$d_{\overline{M}}^0(n-1) \leqslant \max\big\{0, d_{\overline{M}}^0(n) - 1\big\} \quad \text{for all } n \in \mathbb{Z}.$$

As $D^1_{R_+}(H) = H^2_{R_+}(H) = 0$, we have

$$d_{\overline{M}}^0(v) \leqslant d_M^0(v) \leqslant d_N^0(v) = p_N(v) = p(v)$$

and it follows that

$$d_{\overline{M}}^0(n) = 0 \quad \text{for all } n \leqslant -v - p(v) - 1.$$

One consequence of this is, that $T := D^0_{R_+}(\overline{M})$ is a finitely generated R-module. As $H^i_{R_+}(M) \cong H^i_{R_+}(\overline{M})$ for all $i > 1$, we have $\operatorname{reg}^2(T) = \operatorname{reg}^2(\overline{M}) = \operatorname{reg}^2(M)$. As $H^i_{R_+}(T) = 0$ for $i = 0, 1$, we thus get $\operatorname{reg}^2(M) =$

$\operatorname{reg}(T)$. As $T_n = 0$ for all $n \leqslant -v - p(v) - 1$, we finally obtain (see Reminder 2.2(E))

$$\operatorname{reg}^2(M) = \operatorname{reg}(T) \geqslant \operatorname{gendeg}(T) \geqslant \operatorname{beg}(T) \geqslant -v - p(v).$$

This proves statement $(**)$.

Now, we may write

$$\mathcal{C} \subseteq \mathcal{C}_{-\infty} \cup \bigcup_{t=-v-p(v)}^{s} \mathcal{C}_t,$$

where

$$\mathcal{C}_{-\infty} := \left\{(R, M) \in \mathcal{D}^d \;\middle|\; \dim(M) \leqslant 1 \text{ and } \varrho(M) \leqslant p(v)\right\}$$

and, for all $t \in \mathbb{Z}$ with $-v - p(v) \leqslant t \leqslant s$,

$$\mathcal{C}_t := \left\{(R, M) \in \mathcal{D}^d \;\middle|\; \operatorname{reg}^2(M) = t,\ \varrho(M) \leqslant p(v)\right\}.$$

The class $\mathcal{C}_{-\infty}$ clearly is of bounded cohomology.

Now, by Remark 5.2(C) and by Corollary 5.8, each of the classes $\mathcal{C}_t$ is of bounded cohomology. This proves our claim. □

Corollary 5.11. *Let $r \in \mathbb{Z}$ and let $p \in \mathbb{Q}[t]$ be a polynomial. Let $\mathcal{C} \subseteq \mathcal{D}^d$ be the class of all pairs $(R, M) \in \mathcal{D}^d$ satisfying the condition (α) of Theorem* 5.10. *Then, the invariant* reg^2 *bounds cohomology in the class $\mathcal{C}$.*

Proof. This is immediate by Theorem 5.10. □

Corollary 5.12. *Let $r \in \mathbb{Z}$ and let $(R, N) \in \mathcal{D}^d$. If M runs through all graded submodules $M \subseteq N$ with* $\operatorname{reg}^2(M) \leqslant r$*, only finitely many cohomology tables d_M and hence only finitely many Hilbert polynomials p_M occur.*

Proof. This is clear by Theorem 5.10. □

Corollary 5.13. *Let $r \in \mathbb{Z}$ and let $(R, N) \in \mathcal{D}^d$. If M runs through all graded submodules of N with* $\operatorname{reg}^1(M) \leqslant r$ *only finitely many families*

$$\left(h^i_M(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad \textit{and} \quad \left(h^i_{N/M}(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}$$

can occur.

Proof. Let $\mathcal{P}$ be the set of all graded submodules $M \subseteq N$ with $\operatorname{reg}^1(M) \leqslant r$.

Now, for each $M \in \mathcal{P}$ we have the following three relations

$$d^i_M(n) = h^{i+1}_M(n) \quad \text{for all } i \geqslant 1 \text{ and all } n \in \mathbb{Z};$$

$$\begin{cases} h^1_M(n) \leqslant d^0_M(n) & \text{for all } n \in \mathbb{Z}; \\ h^1_M(n) = d^0_M(n) & \text{for all } n < \operatorname{beg}(N); \\ h^1_M(n) = 0 & \text{for all } n \geqslant r, \end{cases}$$

and

$$h^0_M(n) \leqslant h^0_N(n) \quad \text{for all } n \in \mathbb{Z}.$$

So, by Corollary 5.12 the set

$$\mathcal{U} := \left\{ \left(h^i_M(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \;\middle|\; M \in \mathcal{P} \right\}$$

is finite.

For each $M \in \mathcal{P}$ the short exact sequence $0 \to M \to N \to N/M \to 0$ yields that for all $n \in \mathbb{Z}$ and all $i \in \mathbb{N}_0$

$$h^0_{N/M}(n) \leqslant h^0_N(n) + h^1_M(n), \tag{1}$$

$$d^i_{N/M}(n) \leqslant d^i_N(n) + h^{i+2}_M(n). \tag{2}$$

By the finiteness of $\mathcal{U}$ it follows that the set of functions

$$\mathcal{U}_0 := \left\{ \left(h^0_{N/M}(n)\right)_{n\in\mathbb{Z}} \;\middle|\; M \in \mathcal{P} \right\}$$

is finite and that the set of cohomology diagonals

$$\mathcal{W} := \left\{ \left(d^i_{N/M}(-i)\right)_{i=0}^{d-1} \;\middle|\; M \in \mathcal{P} \right\}$$

is finite.

In view of the theorem [6, Theorem 5.4] the finiteness of $\mathcal{W}$ implies that the set

$$\mathcal{U}_1 := \left\{ \left(d^i_{N/M}(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \;\middle|\; M \in \mathcal{P} \right\}$$

is finite. Moreover for all $M \in \mathcal{P}$ we have

$$\operatorname{end}\left(H^1_{R_+}(N/M)\right) < \operatorname{reg}^1(N/M) \leqslant \max\left\{\operatorname{reg}^2(M) - 1, \operatorname{reg}^2(N)\right\} \leqslant \max\left\{r - 1, \operatorname{reg}^1(N)\right\}$$

and

$$h^1_{N/M}(n) \leqslant d^0_{N/M}(n) \quad \text{for all } n \in \mathbb{Z}, \text{ with equality if } n < \operatorname{beg}(N).$$

As $d^i_{N/M} \equiv h^{i+1}_{N/M}$ for all $i > 0$ the finiteness of $\mathcal{U}_0$ and $\mathcal{U}_1$ shows that the set

$$\left\{ \left(h^i_{N/M}(n)\right)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \;\middle|\; M \in \mathcal{P} \right\}$$

is finite, too. □

Corollary 5.14. *Assume that R is a homogeneous Noetherian Cohen–Macaulay ring with Artinian local base ring R_0. Let $s \in \mathbb{Z}$ and let N be a finitely generated graded R-module. If M runs trough all graded submodules* *of N with* gendeg$(M) \leqslant s$ *only finitely many families*

$$\big(h^i_M(n)\big)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}} \quad \textit{and} \quad \big(h^i_{N/M}(n)\big)_{(i,n)\in\mathbb{N}_0\times\mathbb{Z}}$$

may occur.

Proof. By [4, Proposition 6.1] we see that reg(M) finds an upper bound in terms of gendeg(M), reg(N), reg(R), beg(N), dim(R), the multiplicity $e_0(R)$ of R and the minimal number of homogeneous generators of the R-module N. Now, we conclude by Corollary 5.13. □

Remark 5.15. If we apply Corollary 5.13 in the special case where $N = R = K[x_1, \dots, x_r]$ is a polynomial ring over a field, we get back the finiteness result [17, Corollary 14]. Correspondingly, if we apply Corollary 5.14 in this special case, we get back [17, Corollary 20].

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