The semistrong limit of multipulse interaction in a thermally driven optical system

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Abstract

We consider the semistrong limit of pulse interaction in a thermally driven, parametrically forced, nonlinear Schrödinger (TDNLS) system modeling pulse interaction in an optical cavity. The TDNLS couples a parabolic equation to a hyperbolic system, and in the semistrong scaling we construct pulse solutions which experience both short-range, tail–tail interactions and long-range thermal coupling. We extend the renormalization group (RG) methods used to derive semistrong interaction laws in reaction–diffusion systems to the hyperbolic–parabolic setting of the TDNLS system. A key step is to capture the singularly perturbed structure of the semigroup through the control of the commutator of the resolvent and a re-scaling operator. The RG approach reduces the pulse dynamics to a closed system of ordinary differential equations for the pulse locations.

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1. Introduction

Pulse solutions occupy a distinguished place in the study of hyperbolic systems, particularly the nonlinear Schrödinger (NLS) equation and its variants. We investigate the existence and stability of multipulse solutions in a hyperbolic–parabolic system built around the NLS equation.
The system models the mean-field behavior of an optical parametric oscillator (OPO), consisting of a nonlinear medium contained within an optical cavity, operating in a particular limit. Interactions between the pulse and the crystal lead to frequency conversion and also heat the crystal. This is modeled with a parametrically forced nonlinear Schrödinger equation (PNLS) coupled to the heat equation. The result is a two-scale system, which supports spatially localized optical pulses coupled through a broad thermal envelope.

There is a well developed literature addressing the existence and stability of stationary pulses via the geometric singular perturbation theory and the Evans function (see [2,3,9,15,16,20,25] and references therein). There is no such theory for pulse interactions; in particular strong pulse interactions are not yet understood mathematically. Weak pulse interactions, in which well separated pulses are driven by tail overlap, have been well studied (see [6,7,17,19,20]). In the weak regime the pulse coupling is exponentially small, and pulses propagate without changes to shape or stability. Recently the semistrong interaction regime was introduced (see [4,22]) for singularly perturbed systems in which pulse components decay at asymptotically distinct rates so that some components approach the background state between pulses while others components do not. The pulse positions, amplitudes, and shapes all change at leading order, and at rates that are algebraic in the perturbation parameter. Moreover the pulses may bifurcate as they propagate.

For a class of reaction–diffusion systems, the renormalization group (RG) approach has been used to reduce the dynamics of the semistrong pulse interaction regime to a system of ordinary differential equations, up to an asymptotically small correction [5]. A key step is the replacement of the linearization about the semistrong pulse configuration with a reduced linearization which captures the strong component of the pulse interaction via a finite rank operator. This replacement greatly reduces the complexity of the associated spectral analysis and resolvent estimates; in particular it permits the spectral problem for an \(N\)-pulse ansatz to be reduced to a condition on the eigenvalues of an \(N \times N\) matrix whose entries are given in terms of explicit integrals. The secularity, i.e., the difference between the exact and the renormalized linearization, is asymptotically large; however, by exploiting the singularly perturbed structure of the system, the semigroup is seen to be strongly contractive on certain subspaces, and with an appropriate choice of the reduced linearization the secularity can be controlled.

The extension of the RG approach for semistrong pulse interactions to a mixed hyperbolic–parabolic setting is nontrivial. The linearized operators are merely \(C^0\) and in contrast to the analytic semigroup case one cannot appeal to the uniform convergence of the Laplace transform representation to export the singularly perturbed structure of the resolvent estimates into the semigroup. Indeed estimates for \(C^0\) semigroups rely on the “Fourier type 2” property [11,26] of an underlying Hilbert space to obtain convergence, and the semigroup estimates come from the principle of uniform boundedness; any singularly perturbed structure within the semigroup is lost in this argument. We show in Theorem 3.1 that in a general setting this singular structure may be preserved if it is factored out of the resolvent before the semigroup estimates are made, and re-introduced after the estimate is in hand. This program requires uniform control of the commutator of the projected resolvent and certain rescaling operators. The remainder of the proof follows along the lines of the analytic semigroup case, modulo technical issues involved in working with norms with different scalings for the various system components.

The application of the RG approach to the semistrong pulse interaction in hyperbolic–parabolic systems exhibits the flexibility of the RG method in two ways. First, the ability to replace part of the linearization with a finite rank operator, within the equivalence class of strongly contracted secularities, greatly simplifies the linear analysis, leading to a sharp characterization of the point spectrum in terms of inner products of known functions, see Proposition 3.
Secondly, an asymptotic construction of the semistrong pulses, particularly the local pulse velocities, would require a delicate application of the geometric singular perturbation theory, see [4]. The current approach sidesteps this issue entirely, constructing \(N\) pulses though a simple formal argument which does not require an explicit resolution of the pulse velocities, rather only that the residual, the amount by which the pulses fail to be standing solutions, is appropriately small. The local pulse velocities are recovered, a posteriori, through the projection of the residual onto the tangent space of the pulse manifold.

In Section 2 we present the model system, formally construct families of \(N\)-pulses within the semistrong regime, state the main result concerning the reduction of the PDE dynamics to an asymptotically closed family of ODEs for the pulse positions, and introduce the notation for the norms and scaling operators. In Section 3 we present the linear analysis, deriving the linearized and reduced linearized operators, and analyze the point spectrum of the reduced linearization, which leads to a nonlocal eigenvalue problem (NLEP). We also derive the resolvent estimates and prove Theorem 3.1, which establishes the singularly perturbed structure of \(C^0\) semigroups in a general setting. In Section 4 we apply the RG methodology, exploiting the estimates derived in Section 3 to show that the PDE dynamics in a neighborhood of the \(N\)-pulses reduces to a closed set of limiting ODEs for the pulses’ evolution. In particular we show that an antisymmetric two-pulse can form a stable bound pair with pulse separation \(q = \mathcal{O}(\varepsilon |\ln \varepsilon|)\).

2. The thermally detuned PNLS model

We study the thermally detuned, parametrically forced, nonlinear Schrödinger equation (TDNLS) which describes the evolution of the depth averaged optical field intensity, \(u \in \mathbb{C}\), and temperature \(\theta\) over the transverse direction, \(x\), of the crystal,

\[
iu_t + \frac{1}{2}u_{xx} + |u|^2u + (i - a(\theta))u - \gamma u^* = 0, \quad (2.1)
\]

\[
\theta_t - \varepsilon^{-2}\theta_{xx} + h\theta = \varepsilon^{-1}\varrho|u|^2. \quad (2.2)
\]

Here \(\gamma\) and \(a\) represent the parametric forcing and the detuning, \(h > 0\) and \(\varrho > 0\) model radiative cooling and a scaled absorption coefficient, and \(\varepsilon \ll 1\) describes the scaling of the inverse thermal diffusivity. The approximately linear dependence of the refractive index on temperature is reflected in the detuning parameter \(a\), i.e.,

\[
a(\theta) = a_0 - \theta. \quad (2.3)
\]

The model, derived in [13], is an extension of earlier optical-only versions [23,24] to include self-induced heating. In [14] it was shown that for \(a > 0\) and \(\gamma > 1\), and in the case of a spatially uniform temperature \(\theta\), the optical equation (2.1) sustains standing waves with a constant polarization \(\alpha_p\) satisfying \(e^{-2i\alpha_p} = (i \pm \sqrt{\gamma^2 - 1})/\gamma\). These pulses were studied in the context of a prescribed thermal envelope in [12].

The form of the uniform temperature pulses suggests recasting the PNLS as a real system after the polarization has been factored out. Introducing the new dependent variables

\[
\tilde{U} = (U_1, U_2, U_3)^t = (\Re(e^{-i\alpha_p}u), \Im(e^{-i\alpha_p}u), \theta)^t, \quad (2.4)
\]

we obtain

\[
\tilde{U}_t = \mathcal{F}(\tilde{U}), \quad (2.5)
\]
Fig. 1. Evolution of an up–down–up three-pulse ansatz and thermal profile with $a = 2.6$, $\gamma = 1.2$, $h = 1$, $\varepsilon = 0.1$, and $\varrho = 0.3$. The optical pulses move up the thermal profile, decreasing their amplitude and increasing their width, until the pulse–pulse repulsion balances the focusing effect of the thermal gradient, generating a steady-state solution.

where

$$
\mathcal{F}(\vec{U}) = \begin{pmatrix}
0 & -\frac{1}{2} \varrho^2 + \eta_-(U_3) - (U_1^2 + U_2^2) & 0 \\
-(\frac{1}{2} \partial_x^2 + \eta_+(U_3) - (U_1^2 + U_2^2)) & -2 & 0 \\
\varepsilon^{-1} \varrho U_1 & \varepsilon^{-1} \varrho U_2 & \varepsilon^{-2} \partial_x^2 - h
\end{pmatrix} \vec{U},
$$

(2.6)

and $\eta_{\pm}(\theta) = \eta_{0,\pm} - \theta$, with $\eta_{0,\pm} = a_0 \pm \sqrt{\gamma^2 - 1}$.

We state below our main result, which outlines the asymptotic stability of the semistrong $N$-pulse solutions constructed formally in the section below. The pulses are stable in a scaled norm, $\| \cdot \|_{X_{\varepsilon}}$, defined in (2.13), which gives $\varepsilon$-uniform control of the $L^\infty$ norm of each component, as well as of the unscaled $H^1$ norm of the optical components.

**Theorem 2.1.** For all parameters $a$, $\gamma$, and $h$ of (2.5) satisfying $\gamma > 1$, $a > \sqrt{\gamma^2 - 1}$, and $h > 0$, there exists $\varrho_0 > 0$ such that the $N$-pulse ansatz constructed in Lemma 2.2 is admissible, see Definition 2.1, for all $\varrho \in (0, \varrho_0)$. Moreover, there exists $\alpha < \sqrt{105/272}$, see Lemma 3.5, such that for all $a < a_c$ we may choose $\varrho_0 > 0$, $\nu > 0$, $M > 1$, and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\varrho \in (0, \varrho_0)$, and for all $\vec{q} \in K$, the manifold, $\mathcal{M}$, of quasi-steady $N$-pulse solutions (2.30), of the TDNLS equation (2.1)–(2.2) is asymptotically exponentially stable up to $O(\varepsilon)$. Specifically, given an initial value $\vec{U}_0$ sufficiently close to $\mathcal{M}$, the corresponding solution $\vec{U}$ of (2.5) can be decomposed as

$$
\vec{U}(x, t) = \Phi(x, \vec{q}) + W(x, t),
$$

(2.7)

where the $N$-pulse $\Phi$ is given by (2.26), the remainder $W$ satisfies

$$
\| W \|_{X_{\varepsilon}} \leq M \left( e^{-\nu t} \| W_0 \|_{X_{\varepsilon}} + \varepsilon \right),
$$

(2.8)
and the optical pulse positions $\tilde{q}(t)$ evolve according to (4.14). In addition, after the perturbation $W$ has decayed to $O(\varepsilon)$, the pulse evolution is given at leading order by the closed set of ordinary differential equations (4.66).

2.1. Notation and function spaces

Given a Banach space, $X$, with dual $X^*$, the pairing $f \in X$ and $g \in X^*$, is denoted $\langle g, f \rangle$. For $M$ an $l \times m$ matrix of elements from the dual space we define the tensor of $M$, denoted $\otimes M : X \mapsto \mathbb{R}^{l\times m}$, by

$$
(\otimes M) f = \left( \langle M_{11}, f \rangle \cdots \langle M_{1m}, f \rangle \right) \cdots \left( \langle M_{l1}, f \rangle \cdots \langle M_{lm}, f \rangle \right).
$$

For $N$ a $k \times l$ matrix of elements of $X$, the tensor product $N \otimes M : X \mapsto \mathbb{R}^{k\times m}$ acts on $f \in X$ through the matrix product of $N$ and $\otimes M f$.

The operator $S_\varepsilon$ scales the independent variable by $\varepsilon$,

$$
S_\varepsilon f(x) = f(\varepsilon x),
$$

and a vector version, $\vec{S}$, scales the independent variable of the third component of functions $F = (f_1, f_2, f_3)^t$,

$$
(\vec{S}_\varepsilon F)(x) = (f_1(x), f_2(x), f_3(\varepsilon x))^t.
$$

The diagonal operator $\vec{D}_\varepsilon$ scales the magnitude of the third component

$$
\vec{D}_\varepsilon = \text{diag}(1, 1, \varepsilon).
$$

We introduce the scaled norm which is the base norm for controlling the remainder $W$,

$$
\| F \|_{X_\varepsilon} = \| \vec{S}_\varepsilon^{-1} F \|_{H^1},
$$

and the associated, $\varepsilon$ independent, Hilbert space $X$. We observe that the scaled $H^1$ norm uniformly controls the $L^\infty$ norm,

$$
\| f \|_{L^\infty} \leq c\sqrt{\varepsilon \| f \|_{L^2}^2 + \varepsilon^{-1} \| \partial_x f \|_{L^2}^2} = c\| \vec{S}_\varepsilon^{-1} f \|_{H^1},
$$

and also controls the unscaled $H^1$ norm, but not uniformly

$$
\| F \|_{H^1} \leq \| \vec{D}_\varepsilon^{-3/2} \vec{S}_\varepsilon F \|_{X}.
$$

Since the third component of our solution satisfies a parabolic equation while the first two are hyperbolic, it is natural to use different norms on the third component. Given two norms $\| \cdot \|_X$ and $\| \cdot \|_Y$, we introduce the bifurcated norm

$$
\| F \|_{X,Y} = \| f_1 \|_X + \| f_2 \|_X + \| f_3 \|_X + \| f_3 \|_Y.
$$
We also introduce the weighted–windowed norms, which provide control of the derivative of the Fourier transform of functions which are localized near a fixed set of positions \( \vec{q} \), independent of the separation distance \(|q_N - q_1|\). For

\[
\vec{q} \in \mathcal{K}_0 = \left\{ \vec{q} \in \mathbb{R}^N \mid l_0|\ln \varepsilon| < \inf_{j=2,\ldots,N} (q_j - q_{j-1}) \right\},
\]

we partition the real line at the midpoints of the \( \vec{q} \) positions \( \{s_1, \ldots, s_{N-1}\} \), \( s_j = \frac{1}{2}(q_j + q_{j+1}) \) for \( j = 1, \ldots, N - 1 \). We introduce the partition of unity, \( \{\chi_j\}_{j=1}^N \), subordinate to the open cover \( \{(s_{j-1} - 1, s_{j} + 1) \mid j = 1, \ldots, N\} \) where \( s_0 = -\infty \) and \( s_N = \infty \). The \( C^\infty \) function \( \chi_j = 1 \) on the interval \((s_{j-1} - 1, s_{j} + 1)\) and has support within \((s_{j-1} - 1, s_{j} + 1)\) and satisfies \( \| \partial_x \chi_j \|_{L^\infty} \leq 1 \).

For each \( p > 0, j \in \{1, \ldots, N\} \), and \( r \geq 1 \) we introduce the \( p \)-weighted, \( q_j \)-shifted, \( L^r \) norm,

\[
\|f\|_{L^r_{p,j}} = \|\langle x - q_j \rangle_p f\|_{L^r},
\]

(2.17)

with weight \( \langle x \rangle_p = \sqrt{1 + |x|^2}^p \). The associated \( p \) weighted, \( \vec{q} \) windowed, \( L^r \) norm is given by

\[
\|f\|_{L^r_{p,\vec{q}}} = \left( \sum_{j=0}^N \|\chi_j f\|_{L^r_{p,j}}^r \right)^{\frac{1}{r}}.
\]

(2.18)

We remark that for \( j = 1, \ldots, N \) any weighted norm controls the windowed norms,

\[
\|f\|_{L^r_{p,\vec{q}}} \leq 2\|f\|_{L^r_{p,j}},
\]

(2.19)

while the windowed \( L^2_{1,\vec{q}} \) norm controls the \( L^1 \) norm,

\[
\|f\|_{L^1} \leq c\sum_{j=1}^N \|\chi_j f\|_{L^1} \leq c\|f\|_{L^2_{1,\vec{q}}},
\]

(2.20)

For \( r = 2 \) we define the weighted Sobolev norm

\[
\|f\|_{H^1_{p,j}} = \left( \|f\|_{L^2_{p,j}}^2 + \|\partial_x f\|_{L^2_{p,j}}^2 \right)^{\frac{1}{2}},
\]

(2.21)

with the weighted–windowed Hilbert space \( H^1_{p,\vec{q}} \) defined where the norm

\[
\|f\|_{H^1_{p,\vec{q}}} = \left( \|f\|_{L^2_{p,\vec{q}}}^2 + \|\partial_x f\|_{L^2_{p,\vec{q}}}^2 \right)^{\frac{1}{2}},
\]

(2.22)

is finite. When the function within the norm has been rescaled, so must the pulse positions, \( \vec{q} \). We emphasize this when using weighted norms via the notation

\[
\|S_{\varepsilon}^{-1}f\|_{L^r_{p,j\varepsilon}} = \|\langle x - \varepsilon q_j \rangle_p S_{\varepsilon}^{-1} f\|_{L^r},
\]

(2.23)
and similarly for the $H^1_{p,j}$. The norms without $j$ subscript, $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^1_p}$, denote the usual, nonwindowed, $p$-weighted norm with weight centered on the mid-point of the $N$-pulse, $\frac{1}{2}(q_N + q_1)$.

We write $f = g + O(\varepsilon)$ in norm $\| \cdot \|$ if

$$\| f - g \| \leq c\varepsilon.$$  \hfill (2.24)

We denote the mass of a function $f$ by $f = \int_{-\infty}^{\infty} f \, d\xi$. The quantity $[\vec{F}]_j$ will denote the $j$th component of the vector $\vec{F}$ when less cumbersome notation is not available.

### 2.2. Semistrong pulse solutions

We construct the semistrong $N$-pulse ansatz by first assuming that the temperature $\theta$ takes a constant value, $p$, so that the pulse solution in the optical variables takes the form $(\phi, 0)^t$ where

$$\phi(x; p, q) = \sqrt{2\eta_+(p)} \text{sech}\left(\sqrt{2\eta_+(p)}(x - q)\right),$$  \hfill (2.25)

and $q \in \mathbb{R}$ denotes the pulse position. Exploiting the difference in scaling between the optical and thermal components of the system, we construct an approximate $N$-pulse ansatz, parameterized by dynamic pulse positions $\vec{q} \in \mathbb{R}$, writing

$$\Phi(x; \vec{q}) = \left(\begin{array}{c} \Xi \\ \Theta \end{array}\right),$$  \hfill (2.26)

where $\Xi$ is given in terms of the localized pulse solutions $\phi_j \equiv \phi(x; p_j, q_j)$, the pulse center temperatures,

$$p_j \equiv \Theta(x = q_j),$$  \hfill (2.27)

and the pulse signs, $\alpha_j = \pm 1$, via

$$\Xi = \sum_{j=1}^{N} \alpha_j \phi_j.$$  \hfill (2.28)

The temperature profile is taken as the quasi-steady solution induced by the thermal heating

$$\Theta = \varepsilon^{-1} \Theta(-\varepsilon^{-2} \partial^2_x + \hbar)^{-1} |\Xi|^2.$$  \hfill (2.29)

The goal is to determine the parameters $\vec{p} = (p_1, \ldots, p_N)$ in terms of the pulse positions $\vec{q}$, thereby constructing a manifold of semistrong $N$-pulse quasi-steady solutions of the TDNLS equation

$$\mathcal{M} = \{ \Phi(x, \vec{q}) \mid \vec{q} \in \mathcal{K}\},$$  \hfill (2.30)

where

$$\mathcal{K} = \bigg\{ \vec{q} \in \mathbb{R}^N \mid l_0 |\ln \varepsilon| < \inf_{j=2,\ldots,N} (q_j - q_{j-1}) \bigg\}.$$  \hfill (2.31)
for $l_0$ sufficiently large, as specified in Lemma 2.2, that the optical component of the pulses satisfies $|\phi_i(q_j)| = O(\varepsilon)$ for $i \neq j$. Indeed the $\eta_\pm$ control the two exponential decay rates associated to an $N$-pulse ansatz and the associated linearized operators. We impose a condition which keeps the value of $\eta_-(\Theta)$ strictly positive.

**Definition 2.1.** The parameters $\{a, \gamma, \varrho, h, N\}$ are admissible if

$$\eta_{\min,-} = \min_{j=1, \ldots, N} \inf_{q \in \mathcal{K}} \eta_{j,-} > 0.$$  

To resolve the temperature profile, $\Theta$, we introduce the local temperature profile, $\Theta_j$, induced by the $j$th pulse,

$$\Theta_j \equiv \varepsilon^{-1} \varrho \left(-\varepsilon^{-2} \partial_x^2 + h\right)^{-1} |\phi_j|^2,$$

which has the solution

$$\Theta_j(x; p_j, q_j) = \frac{\varrho \eta_{j,+}}{\sqrt{h}} \int_{\mathbb{R}} e^{-\varepsilon \sqrt{h} |x-q_j-s|} \text{sech}^2\left(s \sqrt{2 \eta_{j,+}}\right) ds,$$

where $\eta_{j,\pm} = \eta_{\pm}(p_j) = \eta_0, - p_j$. In the rescaled norm induced by the scaling operator $\mathcal{S}_\varepsilon$, defined in (2.10), the thermal profile is well approximated by the sum of the local temperature profiles.

**Lemma 2.1.** For $\tilde{q} \in \mathcal{K}$, the thermal profile $\Theta$ has the expansion

$$\left\| \mathcal{S}_\varepsilon^{-1} \left( \Theta - \sum_{j=1}^N \Theta_j \right) \right\|_{H^2} = O(\varepsilon).$$

Moreover for $|x - q_j| \geq l_0 |\ln \varepsilon|$, the local thermal profiles take the explicit form

$$\Theta_j(x) = \varrho \sqrt{\frac{2 \eta_{j,+}}{h}} e^{-\varepsilon \sqrt{h} |x-q_j|} \left(1 + O(\varepsilon^2)\right),$$

while

$$\Theta_j(q_j) = \varrho \sqrt{\frac{2 \eta_{j,+}}{h}} - \varepsilon \varrho \ln 2 + O(\varepsilon^2).$$

In addition there exists $c > 0$, which may be chosen independent of $\tilde{q} \in \mathcal{K}$, for which

$$\left\| \partial_x \mathcal{S}_\varepsilon^{-1} \Theta \right\|_{L^2} + \left\| \partial_x \Theta \right\|_{L^\infty} \leq c \varepsilon.$$

**Proof.** Since the pulses are well separated, the tail–tail interactions are small, and we see that $\|(|\Xi|^2 - \sum |\phi_j|^2)\|_{H^1} = O(\varepsilon)$. Thus the $H^1$ norm of the difference between $\Theta$ and the sum of the local thermal profiles is $O(\varepsilon)$ in the rescaled variables for which $\Theta$ and each $\Theta_j$ decay at
Fig. 2. The region of spectral stability of an individual pulse of the PNLS equation with constant detuning parameter $a$. Region I is the stability region for PNLS pulses, crossing into region II coincides with a Hopf bifurcation, see [1], and there are unstable point spectra. In regions III and IV there is unstable essential spectra. Region $I$ is the stability region for PNLS pulses, crossing into region $II$ coincides with a Hopf bifurcation, see [1], and there are unstable point spectra. In regions III and IV there is unstable essential spectra, $\eta_0 < 0$ and the data is inadmissible. In the TDNLS system, the impact of the thermal field is to lower the thermal detuning, $a$, locally about a pulse. Admissibility of the parameters is equivalent to $a(\theta(x))$ remaining above the essential bifurcation curve $a = \sqrt{\gamma^2 - 1}$ for all values of $x \in \mathbb{R}$.

an $\mathcal{O}(1)$ exponential rate as $|x| \to \infty$. The evaluations (2.35) and (2.36) of $\Theta_j$ follow from a straightforward expansion of the integrals (2.33). Taking the derivative of (2.33) we obtain

$$\Theta_j'(x) = -\varepsilon \varrho \eta_j + \int_{\mathbb{R}} H(x - q_j - s) e^{-\varepsilon \sqrt{\eta_0} |x - q_j - s|} \text{sech}^2(s \sqrt{\frac{2}{\eta_j}}) ds,$$

(2.38)

where $H(s) = \text{sign}(s)$. Since the integral is uniformly bounded and $\mathcal{O}(1)$, the result (2.37) follows by summing over $j = 1, \ldots, N$. \hfill \Box

Using (2.34) and (2.35), the $N$ unknown thermal detunings $p_j$ are determined through the pulse positions $q_j$ from (2.27), which, at leading order, results in $N$ coupled quadratic equations for $\tilde{p}$,

$$p_j = \Theta(q_j) = \varrho \sqrt{\frac{2}{h}} \left( \sum_{l=1}^{N} \sqrt{\eta_0 - p_l e^{-\varepsilon \sqrt{\eta_0} |q_j - q_l|}} \right) + \mathcal{O}(\varepsilon),

(2.39)$$

for $j = 1, \ldots, N$. These equations can be solved for arbitrary $\tilde{q} \in \mathcal{K}$ if $\varrho$ is sufficiently small, but still $\mathcal{O}(1)$ with respect to $\varepsilon$, or for arbitrary admissible parameters if the pulses are sufficiently close together, on the order of $\mathcal{O}(\varepsilon^{-1})$. Since admissibility of the parameters also requires $\varrho$ sufficiently small, we concentrate on the former regime, and construct the general $N$-pulse ansatz, depicted in Fig. 1, in the lemma below.

Lemma 2.2. Given $a_0$ and $\gamma$ satisfying $\gamma > 1$ and $a_0 > \sqrt{\gamma^2 - 1}$, equivalently residing in the interior of the union of regions I and II of Fig. 2, and given $h > 0$, and $N \in \mathbb{N}_+$, then there exists $\varrho_0 > 0$, independent of $\varepsilon$, such that for all $\varrho \in (0, \varrho_0)$ the parameters are admissible and there is a unique solution $\tilde{p} = \tilde{p}(\tilde{q}, \varrho)$ to Eq. (2.39). Taking $l_0$ in (2.31) to satisfy,

$$l_0 = 1/\sqrt{2\eta_{\text{min},-}},

(2.40)$$
then the pulse overlaps \( \phi_i(q_j) = O(\epsilon) \) for \( i \neq j \). Moreover for \( \tilde{q}, \tilde{q}_s \in K \) satisfying \( |\tilde{q} - \tilde{q}_s| \leq 1 \) there exists \( c > 0 \) such that

\[
\| \Phi_{\tilde{q}} - \Phi_{\tilde{q}_s} \|_{X_\epsilon} \leq c|\tilde{q} - \tilde{q}_s|,
\]

and

\[
|\nabla_{\tilde{q}} \tilde{p}| \leq c\epsilon.
\] (2.42)

**Proof.** Defining \( G = (G_1, \ldots, G_N)^t \) by \( G_j(\tilde{p}, \tilde{q}, q) = p_j - \Theta(q_j; \tilde{p}, \tilde{q}) \), then the system (2.39) is equivalent to \( G = 0 \). It is clear that \( G(0, \tilde{q}, 0) = 0 \), and since \( \nabla_{\tilde{p}} G(0, \tilde{q}, 0) = I_{N \times N} \), from the implicit function theorem there exists \( \varrho_0 > 0 \) such that \( G(\tilde{p}, \tilde{q}, \varrho) = 0 \) has a unique, smooth solution \( \tilde{p} = \tilde{p}(\tilde{q}, \varrho) \), for all \( \varrho \in (0, \varrho_0) \). Moreover taking the \( \tilde{q} \) gradient of this equation we see that \( \nabla_{\tilde{q}} \tilde{p} = -[\nabla_p G]^\dagger \nabla_q G \). From (2.39) we have \( \| \nabla_{\tilde{q}} G \| = O(\epsilon) \), and (2.42) follows.

From Lemma 2.1 we see that \( \| \Theta_j \|_{L^\infty} \leq \varrho \sqrt{\frac{2\eta_{j,+}}{h}} \leq \varrho \sqrt{\frac{2\varrho_0}{h}} \), and hence from (2.34), we have \( \| \Theta \|_{L^\infty} \leq N \varrho \sqrt{\frac{2\varrho_0}{h}} \). In particular, for \( \varrho \) sufficiently small, in terms of \( a, \gamma, \) and \( N \), we have \( \eta_{j,-} = \eta_{0,-} - p_j > \eta_{0,-} - \| \Theta \|_{L^\infty} \equiv \eta_{\min,-} > 0 \), and by redefining \( \varrho_0 \) we verify the parameters are admissible for all \( \varrho \in (0, \varrho_0) \). Moreover, since \( \eta_{j,+} > \eta_{j,-} \) it follows that \( \eta_{\min,+} \equiv \eta_{0,+} - \| \Theta \|_{L^\infty} \geq \eta_{\min,-} > 0 \) and we may take \( l_0 = 1/\sqrt{2\eta_{\min,-}} \geq 1/\sqrt{2\eta_{\min,+}} \), in the definition of \( K \). Since the fast pulses all decay at a rate faster than \( l_0 \), the interactions between adjacent pulses is at most \( O(\epsilon) \). The result (2.41) follows readily from the continuity of \( \phi_j \), the exponential localization of the subpulses which comprise the first component of \( \Phi \), and the smallness of the derivative of third component of \( \Phi \) given by (2.37). \( \square \)

We remark that the pulse amplitudes which are controlled by \( \tilde{p} \) depend upon the pulse positions, \( \tilde{q} \), and will vary by an \( O(1) \) amount as the pulses evolve. For the two-pulse this dependence is made explicit in the following sub-section.

### 2.3. Two-pulse solutions

In the case of two pulses an explicit solution is possible. By symmetry the thermal amplitudes are equal, \( p_1 = p_2 = p \), and (2.39) reduces to

\[
p = \varrho \sqrt{\frac{2}{\hbar}} \sqrt{\eta_0 - \varrho \left( 1 + e^{-2\varrho \sqrt{\eta_0}} \right)} + O(\epsilon),
\]

(2.43)

where \( q = -q_1 = q_2 \). The thermal amplitude depends upon the pulse position via the relation

\[
p(q) = \frac{\varrho^2}{\hbar} \left( 1 + e^{-2\varrho \sqrt{\eta_0}} \right)^2 \left( 1 + \frac{4\eta_0 \varrho}{2\varrho^2 \left( 1 + e^{-2\varrho \sqrt{\eta_0}} \right)^2 - 1} \right),
\]

(2.44)

which varies by roughly a factor of four as the interpulse distance ranges from \( \Delta \gg \epsilon^{-1} \) to \( \Delta \ll \epsilon^{-1} \). The two pulse ansatz is then given by

\[
\Phi = \left( \begin{array}{ccc}
\alpha_1 \phi(x; p(q), -q) + \alpha_2 \phi(x; p(q), q) \\
0 \\
\Theta_1(x; -q) + \Theta_2(x; q)
\end{array} \right),
\]

(2.45)

where \( \phi \) is given by (2.25) and \( \Theta_j \) is given by (2.33).
3. Linear analysis

We decompose the solution of the TDNLS equation (2.5) as

\[
\tilde{U} = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \Phi(x; \vec{q}(t)) + W(x, t),
\]

(3.1)

where the remainder \( W = (W_1, W_2, W_3) \). In these coordinates the TDNLS equation can be written as

\[
W_t + \frac{\partial \Phi}{\partial \vec{q}} \dot{\vec{q}} = R + L_{\vec{q}} W + \mathcal{N}(W),
\]

(3.2)

where \( R \) is the residual given by

\[
R \equiv F(\Phi) = \begin{pmatrix} 0 \\ \sum_{j=1}^N (\eta_j - \eta(\Theta) + |\Xi|\phi_j^2 - \phi_j^2) \alpha_j \phi_j \end{pmatrix},
\]

(3.3)

and the linearized operator \( L_{\vec{q}} \) is given by

\[
L_{\vec{q}} = \begin{pmatrix} 0 & D & 0 \\ -C & -2 & 0 \\ 2\varepsilon^{-1} q\Xi & 0 & E \end{pmatrix},
\]

(3.4)

where

\[
C = -\frac{1}{2} \partial_x^2 - 3 \sum_{j,k} \alpha_k \alpha_j \phi_k \phi_j + \eta_+(\Theta),
\]

(3.5)

\[
D = -\frac{1}{2} \partial_x^2 - \sum_{j,k} \alpha_k \alpha_j \phi_k \phi_j + \eta_-(\Theta),
\]

(3.6)

\[
E = \varepsilon^{-2} \partial_x^2 - h.
\]

(3.7)

The nonlinearity is given by

\[
\mathcal{N}(W) = \begin{pmatrix} -2W_1W_2\Xi - (W_1^2 + W_2^2)W_2 \\ (3W_1^2 + W_2^2)\Xi + (W_1^2 + W_2^2)W_1 \\ \varepsilon^{-1} q(W_1^2 + W_2^2) \end{pmatrix}.
\]

(3.8)

While the essential spectra of \( C \) and \( D \) are independent of \( \vec{q} \), their point spectra are controlled by the localized operators

\[
\overline{C}_j = -\frac{1}{2} \partial_x^2 - 3\phi_j^2 + \eta_{j,+},
\]

(3.9)

\[
\overline{D}_j = -\frac{1}{2} \partial_x^2 - \phi_j^2 + \eta_{j,-},
\]

(3.10)

for \( j = 1, \ldots, N \).
Lemma 3.1. The operators $C$ and $D$ are self-adjoint with essential spectra consisting of the intervals $[\sqrt{\eta_0}, \infty)$ and $[\sqrt{\eta_0}, -\infty)$ respectively. Moreover if the parameters are admissible, then there exist $\nu_C, \nu_D > 0$ such that the sets $\{ \lambda \in \sigma_p(C) | \Re \lambda < \nu_C \}$ and $\{ \lambda \in \sigma_p(D) | \Re \lambda < \nu_D \}$ agree, up to multiplicity and to within $O(\varepsilon)$, with the union of the point spectra of the operators $\tilde{C}_j$ and $\tilde{D}_j$, respectively. More specifically

$$\sigma_p(C) \cap \{ \Re \lambda < \nu_C \} = \bigcup_{j=1}^{N} \{-3\eta_{j,+}, 0\} + O(\varepsilon), \tag{3.11}$$

$$\sigma_p(D) \cap \{ \Re \lambda < \nu_D \} = \bigcup_{j=1}^{N} \{ -2\sqrt{\gamma^2 - 1} \} + O(\varepsilon), \tag{3.12}$$

with the corresponding eigenfunctions of $C$ and $D$ being comprised, at leading order, of linear combinations of the associated eigenfunctions of the $C_j$ and $D_j$ respectively.

Proof. The functions $\eta_\pm(\Theta)$ vary slowly, with an $O(\varepsilon)$ derivative, over the support of the potentials in (3.5)–(3.6). The preservation of exponential dichotomies for systems with slowly varying coefficients (see Ref. [8] and Appendix A.2 of Ref. [21]), guarantees that the stable (unstable) manifold at $+\infty$ ($-\infty$) for the first order systems associated to $(C - \lambda)W = 0$ and $(D - \lambda)W = 0$ depend smoothly upon the slow length scale parameter $\varepsilon$ as $\varepsilon \to 0$. In particular the point spectrum and associated eigenfunctions of the slowly varying operator may be obtained as a regular perturbation expansion from the associated constant–coefficient operators for those $\lambda$’s where both the slowly varying and constant coefficient operators have analytic exponential dichotomies, i.e., away from the union of the essential spectra.

To investigate in detail the splitting of the eigenvalues and eigenvectors, we consider the operator $D$, for which the sub-operators $\tilde{D}_j$ each have a single, common eigenvalue $-2\sqrt{\gamma^2 - 1}$ with eigenfunction $\phi_j$. Since the parameters are admissible

$$\nu_D = \frac{1}{2} \inf_{j=1, \ldots, N} \sigma_e(\tilde{D}_j) = \frac{1}{2} \inf_{j=1, \ldots, N} \sqrt{2\eta_{j,-}} > 0,$$

and any $\lambda \in (-\infty, \nu_D]$ is an $O(1)$ distance from the essential spectrum of any of $\tilde{D}_j$. For such $\lambda$ any solution to the eigenvalue problem $D \psi = \lambda \psi$ has an expansion $\lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$, and $\psi = \psi_0 + \varepsilon \psi_1 + \cdots$, where $\psi_0 = \sum_{j=1}^{N} \beta_j \phi_j$, and $\lambda_0 = -2\sqrt{\gamma^2 - 1}$. At leading order $\psi_1$ satisfies

$$(D - \lambda_0)\psi_1 = -\varepsilon^{-1}(D - \lambda_0)\psi_0 + \lambda_1 \psi_0. \tag{3.13}$$

The right-hand side of (3.13) can be written as a sum of terms each of which is exponentially localized about $x = q_j$, the left-hand side can then be approximated, at leading order by $\sum_{j=1}^{N} (\tilde{D}_j - \lambda_0)\psi_{1,j}$ where

$$(\tilde{D}_j - \lambda_0)\psi_{1,j} = \varepsilon^{-1}[(\beta_{j+1}\phi_{j+1} + \beta_{j-1}\phi_{j-1} + \beta_j(\alpha_{j-1}\phi_{j-1} + \alpha_{j+1}\phi_{j+1}))\alpha_j \phi_j^2 - \beta_j(\Theta - \Theta(q_j))\phi_j] + \lambda_1 \beta_j \phi_j. \tag{3.14}$$
The unknowns $\vec{\beta} = (\beta_1, \ldots, \beta_N)$ and $\lambda_1$ are solved for by the Fredholm alternative, demanding that the right-hand sides of (3.14) for $j = 1, \ldots, N$ be orthogonal to the kernel, $\phi_j$ of $D_j - \lambda_0$. This leads to the conditions $\lambda_1 \in \sigma(\hat{D})$ with $\vec{\beta} \in \text{Ker}(\hat{D} - \lambda_1)$, where $\hat{D}$ has the tridiagonal form

$$
\hat{D} = -\varepsilon^{-1}(\text{diag}(\|\phi_1\|_{L^2}^2, \ldots, \|\phi_N\|_{L^2}^2))^{-1} \begin{pmatrix}
  d_{11} & \delta_{12} & \cdots & 0 \\
  \delta_{12} & d_{22} & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \delta_{N-1,N} \\
  0 & \cdots & \delta_{N-1,N} & d_{NN}
\end{pmatrix},
$$

(3.15)

where

$$
d_{ii} \equiv 2\alpha_i(\alpha_i - 1)\delta_{i,i-1} + \alpha_{i+1}\delta_{i,i+1} - (\Theta - \Theta(q_j), \phi_j^2)_{L^2},$$

and $\delta_{i,j} \equiv (\phi_i, \phi_j^2)_{L^2}$, for $i, j = 1, \ldots, N$ with $\alpha_0 = \alpha_{N+1} = 0$. For $\vec{q} \in \mathcal{K}$ we have $\delta_{j,j+1} = O(\varepsilon)$ and the $N$ eigenvalues split at $O(\varepsilon)$. We remark that the impact of the thermal perturbation, $(\Theta - \Theta(q_j), \phi_j^2)_{L^2} = O(\varepsilon^2)$ since $\phi_j$ is even about $q_j$ and $\Theta = \Theta(\varepsilon x)$ is slowly varying. The thermal perturbations have only a higher order impact on the eigenvalue splitting, unless the pulse separations are so great that $\delta_{ij} = O(\varepsilon^2)$, in which case the thermal perturbations prevent the eigenvalue splitting from becoming exponentially small.

The essential spectra of $C$ and $D$ depend only upon the limiting operators, $C_0 = -\frac{1}{2} \partial_x^2 + \eta_{0,+}$ and $D_0 = -\frac{1}{2} \partial_x^2 + \eta_{0,-}$, and are comprised of the intervals $[\eta_{0,+}, \infty)$ and $[\eta_{0,-}, \infty)$ respectively.

3.1. The reduced operators

A key step in the renormalization group treatment is the replacement of the exact linearization with a reduced operator whose spectral and semigroup properties are easier to analyze, yet such that the difference between the exact and the reduced operator, the secularity, does not lead to growth of the remainder $W$. Due to the contractivity of the resolvent operator $(E - \lambda)^{-1}$, the $N$-pulse potential which comprises the $L = \Xi$ component of $L_\vec{q}$ can be replaced with a finite-rank operator, $\mathcal{S}_N = \vec{\xi} \otimes \vec{\phi}$ whose range $\vec{\xi}$ consists of $N$ shifted copies of a smooth, exponentially decaying, mass 1 function $\xi$,

$$
\vec{\xi} = (\xi_1 \ldots \xi_N),
$$

(3.16)

with $\xi_j(x) = \xi(x - q_j)$, and

$$
\vec{\phi} = (\alpha_1 \phi_1 \ldots \alpha_N \phi_N)^t.
$$

(3.17)

The reduced linearization is

$$
\tilde{L}_\vec{q} = \begin{pmatrix}
  0 & D & 0 \\
  -C & -2 & -\mathcal{S} \\
  2\varepsilon^{-1}q \mathcal{S}_N & 0 & E
\end{pmatrix}.
$$

(3.18)
3.2. The point spectrum

The $N$-pulse profiles which comprise the manifold $\mathcal{M}$, defined in (2.30), are not stationary solutions, and as such it is not self-consistent to determine their linear stability in terms of the spectrum of the associated linearized operator. We say that the $N$-pulse ansatz $\Phi$ is spectrally compatible with the manifold $\mathcal{M}$ if the spectrum of the associated linear operator can be decomposed into a part strictly contained within the left-half complex plane and a finite-dimensional part whose associated eigenspace approximates the tangent plane of $\mathcal{M}$ at $\Phi_{\vec{q}}$.

The Green’s function of $E$,

$$G_0(x; \lambda) = (E - \lambda)^{-1} \delta,$$  \hspace{1cm} (3.19)

plays a central role in the reduction of the eigenvalue problem. It can be represented in terms of its Fourier transform,

$$\widehat{G}_0(k) = -\frac{1}{\sqrt{2\pi}} \frac{1}{\varepsilon^{-2}k^2 + (\lambda + h)},$$  \hspace{1cm} (3.20)

or in spatial coordinates as

$$G_0(x) = -\frac{\varepsilon}{\sqrt{\lambda + h}} e^{-\varepsilon \sqrt{\lambda + h}|x|},$$  \hspace{1cm} (3.21)

where on the branch cut $B = (-\infty, -h]$ we choose the square root so that $\Re \sqrt{\lambda + h} < 0$ for $\lambda \in \mathbb{C} \setminus B$.

A central role is played in the analysis by the commutator of the resolvent of $E$ and the scaling operator, $S_{\varepsilon}$, and by the strong contractivity of the resolvent on zero-mass functions.

**Lemma 3.2.** Let $\lambda \in \mathbb{C}$ be an $O(1)$ distance from the branch cut $B$, then the following estimates hold uniformly in $\lambda$,

$$\| S_{\varepsilon}^{-1}(E - \lambda)^{-1} f \|_{H^1} \leq c \varepsilon \| f \|_{L^1},$$  \hspace{1cm} (3.22)

$$\| S_{\varepsilon}^{-1}(E - \lambda)^{-1} S_{\varepsilon} f \|_{H^1} \leq c \| f \|_{L^2}.$$  \hspace{1cm} (3.23)

If moreover for some $\vec{q} \in K$, $f$ has small mass in each window, then we have the improved estimate

$$\| S_{\varepsilon}^{-1}(E - \lambda)^{-1} f \|_{H^1} \leq c \varepsilon \left( \sum_{j=1}^{N} |\chi_j f| + \varepsilon^{\frac{1}{2}} \| f \|_{H^1_{\vec{q}}} \right),$$  \hspace{1cm} (3.24)

where $\{\chi_1, \ldots, \chi_N\}$ is the partition of unity associated to $\vec{q}$, and $c > 0$ is independent of $\vec{q}$.

**Proof.** Writing $g = (E - \lambda)^{-1} f$ as a convolution of $f$ with the Green’s function, $g = G_0 * f$, we observe that

$$(S_{\varepsilon}^{-1}g)(x) = \varepsilon^{-1} (S_{\varepsilon}^{-1}G_0) * (S_{\varepsilon}^{-1}f)(x).$$  \hspace{1cm} (3.25)
Taking the $H^1$ norm of (3.25), and observing that $S_e^{-1}G_0$ decays exponentially at an $O(1)$ rate we have

$$
\|S_e^{-1}g\|_{H^1} \leq \varepsilon^{-1}\|S_e^{-1}G_0\|_{H^1} \|S_e^{-1}f\|_{L^1} \leq c\|S_e^{-1}f\|_{L^1} \leq \varepsilon\|f\|_{L^1},
$$

(3.26)

which establishes (3.22). Setting $\tilde{g} = G_0 * (S_e f)$ we see that $S_e^{-1}\tilde{g} = \varepsilon^{-1}(S_e^{-1}G_0) * f$ so that

$$
\|\tilde{g}\|_{H^1} \leq c\varepsilon^{-1}(\|S_e^{-1}G_0\|_{L^1} + \|\partial_x S_e^{-1}G_0\|_{L^1})\|f\|_{L^2} \leq c\|f\|_{L^2},
$$

(3.27)

which establishes (3.23).

For the small mass estimate we decompose $f$ as $f = f_1 + \cdots + f_N$ where $f_j = \chi_j f$. By linearity $g = g_1 + \cdots + g_N$ where $g_j = (E - \lambda)^{-1}f_j$, and $\|f\|_{H^1_{1,q}} = \|f_1\|_{H^1_{1,1}} + \cdots + \|f_N\|_{H^1_{1,N}}$. By assumption each $f_j$ has small mass and may be decomposed as a small mass and a massless part

$$
f_j = \tilde{f}_j \rho_j + \gamma_j',
$$

(3.28)

where $\gamma_j = \rho_0(x - q_j)$, with $\rho_0 = 1$ and $\|\rho_0\|_{H^1} = 1$. Moreover we may choose $\gamma_j$ to satisfy

$$
\|\gamma_j\|_{L^2} + \|\gamma_j'\|_{H^1_{1,j}} \leq c\|f_j\|_{H^1_{1,j}},
$$

(3.29)

since for any function $f_j$ with zero mass satisfying $\gamma_j' = f$ we have the bound

$$
\|\gamma_j\|_{L^2}^2 = \int \gamma_j^2 \partial_x (x - q_j) \, dx = \int (x - q_j)\gamma_j f_j \, dx \leq c\|\gamma_j\|_{L^2}\|f_j\|_{L^1_{1,j}},
$$

(3.30)

where we integrated by parts to obtain the second equality. Using the decomposition (3.28) of $f_j$, we write $g_j = g_{j1} + g_{j0}$ where $g_{j1} = (E - \lambda)^{-1}\tilde{f}_j \rho_j$ is estimated from (3.22) as $\|S_e^{-1}g_{j1}\|_{H^1} \leq \|\tilde{f}_j \rho_j\|_{L^1} \leq c\varepsilon\|\tilde{f}_j\|$. For the zero mass term $g_{j0} = (E - \lambda)^{-1}f'$, we write the convolution as $g_{j0} = G_0 * \gamma_{j0}' = G_0' * \gamma_{j0}$, and

$$
S_e^{-1}g_{j0} = \varepsilon^{-1}(S_e^{-1}G_0') * (S_e^{-1}\gamma_j).
$$

(3.31)

Since $G_0$ is slowly varying, while $S_e^{-1}G_0$ decays at an $O(1)$ exponential rate we have the bounds

$$
\|S_e^{-1}g_{j0}\|_{L^2} \leq c\|S_e^{-1}G_0'\|_{L^1}\|S_e^{-1}\gamma_j\|_{L^2} \leq c\varepsilon\|S_e^{-1}\gamma_j\|_{L^2} \leq c\varepsilon^2\|\gamma_j\|_{L^2}
$$

$$
\leq c\varepsilon^2\|f_j\|_{L^2_{1,j}}.
$$

(3.32)

Moreover taking the Fourier transform of the scaled $g_{j0}$ we have

$$
\hat{S}_e^{-1}g_{j0}(k) = \varepsilon \hat{g}_{j0}(\varepsilon k) = \frac{1}{\sqrt{2\pi}} \frac{\varepsilon^2 k \hat{\gamma}_j(\varepsilon k)}{k^2 + (\lambda + h)},
$$

(3.33)
so that
\[
\| \partial_k \mathcal{S}_e^{-1} g_{j_0} \|_{L^2} = \| k \mathcal{S}_e^{-1} g_{j_0} \|_{L^2} \leq \varepsilon^2 \| \mathcal{S}_e \hat{g}_{j_0} \|_{L^2} \leq c \varepsilon^3 \| \hat{g}_{j_0} \|_{L^2} \leq c \varepsilon^2 \| f_j \|_{L^2_{1,j}}.
\] (3.34)

The inequalities (3.32) and (3.34), taken together yield (3.24).

For \( j = 1, \ldots, N \) we introduce the \( E \)-resolvent of the shifted \( \xi \) functions (3.16),
\[
G_j(x; \lambda) = 2 \varepsilon^{-1} \varrho(E - \lambda)^{-1} \xi_j.
\] (3.35)

**Lemma 3.3.** The functions \( G_j \) are smoothed, scaled, and shifted approximates of the Green’s function of \( E - \lambda \). In particular
\[
\| \mathcal{S}_e^{-1} G_j \|_{H^1_{1,j}} \leq c \varrho,
\] (3.36)

and
\[
G_j(x, \lambda) = 2 \varrho \varepsilon^{-1} G_0(x - q_j) + \mathcal{O}(\varepsilon),
\] (3.37)
in \( L^2 \).

**Proof.** The \( H^1 \) bound on \( G_j \) follows from (3.35) and (3.22) since \( \| \xi_j \|_{L^2_{1,j}} = \mathcal{O}(1) \). It remains to establish the weighted part of the estimate. Without loss of generality we shift \( \xi_j, G_j \), and the weight function to be centered at the origin and drop the \( j \) subscript. Observing that
\[
\hat{\mathcal{S}}_e^{-1} G = \varepsilon \mathcal{S}_e \hat{G} = \frac{1}{\sqrt{2\pi}} \frac{\varepsilon \hat{\xi}(\varepsilon k)}{k^2 + (\lambda + h)},
\] (3.38)
we find
\[
\| |x| \mathcal{S}_e^{-1} G \|_{L^2} = \| \partial_k \mathcal{S}_e^{-1} G \|_{L^2},
\] (3.39)
however
\[
\partial_k \mathcal{S}_e^{-1} G(k) = \frac{1}{\sqrt{2\pi}} \left( \frac{\varepsilon^2 \hat{\xi}'(\varepsilon k)}{k^2 + (\lambda + h)} - \frac{2 \varepsilon k \hat{\xi}(\varepsilon k)}{(k^2 + (\lambda + h))^2} \right),
\] (3.40)
so that
\[
\| |x| \mathcal{S}_e^{-1} G \|_{L^2} \leq c \varepsilon \| \hat{\xi} \|_{L^\infty} + \varepsilon^\frac{3}{2} \| \hat{\xi}' \|_{L^2} \leq c \varepsilon \| \xi \|_{L^2_1} = \mathcal{O}(1).
\] (3.41)

For the second estimate, since \( \xi \) has mass 1, its Fourier transform has the expansion
\[
\hat{\xi}_j(k) = e^{ikq_j} (\hat{\xi}(0) + k \hat{\xi}'(0) + \cdots) = e^{ikq_j} \left( \frac{1}{\sqrt{2\pi}} + k \hat{\xi}'(0) + \cdots \right),
\] (3.42)
locally about \( k = 0 \). Taking the Fourier transform of \( G_j \) we have
\[
\hat{G}_j(k; \lambda) = -2\varrho \frac{\varepsilon^{-1}e^{ikq_j}}{\varepsilon^{-2}k^2 + (\lambda + h)} + \mathcal{O}(\varepsilon),
\]
(3.43)
in the \( L^2 \) norm. Taking the inverse Fourier transform we obtain (3.37) where the first term on the right-hand side is the shifted Green’s function, \((E - \lambda)^{-1}\delta_{x=q_j} \). \( \square \)

To aid in the characterization of the point spectrum of \( \tilde{L} \) we denote by \( P \) the upper-left \( 2 \times 2 \) sub-operator of \( \tilde{L} \) and introduce the localized operators \( \tilde{P}_j \), obtained from \( P \) by replacing \( C \) and \( D \) with \( C_j \) and \( D_j \) for \( j = 1, \ldots, N \),
\[
\tilde{P}_j = \begin{pmatrix} 0 & D_j \\ -\overline{C}_j & -2 \end{pmatrix}.
\]
(3.44)
The point spectrum of the operator \( P \) which is an \( \mathcal{O}(1) \) distance from
\[
\sigma_e(\tilde{P}) \equiv \bigcup_{j=1}^N \sigma_e(\tilde{P}_j),
\]
(3.45)
can be expressed as a regular perturbation expansion of the eigenvalues of
\[
\sigma_p(\tilde{P}) \equiv \bigcup_{j=1}^N \sigma_p(\tilde{P}_j).
\]
(3.46)
We define
\[
\nu_e = -\sup\{ \Re \lambda \mid \lambda \in \sigma_e(\tilde{P}), \overline{q} \in K \},
\]
(3.47)
and remark that the parameters \( a, \gamma, \varrho, h, \) and \( N \) are admissible exactly when the corresponding \( \nu_e > 0 \).

As is typical for the NLEP problem associated with semistrong pulse interactions, see [4,5], the point spectrum of \( \tilde{L} \) consists of two disjoint sets, the unperturbed spectrum, which is also part of \( \sigma_p(P) \) and the perturbed spectrum which interacts with the finite rank operator \( \Sigma_N \), and which corresponds to the kernel of the NLEP operator given in (3.56).

**Proposition 1.** Let the parameters be admissible. For any \( \nu \in (0, \nu_e) \) the point spectrum of \( \tilde{L} = \tilde{L}\overline{q} \) can be decomposed as \( \sigma_p(\tilde{L}) \cap \{ \Re \lambda > -\nu \} = \sigma_p \cup \sigma_M \), where the set \( \sigma_p \) consists of those \( \lambda \in \sigma_p(P) \cap \{ \Re \lambda > -\nu \} \) for which the associated eigenfunction \( (\Psi_1, \Psi_2) \) satisfies \( \otimes \overline{\phi}_1 \Psi_1 = 0 \) while the set \( \sigma_M \) is defined by
\[
\sigma_M = \{ \lambda \mid \Re \lambda > -\nu \text{ and } 1 \in \sigma(M) \},
\]
(3.48)
where the \( N \times N \) matrix \( M \) is given by
\[
M(\lambda) = \otimes \overline{\phi}_1 D(CD + \lambda(\lambda + 2))^{-1} \Sigma \hat{G}.
\]
(3.49)
Proof. Since $\sigma_e(P) \subset \{ \Re \lambda < -\nu_e \}$ it follows that $\sigma(P) \cap \{ \Re \lambda > -\nu_e \} \subset \sigma_p(P)$. The eigenvalue problem for the reduced operator is written as
\[
\begin{pmatrix}
-\lambda & D & 0 \\
-C & -2 - \lambda & -\Sigma \\
2\varepsilon^{-1}q \Sigma_N & 0 & E - \lambda
\end{pmatrix} \Psi = 0. \tag{3.50}
\]
Addressing the third component of this equation we solve for $\Psi_3$,
\[
\Psi_3 = -2\varepsilon^{-1}q(E - \lambda)^{-1} \Sigma_N \Psi_1, \tag{3.51}
\]
which can be written in terms of the smoothed Green’s functions (3.35) as
\[
\Psi_3 = -\tilde{G} \otimes \tilde{\phi} \Psi_1, \tag{3.52}
\]
where we have introduced $\tilde{G} = (G_1, \ldots, G_n)$. Eliminating $\Psi_3$ from the first two rows of (3.50) we arrive at a finite rank perturbation of the eigenvalue problem for $P$, the NLEP equation,
\[
(P - \lambda) \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = -\begin{pmatrix} 0 \\ \Sigma \end{pmatrix} (\tilde{G} \otimes \tilde{\phi}) \Psi_1. \tag{3.53}
\]
As seen in [10,14], $P - \lambda$ has the inverse
\[
(P - \lambda)^{-1} = \begin{pmatrix}
-(\lambda + 2)(DC - z)^{-1} & -D(CD - z)^{-1} \\
C(DC - z)^{-1} & -\lambda(CD - z)^{-1}
\end{pmatrix}, \tag{3.54}
\]
where $z \equiv -\lambda(\lambda + 2)$. Thus either $\lambda \in \sigma_p(P)$ and $\otimes \tilde{\phi} \Psi_1 = 0$ or the eigenvalue system reduces to a scalar equation for $\Psi_1$,
\[
\Psi_1 = D(CD - z)^{-1} \Sigma (\tilde{G} \otimes \tilde{\phi}) \Psi_1. \tag{3.55}
\]
This is the dichotomy which characterizes $\sigma_P$ and $\sigma_M$. We remark that solving Eq. (3.55) can be recast as finding the kernel of the scalar NLEP operator,
\[
\mathcal{L} \Psi = (CD - z)D^{-1} \Psi - \Sigma (\tilde{G} \otimes \tilde{\phi}) \Psi. \tag{3.56}
\]
Acting on (3.55) with $\otimes \tilde{\phi}$ we obtain the matrix system
\[
(I - M) \otimes \tilde{\phi} \Psi_1 = 0, \tag{3.57}
\]
where $M$ is the $N \times N$ matrix given by (3.49), or equivalently
\[
M_{jk} = \langle D(CD - z)^{-1} \Sigma G_j, \alpha_k \phi_k \rangle_{L^2}. \tag{3.58}
\]
Lemma 3.4. Let $\nu_e$ be given by (3.47) and fix $\nu_0 \in (0, \nu_e)$. For any $\nu \in (0, \nu_0)$ the set $\sigma_p(P) \cap \{ \Re \lambda > -\nu \}$ is in one-to-one correspondence, up to multiplicity and to within $O(\varepsilon)$, with the set $\sigma_p(\overline{P}) \cap \{ \Re \lambda > -\nu \}$. □
Proof. For $\nu$ defined as above, the operators $P$ and $P_j$ have only point spectra on the set $\Re \lambda > -\nu$; in particular this set is an $O(1)$ distance from the branch cut $B$ and the set $\sigma_e(P)$. The remainder of the proof follows from the persistence of exponential dichotomies under slowly varying perturbations, along the lines of Lemma 3.1. \qed

The spectrum of the localized operators $P_j$ is bounded in Theorem 3.3 of [14], in particular the PNLS stability region $I$ depicted in Fig. 2 is shown to be nonempty. The lemma below adapts these results to the optical component $P$ of the TDNLS equations.

Lemma 3.5. There exists $a_c > \sqrt{105/272}$ such that for all $\gamma > 1$, $h > 0$, $N \in \mathbb{N}_+$, and $a_0 \in (\sqrt{\gamma^2 - 1}, a_c)$,

$$a_0 \in \left(\sqrt{\gamma^2 - 1}, a_c\right),$$

(3.59)

there exists $\varrho_0 > 0$ and $\nu_p > 0$ such that for all $\varrho \in (0, \varrho_0)$ the parameters are admissible and the point spectrum of the frozen operator satisfies

$$\sigma_p(P) - \{0\} \subset \{\lambda \mid \Re \lambda < -\nu_p\},$$

for all $\bar{q} \in K$. Moreover the eigenvalue of each $P_j$ at $\lambda = 0$ is simple with eigenfunction $(\phi_j^0, 0)^t$. For $a_0 > a_c$, this result holds if $\gamma$ is close enough to 1.

Proof. For a single operator $P_j$ with $p_j = 0$, i.e. $\eta_{j,\pm} = \eta_{0,\pm}$ the result follows from [14]. Since for $\varrho$ sufficiently small each $p_j$ can be made small, independent of $\bar{q} \in K$, the result, particularly the uniformity of $\nu_p$, follows. \qed

Remark 3.1. Numerically it can be shown that if $a_0$ and $\gamma$ lie in region $I$ of Fig. 2, then a single pulse in a constant thermal background is spectrally stable. The admissibility of the parameters is equivalent to the essential spectrum of each operator $P_j$ lying strictly in the left-half complex plane, which is again equivalent to $a_0 - p_j > \sqrt{\gamma^2 - 1}$.

We expand slightly the definition of $\sigma_P$.

Proposition 2. Fix $\nu \in (0, \min\{\nu_e, \nu_p\})$, and let the parameters satisfy (3.59) and $\varrho \in (0, \varrho_0)$ with $\varrho_0$ given by Lemma 3.5. Then

$$\sigma(L) \cap \{\Re \lambda > -\nu\} = \sigma \prime_p \cup \sigma_M,$$

(3.60)

where $\sigma_M$ is as given in (3.48) and $\sigma \prime_p$ consists of eigenvalues from $\sigma_p(P) \cap \{\Re \lambda > -\nu\}$ whose associated eigenfunction $(\Psi_1, \Psi_2)^t$ satisfies $|\otimes \phi \Psi_1| = O(\varepsilon)$. Moreover $\sigma \prime_p$ consists of $N$ eigenvalues within $O(\varepsilon)$ of zero and the corresponding eigenfunctions, $\Psi_j$, of $L$ satisfy

$$\left\| \mathcal{S}^{-1}_\varepsilon \left( \begin{pmatrix} \phi_j^0 \\ 0 \\ 0 \end{pmatrix} \right) \right\|_{H^1, L^1} + \left\| \mathcal{S}^{-1}_\varepsilon \Psi_j \right\|_{H^1, \bar{q}} = O(\varepsilon).$$

(3.61)
Proof. Lemmas 3.4 and 3.5 establish that \( \sigma_P(P) \cap \{ \Re \lambda > -\nu \} \) contains \( N \) eigenvalues, all within \( \varepsilon \) of zero. Moreover, up to \( O(\varepsilon) \) in the \( H^1 \) norm, the associated eigenfunctions \( (\Psi_{j,1}, \Psi_{j,2})' \) are given by \( (\phi_j', 0)' \). Since \( | \otimes \phi_j' | = O(\varepsilon) \), either \( \otimes \phi_j' = 0 \) and \( \lambda_j \in \sigma_P \) or the NLEP equation (3.53) can be treated as a regular perturbation expansion

\[
(P_j + \varepsilon P_{j,1} - \varepsilon \lambda_j) \left( \begin{pmatrix} \phi_j' \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} \right) = \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \left( \begin{pmatrix} \phi_j' \\ 0 \end{pmatrix} + \varepsilon \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} \right),
\]

(3.62)

for which the \( O(\varepsilon) \) terms are given by

\[
P_j \left( \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} \right) = (\lambda_j - P_{j,1}) \begin{pmatrix} \phi_j' \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix} \left( G_{j,1} \phi_j' + \varepsilon \begin{pmatrix} \Psi_{j,1} \\ \Psi_{j,2} \end{pmatrix} \right),
\]

(3.63)

and

\[
\Psi_{j,3} = \tilde{G} \otimes \phi_j (-\varepsilon)^{-1} \phi_j' + \Psi_{j,1}.
\]

(3.64)

The result (3.61) then follows from (3.36) \( \square \)

The following proposition characterizes \( \sigma_M \).

Proposition 3. Fix \( \nu \in (0, \min\{v_v, v_p\}) \), and let the parameters satisfy (3.59) and \( \varphi \in (0, \varphi_0) \) with \( \varphi_0 \) as in Lemma 3.5. Then the parameters are admissible and there exist \( R_0, \mu_0 > 0 \) such that \( \sigma_M \subset B(0, R_0) \) and for all \( \varphi \in K, \) the elements of \( \sigma_M \) are in one-to-one correspondence, up to multiplicity and to within \( O(\mu_0) \), with the set

\[
\sigma_{M_0} = \{ \lambda \mid \Re \lambda > -\nu \text{ and } 1 \in \sigma(M_0(\lambda)) \},
\]

(3.65)

where \( M_0 \) is given by (3.74) and \( \nu \) is as in Proposition 2. Moreover by taking \( \varphi_0 \) sufficiently small, independent of \( \varepsilon, \sigma_M \) is empty for all \( \varphi \in (0, \varphi_0) \) and \( \varphi \in K \).

If \( |\lambda| \gg 1 \) then we see from (3.37) that \( \| G_j \|_\infty \leq \frac{c}{\sqrt{1 + |\lambda|}} \ll 1 \). The entries \( M_{jk} \) of \( M \), given by (3.58) satisfy the bound

\[
|M_{jk}| \leq \| \varepsilon G_j \|_2 \| (C D - z)^{-1} D \phi_j \|_2 \leq \frac{c \| (C D - z)^{-1} \|}{\sqrt{1 + |\lambda|}}.
\]

(3.66)

However from Lemma 4.2 of [14] we know that \( \| (C D - z)^{-1} \| \leq c(1 + |\lambda|)^{-1} \) for \( |\lambda| \gg 1 \). In particular we see that \( |M_{jk}| \to 0 \) as \( |\lambda| \to \infty \) for \( \Re \lambda > -\nu \). Thus there exists an \( R_0 > 0 \), independent of \( \varphi \in K \), such that \( I + M \) is invertible for \( |\lambda| > R_0 \).

We assume that \( |\lambda| < R_0 \) and \( \Re \lambda > -\nu \). In this case \( G_j \), given by (3.37) is a slowly varying function of space. In particular we have

\[
M_{jk} = \sum_{l=1}^{N} \left( \alpha_l \phi_l G_j, \alpha_j (C D - z)^{-1} D \phi_j \right)_2
\]

(3.67)

\[
= \sum_{l=1}^{N} G_{jl} \left( \alpha_l \phi_l, \alpha_j (C D - z)^{-1} D \phi_j \right)_2 (1 + O(\varepsilon)),
\]

(3.68)
where $G_{jj} \equiv G_j(q_j)$. The functions $D\phi_j$ are strongly localized, decaying at an $O(1)$ exponential rate away from $x = q_j$. Similarly if $f$ is strongly localized about $x = q_j$ we have

$$(CD - z)^{-1} f = (C_j D_j - z)^{-1} f,$$  \hspace{1cm} (3.69)

up to $O(\varepsilon)$ in $L^2$ for $z$ at least $O(\varepsilon)$ away from the poles of either side. In particular

$$(\overline{C}_j D_j - z)^{-1} D_j \phi_j, \phi_j)_{L^2} = \begin{cases} O(e^{-\rho|q_j - q_l|}), & j \neq l, \\
(C_j D_j - z)^{-1} D_j \phi_j, \phi_j)_{L^2}, & j = l,
\end{cases}$$  \hspace{1cm} (3.70)

for some $\rho > 0$ and all $\Re \lambda > -\nu$. Since $l_0 > 1/\sqrt{2\eta_{j,+}}$ and $|q_j - q_l| \geq l_0 |\ln \varepsilon|$ we see that $e^{-\rho|q_j - q_l|} \leq e^{-l_0 |\ln \varepsilon|/\rho} \leq c\varepsilon^{\mu_0}$, where $\mu_0 = l_0 \rho$. Changing variables to $y = \sqrt{2\eta_{j,+}}(x - q_j)$ in the $j = l$ inner product yields

$$(\overline{C}_j D_j - z)^{-1} D_j \phi_j, \phi_j)_{L^2} = \eta_{j,+}^4 \left( \frac{1}{\eta_{j,+}^2} \right)^{-1} \overline{D}_0(p_j)\phi_0, \phi_0 \right)_{L^2},$$  \hspace{1cm} (3.71)

where $\overline{C}_0 = -\partial_y^2 - 6\phi_0^2 + 1$, $\overline{D}_0(p) = -\partial_y^2 - 2\phi_0^2 + \mu(p)$, $\mu(p) = \eta_{-}(p)/\eta_{+}(p)$, and $\phi_0(y) = \text{sech}(y)$. Introducing the $\lambda$-analytic function

$$R(\lambda, p) = \left( \overline{C}_0 \overline{D}_0(p_j) + \frac{\lambda(\lambda + 2)}{\eta_{+}^2(p)} \right)^{-1} \overline{D}_0(p_j)\phi_0, \phi_0 \right)_{L^2},$$  \hspace{1cm} (3.72)

we may write the matrix $M$ as

$$M_{jk} = \eta_{j,+}^4 G_{jk} R(\lambda, p_j) + O(\varepsilon^{\mu_0}),$$  \hspace{1cm} (3.73)

or equivalently $M = M_0 + O^{\mu_0}$ where

$$M_0 = G_{N \times N} \begin{pmatrix}
\eta_{1,+}^4 R(\lambda, p_1) & 0 & \cdots & 0 \\
0 & \eta_{2,+}^4 R(\lambda, p_2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \eta_{N,+}^4 R(\lambda, p_N)
\end{pmatrix},$$  \hspace{1cm} (3.74)

and $G_{N \times N}$ denotes the $N \times N$ matrix with entries $G_{jk}$.

Finally we observe that for the data as specified $|p_j| \leq cQ$ and thus $R(\lambda, p_j) \rightarrow R(\lambda, 0)$ as $Q \rightarrow 0$. Moreover from (3.36), $|G_{jk}| \leq cQ$ and as $Q \rightarrow 0$ we have $M_0 \rightarrow \eta_{0,+}^4 R(\lambda, 0) G_{N \times N}$, and $\|M_0\| \leq cQ$. In particular, for $Q$ sufficiently small, $-1 \notin \sigma(M_0)$. \hfill $\square$

To define the spectral projections associated to $\tilde{L}$ we must characterize the small point spectrum of the adjoint, $\tilde{L}^\dagger$.

**Lemma 3.6.** The adjoint eigenfunctions corresponding to the $N$ eigenvalues in $\sigma_p(\tilde{L}) \cap \{\Re \lambda > -\nu\}$ satisfy the estimate

$$\left\| \frac{z}{D^2_e} S_e^{-1} \left( \psi_j^+ - \left( \begin{array}{c}
\frac{1}{2} \phi_j' \\
0
\end{array} \right) \right) \right\|_{H^1} \leq c\varepsilon.$$  \hspace{1cm} (3.75)
Proof. The adjoint eigenvalue problem can be reduced to a $2 \times 2$ system

$$
(P^\dagger - \lambda) \begin{pmatrix} \Psi_1^\dagger \\ \Psi_2^\dagger \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2\varepsilon \Xi^\dagger \Xi e^{-1}(E - \lambda)^{-1} \Psi_2^\dagger,
$$

(3.76)

by solving for $\Psi_3^\dagger = (E - \lambda)^{-1} \Xi \Psi_2^\dagger$. However since $\Xi^\dagger = \tilde{\phi}^t \otimes (\xi_1 \ldots \xi_n)^t$, we may use the self-adjointness of $(E - \lambda)$ to rewrite (3.76) as

$$
(P^\dagger - \lambda) \begin{pmatrix} \Psi_1^\dagger \\ \Psi_2^\dagger \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} 2\varepsilon \tilde{\phi}^t \otimes \tilde{G}^t \Psi_2^\dagger.
$$

(3.77)

The kernels of the localized operators $P_j^\dagger$ are given by

$$
\begin{pmatrix} \Psi_{j,1}^\dagger \\ \Psi_{j,2}^\dagger \end{pmatrix} = \begin{pmatrix} \tilde{D}_j^{-1} \phi_j^t \\ \frac{1}{2} \phi_j^t \end{pmatrix} + \mathcal{O}(\varepsilon),
$$

(3.78)

in the $H^1$ norm. At leading order these functions are also in the kernel of $\otimes \tilde{G}^t$. Thus we can look for the small eigenvalue eigenfunctions of $\tilde{L}^\dagger$ as perturbations of these functions. Turning to the $\Psi_3$ component, the product $\Xi \Psi_2^\dagger$ has $\mathcal{O}(\varepsilon)$ mass in each $\vec{q}$-window and is $\mathcal{O}(1)$ in the $H^1_{1,\vec{q}}$ norm. From (3.24) we see that

$$
\| \tilde{S}_\varepsilon^{-1} \Psi_3^\dagger \|_{H^1} \leq c\varepsilon^{3/2},
$$

(3.79)

and the result (3.75) follows. 

3.3. Resolvent and semigroup estimates

We define the finite rank spectral projection onto the spectral set of $\tilde{L}$ corresponding to its $N$ small eigenvalues,

$$
\pi_{\vec{q}} F = \sum_{j=1}^N \frac{(F, \Psi_j^\dagger)_{L^2}}{(\Psi_j, \Psi_j^\dagger)_{L^2}} \Psi_j.
$$

(3.80)

The complementary spectral projection, $\tilde{\pi}_{\vec{q}} = I - \pi_{\vec{q}}$, has range $X_{\vec{q}}$, which corresponds to the functions which are uniformly contracted under $\tilde{L}_{\vec{q}}$ evolution. The following lemma bounds the commutator of the projection operator and the scaling operator.

Lemma 3.7. There exists $c > 0$ independent of $\varepsilon$ and $\vec{q} \in \mathcal{K}$ such that

$$
\| \pi F \|_{H^1, L^1} \leq c \| \tilde{D}_\varepsilon^3 F \|_{H^1, L^1},
$$

(3.81)

$$
\| \tilde{D}_\varepsilon^{-1} \tilde{S}_\varepsilon^{-1} \rho \tilde{S}_\varepsilon F \|_{H^1} \leq c \| \tilde{D}_\varepsilon^3 F \|_{H^1},
$$

(3.82)
and, if \( F = (0, 0, f_3) \) where \( f_3 \) has small mass in each \( \tilde{q} \) window, then

\[
\| \bar{D}_\varepsilon \pi F \|_{H^{1,1}_{H_1,\tilde{q}}} \leq c \varepsilon \frac{1}{\theta_0} \left( \sum_{j=1}^{N} |\chi_j f_3| + \varepsilon \frac{1}{\theta_0} \| f_3 \|_{H^1_{H_1,\tilde{q}}} \right).
\]  

(3.83)

**Proof.** The bounds (3.61) and (3.75) imply that \( \| \Psi_{j,3} \|_{L^2} \leq c \sqrt{\varepsilon} \) and \( \| \Psi^\dagger_{j,3} \|_{L^2} \leq c \varepsilon \) so that the inner products \((\Psi_j, \Psi^\dagger_j)_{L^2} = \theta_j + \mathcal{O}(\varepsilon)\), where \( \theta_j \) is defined in (4.15). Moreover Lemma 3.2 of [14] shows that the \( \theta_j \) are uniformly bounded from below by some \( \theta_0 > 0 \). From the structure of the projection operator, \( \pi \), we observe that

\[
\| \bar{D}_\varepsilon \pi F \|_{H^{1,1}_{H_1,\tilde{q}}} \leq \frac{c}{\theta_0} \sum_{j=1}^{N} \| \bar{D}_\varepsilon \Psi_j \|_{H^{1,1}_{H_1,\tilde{q}}}|(F, \Psi^\dagger_j)_{L^2}|.
\]

(3.84)

From (3.75) we calculate

\[
|\langle F, \Psi^\dagger_j \rangle_{L^2}| \leq c(\| (f_1, f_2) \|_{L^2} + \| \Psi^\dagger_{j,3} \|_{L^\infty} \| f_3 \|_{L^1})
\]

\[
\leq c(\| (f_1, f_2) \|_{L^2} + \| S_{\varepsilon}^{-1} \Psi^\dagger_{j,3} \|_{H^1} \| f_3 \|_{L^1})
\]

\[
\leq c(\| (f_1, f_2) \|_{L^2} + \varepsilon \frac{2}{\theta_0} \| f_3 \|_{L^1}) \leq c \| \bar{D}_\varepsilon^\dagger F \|_{H^{1,1}_{H_1,\tilde{q}}},
\]

(3.85)

(3.86)

where we used the bound (2.14) and (3.75). From (3.61) and (3.64) we readily see that

\[
\| \Psi_j \|_{H^{1,1}_{H_1,\tilde{q}}} \leq c(1 + \varepsilon (\| G_j \|_{H^1} + \| G_j \|_{L^1})) = \mathcal{O}(1).
\]

(3.87)

Combining this result with (3.86) establishes the estimate (3.81).

Addressing the inequality (3.82) we observe that

\[
|\langle S_{\varepsilon} f_3, \Psi^\dagger_{j,3} \rangle_{L^2}| = \varepsilon^{-1}|\langle f_3, S_{\varepsilon}^{-1} \Psi^\dagger_{j,3} \rangle_{L^2}| \leq c \varepsilon \frac{1}{\theta_0} \| f_3 \|_{L^2},
\]

(3.88)

which implies \(|\langle S_{\varepsilon} F, \Psi^\dagger_{j,3} \rangle_{L^2}| \leq c \| \bar{D}_\varepsilon^\dagger F \|_{H^1}\). From (3.61) we find that \( \| \bar{D}_\varepsilon^{-1} S_{\varepsilon}^{-1} \Psi_j \|_{H^1} = \mathcal{O}(1) \), and the desired result follows.

For the small mass case, with \( F = (0, 0, f_3) \), each \( f_{3,k} = \chi_k f_3 \) can be decomposed

\[
f_{3,k} = \tilde{f}_k \rho_k + \gamma_k,
\]

(3.89)

where \( \rho_k \) and \( \gamma_k \) are as in (3.28). The inner product can be estimated

\[
|\langle f_{3,k}, \Psi^\dagger_{j,3} \rangle_{L^2}| \leq c(|\tilde{f}_k| \| \Psi^\dagger_{j,3} \|_{L^\infty} + \| \gamma_k \|_{L^2} \| \partial_x \Psi^\dagger_{j,3} \|_{L^2})
\]

\[
\leq c(|\tilde{f}_k| \| S_{\varepsilon}^{-1} \Psi^\dagger_{j,3} \|_{H^1} + \varepsilon \| f_{3,k} \|_{H^1_{H_1,\tilde{q}}} \| \Psi^\dagger_{j,3} \|_{L^2})
\]

\[
\leq c \varepsilon \frac{1}{\theta_0} \left( |\tilde{f}_k| + \varepsilon \frac{1}{\theta_0} \| f_{3,k} \|_{H^1_{H_1,\tilde{q}}} \right).
\]

(3.90)
On the other hand from (3.61) and (3.64) we have the estimate

\[ \| \Psi_j \|_{H^1, H_{1, q}^1} \leq c (c + \varepsilon) \| \tilde{G} \|_{H_{1, q}^1} \leq c \varepsilon^{-\frac{1}{2}} \| \tilde{S}^{-1} \tilde{G} \|_{H_{1, q}^1} = \mathcal{O}(\varepsilon^{-\frac{1}{2}}), \]  

(3.91)

where we used (2.15) and (3.36). Together (3.90) and (3.91) yield (3.83). \( \square \)

We bound the resolvent of \( P \) as a map from \( H^1 \times H^1 \) into \( H^1 \times H^1 \). For this we introduce the eigenspaces of \( P \) and \( P^\dagger \) corresponding to eigenvalues with \( \Re \lambda > -\nu \),

\[ \mathcal{V} = \text{span} \{ (\Psi_1, \Psi_2)^t \in \ker(P - \lambda) \mid \lambda \in \sigma_p(P) \cap \{ \Re \lambda > -\nu \} \}, \]  

(3.92)

\[ \mathcal{V}^\dagger = \text{span} \{ (\Psi_1^\dagger, \Psi_2^\dagger)^t \in \ker(P^\dagger - \lambda) \mid \lambda \in \sigma_p(P) \cap \{ \Re \lambda > -\nu \} \}. \]  

(3.93)

**Proposition 4.** Let \( \nu \) be as in Proposition 2. For all \( F \in H^1 \times H^1 \) which are \( L^2 \) orthogonal to \( \mathcal{V}^\dagger \), we have the following bound

\[ \| (P - \lambda)^{-1} F \|_{H^1} \leq c \| F \|_{H^1}, \]  

(3.94)

where \( c > 0 \) may be chosen independent of \( \tilde{q} \in K \).

**Proof.** Since \( F \) is orthogonal to the eigenfunctions of \( P^\dagger \) we know that \( (P - \lambda)^{-1} F \in (H^1)^2 \) and is bounded for \( \Re \lambda > -\nu \). The issue is the uniformity of the bound as \( |\lambda| \to \infty \). Given \( \tilde{q} \in K \), consider the partition of the real line, \( \{s_0, s_1, \ldots, s_N\} \) where \( s_0 = q_1 - \sqrt{\varepsilon}, s_N = q_N + \sqrt{\varepsilon} \), and \( s_j = \frac{1}{2}(q_j + q_{j+1}) \) for \( j = 1, \ldots, N - 1 \). Let \( \chi_j \) be a partition of unity subordinate to the open cover \( \{ (s_j - 1, s_{j+1} + 1) \mid j = -1, \ldots, N + 1 \} \) where \( s_{-1} = -\infty \) and \( s_{N+1} = \infty \).

We decompose \( F = \sum_j F_j \mathcal{X}_j \) where \( F_j = F \mathcal{X}_j \). For \( j = 1, \ldots, N \) the function \( F_j \) is localized about \( q_j \) and hence \( (P - \lambda)^{-1} F_j \) is asymptotically close to \( (\tilde{P}_j - \lambda)^{-1} F_j \). Moreover, from Proposition 4.1 of [14] we know that the resolvent of the localized operators \( \tilde{P}_j \) decays to zero in \( H^1 \) as \( |\lambda| \to \infty \) for \( \Re \lambda > -\nu \). For the terms \( (P - \lambda)^{-1} (F_{-1} + F_{N+1}) \) we see that on the support of \( \mathcal{X}_{-1} \) and \( \mathcal{X}_{N+1} \), \( P \) is a small, compact perturbation of a constant coefficient operator and a slowly varying potential. By further subdividing the support of \( F \) we may approximate \( P \) by boundedly invertible constant coefficient operators on each sub-interval of \( (-\infty, s_{-1} + 1) \) and \( (s_{N+1} - 1, \infty) \). \( \square \)

The estimates on \( E \) and \( P \) in Lemma 3.2 and Proposition 4 permit us to bound the resolvent of \( \hat{L} \) in a rescaled norm, and estimate its commutator with the scaling operator.

**Proposition 5.** Let \( \nu \) be as in Proposition 2. For \( F \in X \) we have the following estimates on the resolvent of \( \hat{L} \), for \( c > 0 \) independent of \( \tilde{q} \in K \) and \( \Re \lambda > -\nu \),

\[ \| \tilde{S}^{-1} (\hat{L} - \lambda)^{-1} \tilde{F} \|_{H^1} \leq c \| \tilde{D}_F \|_{H^1, L^1}, \]  

(3.95)

\[ \| \tilde{S}^{-1} (\hat{L} - \lambda)^{-1} \tilde{F} \|_{H^1} \leq c \| F \|_{H^1}. \]  

(3.96)
If in addition $F = (0, 0, f_3)$ where the mass of $f_3$ is small in each $\tilde{q}$ window, then we have the improved estimate

$$\|S_\varepsilon^{-1}(\tilde{L} - \lambda)^{-1}\tilde{F}\|_{H^1} \leq ce\left(\sum_{j=1}^{N} |\chi_j f_3| + \varepsilon \frac{1}{2} \|f_3\|_{H^1_{\tilde{q}}} \right).$$

(3.97)

**Proof.** First we construct the resolvent of $\tilde{L}$ in terms of the resolvent of $P$. Fixing $F = (f_1, f_2, f_3)^t \in X$ and $\tilde{F} = \tilde{\pi} F = (\tilde{f}_1, \tilde{f}_2, \tilde{f}_3)^t \in X_{\tilde{q}}$ we consider the solution $\Psi \in X_{\tilde{q}}$ of $(\tilde{L} - \lambda)\Psi = \tilde{F}$. Solving for the third component, we find

$$\Psi_3 = \tilde{G} \otimes \tilde{\phi} \Psi_1 + (E - \lambda)^{-1} \tilde{f}_3,$$

(3.98)

and eliminating $\Psi_3$ from the residual, arrive at the $2 \times 2$ system,

$$(P - \lambda)\begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \tilde{f}_2 - \mathbb{E} \tilde{G} \otimes \tilde{\phi} \Psi_1 + \mathbb{E} (E - \lambda)^{-1} \tilde{f}_3 \end{pmatrix}.$$

(3.99)

Inverting $(P - \lambda)$, operating on the first row of the resultant vector with $\otimes \tilde{\phi}$, and solving for $\otimes \tilde{\phi} \Psi_1$ yields

$$\otimes \tilde{\phi} \Psi_1 = (I + M)^{-1} \otimes \tilde{\phi} \left[ (P - \lambda)^{-1} \begin{pmatrix} \tilde{f}_2 + \mathbb{E} (E - \lambda)^{-1} \tilde{f}_3 \end{pmatrix} \right],$$

(3.100)

where the matrix $M$ is as defined in (3.49). Substituting the expression in (3.100) for $\otimes \tilde{\phi} \Psi_1$ into (3.98)–(3.99) yields an explicit expression for $\Psi$ in terms of $F$. From (3.22) we see that

$$\|\mathbb{E} (E - \lambda)^{-1} \tilde{f}_3\|_{H^1} \leq c\|\mathbb{E}\|_{H^1} \left( \|E - \lambda\|_{L^\infty} + \|\partial_x (E - \lambda)^{-1} \tilde{f}_3\|_{L^2} \right)$$

(3.101)

$$\leq c\|S_\varepsilon^{-1}(E - \lambda)^{-1}\tilde{f}_3\|_{H^1} \leq c\varepsilon \|\tilde{f}_3\|_{L^1}.$$

(3.102)

The uniform boundedness of $(P - \lambda)^{-1}$ from $(\mathcal{V}^\dagger)_+ \subset (H^1_{1,\tilde{q}})^2$ into $(H^1_{1,\tilde{q}})^2$ follows from Proposition 4. The operand of $(P - \lambda)^{-1}$ in (3.100) is not in $(\mathcal{V}^\dagger)_+$ for arbitrary $\tilde{F} \in X_{\tilde{q}}$, however the quantity

$$\otimes \tilde{\phi} \left[ (P - \lambda)^{-1} \begin{pmatrix} \tilde{f}_2 + \mathbb{E} (E - \lambda)^{-1} \tilde{f}_3 \end{pmatrix} \right]$$

is a vector comprised of elements

$$\left( (P^\dagger - \lambda)^{-1} \begin{pmatrix} \phi_j \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{f}_2 + \mathbb{E} (E - \lambda)^{-1} \tilde{f}_3 \end{pmatrix} \right)_{L^2},$$

(3.103)

which are bounded since $(\phi_j, 0)^t$ is orthogonal to $\mathcal{V}$. Addressing Eq. (3.99), the choice of $\otimes \tilde{\phi} \Phi_1$ given by (3.100) is exactly what is required to render the right-hand side of (3.99) orthogonal to
\[ \forall. \text{From Proposition 3 we know that } I + M \text{ is uniformly invertible on } \Re \lambda > -\nu, \text{ and it follows that} \]

\[ \| (\Psi_1, \Psi_2)^t \|_{H^1} \leq c \| \tilde{D}_\epsilon \tilde{F} \|_{H^1,L^1}. \tag{3.104} \]

Turning to the third component of \( \Psi \), we act on (3.98) with \( S^{-1}_e \) and take the \( H^1 \) norm,

\[ \| S^{-1}_e \Psi_3 \|_{H^1} \leq c \left( \| S^{-1}_e G \|_{H^1} \| \Psi_1 \|_{L^2} + \| S^{-1}_e (E - \lambda)^{-1} \tilde{f}_3 \|_{H^1} \right) \]

\[ \leq c \left( \| \Psi_1 \|_{L^2} + \epsilon \| \tilde{f}_3 \|_{L^1} \right) \leq c \| \tilde{D}_\epsilon \tilde{F} \|_{H^1,L^1}, \tag{3.105} \]

where we used (3.23) and (3.22). Together (3.104) and (3.105) control the left-hand side of (3.95) by the quantity \( \| \tilde{D}_\epsilon \tilde{F} \|_{H^1,L^1} \). The estimate (3.81) implies that

\[ \| \tilde{D}_\epsilon \tilde{F} \|_{H^1,L^1} = \| \tilde{D}_\epsilon (F - \pi F) \|_{H^1,L^1} \leq c \left( \| \tilde{D}_\epsilon F \|_{H^1,L^1} + \| \tilde{D}_\epsilon^2 F \|_{H^1,L^1} \right), \tag{3.106} \]

which establishes (3.95).

Addressing the estimate (3.96), we set \( \tilde{F} = \tilde{\pi} \tilde{S}_e F = \tilde{S}_e H \) where \( H = (I - \tilde{S}_e^{-1} \pi \tilde{S}_e) F \). Reprising the argument, we find that (3.102) becomes

\[ \| \mathcal{E} (E - \lambda)^{-1} \tilde{f}_3 \|_{H^1} \leq c \left( \| S^{-1}_e (E - \lambda)^{-1} S_e h_3 \|_{H^1} \leq c \| h_3 \|_{L^2}, \right. \tag{3.107} \]

where we used (3.23). We conclude that \( \| (\Psi_1, \Psi_2)^t \|_{H^1} \leq c \| H \|_{H^1} \), and (3.105) becomes

\[ \| S^{-1}_e \Psi_3 \|_{H^1} \leq c \left( \| \Psi_1 \|_{L^2} + \| S^{-1}_e (E - \lambda)^{-1} S_3 h_3 \|_{H^1} \right) \leq c \| H \|_{H^1}, \tag{3.108} \]

where we again used the second member of (3.22). However from bound (3.82) we have \( \| H \|_{H^1} \leq c \| F \|_{H^1} \), which yields (3.96).

Turning to the small mass estimate (3.97), since \( H_{1,q} \) uniformly controls the \( L^1 \) norm, it is sufficient to consider only \( F \) with zero third component mass, a subspace which is invariant under the action of \( \tilde{S}_e \) and \( \tilde{D}_\epsilon \). We repeat the analysis which lead to (3.95) and (3.105), but apply (3.24) in the place of (3.22), picking up an extra factor of \( \epsilon^{\frac{1}{2}} \) multiplying the third component of \( F \). Moreover from (3.83), the projection \( \tilde{\pi} \) respects the \( H_{1,q} \) norm

\[ \| [\tilde{F}]_3 \|_{H_{1,q}} = \| f_3 - [\pi F]_3 \|_{H_{1,q}} \leq c \left( \| f_3 \|_{H_{1,q}} + \epsilon^{\frac{1}{2}} \| f_3 \|_{H_{1,q}} \right), \tag{3.109} \]

which establishes (3.97). \( \square \)

To exploit the contractiveness of the resolvent on subspaces we extend Theorem 2.1 of Xu and Feng [26] on decay estimates for \( C_0 \) semigroups to a class of singularly perturbed operators.

**Theorem 3.1.** Fix \( t_0 > 0 \). For each \( p \) in the index set \( \mathcal{P} \), let \( A_p \) be a closed, densely defined operator on a Hilbert space \( H \) with norm \( \| \cdot \|_H \). Let \( X \subset H \) be a Banach space with norm \( \| \cdot \|_X \). Let \( A_p \) generate a \( C_0 \) semigroup, \( S_p(t) \), let \( \pi_p \) be a spectral projection associated to \( A_p \), and let \( B_p \) and \( C_p \) be bi-continuous, invertible operators from \( H \) into \( H \). If for all \( \sigma > \nu \),
\[
\sup \left\{ \| C_p(A_p - \lambda)^{-1} \pi_p B_p^{-1} \|_{X \to H} + \| C_p(A_p - \lambda)^{-1} \pi_p C_p^{-1} \|_{H \to H} \mid \lambda > \sigma, p \in \mathcal{P} \right\} < \infty, \tag{3.110}
\]
then for all \( \delta > 0 \) there exists \( c_\delta > 0 \), independent of \( p \in \mathcal{P} \) and \( t > t_0 \), such that
\[
\| C_p S_p(t) \pi_p x \|_H \leq c_\delta e^{(\nu + \delta)t} \| B_p x \|_X, \tag{3.111}
\]
for all \( x \in X \).

**Proof.** Choose \( \delta > 0 \) and without loss of generality assume that \( \sigma \equiv \nu + \delta = 0 \). We must show there exists \( c > 0 \) such that \( \| C_p S_p(t) \pi_p x \|_H \leq c \| B_p x \|_X \), or equivalently that \( \| C_p S_p(t) \pi_p B_p^{-1} \|_{X \to H} \) is uniformly bounded for all \( t > t_0 \) and \( p \in \mathcal{P} \). The case when \( \sigma \neq 0 \) is handled by shifting the operator \( A_p \) and using the Cauchy integral theory to shift the contours of integration, see [26] for details.

Let \( x \in X \) and \( y \in H' = H \subset X' \) be arbitrary, and omitting the \( p \) subscript on the operators, we have the equality
\[
(\mathcal{C} S(t) \pi B^{-1} x, y)_H = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} e^{\lambda t} (\mathcal{C}(\lambda - A)^{-1} \pi B^{-1} x, y)_H d\lambda. \tag{3.112}
\]
Integrating by parts we have
\[
(\mathcal{C} S(t) \pi B^{-1} x, y)_H = \frac{1}{2\pi i} \lim_{\tau \to \infty} \left[ \frac{e^{\lambda t}}{t} (\mathcal{C}(\lambda - A)^{-1} \pi B^{-1} x, y)_H |_{0-i\tau}^{0+i\tau} \right.
\]
\[
+ \int_{0-i\tau}^{0+i\tau} \frac{e^{\lambda t}}{t} (\mathcal{C}(\lambda - A)^{-2} \pi B^{-1} x, y)_H d\lambda \right], \tag{3.113}
\]
\[
= \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \frac{e^{\lambda t}}{t} (\mathcal{C}(\lambda - A)^{-2} \pi B^{-1} x, y)_H d\lambda, \tag{3.114}
\]
where we used the property of the Laplace transform, that \( \| (\lambda + i\tau - A)^{-1} x \|_H \to 0 \) as \( \tau \to \infty \pm \infty \). We use the invertability of \( \mathcal{C} \) to reformulate (3.114) as
\[
(\mathcal{C} S(t) \pi B^{-1} x, y)_H = \frac{1}{2\pi i} \int_{0-i\infty}^{0+i\infty} \frac{e^{\lambda t}}{t} (\mathcal{C}(\lambda - A)^{-1} \pi B^{-1} x, C^{-1}(\lambda - A^\dagger)^{-1} \pi^\dagger C^\dagger y)_H d\lambda, \tag{3.115}
\]
so that
\[
\left\| (CS(t)\pi B^{-1}x, y) \right\|_H \leq \frac{c}{t} \int_{0-i\infty}^{0+i\infty} \left( \|C(\lambda - A)^{-1} \pi B^{-1}x\|_H^2 + \|C^{-1\dag}(\lambda - A^{-1}) \pi C^{-1\dag}y\|_H^2 \right) d\lambda.
\]

(3.116)

Since every Hilbert space is Fourier type 2, see [11], these integrals are convergent. Moreover
\[
\left\| C(\lambda - A)^{-1} \pi B^{-1}x \right\|_H \leq \left\| C(\lambda - A)^{-1} \pi B^{-1} \right\|_{X \rightarrow H} \|x\|_X,
\]
while
\[
\left\| C^{-1\dag}(\lambda - A^{-1}) \pi C^{-1}\pi B^{-1}x \right\|_H = \sup_{\|z\|_H = 1} (z, C^{-1\dag}(\lambda - A^{-1}) \pi C^{-1}z)_H
\]
\[
\leq \left\| C(\lambda - A)^{-1} \pi C^{-1} \right\|_{H \rightarrow H} \|z\|_H.
\]

(3.117)

(3.118)

so from (3.110) there exists a function \( \mu(x, y) \), independent of \( \epsilon \) and \( t > t_0 \), such that
\[
\left\| (CS(t)\pi B^{-1}x, y) \right\|_H \leq \frac{\mu(x, y)}{t}.
\]

(3.119)

It follows from the Banach–Steinhaus theorem, see for example Theorem 2.6 of [18], that since the sets \( \{CpSp(t)\pi pB^{-1}x \mid t > t_0, p \in \mathcal{P} \} \) are \( H \)-norm bounded for each \( x \in X \), the set of operators \( \{CpSp(t)\pi pB^{-1} \mid t > t_0, p \in \mathcal{P} \} \) is equicontinuous. In particular there exists a \( c > 0 \) such that
\[
\left\| CpSp(t)\pi pB^{-1} \right\|_{X \rightarrow H} \leq c,
\]
for all \( p \in \mathcal{P} \), which establishes (3.111).

Proposition 6. If the parameters satisfy (3.59) and \( \rho_0 \) is sufficiently small, then for all \( \delta > 0 \) setting \( \nu = \min\{\nu_e, \nu_p\} - \delta > 0 \) there exists \( M > 1 \), independent of \( \tilde{q} \in \mathcal{K} \), \( \epsilon \in (0, \varepsilon_0) \), and \( t > 0 \), such that the semigroup \( S_{\tilde{q}}(t) \) associated to \( \tilde{L}_{\tilde{q}} \) satisfies
\[
\left\| S_{\tilde{q}}(t)F \right\|_{X_\varepsilon} \leq Me^{-\nu t} \left\| D_{t_m}^{-3/4}D_t F \right\|_{H^1, L^1},
\]
\[
\left\| S_{\tilde{q}}(t)F \right\|_{X_\varepsilon} \leq Me^{-\nu t} \|F\|_{X_\varepsilon},
\]

(3.121)

(3.122)

for all \( F \in X_{\tilde{q}} \), where \( t_m = \min\{1, t\} \). If moreover \( F = (0, 0, f_3) \), where \( f_3 \) has small mass in each \( \tilde{q} \) window, we have the improved estimate
\[
\left\| S_{\tilde{q}}(t)F \right\|_{X_\varepsilon} \leq Me^{-\nu t} \varepsilon t_m^{-\frac{3}{4}} \left( \sum_{j=1}^{N} |\overline{X_j f_3}| + \varepsilon^{\frac{1}{2}} \|f_3\|_{H^1_{L_{\tilde{q}}}} \right).
\]

(3.123)

Proof. It is straightforward to see that \( \tilde{L} \) generates a \( C_0 \) semigroup since it is a bounded perturbation of a constant coefficient operator. To apply Theorem 3.1 we take \( \mathcal{P} = \{(\epsilon, \tilde{q}) \mid 0 < \epsilon < \varepsilon_0 \}

and \( \tilde{q} \in \mathcal{K} \), and set \( A_p = \tilde{L}(\tilde{q}, \varepsilon) \), \( B_p = \tilde{D}_\varepsilon \), \( C_p = \tilde{S}_\varepsilon^{-1} \), and \( \pi_p = \tilde{\pi}(\varepsilon, \tilde{q}) \), the spectral projection off of the \( N \) small eigenvalues of \( \tilde{L} \). The bounds (3.95) and (3.96) serve to verify the condition (3.110) and the conclusion, (3.111), is exactly (3.121) for \( t > t_0 \). On the other hand, taking \( B_p = \tilde{S}_\varepsilon^{-1} \), then both terms in the condition (3.110) reduce to (3.96) and the conclusion (3.111) yields (3.122), also for \( t > t_0 \). The small time estimates are automatic, for fixed \( \varepsilon \), since \( \tilde{L} \) generates a \( C_0 \) semigroup and off of a compact subset of \( \mathcal{K} \) the pulses separate into the weak interaction regime in which the \( N \) pulses can be analyzed as separate copies. The key is to show the small time estimates may be made uniform in \( \varepsilon \). The issue is the third component of the solution \( W = (W_1, W_2, W_3)' \) of \( W_t = \tilde{L} W \), with initial data \( W(0) = F = (f_1, f_2, f_3)' \). The function \( W \) satisfies

\[
W_{3,t} = E W_3 + 2\varepsilon^{-1} q \Sigma W_1. \tag{3.124}
\]

Setting \( g(t) = 2q \Sigma W_1 \), which is a smooth function and exponentially localized in each window, we can solve for \( W_3 \) in terms of its Fourier modes,

\[
\hat{W}_3(k) = e^{-[(\varepsilon k)^2 + \varepsilon]t} \hat{f}_3(k) + \varepsilon^{-1} \int_0^t e^{-[(\varepsilon k)^2 + \varepsilon]t-s} \hat{g}(s, k) \, ds. \tag{3.125}
\]

Since we are concerned only with small \( t \), we may take \( \hat{g} \) to be constant in time and obtain the estimate

\[
\hat{W}_3(k) = e^{-[(\varepsilon k)^2 + \varepsilon]t} \hat{f}_3(k) + \frac{\varepsilon \hat{g}(k)}{k^2 + \varepsilon^2} \left( 1 - e^{-[(\varepsilon k)^2 + \varepsilon]t} \right). \tag{3.126}
\]

So that

\[
(S^{-1}_\varepsilon W_3)(k) = \varepsilon \hat{W}_3(\varepsilon k) = e^{-[k^2 + \varepsilon]t} \hat{f}_3(\varepsilon k) + \frac{\hat{g}(\varepsilon k)}{k^2 + \varepsilon} \left( 1 - e^{-[k^2 + \varepsilon]t} \right). \tag{3.127}
\]

For the homogeneous term in (3.127) the most poorly behaved part of the \( H^1 \)-norm is the \( L^2 \) norm of the derivative which blows up like

\[
\left\| ke^{-[k^2 + \varepsilon]t} \hat{f}_3(\varepsilon k) \right\|_{L^2} \leq \left\| ke^{-[k^2 + \varepsilon]t} \right\|_{L^2} \left\| S_\varepsilon \hat{f} \right\|_{L^\infty} \leq \frac{c}{t^{\frac{1}{2}}} \left\| f_3 \right\|_{L^1}, \tag{3.128}
\]

as \( t \to 0^+ \). We conclude

\[
\left\| S^{-1}_\varepsilon W_3 \right\|_{H^1} \leq \frac{\varepsilon}{t^{\frac{1}{2}}} \left\| f_3 \right\|_{L^1} + \left\| \frac{\hat{g}(\varepsilon k)}{k^2 + \varepsilon} \left( 1 - e^{-[k^2 + \varepsilon]t} \right) \right\|_{H^1} \leq c \left( \frac{\varepsilon}{t^{\frac{1}{2}}} \left\| f_3 \right\|_{L^1} + \left\| f_1 \right\|_{H^1} \right). \tag{3.129}
\]

The short time estimates, and the result (3.121) follows. To establish (3.122) we observe that

\[
\left\| e^{-[k^2 + \varepsilon]t} \hat{f}_3(\varepsilon k) \right\|_{H^1} = \left\| e^{-[k^2 + \varepsilon]t} \hat{S}^{-1}_\varepsilon f_3(k) \right\|_{H^1} \leq c \left\| \hat{S}^{-1}_\varepsilon f_3 \right\|_{H^1}, \tag{3.130}
\]
and so there is no blow-up as $t \to 0^+$. To establish the small mass estimate (3.123) we need only consider the case of zero third component mass. Since the operators $\vec{S}_\varepsilon$ and $\vec{D}_\varepsilon$ preserve the zero-mass condition, we take $\mathcal{B}_p = \vec{D}_\varepsilon^{1/2}$, apply (3.97) to show that (3.111) is verified. Since the weighted–windowed norm $H_{1,\vec{q}}^1$ uniformly controls $L^1$, the short-time estimates may be treated as above, yielding (3.97).

4. Nonlinear stability via the renormalization group method

The renormalization group method for pulse interaction was developed in [17] and extended in [5] to include semistrong interactions of pulses in a singularly perturbed, dissipative system. The linear estimates developed in Section 3 for the $C_0$ semigroup of the hyperbolic–parabolic system (3.18) are comparable to the estimates developed on the analytic semigroup corresponding to the dissipative system. The nonlinear analysis is thus similar to that presented in [5] and we highlight the differences, sketching the proof of the basic lemmas.

We fix $\varepsilon \in (0, \varepsilon_0)$ and assume at time $t_0$ that the initial data $\vec{U}_0$ to the system (2.5)–(2.6) satisfies

$$\|\Phi_{\vec{q}_*} - \vec{U}_0\|_{X_\varepsilon} \leq \delta, \quad (4.1)$$

for some $\vec{q}_* \in \mathcal{K}$. The following proposition, adapted from Proposition 2.2 of [17], permits us to choose our base point $\vec{q}_0$ about which we develop our local coordinate system.

**Proposition 7.** Fix $\delta \ll 1$. Given $\vec{U}_0$ and $\vec{q}_* \in \mathcal{K}$ satisfying $\|W_*\|_{X_\varepsilon} \leq \delta$, for $W_* \equiv \Phi_{\vec{q}_*} - \vec{U}_0$, then there exists $M > 0$, independent of $\vec{U}_0$ and $\vec{q}_*$, and a smooth function $\mathcal{H}: X \mapsto \mathcal{K}$ such that $\vec{q} = \vec{q}_* + \mathcal{H}(W_*)$ satisfies

$$W_0 \equiv \vec{U}_0 - \Phi_{\vec{q}} \in X_{\vec{q}}. \quad (4.2)$$

Moreover, if $W_* \in X_{\vec{q}_0}$ for some $\vec{q}_0 \in \mathcal{K}$ with $|\vec{q}_0 - \vec{q}_*| = o(1)$ then

$$\|W_0 - W_*\|_{X_\varepsilon} + |\vec{q} - \vec{q}_*| \leq M_0\|W_*\|_{X_\varepsilon}|\vec{q}_* - \vec{q}_0|. \quad (4.3)$$

**Proof.** Since

$$W_0 = W_* + \Phi_{\vec{q}} - \Phi_{\vec{q}_*}, \quad (4.4)$$

the condition (4.2) is equivalent to the system

$$0 = \Lambda \equiv \pi_{\vec{q}}^* W_0 = \pi_{\vec{q}}^* (W_* + \Phi_{\vec{q}} - \Phi_{\vec{q}_*}). \quad (4.5)$$

The adjoint eigenvectors $\Psi_j^\dagger$ are characterized by (3.75), while the pulse form is given by (2.26), so that the quantities $\Lambda = (\Lambda_1, \ldots, \Lambda_N)^t$, depend, up to $O(\varepsilon)$, only upon the first two components

$$\Lambda_j(\vec{q}, W_*) = (W_{1,*} + \phi_j(\vec{q}) - \phi_j(\vec{q}_*), D_j^{-1}\phi_j'(:, \vec{q}))_{L^2} + (W_{2,*}, \phi_j'(:, \vec{q}))_{L^2} + O(\varepsilon), \quad (4.6)$$
for $j = 1, \ldots, N$. Since $A(\tilde{q}_s, 0) = 0$ and the $\tilde{q}$ gradient of $A$ is diagonally dominant

$$
\nabla_{\tilde{q}} A|_{(\tilde{q} = \tilde{q}_s, W_s = 0)} = -\text{diag}(\|\phi'_1\|_{L^2}^2, \ldots, \|\phi'_N\|_{L^2}^2) + O(\varepsilon),
$$

(4.7)

and hence uniformly invertible, the implicit function theorem guarantees the existence of a smooth function $H$ which provides the solution of (4.2) in a neighborhood about the manifold $\mathcal{M}$ defined in (2.30). The interval of existence of $H$ may be chosen uniformly in $\tilde{q}$ since $K$ is compact.

If in addition we have $W_s \in X_{\tilde{q}_0}$, then $(W_s, \Psi_j^\dagger(\tilde{q}_o))_{L^2} = 0$ for $j = 1, \ldots, N$. We see that

$$
\left|(W_s, \Psi_j^\dagger(\tilde{q}_s))_{L^2}\right| \leq \left|(W_s, \Psi_j^\dagger(\tilde{q}_o) - \Psi_j^\dagger(\tilde{q}_s))_{L^2}\right| \leq M_0\|W_s\|_{L^2}|\tilde{q}_0 - \tilde{q}_s|.
$$

(4.8)

From (4.7) we see that the $\tilde{q}$ gradient of $A$ has an $O(1)$ inverse, and the bound on $|\tilde{q} - \tilde{q}_s|$ in (4.3) follows from the implicit function theorem. The bound on $W_0 - W_s$ in (4.3) follows from (4.4) and (2.41).

4.1. The projected equations

To begin the RG procedure we freeze $\tilde{q} = \tilde{q}_0$ in $X_{\tilde{q}_0}$, where $\tilde{q}_0$ is the base point provided by Proposition 7, and change variables as

$$
\tilde{U}(t) = \Phi_{\tilde{q}} + W,
$$

(4.9)

where $W \in X_{\tilde{q}_0}$, and $\tilde{q} = \tilde{q}(t)$. Inserting this into the evolution equation (2.5), the evolution for the remainder $W$ becomes

$$
W_t + \nabla_{\tilde{q}} \Phi \dot{\tilde{q}} = R + \tilde{L}_{\tilde{q}_0} W + (L_{\tilde{q}} - \tilde{L}_{\tilde{q}_0}) W + \mathcal{N}(W),
$$

(4.10)

$$
W(x, 0) = W_0,
$$

(4.11)

where $W_0 = W_s + \Phi_{\tilde{q}_0} - \Phi_{\tilde{q}_s}$. The terms $\Delta L \equiv L_{\tilde{q}} - \tilde{L}_{\tilde{q}_0}$ include both the approximations made to the linear operator and the secular growth implicit in the sliding of $\tilde{q}$ away from $\tilde{q}_0$.

To enforce $W \in X_{\tilde{q}_0}$ we impose the nondegeneracy condition $\frac{\partial}{\partial \tilde{q}} \pi_0 W = 0$, where $\pi_0 = \pi_{\tilde{q}_0}$ is given by (3.80). Since $\pi_0$ is independent of time, the nondegeneracy condition is equivalent to $\pi_0 W_t = 0$, and moreover as $\pi_0$ commutes with $\tilde{L}_{\tilde{q}_0}$ it follows that $\pi_0 \tilde{L}_{\Gamma_0} W = \tilde{L}_{\Gamma_0} \pi_0 W = 0$. The nondegeneracy condition is thus equivalent to the $N$ equations

$$
\left(\nabla_{\tilde{q}} \Phi \dot{\tilde{q}}, \Psi_j^\dagger\right)_{L^2} = (R + \Delta L W + \mathcal{N}(W), \Psi_j^\dagger)_{L^2}.
$$

(4.12)

From the form (2.26) of the semistrong pulse solutions, we calculate

$$
\nabla_{\tilde{q}} \Phi \dot{\tilde{q}} = \left(\sum_{j=1}^{N} \alpha_j (\partial_{p_j} \phi_j \nabla_{\tilde{q}} p_j \dot{\tilde{q}} - \phi'_j \dot{q}_j)\right)
$$

(4.13)
While \( \| \partial_p \phi_j \|_{H^1} = O(1) \), from (2.42) we see that \( \nabla_{\tilde{q}} \tilde{p} = O(\varepsilon) \). Neglecting the first term on the right-hand side of (4.13), and using the form of the adjoint eigenvector (3.75), the nondegeneracy equations (4.12) for \( \dot{\tilde{q}} \) take the form

\[
\dot{\tilde{q}}_j = -\frac{(R + \Delta L W + N(W), \Psi^*_j)_{L^2}}{\alpha_j \theta_j + O(\varepsilon)},
\]

for \( j = 1, \ldots, N \), where

\[
\theta_j \equiv (\phi_j', D_j^{-1} \phi_j') > \theta_0 > 0,
\]

see Lemma 3.2 of [14].

We introduce the quasi-static residual

\[
\tilde{R} = \tilde{\pi}_{\tilde{q}}(R - \nabla_{\tilde{q}} \Phi_{\tilde{q}}) = \tilde{\pi}_{\tilde{q}} \left( -\sum_{j=1}^N \alpha_j (\partial_p \phi_j \nabla_{\tilde{q}} p_j \dot{\tilde{q}} - \phi_j' \dot{\tilde{q}}_j) \right)
\]

so that the evolution for the remainder \( W \) may be written concisely as

\[
W_t = \tilde{R} + \tilde{L}_0 W + \tilde{\pi}_0 (\Delta L W + N),
\]

\[
W(x, t_0) = W_0,
\]

where \( \tilde{L}_0 = \tilde{L}_{\tilde{q}_0} \) and \( \tilde{\pi}_0 = I - \pi_{\tilde{q}_0} \). The evolution for the remainder \( W \) is dominated by the first two terms on the right-hand side of (4.17), the last two terms are asymptotically irrelevant, until \( \tilde{q} - \tilde{q}_0 \) is driven so large by the pulse dynamics that the secularity implicit in \( \Delta L \) forces an update of \( \tilde{q}_0 \).

The residual and the quasi-steady residual satisfy the following bounds.

**Lemma 4.1.** There exists \( c > 0 \), independent of \( \tilde{q} \in \mathcal{M} \) such that the projected residual takes the form

\[
(R, \Psi^*_j)_{L^2} = -\frac{\alpha_j \Theta'(q_j)}{4} \| \phi_j \|_{L^2}^2 - \frac{1}{2} \int_\mathbb{R} \phi_j^2 \left( \alpha_{j-1} \phi_j'^{j-1} + \alpha_{j+1} \phi_j'^{j+1} \right) dx + O(\varepsilon^2),
\]

and the quasi-steady residual satisfies

\[
\| \tilde{R} \|_{H^1, L^1} \leq c \varepsilon.
\]

**Proof.** From the form of the residual, (3.3) and the adjoint eigenvectors (3.75), and keeping only leading order terms, we see

\[
(R, \Psi^*_j) = \frac{(R_2, \phi_j')}{2}
\]

\[
= \left( \frac{3}{2} \left[ \alpha_{j+1} \phi_j^2 \phi_{j+1} + \alpha_{j-1} \phi_j^2 \phi_{j-1} \right] + \frac{\alpha_j}{2} \Theta'(q_j)(x - q_j)\phi_j \phi_j' \right)_{L^2}.
\]
Integrating (4.21) by parts yields (4.19). The result (4.20) follows from (4.16) and the fact that \( \dot{\vec{q}} = \mathcal{O}(\varepsilon) \) and \( \|\partial_{q_j} \Theta\|_{L^1} = \mathcal{O}(1) \).

4.2. Decay of the remainder

We identify the duration of each renormalization interval, and quantify the decay of the remainder \( W \) over this interval. To control the dynamics we introduce the quantities

\[
T_0(t) = \sup_{t_0 < s < t} e^{\nu(s-t_0)} \|W(s)\|_X, \tag{4.22}
\]

\[
T_1(t) = \sup_{t_0 < s < t} |\vec{q}(s) - \vec{q}_0|. \tag{4.23}
\]

The first enforces the decay of the remainder, \( W \), the second measures the distance the pulse positions have moved from their frozen base point. The variation of constants formula applied to (4.17) yields the solution

\[
W(x, t) = S(\Delta t)W_0 + \int_{t_0}^t S(t-s)(\tilde{R} + \tilde{\pi}_0(\Delta LW + \mathcal{N})) ds, \tag{4.24}
\]

where we have introduced \( \Delta t = t - t_0 \).

We freeze the base point at \( \vec{q}_0 \) and break \( \Delta L \) in secular and reductive parts, \( \Delta L = \Delta L_s + \Delta L_r \) where \( \Delta L_s = L_{\Gamma} - L_{\Gamma 0} \) and \( \Delta L_r = L_{\Gamma 0} - \tilde{L}_{\Gamma 0} \).

**Lemma 4.2.** There exists \( c > 0 \), independent of \( \varepsilon \in (0, \varepsilon_0) \) and \( \vec{q}_0 \in \mathcal{K} \) such that the following estimates on each of the terms in (4.24) hold,

\[
\|\vec{D}_\varepsilon \Delta L_s W\|_{H^1, L^1} \leq cT_1 \|W\|_{X_\varepsilon}, \tag{4.25}
\]

\[
\|\vec{D}_\varepsilon^{\frac{3}{2}} \Delta L_r W\|_{H^1, H^1_{\vec{q}}} \leq c \varepsilon^{\frac{1}{2}} \|W\|_X, \tag{4.26}
\]

\[
\|\vec{D}_\varepsilon \mathcal{N}(W)\|_{H^1, L^1} \leq c \|W\|_{X_\varepsilon}^2. \tag{4.27}
\]

**Proof.** The operator \( \vec{D}_\varepsilon \Delta L_s \) takes the form

\[
\Delta L_s = \begin{pmatrix}
0 & V_D - \tilde{V}_D & 0 \\
V_C - \tilde{V}_C & 0 & \Xi - \tilde{\Xi} \\
\Xi - \tilde{\Xi} & 0 & 0
\end{pmatrix}. \tag{4.28}
\]

where \( V_C \) and \( V_D \) represent the potentials in the operators \( C \) and \( D \). The potentials without overbars are evaluated at the evolving parameters \( \vec{q}(t) \) and those with overbars are evaluated at the frozen parameters \( \vec{q}_0 \). Since the potentials \( V_C \) and \( V_D \) decay exponentially in each window and depend smoothly on \( \vec{q} \), we have the bounds

\[
\| (V_C - \tilde{V}_C) W_1 \|_{H^1} + \| (V_D - \tilde{V}_D) W_2 \|_{H^1} \\
\leq c|\vec{q} - \vec{q}_0| \| (W_1, W_2) \|_{H^1} \leq cT_1 \|W\|_{X_\varepsilon}. \tag{4.29}
\]
Similarly
\[
\| (\mathcal{E} - \mathcal{E}) W_3 \|_{H^1} \leq \| \mathcal{E} - \mathcal{E} \|_{H^1}(\| W_3 \|_{L^\infty} + \| \partial_x W_3 \|_{L^2}) \leq c T_1 \| S_{\epsilon}^{-1} W_3 \|_{H^1}.
\] (4.30)

The quantity \((\mathcal{E}_N - \mathcal{E}_N) W_1\) is a sum of terms \((\eta_j \otimes \phi_j - \bar{\eta}_j \otimes \bar{\phi}_j) W_1\), each of which decays exponentially in each \(\vec{q}\) window. These terms satisfy the bound
\[
\| (\eta_j \otimes \phi_j - \bar{\eta}_j \otimes \bar{\phi}_j) W_1 \|_{H^1 + L^1} \\
\leq c \left( \| \eta_j \|_{H^1, L^1} \| (\phi_j - \bar{\phi}_j, W_1) \| + \| \eta_j - \bar{\eta}_j \|_{H^1, L^1} \| W_1 \|_{L^2} \right)
\] (4.31)
\[
\leq c T_1 \| W_1 \|_{L^2}.
\] (4.32)

Summing over \(j\) we see
\[
\| (\mathcal{E}_N - \mathcal{E}_N) W_1 \|_{H^1, L^1} \leq c T_1 \| W \|_{X_{\epsilon}^2}.
\] (4.33)

Combining (4.29), (4.30), and (4.33) yields (4.25).

For the reductive term, since both \(\mathcal{E}\) and \(\mathcal{E}_N\) are exponentially localized in each \(\vec{q}\) window we have the estimate
\[
\| \mathcal{L}_\epsilon^{\frac{3}{2}} \Delta L_r^1 W \|_{H^1 \setminus H_{\epsilon, \vec{q}}} = \epsilon^{\frac{1}{2}} \| (\mathcal{E} - \mathcal{E}_N) W_1 \|_{H^1_{\epsilon, \vec{q}}} \\
\leq c \epsilon^{\frac{1}{2}} \left( \| W_1 \|_{L^\infty} + \| \partial_x W_1 \|_{L^2} \right) \leq c \epsilon^{\frac{1}{2}} \| W_1 \|_{H^1}.
\] (4.34)

Regarding the nonlinear estimate, from (3.8) we see that \(\mathcal{L}_\epsilon^1\mathcal{N}\) is homogeneous of degree at least two in \(W_1\) and \(W_2\), and is independent of \(\epsilon\). Since \(H^1\) is an algebra and \(\| W_j^2 \|_{L^1} = \| W_j \|_{L^2}^2\) it follows that
\[
\| \mathcal{L}_\epsilon^1\mathcal{N}(W) \|_{H^1 \setminus L^1} \leq c \| (W_1, W_2)^f \|_{H^1}^2 \leq c \| W \|_{X_{\epsilon}^2}^2,
\] (4.35)
which establishes (4.27).

To estimate the distance that the pulse locations, \(\vec{q}\), have moved from the base point, \(\vec{q}_0\), we examine Eqs. (4.14), controlling the drift of the pulses by their speed. Regarding the residual, it is straightforward to estimate from (4.19) that \(|(R, \Psi_j^\dagger)_{L^2}| = O(\epsilon)\), while for the other terms we observe that
\[
\left| (F, \Psi_j^\dagger)_{L^2} \right| \leq c \left( \| (f_1, f_2)^f \|_{L^2} + \| f_3 \|_{L^1} \| \Psi_j^\dagger \|_{L^\infty} \right) \leq c \| \mathcal{L}_\epsilon^{\frac{3}{2}} F \|_{L^2 \setminus L^1}.
\] (4.36)

Using these estimates and Lemma 4.2 we obtain
\[
T_1(t) \leq \int_{t_0}^{t_0 + \Delta t} \left| \dot{\vec{q}}(s) \right| ds
\] (4.37)
\[
\begin{align*}
\int_{t_0}^{t_0 + \Delta t} & c(\| R_2(s) \|_{L^2} + (T_1(s) + \varepsilon^{\frac{1}{2}})) \| W(s) \|_{X_{\varepsilon}} + \| W(s) \|_{X_{\varepsilon}}^2) \, ds \\
& \leq \int_{t_0}^{t_0 + \Delta t} c(\varepsilon + e^{-\nu(s-t_0)}(T_1(t) + \varepsilon^{\frac{1}{2}})T_0(t) + e^{-2\nu(s-t_0)}T_0^2(t)) \, ds \\
& \leq c(\varepsilon \Delta t + \varepsilon^{\frac{1}{2}} + T_1)T_0 + T_0^2).
\end{align*}
\]

For \( T_0 \) small enough we can eliminate \( T_1 \) from the right-hand side, obtaining
\[
T_1 \leq c(\varepsilon \Delta t + T_0^2). \tag{4.41}
\]

The evolution for the remainder is given by (4.24). To bound the terms in this equality we combine the semigroup estimates developed in Proposition 6 with the bounds of Lemma 4.2.

\[
\begin{align*}
\| W(t) \|_{X_{\varepsilon}} & \leq \| S(\Delta t)W_0 \|_{X_{\varepsilon}} + \int_{t_0}^{t} (\| S(t-s)\tilde{R} \|_{X_{\varepsilon}} + \| S(t-s)\tilde{\pi}_0\Delta L_sW \|_{X_{\varepsilon}}) \, ds \\
& \leq \int_{t_0}^{t} e^{-\nu \Delta t} \| W_0 \|_{X_{\varepsilon}} + \int_{t_0}^{t} e^{-\nu(s-t_0)}(\| \tilde{\mathcal{D}}_\varepsilon \tilde{R} \|_{H^1,L^1} + \| \tilde{\mathcal{D}}_\varepsilon \Delta \tilde{L}_r W \|_{H^1,L^1}) \, ds \\
& \quad + \| \tilde{\mathcal{D}}_\varepsilon \Delta \tilde{L}_r W \|_{H^1,H^1_{\varepsilon}} + \| \tilde{\mathcal{D}}_\varepsilon \tilde{\pi}_0 \mathcal{N}(W) \|_{H^1,L^1} \, ds \\
& \leq \int_{t_0}^{t} e^{-\nu \Delta t} \| W_0 \|_{X_{\varepsilon}} + \int_{t_0}^{t} e^{-\nu(s-t_0)}(\varepsilon + (T_1 + \varepsilon^{\frac{1}{2}})) \| W(s) \|_{X_{\varepsilon}} + \| W(s) \|_{X_{\varepsilon}}^2) \, ds.
\end{align*}
\]

To estimate the decay of \( \| W(t') \|_X \) for \( t' \in (t_0, t) \) we evaluate (4.44) at \( t = t' \), multiply by \( e^{\nu(t'-t_0)} \), and take the sup over \( t' \in (t_0, t) \) obtaining
\[
T_0(t) \leq M_0 \left( T_0(t_0) + \int_{t_0}^{t} \left( (e^{\nu(s-t_0)} + (T_1 + \varepsilon^{\frac{1}{2}}))T_0(t) + e^{-\nu(s-t_0)}T_0^2(t) \right) ds \right) \tag{4.45}
\]
\[
\leq M \left( T_0(t_0) + \varepsilon e^{\nu \Delta t} + (T_0(t_0)) \Delta tT_0(t) + T_0^2(t) \right). \tag{4.46}
\]

From (4.41) we may eliminate \( T_1 \) from the \( T_0 \) estimate, and neglecting orders higher than \( T_0^2 \) obtain
\[
T_0(t) \leq M \left( T_0(t_0) + \varepsilon e^{\nu \Delta t} + (\varepsilon^{\frac{1}{2}} + \varepsilon \Delta t)T_0(t) + T_0^2(t) \right). \tag{4.47}
\]
For \( \varepsilon \) small enough and for \( \Delta t \ll \varepsilon^{-1} \), the term \( M (\varepsilon^{\frac{1}{2}} + \varepsilon \Delta t) < \frac{1}{2} \) and we may eliminate the term linear in \( T_0 \) from the right-hand side of (4.47) yielding

\[
T_0 \leq 2M\left(T_0(t_0) + \varepsilon e^{\nu \Delta t} + T_0^2\right). \tag{4.48}
\]

The quadratic equation in \( T_0 \)

\[
0 = T_0(t_0) + \varepsilon e^{\nu \Delta t} - \frac{1}{2M} T_0 + T_0^2,
\]

has two positive real roots so long as \( T_0(t_0) + \varepsilon e^{\nu \Delta t} \ll 1 \). The smaller of these roots, \( r_0 \) takes the form

\[
r_0 = 2M\left(T_0(t_0) + \varepsilon e^{\nu \Delta t}\right) + O\left(T_0(t_0) + \varepsilon e^{\nu \Delta t}\right)^2, \tag{4.50}
\]

while the larger is

\[
r_1 = \frac{1}{2M} + O\left(T_0(t_0) + \varepsilon e^{\nu \Delta t}\right). \tag{4.51}
\]

Thus if \( T_0(t_0) \ll 1 \) and \( \varepsilon e^{\nu \Delta t} \ll 1 \) then there is an excluded region, either \( 0 < T_0 < r_0 \) or \( r_1 < T_0 < \infty \). Since \( T_0(t_0) < r_0 \) and \( T_0 \) is continuous in \( t \), we see that

\[
T_0(t) \leq r_0 \leq M\left(T_0(t_0) + \varepsilon e^{\nu \Delta t}\right) \tag{4.52}
\]

so long as

\[
\Delta t \leq \frac{\beta |\log \varepsilon|}{\nu} \tag{4.53}
\]

for any fixed \( \beta < 1 \). This condition on \( \Delta t \) prevents the secularity from dominating the linear operator, in particular it is a stronger condition on \( \Delta t \) than that imposed after Eq. (4.47). This implies that

\[
\|W(t)\|_{X_\varepsilon} \leq M(\varepsilon e^{\nu(t-t_0)} \|W(t_0)\|_X + \varepsilon) \quad \text{for} \quad t \in \left(t_0, t_0 + \frac{\beta |\log \varepsilon|}{\nu}\right). \tag{4.54}
\]

and in particular for \( t_1 = t_0 + \Delta t \) we have

\[
\|W(t_1)\|_X \leq M(\varepsilon^\beta \|W(t_0)\|_X + \varepsilon). \tag{4.55}
\]

4.3. The RG iteration

We break the time evolution into a series of initial value problems, tracking the decay of the remainder over the long-time scale of many RG iterations. We fix \( \beta < 1 \) and \( \Delta t = \frac{\beta |\log \varepsilon|}{\nu} \). The renormalization times are defined sequentially

\[
t_n = t_{n-1} + \Delta t. \tag{4.56}
\]
We break the evolution of $W$ into disjoint intervals $I_n = [t_n, t_{n+1})$. On each interval $I_n$ we solve the initial value problem (4.17) with initial data $W(t_n) \in X_\vec{q}$, with the quantities $T_{0,n}$ and $T_{1,n}$ corresponding to (4.22)–(4.23) over $I_n$. The renormalization map, $G$, takes the initial data, $W_{n-1} = W(t_{n-1})$ for the initial value problem on interval $I_{n-1}$ and returns the initial data $W_n = W(t_n)$ for the initial value problem on the interval $I_n$,

$$GW_{n-1} = W_n.$$  \hfill (4.57)

Arguing inductively, the initial data and the new base point $\vec{q}_n$ are obtained from $W(t_n^-)$, the end-value of the evolution of $W$ over $I_{n-1}$, by applying Proposition 7. Indeed we know that $W(t_n^-) \in X_{\vec{q}_n}$ and so from (4.3) we have

$$|\vec{q}_n - \vec{q}(t_n^-)| \leq M_0 \|W(t_n^-)\|_{X_\varepsilon} |\vec{q}(t_n^-) - \vec{q}(t_{n-1})| \leq M_0 \|W(t_n^-)\|_{X_\varepsilon} T_{1,n-1}(t).$$ \hfill (4.58)

From the estimates on $\Delta t$, and $T_{1,n-1}$, we bound the jump in $\vec{q}$ at renormalization by

$$|\vec{q}_n - \vec{q}(t_n^-)| \leq M_0 (|\log \varepsilon|_{\varepsilon} + T_{0,n-1}^2) \|W(t_n^-)\|_{X_\varepsilon}.$$ \hfill (4.59)

The solution at time $t = t_n$ is independent of the decomposition,

$$\vec{U}(t_n) = \Phi_{\vec{q}(t_n)} + W(t_n^-) = \Phi_{\vec{q}_n} + W_n,$$ \hfill (4.60)

and from (4.3) we may bound the jump in $W$ at each renormalization

$$\|W(t_n^-) - W(t_n)\|_X = M_0 \|W(t_n^-)\|_{X_\varepsilon} |\vec{q}(t_n^-) - \vec{q}_{n-1}| 
\leq M_0 (|\log \varepsilon|_{\varepsilon} + T_{0,n-1}^2) \|W(t_n^-)\|_{X_\varepsilon},$$ \hfill (4.61)

where we used (4.41). From (4.52), using the equality $T_{0,n-1}(t_{n-1}) = \|W_{n-1}\|_X$, we have the estimate

$$T_{0,n-1} \leq M_1 (\|W_{n-1}\|_X + \varepsilon^{(1-\beta)}).$$ \hfill (4.62)

Combining the estimates (4.61) and (4.62) with (4.55), we obtain a bound on $GW_{n-1} = W_n$,

$$\|GW_{n-1}\|_X \leq (1 + M_0[|\log \varepsilon|_{\varepsilon} + M_1^2 (\|W(t_{n-1})\|_X + \varepsilon^{(1-\beta)})^2]) M (\varepsilon^\beta \|W(t_{n-1})\|_X + \varepsilon^3).$$ \hfill (4.63)

Neglecting the terms involving positive powers of $\varepsilon$ within the first parenthesis on the left-hand side, we may bound $\|W(t_n)\|_X$ by $\eta_n$, the solution of the map

$$\eta_n = \Phi_{\eta_n - \varepsilon^3} (\varepsilon^\beta \eta_{n-1} + \varepsilon),$$ \hfill (4.64)

with $\eta_0 = \|W(\cdot, t_0)\|_X$, and $M_2 = M_0 M_1^2$. It is easy to see for $\eta_0 = O(1)$ and $\varepsilon$ sufficiently small that

$$\eta_n \to \frac{M}{1 - \varepsilon^\beta M} \varepsilon,$$ \hfill (4.65)

as $n \to \infty$. Since $\|W(\cdot, t_n)\|_X \leq \eta_n$, the estimate (4.55) yields the result (2.8) in Theorem 1.1.
4.4. Long-time asymptotics

To recover the asymptotic pulse motion, we consider the situation where \( t \) is sufficiently large that \( \|W\|_X \leq M\varepsilon \). In this regime we have the following result.

**Proposition 8.** For \( t \) sufficiently large that \( \|W\|_X \leq M\varepsilon \) then the pulse dynamics are given at leading order by

\[
\ddot{q}_j = \frac{\Theta'(q_j)}{4\theta_j} \|\phi_j\|_{L^2}^2 + \frac{\alpha_j}{2\theta_j} \int_{\mathbb{R}} \phi_j^3(\alpha_{j-1}\phi_{j-1} + \alpha_{j+1}\phi_{j+1}) \, dx + \mathcal{O}(\varepsilon^2). \tag{4.66}
\]

**Proof.** From (4.41) we see that \( T_1 \leq c\varepsilon^2 |\log \varepsilon| \), and hence from (4.25)–(4.26) and (4.36) that

\[
|\langle \Delta L W, \Psi_j^\dagger \rangle|_{L^2} \leq c\varepsilon^\frac{3}{2}. \tag{4.67}
\]

Moreover from (4.27) we verify that

\[
|\langle N(W), \Psi_j \rangle|_{L^2} \leq c\|W\|^2_X = \mathcal{O}(\varepsilon^2). \tag{4.68}
\]

In this regime the estimates (4.67) and (4.68) on the secularity and the nonlinearity show that the remainder \( W \) has an asymptotically small influence on the pulse evolution equations, (4.14), which from (4.19) reduce to

\[
\ddot{q}_j = -\frac{\alpha_j}{\theta_j} (R, \Psi_j^\dagger)_{L^2} + \mathcal{O}(\varepsilon^\frac{3}{2}), \tag{4.69}
\]

which is equivalent, at leading order, to (4.66). From (2.37) we see that first term is generically \( \mathcal{O}(\varepsilon) \) while the integral terms scale with the pulse spacing like \( \varepsilon^\sqrt{2\eta_j + |q_j - q_{j+1}|} \), which with our choice of \( l_0 \) from (2.40) can be as large as \( \mathcal{O}(\varepsilon) \). \( \Box \)

In the case of a two-pulse ansatz with pulses located at \( \pm q \), the pulse construction is explicit, see (2.45), and we simplify the pulse dynamics (4.66) by approximating (2.38) for \( q \ll \varepsilon^{-1} \) and evaluating the tail–tail interaction integral, yielding

\[
\dot{q} = -\frac{1}{\theta_1} (\varepsilon \rho \eta_+ + 8\alpha_1 \alpha_2 \eta_+^2 e^{-2q\sqrt{2\eta_+}}). \tag{4.70}
\]

For an antisymmetric two pulse with \( \alpha_1 \alpha_2 = -1 \), the long-range thermal gradient will move the pulses together until a balance is achieved with the pulse–pulse repulsion, forming a stable, bound two-pulse ansatz, with pulse separation \( q_s \approx |\ln \varepsilon| \). For this pulse separation we may approximate the two-pulse thermal detuning \( p \) given by (2.44) as

\[
p = \frac{4}{h} \Theta \left( \sqrt{1 + \frac{\eta_0 h}{2\theta^2}} - 1 \right) + \mathcal{O}(\varepsilon |\ln \varepsilon|). \tag{4.71}
\]
and hence $\eta_+ = \eta_{+,0} - p$ is independent of $q$ to leading order. The steady two-pulse separation is then given by

$$q_s = \frac{1}{2\sqrt{2\eta_+}} \ln \frac{8}{\varepsilon Q} + \mathcal{O}(\varepsilon |\ln \varepsilon|).$$

(4.72)

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References


