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Interval Criteria for Oscillation of Second-Order Functional Differential Equations

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Abstract—By using averaging functions, new interval oscillation criteria are established for the second-order functional differential equation,

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + F\left(t, x(t), x(\tau(t)), x'(t), x'(\tau(t))\right) = 0, \qquad t \geq t_0,$$

that are different from most known ones in the sense that they are based on information only on a sequence of subintervals of $[t_0, \infty)$, rather than on the whole half-line. Our results can be applied to three cases: ordinary, delay, and advance differential equations. In the case of half-linear functional differential equations, our criteria implies that the $\tau(t) \leq t$ delay and $\tau(t) \geq t$ advance cases do not affect the oscillation. In particular, several examples are given to illustrate the importance of our results.

Keywords—Quasilinear differential equation, Oscillation, Interval criteria, Generalized Riccati technique, Integral averaging method.

1. INTRODUCTION AND PRELIMINARIES

This paper is concerned with the problem of oscillation of second-order functional differential equations,

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + F\left(t, x(t), x(\tau(t)), x'(t), x'(\tau(t))\right) = 0, \tag{1.1}$$

for $t \geq t_0 > 0$, where

(i) $\alpha > 0$ is a constant;
(ii) $F : [t_0, \infty) \times R \times R \times R \times R \to R$ is a continuous function;
(iii) $r : [t_0, \infty) \to (0, \infty)$ is a continuous function;
(iv) $\tau : [t_0, \infty) \to (0, \infty)$ is a continuous function and $\lim_{t\to\infty} \tau(t) = \infty$.

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We restrict our attention to those solutions $x(t)$ of (1.1) which exist on $[t_0, \infty)$ and satisfy $\sup\{|x(t)| : t \geq t_x\} > 0$, for any $t_x \geq t_0$. By a solution of (1.1), we mean a function $x(t) : [t_x, \infty) \to R$ which is continuously differentiable on $[t_x, \infty)$ together with $r(t)|x'(t)|^{\alpha-1}x'(t)$ and satisfies (1.1) at every point of $[t_x, \infty)$. A solution is said to be oscillatory if it has a sequence of zeros clustering at ∞ and nonoscillatory otherwise.

The oscillation problem for equation (1.1) and for less general equations has been studied by numerous authors. For recent contributions, we refer the reader to [1–15] and references therein. In all of these works, the conditions in terms of the coefficients involving integral averages over the whole half-line $[t_0, \infty)$ are used.

As pointed out in earlier works [16,17], oscillation is an interval property, that is, it is more reasonable to investigate solutions on an infinite set of bounded intervals. Therefore, the problem is to find oscillation criteria which use only the information about the involved functions on these intervals; outside of these intervals the behavior of the functions is irrelevant. Such type of oscillation criteria are referred to as interval oscillation criteria, see [16–27]. The first attempt in this direction was due to El-Sayed [16], who investigated the linear case of (1.1).

In [8], Mahfoud considered the Euler differential equation with constant delay of the form, $\tau(t) = t - \sigma$, that is,

$$x''(t) + \frac{\mu}{t^2} x(t-\sigma) = 0, \qquad t \geq t_0 > \sigma > 0, \tag{1.2}$$

and conclude that the delay does not effect the oscillation. In other words, this equation oscillates if and only if the corresponding equation without delay,

$$x''(t) + \frac{\mu}{t^2} x(t) = 0, \qquad t \geq t_0, \tag{1.3}$$

oscillates, that is, if and only if

$$\mu > \frac{1}{4}.$$

Li and Yeh [28,29] proved further that $\tau(t) = t + \sigma$ in equation (1.2) does not effect the oscillation, where σ is a positive constant. For other related results, refer to [30].

On the other hand, in 1973 and 1984, Erbe [31] and Ohriska [10] addressed the oscillatory behavior of the second-order linear functional differential equation,

$$x''(t) + p(t)\, x(\tau(t)) = 0.$$

In [5, Chapter 4], Ladde, Lakshmikantham and Zhang considered the second-order nonlinear functional differential equation,

$$(r(t)\, x'(t))' + f(t, x(t), x(\tau(t)), x'(t), x'(h(t))) = 0,$$

where $\tau, h \in C[R_+, R]$, $\tau(t) \leq t$, $\lim_{t\to\infty} \tau(t) = \infty$, $f \in C[R_+ \times R^4, R]$, and $uf(t,u,v,w,z) > 0$, for $uv > 0$, $t \geq t_0$, and obtained several interesting oscillatory criteria.

In 1997, under certain assumptions on f, τ, and α, Wang [32] studied the oscillatory and nonoscillatory behavior of quasilinear functional differential equations of the type,

$$\left(|x'(t)|^{\alpha-1} x'(t)\right)' + f(t, x(\tau(t))) = 0.$$

Agarwal *et al.* [1] investigated the following retarded differential equation,

$$\left(r(t)\, |x'(t)|^{\alpha-1} x'(t)\right)' + \left(p(t)\, |x(\tau(t))|^{\beta-1} x(\tau(t))\right) = 0,$$

where α and β are positive constants, and $p, \tau, r \in C([t_0, \infty), R)$ satisfy some suitable conditions. Their results generalize those of Erbe, Ohriska and Lakshmikantham.

Recently, Li [22] obtained some new oscillation criteria for the case of half-linear functional differential equation of the special form,

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + p(t)\left|x(\tau(t))\right|^{\alpha-1}x(\tau(t)) = 0, \tag{1.4}$$

where $p(t)$ is a positive continuous function on $[t_0,\infty)$, $r(t)$ is an eventually positive function; $\tau(t)$ is a positive continuously differentiable function on $[t_0,\infty)$, such that $\tau'(t) > 0$, for $t \ge t_0$ and $\lim_{t\to\infty}\tau(t) = \infty$. Li showed that his results imply that the delay $\tau(t) = t \pm \sigma$ does not affect the oscillation and can be applied to extreme case such as $\int_{t_0}^{\infty} p(s)\,ds = -\infty$. However, because of the sign condition on $p(t)$ and the proof of his main theorem, the above extreme case and the $\tau(t) \ge t$ advance case are impossible.

Motivated by the ideas in [5,7,17,22], by employing the Riccati technique and the integral averaging method, we shall establish several new interval criteria for oscillation of equation (1.1), that is, criteria given by the behavior of equation (1.1) (or r, p, and α) only on a sequence of subintervals of $[t_0,\infty)$. In fact, by choosing appropriate functions H and ρ, we shall present several easily verifiable oscillation criteria. Our conditions do not require the sign of $\rho'(t)$ and $\tau'(t)$. For the equation (1.4), our results imply that the $\tau(t) \le t$ delay and $\tau(t) \ge t$ advance cases do not effect the oscillation, and improve and extend the results of [4,11,17,28,29,33], and can be applied to extreme cases such as $\int_{t_0}^{\infty} p(s)\,ds = -\infty$ for ordinary differential equations. Finally, several interesting examples that point out the importance of our results are included.

We begin with a preparatory lemma which we will heavily rely on in the proofs of our theorems. First, we recall a class of functions defined on $D = \{(t,s) : t \ge s \ge t_0\}$. A function $H \in C(D,R)$ is said to belong to the class $\mathcal{P}$ if

(H1) $H(t,t) = 0$, for $t \ge t_0$ and $H(t,s) > 0$, when $t \ne s$;

(H2) $H(t,s)$ has partial derivatives on D, such that

$$\frac{\partial H(t,s)}{\partial t} = h_1(t,s)\sqrt{H(t,s)}, \qquad \frac{\partial H(t,s)}{\partial s} = -h_2(t,s)\sqrt{H(t,s)},$$

for some locally integrable functions h_1 and h_2.

LEMMA 1.1. *Let A_0, A_1, $A_2 \in C([t_0,\infty),R)$ with $A_2 > 0$, and $w \in C^1([t_0,\infty),R)$. If there exist $(a,b) \subset [t_0,\infty)$ and $c \in (a,b)$, such that*

$$w' \le -A_0(s) + A_1(s)w - A_2(s)|w|^{(\alpha+1)/\alpha}, \qquad s \in (a,b), \tag{1.5}$$

then

$$\begin{aligned}&\frac{1}{H(c,a)}\int_a^c\left[H(s,a)A_0(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{|\phi_1(s,a)|^{\alpha+1}}{(A_2(s))^\alpha}\right]ds\\&+\frac{1}{H(b,c)}\int_c^b\left[H(b,s)A_0(s) - \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{|\phi_2(b,s)|^{\alpha+1}}{(A_2(s))^\alpha}\right]ds \le 0\end{aligned} \tag{1.6}$$

for every $H \in \mathcal{P}$, where

$$\phi_1(s,a) = \frac{h_1(s,a)\sqrt{H(s,a)} + A_1(s)H(s,a)}{(H(s,a))^{\alpha/(\alpha+1)}},$$

$$\phi_2(b,s) = \frac{-h_2(b,s)\sqrt{H(b,s)} + A_1(s)H(b,s)}{(H(b,s))^{\alpha/(\alpha+1)}}.$$

PROOF. Multiplying (1.5) by $H(s,t)$ and integrating with respect to s from t to c for $t \in (a,c]$, we have

$$\begin{aligned}\int_t^c H(s,t)A_0(s)\,ds \le &-\int_t^c H(s,t)w'(s)\,ds + \int_t^c H(s,t)A_1(s)w(s)\,ds\\&-\int_t^c H(s,t)A_2(s)|w(s)|^{(\alpha+1)/\alpha}\,ds.\end{aligned} \tag{1.7}$$

In view of (H1) and (H2), we see that

$$\int_t^c H(s,t)\, w'(s)\, ds = H(c,t)\, w(c) - \int_t^c h_1(s,t)\sqrt{H(s,t)}w(s)\, ds. \tag{1.8}$$

Using (1.8) in (1.7) leads to

$$\begin{aligned}\int_t^c H(s,t)\, A_0(s)\, ds \le {} & -H(c,t)\, w(c) \\ & + \int_t^c \Big(h_1(s,t)\sqrt{H(s,t)} \\ & + A_1(s)\, H(s,t)\Big)\, w(s)\, ds \\ & - \int_t^c H(s,t)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\, ds \\ \le {} & -H(c,t)\, w(c) \\ & + \int_t^c \Big[\Big|h_1(s,t)\sqrt{H(s,t)} + A_1(s)\, H(s,t)\Big|\, |w(s)| \\ & - H(s,t)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\Big]\, ds.\end{aligned} \tag{1.9}$$

For given t and s (t not equal s), set

$$F_1(u) := \Big|h_1\sqrt{H} + A_1 H\Big|\, u - A_2 H u^{(\alpha+1)/\alpha}, \qquad u > 0.$$

$F_1(u)$ attains its maximum at

$$u := \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{\Big|h_1\sqrt{H} + A_1 H\Big|^{\alpha}}{(A_2 H)^{\alpha}}$$

and

$$F_1(u) \le F_{1\max} = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\Big|h_1\sqrt{H} + A_1 H\Big|^{\alpha+1}}{(A_2 H)^{\alpha}}. \tag{1.10}$$

Then, by using (1.10) in (1.9), we get

$$\begin{aligned}\int_t^c H(s,t)\, A_0(s)\, ds \le {} & -H(c,t)\, w(c) \\ & + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \int_t^c \frac{\Big|h_1(s,t)\sqrt{H(s,t)} + A_1(s)\, H(s,t)\Big|^{\alpha+1}}{(A_2(s)\, H(s,t))^{\alpha}}\, ds \\ = {} & -H(c,t)\, w(c) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \int_t^c \frac{|\phi_1(s,t)|^{\alpha+1}}{(A_2(s))^{\alpha}}\, ds.\end{aligned} \tag{1.11}$$

Similarly, if (1.5) is multiplied by $H(t,s)$ and then integrated from c to t for $t \in [c,b)$, then one gets

$$\begin{aligned}\int_c^t H(t,s)\, A_0(s)\, ds \le {} & H(t,c)\, w(c) + \int_c^t \big[-h_2(t,s)\sqrt{H(t,s)} \\ & + A_1(s)\, H(t,s)\big] w(s)\, ds - \int_c^t H(t,s)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\, ds \\ \le {} & H(t,c)\, w(c) + \int_c^t \Big[\Big|-h_2(t,s)\sqrt{H(t,s)} + A_1(s)\, H(t,s)\Big|\, |w(s)| \\ & - H(t,s)\, A_2(s)\, |w(s)|^{(\alpha+1)/\alpha}\Big]\, ds.\end{aligned} \tag{1.12}$$

For given t and s (t not equal s), set

$$F_2(u) := \left|-h_2\sqrt{H} + A_1 H\right| u - A_2 H u^{(\alpha+1)/\alpha}, \qquad u > 0.$$

$F_2(u)$ attains its maximum at

$$u := \left(\frac{\alpha}{\alpha+1}\right)^{\alpha} \frac{\left|-h_2\sqrt{H} + A_1 H\right|^{\alpha}}{(A_2 H)^{\alpha}}$$

and

$$F_2(u) \le F_{2\max} = \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\left|-h_2\sqrt{H} + A_1 H\right|^{\alpha+1}}{(A_2 H)^{\alpha}}. \tag{1.13}$$

Then, by using (1.13) in (1.12), we get

$$\int_c^t H(t,s) A_0(s)\, ds \le H(t,c)\, w(c) + \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \int_c^t \frac{|\phi_2(t,s)|^{\alpha+1}}{(A_2(s))^{\alpha}}\, ds. \tag{1.14}$$

Letting $t \to a^+$ in (1.11) and $t \to b^-$ in (1.14) and adding the resulting inequalities, we have equation (1.6).

2. MAIN RESULTS

We are now able to state the main results.

THEOREM 2.1. *Suppose that Conditions (i)–(iv) are satisfied. Suppose also that there exists an interval $(a,b) \subset [t_0,\infty)$ and there exist $c \in (a,b)$, $p \in C([t_0,\infty))$, $H \in \mathcal{P}$, $\rho \in C^1([t_0,\infty))$, such that $\rho(t) > 0$,*

(v) *$F(t,x,u,v,w)/|x|^{\alpha-1}x \ge p(t)$ holds, for $t \ge t_0$ and $x \ne 0$, u, v, $w \in R$,*

and

$$\begin{aligned} &\frac{1}{H(c,a)} \int_a^c \rho(s) \left[H(s,a)\, p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s) \left|\phi_1(s,a)\right|^{\alpha+1}\right] ds \\ &+ \frac{1}{H(b,c)} \int_c^b \rho(s) \left[H(b,s)\, p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s) \left|\phi_2(b,s)\right|^{\alpha+1}\right] ds > 0, \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \phi_1(s,a) &= \frac{h_1(s,a)\sqrt{H(s,a)} + (\rho'(s)/\rho(s))\, H(s,a)}{(H(s,a))^{\alpha/(\alpha+1)}}, \\ \phi_2(b,s) &= \frac{-h_2(b,s)\sqrt{H(b,s)} + (\rho'(s)/\rho(s))\, H(b,s)}{(H(b,s))^{\alpha/(\alpha+1)}}. \end{aligned} \tag{2.2}$$

Then every solution of (1.1) has a zero in (a,b).

PROOF. Otherwise, $x(t) \ne 0$, for all $t \in (a,b)$. Define

$$w(t) = \frac{\rho(t)\, r(t)\, |x'(t)|^{\alpha-1} x'(t)}{|x(t)|^{\alpha-1} x(t)}, \qquad t \in (a,b). \tag{2.3}$$

Differentiating (2.3) and making use of (1.1) and Condition (v), it follows that

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\rho(t)\, F(t, x(t), x(\tau(t)), x'(t), x'(\tau(t)))}{|x(t)|^{\alpha-1} x(t)} - \frac{\alpha \rho(t)\, r(t)\, |x'(t)|^{\alpha+1}}{|x(t)|^{\alpha+1}} \\ &\le -\rho(t)\, p(t) + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha\, |w(t)|^{(\alpha+1)/\alpha}}{(r(t)\,\rho(t))^{1/\alpha}}. \end{aligned} \tag{2.4}$$

Comparing inequalities (1.5) and (2.4), we identify that

$$A_0(t) = \rho(t)\,p(t), \qquad A_1(t) = \frac{\rho'(t)}{\rho(t)}, \qquad \text{and} \qquad A_2(t) = \frac{\alpha}{(r(t)\,\rho(t))^{1/\alpha}}.$$

Applying Lemma 1.1 to (2.4), we see that inequality (2.1) fails to hold. ∎

If the conditions of Theorem 2.1 hold for a sequence $\{(a_n, b_n)\}$ of intervals, such that

$$\lim_{n\to\infty} a_n = \infty,$$

then we may conclude that (1.1) is oscillatory. That is, we have the following corollary.

COROLLARY 2.1. *If for a given $T \ge t_0$ there exists an interval $(a,b) \subset [T,\infty)$ for which the conditions of Theorem 2.1 are satisfied, then (1.1) is oscillatory.*

The following theorem is also a consequence of Theorem 2.1.

THEOREM 2.2. *Let Conditions (i)–(v) hold. Suppose that there exist $H \in \mathcal{P}$ and $\rho \in C^1([t_0,\infty))$, such that $\rho(t) > 0$,*

$$\limsup_{t\to\infty} \int_l^t \rho(s)\left[H(s,l)\,p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_1(s,l)|^{\alpha+1}\right] ds > 0, \tag{2.5}$$

and

$$\limsup_{t\to\infty} \int_l^t \rho(s)\left[H(t,s)\,p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_2(t,s)|^{\alpha+1}\right] ds > 0, \tag{2.6}$$

for each $l \ge t_0$, where ϕ_1 and ϕ_2 are as in (2.2). Then, (1.1) is oscillatory.

PROOF. Suppose that $x(t) \neq 0$, for all $t \in [t_1,\infty)$, for some $t_1 \ge t_0$. Set $l = a \ge t_1$ in (2.5). Clearly, we see from (2.5) that there exists $c > a$, such that

$$\int_a^c \rho(s)\left[H(s,a)\,p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_1(s,a)|^{\alpha+1}\right] ds > 0. \tag{2.7}$$

Similarly, setting $l = c \ge t_1$ in (2.6), it follows that there exists $b > c$, such that

$$\int_c^b \rho(s)\left[H(b,s)\,p(s) - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_2(b,s)|^{\alpha+1}\right] ds > 0. \tag{2.8}$$

From (2.7) and (2.8), we see that (2.1) is satisfied. Therefore, in view of Corollary 2.1, we may conclude that (1.1) is oscillatory. ∎

THEOREM 2.3. *Suppose that (i)–(iv) are satisfied. Further,*

(vi) $\operatorname{sgn} F(t,x,u,v,w) = \operatorname{sgn} x$ *for each $t \ge t_0$ and $x,u,v,w \in R$;*
(vii) *$r(t)$ is a nondecreasing, differentiable function, and satisfies $\int^\infty (ds/r^{1/\alpha}(s)) = \infty$.*

Suppose also that there exists $(a,b) \subset [t_0,\infty)$ and there exist $c \in (a,b)$, $p \in C([t_0,\infty))$, $H \in \mathcal{P}$, $\rho \in C^1([t_0,\infty))$, such that $\rho(t) > 0$,

(viii) *$F(t,x,u,v,w)/|u|^{\alpha-1}u \ge p(t)$, for $t \ge t_0$, $x \neq 0$, $u \neq 0$, $v,w \in R$,*

and

$$\begin{aligned}&\frac{1}{H(c,a)} \int_a^c \rho(s)\left[H(s,a)\,p(s)\left[\frac{k\tau_*(s)}{s}\right]^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_1(s,a)|^{\alpha+1}\right] ds \\ &+\frac{1}{H(b,c)} \int_c^b \rho(s)\left[H(b,s)\,p(s)\left[\frac{k\tau_*(s)}{s}\right]^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} r(s)\,|\phi_2(b,s)|^{\alpha+1}\right] ds > 0,\end{aligned} \tag{2.9}$$

where k is a constant, $k \in (0,1)$, $\tau_*(t) = \min\{t, \tau(t)\}$, and ϕ_1 and ϕ_2 are as in (2.2). Then, every solutions of (1.1) has a zero in (a,b).

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ and $x(\tau(t)) > 0$, for $t \geq T_0 \geq t_0$. Because, from equation (1.1) and Condition (vi), a similar analysis holds for $x(t) < 0$ and $x(\tau(t)) < 0$. Using Conditions (iii), (vi), and (vii), from equation (1.1), it is easy to prove that $x''(t) \leq 0$ and $x'(t) > 0$, for $t \geq T_1 \geq T_0$. Hence, by Lemma 2.1 in [10], for any $k \in (0,1)$, there exists a $T_2 \geq T_1$, such that

$$x(\tau(t)) \geq \frac{k\tau_*(t)}{t} x(t), \qquad \text{for all } t \geq T_2,$$

where $\tau_*(t) = \min\{t, \tau(t)\}$. Let us define $w(t)$ again by (2.3). Then, we may obtain

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\rho(t) F(t, x(t), x(\tau(t)), x'(t), x'(\tau(t)))}{(x(t))^\alpha} - \frac{\alpha |w(t)|^{(\alpha+1)/\alpha}}{(r(t)\rho(t))^{1/\alpha}} \\
&= \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\rho(t) F(t, x(t), x(\tau(t)), x'(t), x'(\tau(t)))}{(x(\tau(t)))^\alpha} \left(\frac{x(\tau(t))}{x(t)}\right)^\alpha \\
&\quad - \frac{\alpha |w(t)|^{(\alpha+1)/\alpha}}{(r(t)\rho(t))^{1/\alpha}} \\
&\leq -\rho(t) p(t) \left[\frac{k\tau_*(t)}{t}\right]^\alpha + \frac{\rho'(t)}{\rho(t)} w(t) - \frac{\alpha |w(t)|^{(\alpha+1)/\alpha}}{(r(t)\rho(t))^{1/\alpha}}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 2.1, hence, is omitted. ∎

COROLLARY 2.2. *If for a given $T \geq t_0$ there exists an interval $(a,b) \subset [T,\infty)$ for which the conditions of above theorem are satisfied, then (1.1) is oscillatory.*

The following theorem is also a consequence of Theorem 2.3.

THEOREM 2.4. *Let Conditions (i)–(iv) and (vi)–(viii) hold. Suppose that there exist $H \in \mathcal{P}$ and $\rho \in C^1([t_0,\infty))$, such that $\rho(t) > 0$,*

$$\limsup_{t\to\infty} \int_l^t \rho(s) \left[H(s,l) p(s) \left[\frac{k\tau_*(s)}{s}\right]^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} r(s) |\phi_1(s,l)|^{\alpha+1}\right] ds > 0,$$

and

$$\limsup_{t\to\infty} \int_l^t \rho(s) \left[H(t,s) p(s) \left[\frac{k\tau_*(s)}{s}\right]^\alpha - \frac{1}{(\alpha+1)^{\alpha+1}} r(s) |\phi_2(t,s)|^{\alpha+1}\right] ds > 0,$$

for each $l \geq t_0$, where k is a constant, $k \in (0,1)$, $\tau_*(t) = \min\{t, \tau(t)\}$, and ϕ_1 and ϕ_2 are as in equation (2.2). Then, equation (1.1) is oscillatory.

PROOF. The proof of this theorem is similar to the proof of Theorem 2.2. ∎

The above theorems are presented in the form of a high degree of generality and deriving different explicit oscillation criteria for (1.1) with appropriate choices of $H(t,s)$ and $\rho(s)$. For the case where $H := H(t-s) \in \mathcal{P}$, we have that $h_1(t-s) = h_2(t-s)$ and denote them by $h(t-s)$. The subclass of $\mathcal{P}$ containing such $H(t-s)$ is denoted by $\mathcal{P}_0$. Applying Corollary 2.1 and 2.2, we obtain the following results, respectively.

THEOREM 2.5. *Suppose that (i)–(v) hold. If for each $T \geq t_0$, there exist $H \in \mathcal{P}$, $\rho \in C^1[t_0,\infty)$ and $a, c \in R$, such that $T \leq a < c$, $\rho(t) > 0$ and*

$$\int_a^c H(s-a)\left(\rho(s)p(s) + \rho(2c-s)p(2c-s)\right) ds$$
$$> \frac{1}{(\alpha+1)^{\alpha+1}} \int_a^c \left(\frac{r(s)\rho(s)\left|h(s-a)\sqrt{H(s-a)} + (\rho'(s)/\rho(s))H(s-a)\right|^{\alpha+1}}{(H(s-a))^{\alpha}} ds \right. \tag{2.10}$$
$$\left. + \frac{r(2c-s)\rho(2c-s)\left|-h(s-a)\sqrt{H(s-a)} + (\rho'(2c-s)/\rho(2c-s))H(s-a)\right|^{\alpha+1}}{(H(s-a))^{\alpha}} \right) ds,$$

then equation (1.1) is oscillatory.

THEOREM 2.6. *Suppose that Conditions (i)–(iv) and (vi)–(viii) hold. If for each $T \geq t_0$, there exist $H \in \mathcal{P}$, $\rho \in C^1[t_0,\infty)$ and $a, c \in R$, such that $T \leq a < c$, $\rho(t) > 0$, and*

$$\int_a^c H(s-a)\left(\rho(s)p(s)\left[\frac{k\tau_*(s)}{s}\right]^{\alpha} + \rho(2c-s)p(2c-s)\left[\frac{k\tau_*(2c-s)}{2c-s}\right]^{\alpha}\right) ds$$
$$> \frac{1}{(\alpha+1)^{\alpha+1}} \int_a^c \left(\frac{r(s)\rho(s)\left|h(s-a)\sqrt{H(s-a)} + (\rho'(s)/\rho(s))H(s-a)\right|^{\alpha+1}}{(H(s-a))^{\alpha}} \right.$$
$$\left. + \frac{r(2c-s)\rho(2c-s)\left|-h(s-a)\sqrt{H(s-a)} + (\rho'(2c-s)/\rho(2c-s))H(s-a)\right|^{\alpha+1}}{(H(s-a))^{\alpha}} \right) ds,$$

where k is a constant, $k \in (0,1)$, and $\tau_*(t) = \min\{t, \tau(t)\}$. Then, equation (1.1) is oscillatory.

Next, we define

$$R(t) = \int_l^t r^{-1/\alpha}(s)\, ds, \qquad t \geq l \geq t_0, \tag{2.11}$$

and let

$$H(t,s) = [R(t) - R(s)]^{\lambda}, \qquad t \geq s \geq t_0, \tag{2.12}$$

where $\lambda > \max\{1, \alpha\}$ is a constant. By Theorems 2.2 and 2.4 for $\rho(t) \equiv 1$, respectively, we have the following oscillation criteria.

THEOREM 2.7. *Let Conditions (i)–(v) hold. Assume that $\lim_{t\to\infty} R(t) = \infty$. Then, equation (1.1) is oscillatory provided that for each $l \geq t_0$ and, for some $\lambda > \max\{1,\alpha\}$, the following two inequalities hold,*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(s) - R(l)]^{\lambda} p(s)\, ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)} \tag{2.13}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(t) - R(s)]^{\lambda} p(s)\, ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)}. \tag{2.14}$$

THEOREM 2.8. *Let Conditions (i)–(iv) and (vi)–(viii) hold, and assume that $\lim_{t\to\infty} R(t) = \infty$. Then, equation (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > \max\{1,\alpha\}$, the following two inequalities hold,*

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(s) - R(l)]^{\lambda} p(s)\left[\frac{\tau_*(s)}{s}\right]^{\alpha} ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)} \tag{2.15}$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(t)-R(s)]^\lambda p(s) \left[\frac{\tau_*(s)}{s}\right]^\alpha ds > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)}, \tag{2.16}$$

where $\tau_*(t) = \min\{t, \tau(t)\}$.

REMARK 2.1. In the case when $F(t,x,u,v,w) = p(t)|x|^{\alpha-1}x$, Theorems 2.1, 2.2, 2.5, and 2.7 reduce to corresponding all theorems given in [34].

REMARK 2.2. In the case when $F(t,x,u,v,w) = p(t)|u|^{\alpha-1}u$, our results are better than corresponding results of [22], since our conditions do not require the existence of $\rho'(t) \geq 0$ and $\tau'(t) > 0$.

REMARK 2.3. Let $F(t,x,u,v,w) = p(t)x$. If we take $\rho(t) \equiv 1$, $\alpha = 1$, and $r(t) \equiv 1$, Theorem 2.7 (or Theorem 2.8 with $\tau(t) \equiv t$) reduces to the part of Theorem 2.3 in [17].

REMARK 2.4. Taking $F(t,x,u,v,w) = (\mu/t^2)u$, $\rho(t) \equiv 1$, $\alpha = 1$, $r(t) \equiv 1$, and $\tau(t) = t \pm \sigma$, then equation (1.1) reduces to the Euler equation,

$$x''(t) + \frac{\mu}{t^2}x(t\pm\sigma) = 0. \tag{2.17}$$

It is well known that [26,28,29] equation (2.17) is oscillatory if $\mu > 1/4$ and nonoscillatory if $\mu \leq 1/4$. Applying Theorem 2.8 to equation (2.17), we also see that equation (2.17) is oscillatory if $\mu > 1/4$ (see Examples 3.2 and 3.3 in Section 3). This implies that our results are sharp.

3. EXAMPLES

In this section, we will show the application of our oscillation criteria in three examples.

EXAMPLE 3.1. Let $r(t) \in C([0,\infty),(0,1])$ and

$$p(t) = \begin{cases} \eta(t-3k), & 3k \leq t \leq 3k+1, \\ \eta(-t+3k+2), & 3k+1 < t \leq 3k+2, \\ p_0(t), & 3k+2 < t \leq 3k+3, \end{cases}$$

for $k \in \{1,2,\dots\}$ where p_0 is any continuous function which makes p a continuous function, and

$$\eta > \frac{(\lambda+2)\lambda^{\alpha+1}}{(\lambda-\alpha)(\alpha+1)^{\alpha+1}}$$

is a constant for fixed $\lambda > \max\{1,\alpha\}$. Consider the equation,

$$\left(r(t)|x'(t)|^{\alpha-1}x'(t)\right)' + p(t)|x(t)|^{\alpha-1}x(t) = 0, \qquad t \geq 0, \tag{3.1}$$

with $\alpha > 0$ a constant. Let $a = 3k$, $c = 3k+1$, $H(t,s) = (t-s)^\lambda$, and $\rho(t) \equiv 1$. Note that for $h(t,s) = \lambda(t-s)^{(\lambda-2)/2}$,

$$\begin{aligned} &\int_a^c (s-a)^\lambda [p(s)+p(2c-s)]\,ds - \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \int_a^c [r(s)+r(2c-s)](s-a)^{\lambda-(\alpha+1)}\,ds \\ &\geq 2\int_{3k}^{3k+1} \eta(s-3k)^{\lambda+1}\,ds - \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \int_{3k}^{3k+1} 2(s-3k)^{\lambda-(\alpha+1)}\,ds \\ &= \frac{2\eta}{\lambda+2} - \frac{2\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)} > 0, \end{aligned}$$

i.e., (2.10) holds. Thus, equation (3.1) is oscillatory by Theorem 2.5. It is worth mentioning that by a suitable choice of $p_0(t)$ (for example, $p_0(t) = -k\sin\pi t$), we can make $\int_{t_0}^\infty p(s)\,ds = -\infty$, meaning that the results of this paper are applicable for such extreme cases. In this case, the known results such as in [2,9,13,15] fail to apply to this equation.

Now, we shall construct two examples including the Euler equations (1.2) and (1.3) as special cases.

EXAMPLE 3.2. Consider the second-order differential equation,

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + \frac{\mu r^{-(1/\alpha)}(t)}{R^{\alpha+1}(t)}\left|x(\tau(t))\right|^{\alpha-1}x(\tau(t)) = 0, \qquad \alpha > 0, \tag{3.2}$$

for $t \geq t_0 > 0$, where $\mu > 0$ and $\alpha > 0$ are constants, $\tau \in C([t_0,\infty),(0,\infty))$, $\tau(t) \geq t$, $\lim_{t\to\infty}\tau(t) = \infty$, $R(t)$ is defined as in (2.11) and $\lim_{t\to\infty}R(t) = \infty$. Let $\alpha_0 := \max\{1,\alpha\}$. Then, we can verify that equation (3.2) is oscillatory for $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$ by Theorem 2.8. In ordinary case, that is $\tau(t) \equiv t$, this example examined in [34].

Let $H(t,s) = [R(t) - R(s)]^\lambda$, $\lambda > \alpha_0 \geq 1$, and $\rho(t) \equiv 1$, for $t \geq t_0$. As also given in [34], we have

$$[R(s) - R(l)]^\lambda \geq R^\lambda(s) - \lambda R(l)R^{\lambda-1}(s), \qquad \text{for } s \geq l \geq t_0. \tag{3.3}$$

It follows from $R'(t) = r^{-1/\alpha}(t)$ and $\tau_*(t) = t$ that, for each $l \geq t_0$

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t [R(s)-R(l)]^\lambda \frac{\mu r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}\left[\frac{\tau_*(s)}{s}\right]^\alpha ds\\ &\geq \lim_{t\to\infty}\frac{1}{R^{\lambda-\alpha}(t)}\int_l^t \left[R^\lambda(s) - \lambda R(l)R^{\lambda-1}(s)\right]\frac{\mu r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}\,dR(s) = \frac{\mu}{\lambda-\alpha}.\end{aligned} \tag{3.4}$$

For any $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$, there exists $\lambda > \alpha_0$, such that

$$\frac{\mu}{\lambda-\alpha} > \frac{\lambda^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\lambda-\alpha)}.$$

This means that (2.15) holds.

Next, we will prove that

$$\int_l^t [R(t)-R(s)]^\lambda \frac{r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}\,ds \geq \int_l^t [R(s)-R(l)]^\lambda \frac{r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}\,ds. \tag{3.5}$$

Let

$$F(t) := \int_l^t \left\{[R(t)-R(s)]^\lambda - [R(s)-R(l)]^\lambda\right\}\frac{r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}\,ds.$$

Then, $F(l) = 0$, and for $t \geq l$,

$$\begin{aligned}F'(t) &= \int_l^t \lambda[R(t)-R(s)]^{\lambda-1}R'(t)\frac{r^{-1/\alpha}(s)}{R^{\alpha+1}(s)}ds - [R(t)-R(l)]^\lambda\frac{r^{-1/\alpha}(t)}{R^{\alpha+1}(t)}\\ &\geq \frac{R'(t)}{R^{\alpha+1}(t)}\int_l^t \lambda[R(t)-R(s)]^{\lambda-1}r^{-1/\alpha}(s)\,ds - [R(t)-R(l)]^\lambda\frac{r^{-1/\alpha}(t)}{R^{\alpha+1}(t)} = 0.\end{aligned}$$

Hence, $F(t) \geq F(t) = 0$ for $t \geq l$, i.e., (3.5) holds. By (3.4) and (3.5), condition (2.16) holds for the same λ. Applying Theorem 2.8, we find that equation (3.2) is oscillatory if $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$.

Taking $\rho(t) \equiv 1$, $\alpha = 1$, $r(t) \equiv 1$, and $\tau(t) = t + \sigma$, then equation (3.2) reduces to the Euler equation (2.17). It is well known that [28,29] equation (2.17) is oscillatory if $\mu > 1/4$ and nonoscillatory if $\mu \leq 1/4$. Since $\alpha_0 = \alpha = 1$, we also see that equation (2.17) is oscillatory if $\mu > 1/4$.

When $r(t) \equiv 1$, for $t \geq t_0$, $\tau(t) = t$ and $\alpha = 1$, then equation (3.2) reduces to equation (1.3). In this case, $\alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1} = 1/4$ and Example 3.2 is consistent with the well known result of (1.3) that equation (1.3) is oscillatory if $\mu > 1/4$ and to a certain extent it also reveals the construction of the Euler equation (1.3).

EXAMPLE 3.3. Consider the differential equation,

$$\left(|x'(t)|^{\alpha-1} x'(t)\right)' + \frac{\mu}{t^{\alpha+1}} |x(t-\sigma)|^{\alpha-1} x(t-\sigma) = 0, \qquad t \geq t_0 > 0, \tag{3.6}$$

where $t \geq \sigma$ and σ, μ and α are constants. Let $\alpha_0 := \max\{1, \alpha\}$. Then, we can see that equation (3.6) is oscillatory for $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$ by Theorem 2.8.

Let $\lambda > \alpha_0 \geq 1$ and $\rho(t) \equiv 1$ for $t \geq t_0$. It follows from $R(t) = \int_l^t ds = t - l$ and $\tau_*(t) = t - \sigma$ that, for each $l \geq t_0$,

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(s) - R(l)]^\lambda \left[\frac{\tau_*(s)}{s}\right]^\alpha p(s)\, ds \\
&= \lim_{t\to\infty} \frac{1}{(t-l)^{\lambda-\alpha}} \int_l^t (s-l)^\lambda \left(\frac{s-\sigma}{s}\right)^\alpha \frac{\mu}{s^{\alpha+1}}\, ds \\
&= \lim_{t\to\infty} \frac{(t-l)^\lambda}{(\lambda-\alpha)(t-l)^{\lambda-\alpha-1}} \left(\frac{t-\sigma}{t}\right)^\alpha \frac{\mu}{t^{\alpha+1}} = \frac{\mu}{\lambda-\alpha}.
\end{aligned} \tag{3.7}$$

On the other hand, from (3.3), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{R^{\lambda-\alpha}(t)} \int_l^t [R(t) - R(s)]^\lambda \left(\frac{\tau_*(s)}{s}\right)^\alpha p(s)\, ds \\
&= \limsup_{t\to\infty} \left(\frac{t-\sigma}{t}\right)^\alpha \frac{\mu}{(t-l)^{\lambda-\alpha}} \int_\xi^t (t-s)^\lambda \frac{1}{s^{\alpha+1}}\, ds \qquad \text{for } \xi \in (l,t), \\
&\geq \lim_{t\to\infty} \left(\frac{t-\sigma}{t}\right)^\alpha \frac{\mu}{(t-l)^{\lambda-\alpha}} \int_\xi^t \left(t^\lambda - \lambda s t^{\lambda-1}\right) \frac{1}{s^{\alpha+1}}\, ds = \infty.
\end{aligned} \tag{3.8}$$

By (3.7) and (3.8), for any $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$ there exists $\lambda > \alpha_0$, such that

$$\frac{\mu}{\lambda-\alpha} > \left(\frac{\lambda}{\alpha+1}\right)^{\alpha+1} \frac{1}{\lambda-\alpha}.$$

Then, applying Theorem 2.8, we find that (3.6) is oscillatory for $\mu > \alpha_0^{\alpha+1}/(\alpha+1)^{\alpha+1}$. In particular, if $\alpha = 1$, then equation (3.6) reduces to the Euler equation (1.2) and $\alpha^{\alpha+1}/(\alpha+1)^{\alpha+1} = 1/4$. It is well known that equation (1.2) is oscillatory if $\mu > 1/4$.

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