# Existence of Limit Cycles for Generalized Liénard Equations 

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## 1. Introduction

We consider the generalized Liénard system

$$
\begin{align*}
& \dot{x}=h(y)-F(x)  \tag{1}\\
& \dot{y}=-g(x)
\end{align*}
$$

where $F, g$, and $h$ satisfy assumptions to be listed below. Under those assumptions we prove the existence of one or several limit cycles. Our assumptions are, e.g., satisfied by the systems

$$
\begin{align*}
& \dot{x}=y-a \sin x \\
& \dot{y}=-x, \tag{2}
\end{align*}
$$

and

$$
\begin{align*}
& \dot{x}=y-a F(x)  \tag{3}\\
& \dot{y}=-x,
\end{align*}
$$

when $a \neq 0, F(x)=x^{5}-\mu x^{3}+v x$, and $\mu \geqslant 2.32 v^{1^{1 / 2}}$, but also satisfied by the van der Pol system.

The system (2) has been studied by several authors [ $1,2,6]$, but most notably by Zhang Zhifen [7]. Her theorem implies that (2) has exactly $n$ limit cycles in the strip $|x| \leqslant(n+1) \pi$. The system (3) was considered by Rychkov [4,5], who proved that (3) has exactly 2 limit cycles when $\mu>2.5$ and $v=1$.

## 2. Assumptions

For reference purposes we list over assumptions below. Some will be used throughout and some will not.

We shall always assume that $F, g, h \in C(\mathbf{R})$, and also
(F1) There is a positive number $a_{1}$ such that $F$ does not change sign on $\left[0, a_{1}\right]$ and is not identically zero there, but $F(0)=F\left(a_{1}\right)=0$,
$(g 1) \quad g(x)>0$ for $x>0$,
( $h 1$ ) $h$ is strictly increasing and odd, i.e., $h(-x)=-h(x)$ for all $x \in \mathbf{R}$, and

$$
(h F) \quad h(\mathbf{R}) \supset F(\mathbf{R})
$$

We shall often require that $F$ oscillates to some degree: Let $I=[a, c]$ and $J=[c, b]$ where $0 \leqslant a<c<b$.
(g2) $\phi: I \rightarrow J$ is weakly increasing, absolutely continuous, and $g(\phi(x)) \phi^{\prime}(x) \geqslant g(x)$ for a.e. $x \in I$.
$(F 2)$ With a function $\phi$ satisfying (g2) we have $\operatorname{sgn} F(\phi(x))=$ $-\operatorname{sgn} F(x), \operatorname{sgn} F(-\phi(x))=-\operatorname{sgn} F(-x),|F(\phi(x))| \geqslant|F(x)|$, and $|F(-\phi(x))|$ $\geqslant|F(-x)|$ for all $x \in I$.

Furthermore, $F$ does not change sign on $J$ nor on $-J$ (relevant only when $\phi(I) \neq J)$.
(The part of $(F 2)$ dealing with $F(x)$ for $x \geqslant 0$ will be referred to in the sequel as $(F 2+)$.)

In some of the proofs we use a stronger one-sided version of $(F 2)$ :
$(F 3)$ With a function $\phi$ satisfying (g2) with $\phi(I)=J$ we have $F(\phi(x))=-F(x)$ for all $x \in I$.

For completeness we mention also here that $F$ and $g$ in the Theorem will be assumed odd on some symmetric interval.

## 3. Results

Theorem. Assume that the system (1) satisfies (F1), (g1), (h1), and ( $h F$ ).

Let $0=a_{0}<a_{1}<a_{2}<\cdots<a_{q}$, where $q \geqslant 2$, and assume that (F2) is satisfied on each of the intervals $I_{i}=\left[a_{i-1}, a_{i}\right], 1 \leqslant i \leqslant q-1$, with some $\phi_{i}$ satisfying (g2) and with $J_{i}=\left[a_{i}, a_{i+1}\right]$. Finally assume that $F$ is odd on $\left[-a_{q-1}, a_{q-1}\right]$ and that $g$ is odd on $\left[-a_{q}, a_{q}\right]$.

Then the system (1) has for each $i$ with $1 \leqslant i \leqslant q-1$ at least one limit cycle passing through the open interval $\left(a_{i}, \phi_{i}\left(a_{i}\right)\right)\left(\subset\left(a_{i}, a_{i+1}\right)\right)$ on the
$x$-axis. There are no closed orbits passing through the interval $\left[\phi_{i}\left(a_{i}\right), a_{i+1}\right]$ for $1 \leqslant i \leqslant q-2$ and none for $i=q-1$ either, if $F$ is odd on all of $\left[-a_{q}, a_{q}\right]$.

We postpone the proof to make some remarks, state some corollaries. and give some examples.

Remarks. (1) The limit cycles are contained in the region $G(x)+H(y)>G\left(a_{1}\right)$, where $G$ and $H$ are defined by

$$
\begin{equation*}
G(x)=\int_{0}^{1} g(s) d s \quad \text { and } \quad H(y)=\int_{0}^{1} h(s) d s \tag{4}
\end{equation*}
$$

(2) If $F \geqslant 0$ on $\left[0, a_{1}\right]$ then the origin is stable. If $F$ is not identically zero in any neighborhood of 0 then the origin is asymptotically stable.
(3) If $F \geqslant 0$ on $\left[0, a_{1}\right]$ and $i$ is even (odd) then there is through $\left(a_{i}, \phi_{i}\left(a_{i}\right)\right)$ a limit cycle that is (un)stable from the inside and one (possibly the same) that is (un)stable from the outside.
(4) If the assumptions of the Theorem are satisfied for the system (1) then they are also satisfied when in (1), F(x) is replaced by $p(F(x))$, provided that $p$ is an odd, weakly increasing continuous function such that $p \triangleleft F$ is not identically zero on $\left[0, a_{1}\right]$ and such that $h(\mathbf{R}) \supset p(F(\mathbf{R}))$.

Our first corollary applies to the systems considered by Zhang Zhifen [7] and therefore in particular to (2).

Corollary 1. Assume that the system (1) satisfies $(F 1),(g 1),(h 1)$, and $(h F)$. Assume that $F(x)=A(x) F_{1}(x)$ for all $x \in \mathbf{R}$ where $A, F_{1} \in C(\mathbf{R}), A$ is even, $F_{1}$ is odd, $A$ is weakly increasing for $x \geqslant 0$ with $A(0) \geqslant 0$, and $F_{1}$ is periodic with period $2 a_{1}$ and satisfies $F_{1}\left(x+a_{1}\right)=-F_{1}(x)$ for all $x \in \mathbf{R}$. Assume finally that $g$ is odd and satisfies $g\left(x+a_{1}\right) \geqslant g(x)$ for all $x \geqslant 0$.

Then the system (1) has for each $n \geqslant 1$ at least one limit cycle passing through the open interval $\left(n a_{1},(n+1) a_{1}\right)$ on the $x$-axis.

Proof. It is easily verified that the assumptions of the Theorem are satisfied when we take $a_{n}=n a_{1}$ for $n \geqslant 0$ and $\phi_{n}(x)=a_{1}+x$ for all $n \geqslant 1$.

Corollary 2. Suppose the assumptions of Corollary 1 are satisfied and that $g(x)=x$ for all $x \in \mathbf{R}$. Suppose further that $F_{1}$ is monotone on $\left[0, a_{1} / 2\right]$. Then the system (1) has for each $n \geqslant 1$ at least one limit cycle passing through the open interval $\left(n a_{1}, n a_{1}(1+2 / n)^{1 \cdot 2}\right)$ on the $x$-axis and has no closed orbits through $\left[n a_{1}(1+2 / n)^{1: 2},(n+1) a_{1}\right]$.

Proof. With $a_{n}=n a_{1}$ for $n \geqslant 0$ and $\phi_{n}(x)=a_{1}+x$ for $x \in\left[a_{n-1}, a_{n}-a_{1} / 2\right]$ and $\phi_{n}(x)=\left(x^{2}+2 n a_{1}^{2}\right)^{1,2}$ for $x \in\left[a_{n}-a_{1} / 2, a_{n}\right]$, we check that the assumptions of the Theorem are satisfied. It is easily seen that $\phi_{n}$ satisfies
(g2) on $\left[a_{n-1}, a_{n}\right]$ and that $(F 2)$ is satisfied on the first half of that interval. For the other half we use that it follows from the assumptions that $\left|F_{1}\right|$ is decreasing on $\left[a_{n}-a_{1} / 2, a_{n}\right]$ for all $n$. Since $a_{n}-a_{1} / 2 \leqslant$ $\left(x^{2}+2 n a_{1}^{2}\right)^{1 / 2}-a_{1} \leqslant x$ for all $x \in\left[a_{n}-a_{1} / 2, a_{n}\right]$ it follows that

$$
\begin{aligned}
\left|F\left(\left(x^{2}+2 n a_{1}^{2}\right)^{12}\right)\right| & =A\left(\left(x^{2}+2 n a_{1}^{2}\right)^{1 \cdot 2}\right)\left|F_{1}\left(\left(x^{2}+2 n a_{1}^{2}\right)^{1: 2}-a_{1}\right)\right| \\
& \geqslant A(x)\left|F_{1}(x)\right|=|F(x)| .
\end{aligned}
$$

The conclusion now follows from the Theorem since $\phi_{n}\left(a_{n}\right)=$ $n a_{1}(1+2 / n)^{12}$.

Corollary 3. Suppose that the system (1) satisfies the assumptions of the Theorem for some $q \geqslant 2$. Suppose further that $F$ and $g$ are odd on all of $\mathbf{R}$ and that $F$ does not change sign on $\left(a_{q-1}, \infty\right)$. Then (1) has no closed orbits through the interval $\left[\phi_{4-1}\left(a_{4-1}\right), \infty\right)$ on the $x$-axis.

Proof. The corollary is a direct consequence of the Theorem since $a_{4}$ may be taken arbitrarily large.

## 4. Examples

Example 1. The system (2) has (at least) one limit cycle through each of the intervals $\left(n \pi, n \pi(1+2 / n)^{1 / 2}\right.$ ) on the $x$-axis $(n \geqslant 1)$. Zhang Zhifen [7] also proved uniqueness. We get the same conclusion for the system

$$
\dot{x}=y-A(x) \sin x, \quad \dot{y}=-x
$$

if only $A \in C(\mathbf{R})$ is even, $|A|$ is weakly increasing, for $x \geqslant 0$, and if $A$ is not identically zero on $[0, \pi]$. The example $A(x)=-x^{2}$ was considered in [6, pp. 96, 160].

Example 2. Consider the van der Pol system

$$
\dot{x}=y-\varepsilon\left(x^{3}-a_{1}^{2} x\right), \quad \dot{y}=-x,
$$

where $a_{1}>0$ and $\varepsilon \neq 0$.
Let $\phi(x)=\alpha x+a_{1}$ for $x \in\left[0, a_{1}\right]$, where we take $\alpha=\left(5^{1.2}-1\right) / 2$. The assumptions of the Theorem are easily verified with $q=2$. Thus there is (at least) one limit cycle through the interval $\left(a_{1},(\alpha+1) a_{1}\right)$ on the $x$-axis and by Corollary 3 there are no closed orbits through $\left[(\alpha+1) a_{1}, \infty\right)$. It is well known that the limit cycle is unique. The statement about the location of the limit cycle can be improved somewhat by considering a more complicated $\phi$.

Example 3. Consider the system (3), i.e., $\dot{x}=y-a F(x), \dot{y}=-x$, where $F(x)=x^{5}-\mu x^{3}+v x, \mu, v>0$, and $a \neq 0$.

Rychkov [4,5] showed that the system has exactly 2 limit cycles when $\mu>2.5$ and $v=1$. We shall show that $\mu \geqslant \mu_{0} v^{1.2}$ suffices, where $\mu_{0}=1+\alpha+$ $(1+\alpha)^{-1}$ and $\alpha$ is the positive root of $2 \alpha^{3}+3 \alpha^{2}-2 \alpha-1=0$, thus $\mu_{0} \approx 2.3178$.

Let $a_{1}$ and $a_{2}$ be the two positive zeros of $F\left(a_{1}<a_{2}\right)$. Define $\phi_{1}$ and $\phi_{2}$ by $\phi_{1}(x)=\alpha x+a_{1}$ for $x \in\left[0, a_{1}\right]$ and $\phi_{2}(x)=\beta\left(x-a_{1}\right)+a_{2}$ for $x \in\left[a_{1}, a_{2}\right]$, where $\beta>0$ will be picked later. For $\mu \geqslant \mu_{0} v^{1,2}$ we show that the assumptions of the Theorem are satisfied with $q=3$ and $a_{3}=\phi_{2}\left(a_{2}\right)$. We conclude that the system has a limit cycle passing through the interval $\left(a_{1}, a_{1}(1+\alpha)\right)$ on the $x$-axis and also one through $\left(a_{2}, a_{3}\right)$. Uniqueness follows from a theorem by Rychkov [5].

We first consider $\phi_{1}$. The assumption (g2) is easily verified. We let $P(x)=-\left(F(x)+F\left(\phi_{1}(x)\right)\right) / x$, a polynomial of degree 4 . We require $P(0) \geqslant 0$ and $P\left(a_{1}\right) \geqslant 0$. Those inequalities are satisfied if $\mu \geqslant \mu_{0} v^{12}$. For such values of $\mu$ we have $P^{\prime}(0) \geqslant 0$. Since we also have $P^{\prime \prime \prime}(x)<0$ for all $x \geqslant 0$, it follows that $P(x) \geqslant 0$ for all $x \in\left[0, a_{1}\right]$. Thus $(F 2)$ is satisfied on $\left[0, a_{1}\right]$.

Next we consider $\phi_{2}$. The requirement (g2) is satisfied when $\beta$ is not less than $\frac{1}{2}\left(-1+\left(5-4 a_{1}^{2} v^{-12}\right)^{1 / 2}\right) /\left(1-a_{1}^{2} v^{-1,2}\right)$. Let $\beta$ have that value. Let $Q(x)=\left(F(x)+F\left(\phi_{2}(x)\right)\right) /\left(x-a_{1}\right)$, again a polynomial of degree 4. This time we see that $Q\left(a_{1}\right) \geqslant 0, Q^{\prime}\left(a_{1}\right) \geqslant 0$, and $Q^{\prime \prime}\left(a_{1}\right) \geqslant 0$. Since also $Q^{\prime \prime \prime}(x)>0$ for all $x \in\left[a_{1}, a_{2}\right]$ it follows that $Q(x) \geqslant 0$ for all $x \in\left[a_{1}, a_{2}\right]$, so that $(F 2)$ is satisfied on $\left[a_{1}, a_{2}\right]$.

A classical perturbation method (see [2]) applied to this example shows the existence of two closed orbits when $\mu>2(10 r)^{1 / 2} / 3 \approx 2.108 v^{12}$ and when $a$ is sufficiently small. The assumption ( $F 2$ ) in the Theorem implies that the maximum of $|F|$ on $\left[a_{1}, a_{2}\right.$ ] is not less than it is on [ $0, a_{1}$ ], thus we could not expect any better than $\mu \geqslant(5 v)^{1: 2} \approx 2.236 v^{12}$. See also Section 6 for a remark about large perturbations, i.e., large values of $a$.

## 5. The Proof of the Theorem

We first prove a lemma which essentially says that the orbits of (1) spiral around the origin.

Lemma 1. Assume $(g 1),(h 1),(h F)$, and that $F(0)=0$. Let $x_{0}>0$ and $h\left(y_{0}\right) \leqslant F\left(x_{0}\right)$. Then the forward orbit from $\left(x_{0}, y_{0}\right)$ either lies entirely in the right half-plane and approaches the origin as $t \rightarrow+\infty$, or it meets the $y$-axis the first time at some point $(0,-\beta)$ with $\beta>0$ and does not intersect the curve $h(y)=F(x)$ in between. Similarly, if $x_{0}>0$ and $h\left(y_{0}\right) \geqslant F\left(x_{0}\right)$ then the
backward orbit from $\left(x_{0}, y_{0}\right)$ either lies entirely in the right half-plane and approaches the origin as $t \rightarrow-\infty$, or it meets the $y$-axis at some point $(0, \alpha)$ with $\alpha>0$, without intersecting the curve $h(y)=F(x)$ in between.

Proof. We prove the first assertion in which $h\left(y_{0}\right) \leqslant F\left(x_{0}\right)$. Clearly the orbit cannot meet the curve $h(y)=F(x)$ as long as $x \geqslant 0$. Suppose the forward orbit lies entirely in the right half-plane, but that it does not approach the origin as $t \rightarrow+\infty$. Then it would be the graph of an increasing function $x=x(y)$ defined on $\left(-\infty, y_{0}\right]$, and $x(y)$ would tend to some $x_{1} \geqslant 0$ as $y \rightarrow-\infty$. If $h(-\infty)>-\infty$ then $h(-\infty) \notin h(\mathbf{R})$ and therefore by $(h F)$ we have $c \equiv F\left(x_{1}\right)-h(-\infty)>0$. If $g_{\text {max }}$ denotes the maximum of $g$ on $\left[0, x_{0}\right]$, then we find $d x / d y=(F(x)-h(y)) / g(x) \geqslant(c / 2) g_{\text {max }}>0$ for $|y|$ sufficiently large. We have a similar result if $h(-x)=-x$. Thus we cannot have $x(y) \rightarrow x_{1}$ as $y \rightarrow-x$, which is a contradiction.

Now we present the key lemma.

Lemma 2. Assume ( $F 1$ ) with $F \geqslant 0$ on $\left[0, a_{1}\right],(g 1),(h 1),(h F)$, and $(F 2+)$ on $\left[0, a_{1}\right]$. Let $u$ and $w$ satisfy $h(u) \geqslant F\left(a_{2}\right) \geqslant h(w)$ and $|w| \geqslant|u|$. Let $\eta+$ be the backward orbit from $\left(a_{2}, u\right)$ to its first crossing with the $y$-axis at $(0, \alpha)$. Let $\gamma$ - be the forward orbit from $\left(a_{2}, w\right)$ to its first crossing with the $y^{\prime}$-axis at $(0,-\beta)$.

Then we have $0<\alpha<\beta$.
Proof. We first prove the lemma when $(F 3)$ is satisfied in place of $(F 2+)$. We do this indirectly. Thus suppose $\alpha \geqslant \beta$.

Let $\gamma=\gamma_{+}+\gamma_{-}$and let $V(x, y)=G(x)+H(y)$, where $G$ and $H$ are defined in (4). Then

$$
\begin{equation*}
\dot{V} \equiv(d / d t) V(x(t), y(t))=-g(x) F(x) \tag{5}
\end{equation*}
$$

and with $\int_{y_{+}} \dot{V}=\int_{t_{+}}^{0} \dot{V} d t\left(t_{+}<0, x\left(t_{+}\right)=0\right)$ and $\int_{:-} \dot{V}=\int_{0}^{t_{-}} \dot{V} d t\left(t_{-}>0\right.$, $x(t)=0$ ) we find

$$
\begin{aligned}
\int_{; i} \dot{V} & \equiv \int_{i+} \dot{V}+\int_{i-} \dot{V}=V\left(a_{2}, u\right)-V(0, \alpha)+V(0,-\beta)-V\left(a_{2}, w\right) \\
& =H(u)-H(w)+H(\beta)-H(\alpha) \leqslant H(\beta)-H(\alpha) \leqslant 0,
\end{aligned}
$$

where we have used that $|u| \leqslant|w|$ and that $\alpha \geqslant \beta$. However, we shall use (5) and the supposition $\alpha \geqslant \beta$ to show that $\int_{i} \dot{V}>0$, whereby we have a contradiction. Thus necessarily $\alpha<\beta$.

Let $\gamma_{+}$and $\gamma_{-}$be given by $y=y_{+}(x)$ and $y=y_{-}(x)$, respectively, $x \in\left[0, a_{2}\right]$.

We have

$$
\begin{aligned}
\int_{\gamma} \dot{V} & =\int_{\gamma+} \dot{V}+\int_{\gamma-} \dot{V}=\int_{0}^{\alpha_{2}} \frac{\dot{V}}{\dot{x}} d x-\int_{0}^{\alpha_{2}} \frac{\dot{V}}{\dot{x}} d x \\
& =\int_{0}^{\alpha_{7}} \frac{-g(x) F(x)}{h\left(y_{+}(x)\right)-F(x)} d x-\int_{0}^{\alpha_{2}} \frac{-g(x) F(x)}{h\left(y_{-}(x)\right)-F(x)} d x .
\end{aligned}
$$

In the estimate below the contributions from the $\operatorname{arcs} A B$ and $E D$ (see Fig. 1) are combined and so are the contributions from $B C$ and $F E$. Thus we find

$$
\begin{aligned}
& \int_{\vartheta} \dot{V}= \int_{0}^{a_{1}}\left(\frac{-g(x) F(x)}{h\left(y_{+}(x)\right)-F(x)}+\frac{g(\phi(x)) F(\phi(x)) \phi^{\prime}(x)}{h\left(y_{-}(\phi(x))\right)-F(\phi(x))}\right) d x \\
&+\int_{0}^{a_{1}}\left(\frac{-g(\phi(x)) F(\phi(x)) \phi^{\prime}(x)}{h\left(y_{+}(\phi(x))\right)-F(\phi(x))}+\frac{g(x) F(x)}{h\left(y_{-}(x)\right)-F(x)}\right) d x \\
& \geqslant \int_{0}^{a_{1}} g(x) F(x)\left(\left(-h\left(y_{-}(\phi(x))\right)+F(\phi(x))\right)^{-1}\right. \\
&\left.\quad-\left(h\left(y_{+}(x)\right)-F(x)\right)^{-1}\right) d x \\
&+\int_{0}^{a_{1}} g(x) F(x)\left(\left(h\left(y_{+}(\phi(x))\right)-F(\phi(x))\right)^{-1}\right. \\
&\left.\quad-\left(-h\left(y_{-}(x)\right)+F(x)\right)^{-1}\right) d x .
\end{aligned}
$$



Fig. 1. This illustrates the case when $(F 3)$ is assumed.

In this estimate we have used the weaker assumption ( $F 2+$ ) only. If, however, we assume ( $F 3$ ) (as we have temporarily done) then it suffices to prove the inequality

$$
\begin{equation*}
-y_{-}(\phi(x)) \leqslant y_{+}(x) \quad \text { for all } \quad x \in\left[0, a_{1}\right] \tag{6}
\end{equation*}
$$

and the sharp inequality

$$
\begin{equation*}
y_{+}(\phi(x))<-y_{-}(x) \quad \text { for all } x \in\left[0, a_{1}\right] . \tag{7}
\end{equation*}
$$

The inequality (6) is satisfied for $x=0$ since $-y_{-}\left(a_{1}\right) \leqslant-y_{-}(0)=\beta \leqslant \alpha=$ $y_{+}(0)$. We now show that if (6) is satisfied for some $x \in\left[0, a_{1}\right]$ then $(d / d x)$ $\left(-y_{-}(\phi(x))\right) \leqslant(d / d x) y_{+}(x)$. This will prove (6).

We find by using ( $F 2+$ )

$$
\begin{aligned}
-\frac{d}{d x}\left(y_{-}(\phi(x))\right) & =\frac{g(\phi(x)) \phi^{\prime}(x)}{h\left(y_{-}(\phi(x))\right)-F(\phi(x))} \leqslant \frac{g(\phi(x)) \phi^{\prime}(x)}{h\left(y_{-}(\phi(x))\right)+F(x)} \\
& \leqslant \frac{g(\phi(x)) \phi^{\prime}(x)}{-h\left(y_{+}(x)\right)+F(x)} \leqslant \frac{g(x)}{-\left(h\left(y_{+}(x)\right)-F(x)\right)}=\frac{d}{d x} y_{+}(x) .
\end{aligned}
$$

Next we consider (7). We have for $x \in\left[0, a_{1}\right]$

$$
y,(\phi(x)) \leqslant y,\left(a_{1}\right)<\left|y \quad\left(a_{1}\right)\right| \leqslant|y \quad(x)|=-y \quad(x)
$$

where the inequality $y_{+}\left(a_{1}\right)<\left|y_{-}\left(a_{1}\right)\right|$ follows from the fact that $V$ by (5) is (weakly) increasing along any orbit in the strip $x \in\left[a_{1}, a_{2}\right]$ (but $V$ is not a constant). Thus we get by going from $B$ to $C$ along $\gamma_{+}$that $G\left(a_{1}\right)+$ $H\left(y_{+}\left(a_{1}\right)\right)<G\left(a_{2}\right)+H(u)$. Going from $D$ to $E$ along $\gamma$. we find that $G\left(a_{2}\right)+H(w)<G\left(a_{1}\right)+H\left(y_{-}\left(a_{1}\right)\right)$. Since $|u| \leqslant|w|$ we find that $H\left(y_{+}\left(a_{1}\right)\right)<H\left(y_{-}\left(a_{1}\right)\right)$, so $y_{+}\left(a_{1}\right)<\left|y_{-}\left(a_{1}\right)\right|$. This proves the lemma when $(F 3)$ is assumed in place of $(F 2+)$.

To complete the proof of Lemma 2 we state a simple comparison lemma.

Lemma 3. Consider the system (1) where $F, g$, and $h$ satisfy the assumptions ( $g 1$ ), ( $h 1$ ), and ( $h F$ ). Consider also the system

$$
\begin{equation*}
\dot{x}=h(y)-F^{*}(x), \quad \dot{y}=-g(x) \tag{8}
\end{equation*}
$$

where $F^{*} \in C(\mathbf{R}), F^{*}(\mathbf{R}) \supset h(\mathbf{R})$, and for some $b>0, F^{*}(x) \geqslant F(x)$ for $x \in[0, b]$. Let $u$ and $u^{*}$ satisfy, $h\left(u^{*}\right) \geqslant F^{*}(b)$ and $h\left(u^{*}\right) \geqslant h(u) \geqslant F(b)$. Let the backward orbits through $(b, u)$ and $\left(b, u^{*}\right)$ for the systems (1) and (8) be given respectively $b y=y_{+}(x)$ and $y=y_{+}^{*}(x)$ for $x \in[0, b]$. Then we have

$$
\begin{equation*}
y_{+}(x) \leqslant y_{+}^{*}(x) \quad \text { for } \quad x \in[0, b] . \tag{9}
\end{equation*}
$$

Similarly, if $F^{*}(b) \geqslant h\left(w^{*}\right) \geqslant h(w)$ and $F(b) \geqslant h(w)$, and if $y=y_{-}(x)$ and $y^{\prime}=y_{-}^{*}(x)$ are the forward orbits through $(b, w)$ and $\left(b, w^{*}\right)$, then

$$
\begin{equation*}
y_{-}(x) \leqslant y^{*}(x) \quad \text { for } \quad x \in[0 . b] . \tag{10}
\end{equation*}
$$

Before proving Lemma 3 we complete the proof of Lemma 2. Thus suppose $F$ satisfies the original assumptions of Lemma 2, but suppose first that $\phi\left(a_{1}\right)=a_{2}$. Let $F^{*}(x)=F(x)$ for $x \in\left[0, a_{1}\right]$ and $F^{*}(\phi(x))=-F(x)$ for $x \in\left[0, a_{1}\right]$. Then $F^{*}$ satisfies $(F 3)$ so that $y_{+}^{*}(0)<-y_{-}^{*}(0)$. Apply Lemma 3 with $u^{*}=u \geqslant h^{-1}\left(F^{*}\left(a_{2}\right)\right)=0$ and $w^{*}=u \leqslant h^{-1}\left(F\left(a_{2}\right)\right)$. (Clearly it suffices to consider $u \geqslant 0$.)

Thus we find

$$
y_{+}(0) \leqslant y_{+}^{*}(0)<-y_{-}^{*}(0) \leqslant-y_{-}(0) .
$$

This proves Lemma 2 when $\phi$ is onto [ $a_{1}, a_{2}$ ]. If $\phi\left(a_{1}\right)<a_{2}$ then we use that $V$ is increasing along any orbit in the strip $\left[\phi\left(a_{1}\right), a_{2}\right]$. Thus

$$
\begin{aligned}
& G\left(\phi\left(a_{1}\right)\right)+H\left(y_{+}\left(\phi\left(a_{1}\right)\right)\right) \\
& \quad \leqslant G\left(a_{2}\right)+H(u) \leqslant G\left(a_{2}\right)+H\left(w^{\prime}\right) \leqslant G\left(\phi\left(a_{1}\right)\right)+H\left(y_{-}\left(\phi\left(a_{1}\right)\right)\right)
\end{aligned}
$$

and so we have that $y_{+}\left(\phi\left(a_{1}\right)\right) \leqslant-y_{-}\left(\phi\left(a_{1}\right)\right)$ from which we conclude by the results above that $y_{+}(0)<-y_{-}(0)$. This completes the proof of I emma 2.

Proof of Lemma 3. Suppose (9) were not satisfied for all $x \in[0, b]$. Let $x_{1} \in[0, b)$ be any point for which $y_{+}\left(x_{1}\right)>y_{+}^{*}\left(x_{1}\right)$. Then we have at $x=x_{1}$ that

$$
\frac{d y_{+}}{d x}=\frac{-g\left(x_{1}\right)}{h\left(y_{+}\left(x_{1}\right)\right)-F\left(x_{1}\right)}>\frac{-g\left(x_{1}\right)}{h\left(y_{+}^{*}\left(x_{1}\right)\right)-F^{*}\left(x_{1}\right)}=\frac{d y_{+}^{*}}{d x}
$$

It follows that $y_{+}(x)>y_{+}^{*}(x)$ for all $x \geqslant x_{1}$ contradicting $y_{+}(b)=u \leqslant u^{*}=$ $y_{+}^{*}(b)$.

The argument for (10) is precisely the same.
Our final lemma paves the way for the proof of the Theorem.
Lemma 4. Assume that the system (1) satisfies $(F 1),(g 1),(h 1)$, and ( $h F$ ). Suppose $0=a_{0}<a_{1}<a_{2}<\cdots<a_{q}$, where $q \geqslant 2$. Assume that $(F 2+$ ) is satisfied on each of the intervals $I_{i}=\left[\begin{array}{ll}a_{1} & 1\end{array} a_{1}\right], 1 \leqslant i \leqslant q-1$, with some $\phi_{1}$ satisfying (g2) with $J_{i}=\left[a_{i}, a_{i+1}\right]$. Suppose further that $F \geqslant 0$ on $\left[0, a_{1}\right]$. Let $2 \leqslant n \leqslant q$ and let $\gamma_{n}$ be the orbit through $\left(a_{n}, h^{-1}\left(F\left(a_{n}\right)\right)\right.$ ). (When $n \leqslant q-1$, this is just $\left(a_{n}, 0\right)$.) Let its first crossings with the $y$-axis (forward and backward in time) be at $\left(0,-\beta_{n}\right)$ and $\left(0, \alpha_{n}\right)$, respectively. Then we have $0<\alpha_{n}<\beta_{n}$ if $n$ is even, and $0<\beta_{n}<\alpha_{n}$ if $n$ is odd.

Proof. First suppose that $n$ is even. Make the change of variables $\xi=x-a_{n-2}$ in (1). We find

$$
\begin{equation*}
\dot{\zeta}=h(y)-F\left(\xi+a_{n-2}\right), \quad \dot{y}=-g\left(\xi+a_{n-2}\right) . \tag{11}
\end{equation*}
$$

With $F_{1}(\xi)=F\left(\xi+a_{n-2}\right)$ and $g_{1}(\xi)=g\left(\xi+a_{n-2}\right)$ we see that $g_{1}$ satisfies $(g 1), F_{1}$ satisfies $(F 1)$ on $\left[0, a_{n-1}-a_{n-2}\right]$ with $F_{1} \geqslant 0$ there, $g_{1}$ satisfies (g2) with $\phi(\xi)=\phi_{n}\left(\xi+a_{n-2}\right)-a_{n-2}$ mapping $\left[0, a_{n-1}-a_{n-2}\right]$ into $\left[a_{n} 1_{1}-a_{n} 2_{2}, a_{n}-a_{n-2}\right.$ ], and with that function $\phi$ we see that $F_{1}$ satisfies $(F 2+)$ on $\left[0, a_{n-1}-a_{n-2}\right]$. Thus Lemma 2 applies to (11) so we conclude that $0<y_{+}\left(a_{n-2}\right)<-y_{-}\left(a_{n-2}\right)$.

If $n \geqslant 4$ we may then (again after a change of variable) use Lemma 2 to conclude that $0<y_{+}\left(a_{n-4}\right)<-y_{-}\left(a_{n-4}\right)$. Continuing this way we find $0<\alpha_{n}=y_{+}(0)<-y_{-}(0)=\beta_{n}$.

If $n$ is odd we make the change of variables $\xi=x-a_{1}, \eta=-y$, and $\tau=-t$. By (1) we find

$$
\dot{\xi} \equiv(d / d \tau) \xi=h(\eta)+F\left(\xi+a_{1}\right), \quad \dot{\eta} \equiv(d / d \tau) \eta=-g\left(\xi+a_{1}\right)
$$

With $F^{*}(\xi)=-F\left(\xi+a_{1}\right)$ and $g^{*}(\xi)=g\left(\xi+a_{1}\right)$ we see that $g^{*}$ satisfies $(g 1), F^{*}$ satisfies $(F 1)$ on $\left[0, a_{2}-a_{1}\right]$ with $F^{*} \geqslant 0$ there, $g^{*}$ satisfies (g2) with $\phi_{1}^{*}(\xi)=\phi_{i+1}\left(\xi+a_{1}\right)-a_{1}$ mapping $\left[a_{1}-a_{1}, a_{t+1}-a_{1}\right]$ into [ $a_{i+1}-a_{1}, a_{i+2}-a_{1}$ ] for $1 \leqslant i \leqslant n-2$, and with those functions $\phi_{i}^{*}$ we see that $F^{*}$ satisfies $(F 2+)$. Thus the result just proven for $n$ even can be applied. With an obvious definition of $\eta_{+}(\xi)$ and $\eta_{-}(\xi)$ we conclude that $0<-y_{-}\left(a_{1}\right)=\eta_{+}(0)<-\eta,(0)=y_{+}\left(a_{1}\right)$. Since $V=G(x)+H(y)$ is decreasing along the orbits of (1) as long as $x \in\left[0, a_{1}\right]$, we find

$$
\begin{aligned}
H\left(\alpha_{n}\right)=V\left(0, \alpha_{n}\right) & \geqslant V\left(a_{1}, y_{+}\left(a_{1}\right)\right)=G\left(a_{1}\right)+H\left(y_{+}\left(a_{1}\right)\right) \\
& >G\left(a_{1}\right)+H\left(y_{-}\left(a_{1}\right)\right)=V\left(a_{1}, y_{-}\left(a_{1}\right)\right) \\
& \geqslant V\left(0,-\beta_{n}\right)=H\left(\beta_{n}\right) .
\end{aligned}
$$

Thus $\alpha_{n}>\beta_{n}$.
This completes the proof of Lemma 4. We can now prove the Theorem.
Proof of the Theorem. It suffices to consider $F \geqslant 0$ on $\left[0, a_{1}\right]$. Let $V(x, y)=G(x)+H(y)$. Then $V(x, y)>0$ for $(x, y) \neq(0,0)$. Since $\dot{V}=$ $-g(x) F(x) \leqslant 0$ for $(x, y) \in\left[-a_{1}, a_{1}\right] \times \mathbf{R}$ it follows that $(0,0)$ is a stable equilibrium point and that the region defined by

$$
\begin{equation*}
V(x, y) \leqslant V\left(a_{1}, 0\right)=G\left(a_{1}\right), \quad x \in\left[-a_{1}, a_{1}\right] \tag{12}
\end{equation*}
$$

is positively invariant.
If $F$ is not identically zero in any neighborhood of 0 then the set

$$
\left\{(x, y) \mid \dot{V}(x, y)=0, x \in\left[-a_{1}, a_{1}\right],(x, y) \neq(0,0)\right\}
$$

contains no complete positive semiorbit. A theorem by Krasovskii [3, p. 67] then applies and we conclude that $(0,0)$ is asymptotically stable and its basin of attraction contains the region defined by (12). This proves the assertions made in Remarks 1-2 after the Theorem.

We shall now construct a sequence of annular regions which are alternately negatively and positively invariant. The Poincaré-Bendixson Theorem will then give the existence of at least one limit cycle in each one of those regions and with the stability properties mentioned in Remark 3 after the Theorem.

We have somewhat weaker requirements on the intervals $\left[a_{q-1}, a_{q}\right]$ and $\left[-a_{q},-a_{q} \quad 1\right]$, so first suppose $2<q$. Let $\gamma_{R_{2}}$ be that part of the orbit through $\left(a_{2}, 0\right)$ which begins and ends on the $y$-axis at points $\left(0, \alpha_{2}\right)$ and $\left(0,-\beta_{2}\right)$. By Lemma 4, $\alpha_{2}<\beta_{2}$. Sice $(-x(t),-y(t))$ also solves the system (1) we find that the orbit $\gamma_{L_{2}}$ starting at $\left(0,-\alpha_{2}\right)$ crosses the $y$-axis the first time at $\left(0, \beta_{2}\right)$. We let $\gamma_{L 2}$ end there. Now let $S_{1}$ be the region whose inner boundary is given by $V(x, y)=G\left(a_{1}\right)$ and whose outer boundary is the closed curve composed of $\gamma_{R 2}, \gamma_{L 2}$, and the two intervals on the $y$-axis, $\left[-\beta_{2},-\alpha_{2}\right]$ and $\left[\alpha_{2}, \beta_{2}\right]$. See Fig. 2. Then $S_{1}$ is negatively invariant.

Suppose also $3<q$. Let $\gamma_{R 3}$ be that part of the orbit through $\left(a_{3}, 0\right)$ which begins and ends on the 1 -axis at points $\left(0, \alpha_{3}\right)$ and $\left(0,-\beta_{3}\right)$. By Lemma $4, \alpha_{3}>\beta_{3}$. Define $\gamma_{L 3}$ as the orbit starting at $\left(0,-\alpha_{3}\right)$ and ending


Fig. 2. Only the outer boundary of $S_{1}$ is shown.
at $\left(0, \beta_{3}\right)$. Let $S_{2}$ be the annular region whose inner boundary is the outer boundary of $S_{1}$ and whose outer boundary is the closed curve composed of $\gamma_{R 3}, \gamma_{L 3}$, and the two intervals on the $y$-axis, $\left[-\alpha_{3},-\beta_{3}\right]$ and $\left[\beta_{3}, \alpha_{3}\right]$. Then $S_{2}$ is positively invariant.

In this way we can for $2 \leqslant n<q$ construct an annular region $S_{n-1}$ that is negatively invariant if $n$ is even and positively invariant if $n$ is odd.

To construct $S_{q-1}$ we first notice that if $F$ is odd on all of [ $-a_{q}, a_{q}$ ] then we can let $\gamma_{R q}$ be the orbit through $\left(a_{q}, h^{-1}\left(F\left(a_{q}\right)\right)\right.$ ). Lemma 4 gives the appropriate inequality between $\alpha_{q}$ and $\beta_{q} \cdot \gamma_{L q}$ is defined by symmetry as before. If $F$ is odd on $\left[-a_{4-1}, a_{\varphi-1}\right]$ only, then we shall compare $F$ with a function $F^{*}$ which is odd on $\left[-\phi_{q-1}\left(a_{q-1}\right), \phi_{q-1}\left(a_{q-1}\right)\right]$ $\left(\subset\left[-a_{q}, a_{\varphi}\right]\right)$. Let $F^{*}(x)=F(x)$ for $x \in\left[-a_{\varphi-1}, a_{q-1}\right], F^{*}\left(\phi_{\varphi-1}(x)\right)=$ $-F(x)$, and $F^{*}\left(-\phi_{q-1}(x)\right)=-F(-x)$ for $x \in\left[a_{q-2}, a_{q-1}\right]$.

Then $F^{*}$ satisfies all the assumptions of the Theorem and is odd on $\left[-a_{q}^{*}, a_{q}^{*}\right]$, where $a_{q}^{*}=\phi_{\varphi-1}\left(a_{\varphi-1}\right)$. Let $\gamma_{R q}^{*}$ be the orbit (for (1) with $F$ replaced by $F^{*}$ ) through $\left(a_{q}^{*}, h^{-1}\left(F^{*}\left(a_{q}^{*}\right)\right)\right)$ and define $\alpha_{4}^{*}, \beta_{q}^{*}$, and $\gamma_{L q}^{*}$ as above. For the system (1) with the original $F$ consider the orbits $\gamma_{R_{4}}$ and $\gamma_{L q}$ through $\left(a_{q}^{*}, h^{-1}\left(F\left(a_{q}^{*}\right)\right)\right.$ ) and ( $-a_{q}^{*}, h^{-1}\left(F\left(-a_{q}^{*}\right)\right)$ ), respectively. Let $\left(0,-\beta_{q}\right)$ and $\left(0, \alpha_{q}\right)$ be the endpoints of $\gamma_{R_{4}}$. By Lemma 3 we find for $q$ even that $\beta_{q} \geqslant \beta_{q}^{*}$ and $\alpha_{q} \leqslant \alpha_{q}^{*}$. Thus the points $\left(0,-\beta_{q}\right)$ and $\left(0, \alpha_{q}\right)$ have moved down relative to $\left(0,-\beta_{q}^{*}\right)$ and $\left(0, \alpha_{q}^{*}\right)$. For $i_{L q}$ the corresponding points move up. Thus we may define a negatively invariant region $S_{\varphi-1}$ as before. When $q$ is odd the inequalities are reversed and we can define a positively invariant region $S_{q-1}$.

It remains to prove that there are no closed orbits through the interval $\left[\phi_{i}\left(a_{i}\right), a_{i+1}\right]$ on the $x$-axis for $1 \leqslant i \leqslant q-2$ (and for $i=q-1$ if $F$ is odd on all of $\left[-a_{q}, a_{q}\right]$ ). Consider the interval $I=\left[\phi_{n}\left(a_{n}\right), a_{n+1}\right]$ for $n \leqslant q-1$ (or $n \leqslant q$ ). Let $b \in I$. We apply Lemma 4 with the sequence $0=a_{0}<a_{1}<\cdots<$ $a_{n}<b$ and conclude that the orbit through $\left(b, h^{-1}(F(b))\right)$ is not closed since a closed orbit would intersect the $y$-axis at symmetric points $(0, \alpha)$ and $(0,-\alpha)$. For $n \leqslant q-2$ we have $F\left(a_{n+1}\right)=0$ and the result above holds for all $b \in I$, thus for all $b \in I$ it follows that the orbit through $(b, 0)$ is not closed either. (For $n=q-1$ we may either assume that $a_{q}$ has been chosen so that $F\left(a_{q}\right)=0$ or we have $F(x) \geqslant 0$ (or $\leqslant 0$ ) for all $x>a_{q-1}$, in which case the conclusion also follows.)

This completes the proof of the Theorem.

## 6. A Remark about Large Perturbations

We remarked after Example 3 that ( $F 2$ ) implies that max $|F|$ cannot decrease from one interval $\left[a_{i}, a_{i+1}\right]$ to the next. We shall prove a result about the absence of limit cycles when this requirement is violated.

Consider the system (1) where $F, g, h \in C(\mathbf{R})$, and satisfy $(h 1),(h F)$, and also the following three requirements
(i) $\operatorname{xg}(x)>0$ for $x \neq 0$,
(ii) there exist positive numbers $a_{1}$ and $a_{1}^{\prime}$ such that $F$ is nonnegative on $\left[0, a_{1}\right]$ and non-positive on $\left[-a_{1}^{\prime}, 0\right]$,
(iii) $F$ is not identically zero in any neighborhood of 0 .

Let $A$ be the maximum of $F$ on $\left[0, a_{1}\right]$, let $B$ be the maximum of $-F$ on $\left[-a_{1}^{\prime}, 0\right]$, and let $\beta=\min \{A, B\}$. Let $G$ be defined as in (4). Then we have the following result.

Proposition. Suppose that for some $b>0$ we have

$$
\begin{equation*}
h^{-1}\left(F(x)+b[2 G(x)]^{1,2}\right) \leqslant h^{-1}(\beta)-[2 G(x)]^{1,2} / b \tag{13}
\end{equation*}
$$

for all $x \in\left[x_{0}, 0\right]$ where $x_{0}<0$, and suppose that $(x(t), y(t))$ is a maximal solution of (1) which starts at the point $\left(x_{0}, y_{0}\right)$, where $h\left(y_{0}\right)=F\left(x_{0}\right)$. Then $(x(t), y(t)) \rightarrow(0,0)$ as $t \rightarrow+\infty$.

Proof. If $x(t)$ remains negative then the solution tends to the origin as $t \rightarrow+\infty$ (by Lemma 1). If not, then let ( $0, y_{1}$ ) be the first intersection with
 graph of a function $y=y(x), x \in\left[x_{0}, 0\right]$. We show that $h\left(y_{1}\right) \leqslant \beta$. For that purpose let $x_{1}=\sup \left\{x \in\left[x_{0}, 0\right] \mid h(y(x))-F(x) \leqslant b[2 G(x)]^{1 / 2}\right\}$. Since $y_{1}>0$ and $F(0)=G(0)=0$ we have $x_{1}<0$. For $x \in\left[x_{1}, 0\right)$ we get the estimate $d y / d x=-g(x)(h(y)-F(x))^{-1} \leqslant-g(x)\left[(2 G(x))^{1.2} b\right]^{-1}$. Integration from $x_{1}$ to 0 yields $y_{1}-y\left(x_{1}\right) \leqslant\left[2 G\left(x_{1}\right)\right]^{1.2} / b$, but we also have $y^{\prime}\left(x_{1}\right) \leqslant h^{-1}\left(F\left(x_{1}\right)+h\left[2 G\left(x_{1}\right)\right]^{1 ; 2}\right) \leqslant h^{-1}(\beta)-[2 G(x)]^{1,2} / b$ by (13). Thus $y_{1} \leqslant h^{-1}(\beta)$.

Now consider the fate of the solution in the right half-plane from its new start at $\left(0, y_{1}\right)$. Since $0<h\left(y_{1}\right) \leqslant \beta \leqslant A$ the solution intersects the curve $h(y)=F(x)$ while in the strip $0<x<a_{1}$ and therefore remains there for the length of its stay in the right half-plane. If it stays there forever, then it converges to the origin as $t \rightarrow+\infty$ (by Lemma 1). If not, then it crosscs the negative $y$-axis at some point $\left(0, y_{2}\right)$. The function $V(x, y)=G(x)+H(y)$ is weakly decreasing along the orbit since by $(5), \dot{V}=-g(x) F(x) \leqslant 0$. Thus $H\left(y_{2}\right) \leqslant H\left(y_{1}\right)$ so that by $(h 1),\left|y_{2}\right| \leqslant y_{1} \leqslant h^{-1}(\beta) \leqslant h^{-1}(B)$. With the appropriate changes we can now repeat the argument and conclude that unless the solution remains in the left half-plane (in which case it tends to the origin as $t \rightarrow+\infty)$ it intersects the positive $y$-axis at some point $\left(0, y_{3}\right)$ with $y_{3} \leqslant\left|y_{2}\right| \leqslant y_{1}$. By the assumption (iii) we have $y_{3}<y_{1}$.

The argument about the behavior of the solution from $\left(0, y_{1}\right)$ to $\left(0, y_{3}\right)$ depended on the inequality $0<y_{1} \leqslant \beta$ only. Thus it follows that the
solution cannot approach a limit cycle and must consequently approach the origin as $t \rightarrow+\infty$. This completes the proof.

Remark. When $h(y)=y$ the requirement (13) simply reads $F(x) \leqslant$ $\beta-2[2 G(x)]^{12}$ for all $x \in\left[x_{0}, 0\right]$ since 1 is the optimal choice for $b$.

Example 3 (revisited). There are clearly no limit cycles for the system (3) when $\mu^{2} \leqslant 4 v$ since then $\dot{V}=-x F(x) \leqslant 0$ for all $x \in \mathbf{R}$ so that the origin is globally asymptotically stable. Suppose then that $\mu^{2}>4 v$ and let $b_{1}$ and $b_{2}$ be the positive roots of $F^{\prime}(x)=0$. The requirement (13), i.e., $a F(x) \leqslant$ $a \beta-2|x|$ for all $x \leqslant 0$ where $F(x)=x^{5}-\mu x^{3}+v x$ and $\beta=F\left(b_{1}\right)$, is satisfied for some $a>0$ iff $F\left(-b_{2}\right)<F\left(b_{1}\right)$. The latter inequality is satisfied iff $\mu<(5 v)^{12} \approx 2.236 v^{12}$. By the Proposition we conclude that the system (3) does not have any limit cycles when $4 v<\mu^{2}<5 y$ and $a \geqslant a_{0}$ where $a_{0}=2 / F^{\prime}(x)$ and $\alpha$ is the negative root of the equation $F(x)-x F^{\prime}(x)-$ $F\left(b_{1}\right)=0$.

## 7. Concluding Remarks

(a) The Theorem is an existence theorem only. It is desirable to add conditions to ensure that precisely one limit cycle passes through each interval $\left[a_{t}, a_{t+1}\right]$ on the $x$-axis.
(b) It follows from the assumption (g2) that $G\left(a_{n+1}\right)+G\left(a_{n-1}\right) \geqslant$ $2 G\left(a_{n}\right)$ for all $n \geqslant 1$. The Theorem therefore does not apply to, e.g., the system $\dot{x}=y+x^{2} \sin x, \dot{y}=-2 x\left(1+x^{2}\right)^{-2}$, which is considered in [6, p. 96].
(c) The proof of the Theorem relies strongly on $F, g$, and $h$ being odd on some interval.

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