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An algebraic theory of normal forms

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Abstract

In this paper we present *a general theory of normal forms*, based on a categorial result (Dubuc, 1974) for the free monoid construction. We shall use the theory mainly for proposictional modal logic, although it seems to have a wider range of applications. We shall formally represent normal forms as *combinatorial* objects, basically labelled trees and forests. This geometric conceptualization is implicit in (Fine, 1975) and our approach will extend it to other cases and make it more direct: operations of a purely geometric and combinatorial nature (cuts of leaves and roots, renaming labels and more generally segment-by-label replacements) will be introduced in order to give a mathematical description of the basic logical/algebraic constructions (free algebras, morphisms among them, canonical models, the lattice of varieties).

We begin (Section 1) by recalling the above-mentioned categorial construction: we need a careful inspection of it because in the various examples considered later (Sections 2 and 3) we plan to *deduce from it in a uniform way* the normal forms and the description of finitely generated free algebras. This method always works whenever we can describe the category of algebras corresponding to the logic under consideration as a *T-objects category*. When this simple description seems not to be available, still the general theory might be of some interest, because a description of the category of algebras as a T-objects category *plus equation* is possible (we shall give examples in Section 5).

The central part of the paper (Sections 4 and 5) is more advanced and specific: we show how the general approach presented here can provide some insights even in the basic case of the modal system K. Section 4 contains a contribution to the theory of normal forms, namely the description of *the uniform substitution*. This result will enable us to give a *duality theorem* for the category of finitely generated free modal algebras and in Section 5 to provide a characterization of the collections of normal forms which happen to be normal forms for a logic, thus giving a description of the *lattice of modal logics*.

Section 6 (that can be read independently on Section 5) deals with some applications: we shall show how to use normal forms in order to prove for the modal system K the definability of *higher-order propositional quantifiers* and of the *tense operator F* (the parallel results for intuitionistic logic are in Pitts, 1992; Ghilardi, 1992; Ghilardi and Zawadowski, 1993).

As to the prerequisites, the paper is almost self-contained. The reader is only assumed to have familiarity with standard techniques in algebraic logic (a possible reference is Rasiowa (1974)). Knowledge of the basic facts about adjoint functors is required too, see e.g. McLane (1971) or the appendix.

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1. Free T-objects

To begin with, let us take for example the minimum classical normal modal system K (axioms: tautologies plus $\Box(\alpha_1 \to \alpha_2) \to (\Box\alpha_1 \to \Box\alpha_2)$, rules: modus ponens and necessitation $\vdash \alpha \Rightarrow \vdash \Box\alpha$) and let us examine the problem of the description of finitely zgenerated free modal algebras, the algebraic counterparts of the modal system K.[1] We recall (see e.g. [22]) that a modal algebra is a Boolean algebra $B = \langle \underline{B}, \wedge, \top, \vee, \bot, \neg \rangle$ endowed with a "necessity" operator, i.e. with an endo-hemimorphism $\Box : B \to B$ (an hemimorphism between two Boolean algebras is a function preserving only $\wedge$, $\top$). A morphism between modal algebras will be a Boolean morphism preserving the additional operator. We obtain a category, the category **Ma** of modal algebras. This is the traditional definition, there is however another, equivalent, way of introducing modal algebras.

A (meet)-*semilattice* (with unity) $R = \langle \underline{R}, \wedge, \top \rangle$ is a commutative idempotent monoid (equivalently, it is a poset with maximum element and infs of pairs, see [12]); a morphism between semilattices is a function preserving the additional structure. We thus have a category **SemiL** and an obvious forgetful functor $|-|$: **Boole** $\to$ **SemiL**, associating with a Boolean algebra $B = \langle \underline{B}, \wedge, \top, \vee, \bot, \neg \rangle$ the semilattice $\langle \underline{B}, \wedge, \top \rangle$ (the functor operates identically on morphisms). This forgetful functor does have a left adjoint P: **SemiL** $\to$ **Boole**: in Section 2, this adjoint will be completely characterized, at least as far as finite semilattices are concerned, actually we content ourselves of the mere fact that it exists.[2] Hence, we can take into consideration Boolean algebras B endowed with a Boolean morphism $p : P|B| \to B$. In this way we enter in the general context of T-objects categories. Given a category **C** and an endofunctor T: **C** $\to$ **C**, we may form another category **TObj** called the category of T-objects in the following way. Objects of **TObj** are pairs $\langle X, x \rangle$, where X is an object from **C** and $x : T(X) \to X$ is a morphism from **C**. Arrows $\mu : \langle X, x \rangle \to \langle Y, y \rangle$ in **TObj** are exactly the arrows $\mu : X \to Y$ in **C** such that the square

$$\begin{tikzcd} T(X) \arrow[r, "T(\mu)"] \arrow[d, "x"'] & T(Y) \arrow[d, "y"] \\ X \arrow[r, "\mu"'] & Y \end{tikzcd}$$

commutes. Identity and composition in **TObj** are like in **C**. Now it is easily seen that **Ma** is isomorphic to the T-objects category for $T = |-|P$: **Boole** $\to$ **Boole**. We prove this in details, although it is elementary, leaving to the reader the angalogous computations for the other examples below.[3] According to the definition of adjoint

[1] See [3] for the classical approach to this question.

[2] There is a general theorem by Lawvere of existence of left adjoints for algebraic functors which applies to our case. One may also directly use some adjoint functor theorem (see [17]).

[3] This will be our trend for the rest of the paper: we give full details of the general theory and of its applications to the basic example of modal algebras, whereas for the other cases we give only the final results. To get them from the general theorems one needs purely routine work, that can only be summarized here for evident reasons.

functors, we have for every Boolean algebra B and for every semilattice R a "transposition" bijection

$$(-)^{\mathrm{T}}: \mathbf{SemiL}[R, |B|] \to \mathbf{Boole}[P(R), B]$$

(depending of course on R and B, but we usually omit such subscripts) satisfying the following naturality conditions

$$(h|\mu|)^{\mathrm{T}} = h^{\mathrm{T}}\mu, \qquad (kh)^{\mathrm{T}} = P(k)h^{\mathrm{T}}$$

for $h: R \to |B|$ and $k: S \to R$ in **SemiL** and for $\mu: B \to C$ in **Boole**. The inverse map of $(-)^{\mathrm{T}}$ is denoted by $(-)^{\mathrm{t}}$ and enjoys the following naturality conditions (equivalent to the above ones):

$$\varphi^{\mathrm{t}}|\mu| = (\varphi\mu)^{\mathrm{t}}, \qquad k\varphi^{\mathrm{t}} = (P(k)\varphi)^{\mathrm{t}}$$

for appropriate φ, μ, k. Now, in order to show that **Ma** is isomorphic to the claimed T-objects category, we associate with a modal algebra $\langle B, \square\rangle$ the T-object $\langle B, \square^{\mathrm{T}}\rangle$. This passage operates identically on morphisms and is a functor. In fact a Boolean morphism $f: \langle B, \square\rangle \to \langle B', \square'\rangle$ is a morphism between modal algebras iff $\square|f| = |f|\square'$, i.e. iff $(\square|f|)^{\mathrm{T}} = (|f|\square')^{\mathrm{T}}$ which is equivalent to $\square^{\mathrm{T}}f = P|f|(\square')^{\mathrm{T}}$, i.e. to $\square^{\mathrm{T}}f = T(f)(\square')^{\mathrm{T}}$, which expresses the fact that f is a morphism between the T-objects $\langle B, \square^{\mathrm{T}}\rangle$ and $\langle B', (\square')^{\mathrm{T}}\rangle$. Conversely, given a T-object $\langle B, p\rangle$, we associate with it the modal algebra $\langle B, p^{\mathrm{t}}\rangle$. This passage is supposed to operate identically on morphisms too and the above chain of equivalences (read in the opposite sense with p^{t} and $(p')^{\mathrm{t}}$ instead of $\square$ and $\square'$) shows that it is well-defined, hence a functor. That the two passages are inverse is evident, so they define an isomorphism of categories. We give further examples of this situation.

Example I. Let us consider Boolean algebras with an endofunction. These algebras correspond to the minimum congruential modal system C in which we only have (in addition to tautologies and modulus ponens) the rule $\vdash \alpha \leftrightarrow \beta \Rightarrow \square\alpha \leftrightarrow \square\beta$. These algebras are the T-objects for the endofunctor on **Boole** that associates with a Boolean algebra the free Boolean algebra generated by its underlying set.

Example II. Let us consider Boolean algebras endowed with finitely many hemimorphisms, say $\square_1, \ldots, \square_n$. They correspond to the polymodal version of K and are T-algebras for the endofunctor on **Boole** that associates with a Boolean algebra B, the coproduct of n-copies of $P|B|$ (the value of the endofunctor on arrows is the obvious one).

Example III. Let us consider Boolean algebras B endowed with a binary modal operator

$$[-, -]: B \times B \to B,$$

which is assumed to be normal in each variable separately, that is the following equations are supposed to hold:

$$[x_1 \wedge x_2, y] = [x_1, y] \wedge [x_2, y], \qquad [\top, y] = \top,$$

$$[x, y_1 \wedge y_2] = [x, y_1] \wedge [x_2, y], \qquad [x, \top] = \top.$$

These algebras correspond to the binary version of K and are T-objects for the endofunctor on **Boole** that associates with a Boolean algebra B the Boolean algebra freely generated by the tensor product of $|B|$ with itself. In fact if we call linear a finite-meet preserving map between semilattices and bilinear a map that is linear in each variable separately, then we may speak of universal bilinear maps and show the existence of tensor products, exactly as it happens in commutative algebra with modules (formal properties are indeed the sme, here we imitate [14] where these ideas are applied to $\bigvee$-lattices). In this way tensor product becomes a bifunctor on semilattices. To summarize the present example, we say that Boolean algebras with binary binormal modal operators are T-objects for the endofunctor T that associates with a Boolean algebra B the Boolean algebra $P(|B| \otimes |B|)$.

Example IV. Also semantic categories can be treated within our framework. We recall that a *Kripke frame* $\langle W, R\rangle$ is a directed graph, i.e. a set W endowed with a binary relation R. An open morphism $f: \langle W, R\rangle \to \langle W', R'\rangle$ between Kripke frames is a map that preserves the relation and is such that $f(w)R'w' \Rightarrow \exists v \in W(wRv \,\&\, f(v) = w')$ for all $w \in W$, $w' \in W'$. In other words, an open morphism is a function $f: W \to W'$ such that the following square of relations commute:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "R"'] & W' \arrow[d, "R'"] \\ W \arrow[r, "f"'] & W' \end{tikzcd}$$

Kripke frames and open morphisms form a category **Kfr** which is the T-coobjects category for the covariant power set functor[4] from **Set** into itself. In fact, the commutativity of the above square corresponds exactly to the commutativity of the following one:

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\hat{R}"'] & W' \arrow[d, "\hat{R}'"] \\ \mathscr{P}(W) \arrow[r, "\exists_f"'] & \mathscr{P}(W') \end{tikzcd}$$

[4] T-co-objects are defined dually to T-objects: they are pairs $\langle B, p\rangle$, where $p: B \to T(B)$, etc. When we speak of the power set functor, we mean the functor that when applied to objects gives the power set as a result and when applied to arrows gives the inverse image function. This is obviously contravariant; the covariant power set functor on the other hand, when applied to arrows gives direct image as a result.

where the function $\hat{R}$ is so defined: $\hat{R}(w) = \{w' \mid wRw'\}$ and $\exists_f$ is the direct image function (we recall that the correspondence associating $\hat{R}$ with R is bijective).

The examples show that a general theory for free algebras and normal forms applying results arisen in the context of T-objects categories might be of some interest. This is what we are going to do in this paper, by making the connections between known categorial results on one hand and some logical problems on the other, that have been studied and often solved quite independently. We point out that in many interesting cases we have to add *equations* to the pure T-objects structure. Even in these cases, the general theory is of some help because it does automatically some amount of work, indeed we only have to take a quotient for the extra equations (we shall give examples afterwards, still within the context of modal logic).

What we need here is a result concerning the construction of the internal free monoid, which is given in [6]; a simplified case is however sufficient for us. We follow a formulation that appears in [1] (and also in [2] which contains many additional facts) and we report the full proof because we plan to deduce uniformly from it normal forms and combinatorial description of free algebras in all the above examples (except the last one, which has a particular status). We also explore the proof from the point of view of the logician, by relating the various steps of the construction with syntactic aspects, like the modal degree of formulas and proofs.

Free algebras are usually built by Lindenbaum quotients: if G is the set of generators, one takes the set of formulas in the language with G as the set of propositional letters, then the equivalence relation

$$\alpha_1 \approx \alpha_2 \text{ iff } \vdash \alpha_1 \leftrightarrow \alpha_2$$

is defined and the algebraic operations corresponding to the connectives are introduced on the representative elements of each equivalence class. This construction indeed works in all the above syntactic examples but we would like to have an intrinsic, i.e. language-free, description. We may get Lindenbaum algebra in a segmented way, that is in a way that takes the complexity of the construction by successive approximations into account. In the case of modal system K this is done as follows: let F_i be the set of formulas of modal degree at most i (in the propositional letters G).[5] Define on it the equivalence relation

$$\alpha_1 \approx_i \alpha_2 \text{ iff } \vdash_i \alpha_1 \leftrightarrow \alpha_2,$$

where $\vdash_i$ means existence of a proof of modal degree at most i.[6] We thus have, varying i in the set of natural numbers N, Boolean algebras $B_i = F_i / \approx_i$, Boolean

[5] The concept of *modal degree* d of a formula is defined inductively as follows: $d(p) = 0$ for $p \in G$ or $p = \bot$, $\top$, $d(\alpha_1 * \alpha_2) = \max(d(\alpha_1), d(\alpha_2))$ for $* = \vee, \wedge$, $d(\neg\alpha) = d(\alpha)$, $d(\Box\alpha) = d(\alpha) + 1$.

[6] The modal degree $d(\pi)$ of a proof $\pi = \langle \alpha_1, \ldots, \alpha_n \rangle$ is so defined in terms of the modal degree of the proof $\pi' = \langle \alpha_1, \ldots, \alpha_{n-1} \rangle$: $d(\pi) = \max(d(\alpha_n), d(\pi'))$ if α_n is the instance of an axiom (of course, $d(\pi') = 0$ if $n - 1 = 0$), $d(\pi) = d(\pi')$ if α_n is obtained through modus ponens, $d(\pi) = d(\pi') + 1$ if α_n is obtained through necessitation.

morphisms ε_i: $B_i \to B_{i+1}$[7] and hemimorphisms $\Box_i: |B_i| \to |B_{i+1}|$ satisfying the conditions

(Gr) $\quad |\varepsilon_i|\Box_{i+1} = \Box_i|\varepsilon_{i+1}|$

meaning that the squares

$$\begin{array}{ccc} |B_i| & \xrightarrow{\Box_i} & |B_{i+1}| \\ {\scriptstyle |\varepsilon_i|}\downarrow & & \downarrow{\scriptstyle |\varepsilon_{i+1}|} \\ |B_{i+1}| & \xrightarrow[\Box_{i+1}]{} & |B_{i+2}| \end{array}$$

commute. We call such a structure $\mathscr{B} = \langle \{B_i\}_i, \{\varepsilon_i\}_i, \{\Box_i\}_i \rangle$ a *graded modal algebra.* A morphism between two graded modal algebras (to be called a *graded morphism*) $\mathscr{B}$ and $\mathscr{C}$ is a succession of Boolean morphisms $\mu = \{\mu_i: B_i \to C_i\}_i$ such that the following squares commute:[8]

$$\begin{array}{ccc} B_i & \xrightarrow{\mu_i} & C_i \\ {\scriptstyle \varepsilon_i}\downarrow & & \downarrow{\scriptstyle \varepsilon_i} \\ B_{i+1} & \xrightarrow[\mu_{i+1}]{} & C_{i+1} \end{array} \qquad \begin{array}{ccc} |B_i| & \xrightarrow{|\mu_i|} & |C_i| \\ {\scriptstyle \Box_i}\downarrow & & \downarrow{\scriptstyle \Box_i} \\ |B_{i+1}| & \xrightarrow[|\mu_{i+1}|]{} & |C_{i+1}| \end{array}$$

Graded modal algebras and graded morphisms are a category, that we call **Gma**. The reader is asked to check that the graded Lindenbaum algebra introduced above is in fact the free graded modal algebra generated by the set G: this is done in a standard way; however, notice that the limitation on the modal degree of proofs is essential, because a proof of degree n does not have any algebraic interpretation at lower levels, even if the proved formula has lower degree. Otherwise said, there is noting in the definition of a graded modal algebra that forces the ε_i morphisms to be injective. The fact that they may or may not be such in the *free* graded algebra is equivalent to a good property that a logical system may have or not and that *depends on the logical/algebraic axiomatization* chosen: we mean the property that provability of a formula can be established using only proofs not exceeding its degree. In fact our algebraic framework can give us information of that kind (for instance, the usual axiomatization of $S4$ is bad from this point of view, see Section 5). Given a graded modal algebra $\mathscr{B}$, we may take the colimit of the directed system of Boolean algebras $\{\varepsilon_i: B_i \to B_{i+1}\}_i$ and define on it an endo-emimorphism, thanks to condition (Gr), thus getting a modal algebra $\varinjlim \mathscr{B}$ (we shall turn on this construction afterwards). If we apply this procedure to the case of the algebras $F_i/\approx_i$'s above, we get again the usual Lindenbaum algebra. What is the advantage? Now, if the set of generators G is finite,

[7] These morphisms are defined by mapping the equivalence class of, say α, into the equivalence class of α at the next level.

[8] From now on, for simplicity, the corresponding ε_i and $\Box_i$ of different structures will be indicated in the same way. Similar conventions will be applied without explicit mention when confusion does not arise.

then the $F_i/\approx_i$ are finite too (because finitely generated Boolean algebras are finite), which means that they can be described as power sets algebras for finite sets A_i of atoms; moreover, the ε_i are inverse image morphisms along functions λ_i: $A_{i+1} \to A_i$ and the hemimorphisms $\Box_i$ are determined by relations R_i: $A_{i+1} \to A_i$ (i.e. R_i is a subset of $A_{i+1} \times A_i$) in the well-known obvious way. In this way we may hope to get a description of the whole construction in terms of a combinatorial mechanism operating on the set G. Normal forms for formulas will automatically arise.

Let us put all this in the context of T-objects categories, in order to gain the good degree of generality. A *graded T-object* (we suppose that an endofunctor T: $\mathbf{C} \to \mathbf{C}$ is given)

$$B = \langle \{B_i\}_i, \{\varepsilon_i\}_i, \{p_i\}_i \rangle$$

is a succession $\{B_i\}_i$ of objects from $\mathbf{C}$ endowed with morphisms (also from $\mathbf{C}$)

$$\varepsilon_i: B_i \to B_{i+1}, \qquad p_i: T(B_i) \to B_{i+1}$$

such that the following squares

$$\begin{tikzcd} T(B_i) \arrow[r, "p_i"] \arrow[d, "T(\varepsilon_i)"'] & B_{i+1} \arrow[d, "\varepsilon_{i+1}"] \\ T(B_{i+1}) \arrow[r, "p_{i+1}"'] & B_{i+2} \end{tikzcd}$$

commute. A morphism between two graded T-objects $\mathscr{B}$ and $\mathscr{C}$ is a succession of morphisms in $\mathbf{C}$ $\mu = \{\mu_i: B_i \to C_i\}_i$ such that the following squares commute:

$$\begin{tikzcd} B_i \arrow[r, "\mu_i"] \arrow[d, "\varepsilon_i"'] & C_i \arrow[d, "\varepsilon_i"] \\ B_{i+1} \arrow[r, "\mu_{i+1}"'] & C_{i+1} \end{tikzcd} \qquad \begin{tikzcd} T(B_i) \arrow[r, "T(\mu_i)"] \arrow[d, "p_i"'] & T(C_i) \arrow[d, "p_i"] \\ B_{i+1} \arrow[r, "\mu_{i+1}"'] & C_{i+1} \end{tikzcd}$$

Graded T-objects and graded morphisms are a category, that we call **GTObj**. It is easily seen that this definition matches with that of graded modal algebras in case $T = |-|P$ (the two categories are indeed isomorphic, by the usual transposition argument).

There are obvious functors

$$Con: \mathbf{TObj} \to \mathbf{GTObj}, \qquad (-)_0: \mathbf{GTObj} \to \mathbf{C},$$

the former is the constant functor (i.e. for a T-object $\langle B, p\rangle$, $Con(\langle B, p\rangle) = \langle \{B\}_i, \{1_B\}_i, \{p\}_i\rangle$), whereas the latter is the 0th component functors. The composite functor $Con(-)_0$ is the foregetful functor from **TObj** into **C**, that is the functor that associates B with the T-object $\langle B, p\rangle$ and that operates identically on morphisms (in the case of modal algebras it is the functor that forgets the modal operators). We want to build its left adjoint, if it is possible. The value of this left adjoint at an object B of **C** is called the *free T-object* generated by B (in the case of modal algebras, it will be the modal algebra freely generated by the Boolean algebra B). From now on the following assumption (satisfied in all the above examples) is made on **C**: **C** has *binary coproducts*. We begin with the description of the left adjoint to $(-)_0$.

We define, given an object B from **C**, a succession of objects $\{B_i\}_i$ as follows (the symbol $+$ indicates the coproduct in **C**, ι_1 and ι_2 will be the two canonical injections):

$$B_0 = B, \qquad B_{i+1} = B + T(B_i). \tag{1}$$

We define also two successions of morphisms in **C** $\{\varepsilon_i\colon B_i \to B_{i+1}\}_i$ and $\{p_i\colon T(B_i) \to B_{i+1}\}_i$ by

$$\begin{aligned} \varepsilon_0 &= \iota_1\colon B \to B + T(B_0),\\ \varepsilon_{i+1} &= 1_B + T(\varepsilon_i)\colon B + T(B_i) \to B + T(B_{i+1}),\\ p_i &= \iota_2\colon T(B_i) \to B + T(B_i). \end{aligned} \tag{2}$$

Proposition 1.1. *For every object B,*

$$Gr(B) = \langle \{B_i\}_i, \{\varepsilon_i\}_i, \{p_i\}_i \rangle,$$

defined as above, is a graded T-object.

Proof. In fact, $p_i\varepsilon_{i+1} = \iota_2(1_B + T(\varepsilon_i)) = T(\varepsilon_i)\iota_2 = T(\varepsilon_i)p_{i+1}$. □

Proposition 1.2. *For every object B, the pair given by $Gr(B)$ and by the identity map 1_B: $B \to (Gr(B))_0$ is universal from B to the functor $(-)_0$.*

Proof. The statement of the proposition says that for every graded T- object $\mathscr{A} = \langle \{A_i\}_i, \{\varepsilon_i\}_i, \{p_i\}_i \rangle$ and for every morphism $\mu_0\colon B \to A_0$, there exists a unique graded morphism $\mu\colon Gr(B) \to \mathscr{A}$, whose 0th component is μ_0. This means that we must define unique μ_{i+1}'s (for $i \geqslant 0$) in such a way that the following squares commute:

$$\begin{tikzcd} B_0 \arrow[r, "\mu_0"] \arrow[d, "\varepsilon_0"'] & A_0 \arrow[d, "\varepsilon_0"] \\ B_1 \arrow[r, "\mu_1"'] & A_1 \end{tikzcd} \qquad \begin{tikzcd} T(B_0) \arrow[r, "T(\mu_0)"] \arrow[d, "p_0"'] & T(A_0) \arrow[d, "p_0"] \\ B_1 \arrow[r, "\mu_1"'] & A_1 \end{tikzcd}$$

$$\vdots$$

$$\begin{tikzcd} B_{i+1} \arrow[r, "\mu_{i+1}"] \arrow[d, "\varepsilon_{i+1}"'] & A_{i+1} \arrow[d, "\varepsilon_{i+1}"] \\ B_{i+2} \arrow[r, "\mu_{i+2}"'] & A_{i+2} \end{tikzcd} \qquad \begin{tikzcd} T(B_{i+1}) \arrow[r, "T(\mu_{i+1})"] \arrow[d, "p_{i+1}"'] & T(A_{i+1}) \arrow[d, "p_{i+1}"] \\ B_{i+2} \arrow[r, "\mu_{i+2}"'] & A_{i+2} \end{tikzcd}$$

$$\vdots$$

According to the definition of coproduct, μ_{i+1} must be of the kind $[\nu_{i+1}, \xi_{i+1}]$ for unique ν_{i+1}, ξ_{i+1}, moreover the commutativity of the squares on the right means exactly that $\xi_{i+1} = T(\mu_i)p_i$. As μ_0 is given, the commutativity of left square at the top determines uniquely the definition of ν_1 as $\mu_0\varepsilon_0$.

We only have to find the definition of ν_{i+2}, for $i \geqslant 0$. The commutativity condition of the left square at the bottom is $[\nu_{i+1}, \xi_{i+1}]\varepsilon_{i+1} = (1_B + T(\varepsilon_i))[\nu_{i+2}, \xi_{i+2}]$, that is

$$\nu_{i+1}\varepsilon_{i+1} = \nu_{i+2}, \qquad \xi_{i+1}\varepsilon_{i+1} = T(\varepsilon_i)\xi_{i+2}.$$

We can take the former as a definition, but the latter must be proved using the inductive hypothesis $\mu_i\varepsilon_i = \varepsilon_i\mu_{i+1}$.[9] We have that

$$\begin{aligned}
\xi_{i+1}\varepsilon_{i+1} &= T(\mu_i)p_i\varepsilon_{i+1} \\
&= T(\mu_i)T(\varepsilon_i)p_{i+1} = T(\mu_i\varepsilon_i)p_{i+1} \\
&= T(\varepsilon_i, \mu_{i+1})p_{i+1} = T(\varepsilon_i)T(\mu_{i+1})p_{i+1} \\
&= T(\varepsilon_i)\xi_{i+2},
\end{aligned}$$

where we used the characterization of the ξ_i's just found, the graded T-object definition and the inductive hypothesis. The proof of Proposition 2.2 is now complete; we make an additional remark: by induction, one can easily prove that

$$\mu_{i+1} = [\mu_0\varepsilon_0\varepsilon_1 \cdots \varepsilon_i, T(\mu_i)p_i]. \tag{3}$$

This recursive formula for μ_i (we recall that μ_0 is given) will be useful in the next sections (normal forms are hidden in it). □

To continue, we need an extra additional hypothesis: *chain colimits exist in* **C** *and* *T preserves them.* We have to take in mind this extra hypothesis and check it before applying the results below to our examples (as we shall see the hypothesis fails only in the case of Kripke frames, where however Proposition 1.2 still can be usefully applied).

Given a graded T-object $\mathscr{B} = \langle\{B_i\}_i, \{\varepsilon_i\}_i, \{p_i\}_i\rangle$, let $\varinjlim \mathscr{B}$ be the colimit in **C** of the chain diagram formed by the B_i's and the ε_i's with injection morphisms η_i: $B_i \to \varinjlim \mathscr{B}$. By the definition of graded T-object the following diagrams commute

$$\begin{tikzcd}
T(B_i) \arrow[dd, "p_i\eta_{i+1}"'] \arrow[dr, "T(\varepsilon_i)"] & \\
& T(B_{i+1}) \arrow[dl, "p_{i+1}\eta_{i+2}"] \\
\varinjlim \mathscr{B} &
\end{tikzcd}$$

[9] Notice that for $i = 0$, the equation $\mu_0\varepsilon_0 = \varepsilon_0\mu_1$ is just the above definition of ν_1 as $\mu_0\varepsilon_0$.

(in fact, $T(\varepsilon_i)p_{i+1}\eta_{i+2} = p_i\varepsilon_{i+1}\eta_{i+2} = p_i\eta_{i+1}$). Hence, by the chain colimit preservation property, there exists a unique morphism $p: T(\varinjlim\mathscr{B}) \to \varinjlim\mathscr{B}$ such that for every $i \in N$, $T(\eta_i)p = p_i\eta_{i+1}$. We use the notation

$$p = [p_k\eta_{k+1}]_k \tag{4}$$

or also $p = [T(\eta_k): p_k\eta_{k+1}]_k$ to be able to reconstruct the property that identifies p (i.e. p is the only arrow such that, for every k, composed on the left with $T(\eta_k)$ gives $p_k\eta_{k+1}$). We thus have a T-object $\langle\varinjlim\mathscr{B}, p\rangle$ and a graded morphism $\eta = \{\eta_i\}_i: \mathscr{B} \to Con(\langle\varinjlim\mathscr{B}, p\rangle)$.

Proposition 1.3. *For every graded T-object $\mathscr{B}$, the pair given by $\langle\varinjlim\mathscr{B}, p\rangle$ and by the graded morphism $\eta: \mathscr{B} \to Con(\langle\varinjlim\mathscr{B}, p\rangle)$ defined above is universal from $\mathscr{B}$ to the functor Con.*

Proof. The statement of the proposition is the following: given a T-object $\langle B, q\rangle$ and a graded morphism $\xi = \{\xi_i\}_i: \mathscr{B} \to Con(\langle B, q\rangle)$, we are asked to show that there exists a unique T-objects morphism $\xi^{\mathrm{T}}: \langle\varinjlim\mathscr{B}, p\rangle \to \langle B, p\rangle$ such that $\eta Con(\xi^{\mathrm{T}}) = \xi$, i.e. such that $\eta_i\xi^{\mathrm{T}} = \xi_i$ for every i. The fact that ξ is a graded morphism means that for every i

$$\xi_i = \varepsilon_i\xi_{i+1} \quad \text{and} \quad T(\xi_i)q = p_i\xi_{i+1}.$$

In particular, from the former equation, we realize that $\langle B, \{\xi_i\}_i\rangle$ is a cone for the diagram $\langle\{B_i\}_i, \{\varepsilon_i\}_i\rangle$, hence $\xi^{\mathrm{T}}: \varinjlim\mathscr{B} \to B$ must be defined as $\xi^{\mathrm{T}} = [\eta_k: \xi_k]_k$. We only have to show that ξ^{T} is a T-object morphism, i.e. that $T([\eta_k: \xi_k]_k)q = p[\eta_k: \xi_k]_k$,

$$\begin{tikzcd} T(\varinjlim\mathscr{B}) \arrow[r, "p"] \arrow[d, "T([\xi_k]_k)"'] & \varinjlim\mathscr{B} \arrow[d, "[\xi_k]_k"] \\ T(B) \arrow[r, "q"'] & B \end{tikzcd}$$

where we recall that, according to (4), p is defined as $[T(\eta_k): p_k\eta_{k+1}]_k$.

Because of the chain colimits preservation property of T, we have that $\langle T(\varinjlim\mathscr{B}), \{T(\eta_i)\}_i\rangle$ is a colimit cone, hence it is sufficient to show that for every $i \in N$, $T(\eta_i)T([\eta_k: \xi_k]_k)q = T(\eta_i)[T(\eta_k): p_k\eta_{k+1}]_k[\eta_k: \xi_k]_k$. Indeed, the second member is $p_i\xi_{i+1}$ and the first one is $T(\xi_i)q$: they are equal because ξ is a graded morphism. $\square$

The following theorem follows immediately from Propositions 1.2 and 1.3.

Theorem 1.4. *For every B, $\langle\varinjlim Gr(B), p\rangle$ is the free T-object generated by B (the canonical embedding of B into $\varinjlim Gr(B)$ is the injection η_0 into the colimit).*

We recall that $Gr(B)$ is defined by formulas (1) and (2) and p is defined by formula (4).

2. Modal algebras

In this section we examine the basic case of modal algebras: they are T-objects for the endofunctor associating with a Boolean algebra the Boolean algebra freely generated by its underlying meet-semilattice. Such a functor is the composition of two functors, namely $|-|$ and P, both preserving chain colimits (this is automatic for P which is a left adjoint and trivial for $|-|$). Hence Theorem 1.4 can be applied and consequently we have the opportunity of extracting algorithms from the above proofs and of computing what we are interested in (i.e. the free modal algebra generated by a finite Boolean algebra), provided we characterize in an easy way the value of the left adjoint P: **SemiL** $\to$ **Boole** on finite semilattices.

Proposition 2.1. *For every finite similattice $R = \langle \underline{R}, \wedge, \top \rangle$, the pair given by the power set Boolean algebra of $\underline{R}$ and by the semilattice morphism $\downarrow: R \to |\mathscr{P}(\underline{R})|$ defined by $\downarrow(x) = \{y \mid y \leqslant x\}$ is universal from R to the forgetful functor $|-|$.*

We give two proofs of the proposition: the former is more conceptual, the latter is more direct.

First Proof. Clearly, $\downarrow$ is a semilattice morphism. From the general theory of adjoint functors, we know that the pair $\langle P(R), 1^{t}_{P(R)} \rangle$ is universal from R to the functor $|-|$ and that $\downarrow^{\mathrm{T}}$ is the unique Boolean morphism such that the following triangle

$$\begin{tikzcd} & R \arrow[dl, "1^{t}_{P(R)}"'] \arrow[dr, "\downarrow"] & \\ |P(R)| \arrow[rr, "|\downarrow^{\mathrm{T}}|"'] & & |\mathscr{P}(\underline{R})| \end{tikzcd}$$

commutes. We simply show that $\downarrow^{\mathrm{T}}$ is a Boolean isomorphism. This happens if and only if (taking the representable functor **Boole**$[-, \mathbf{2}]$, where $\mathbf{2}$ is the two-element Boolean algebra)

$$\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}] : \mathbf{Boole}[\mathscr{P}(\underline{R}), \mathbf{2}] \to \mathbf{Boole}[P(R), \mathbf{2}]$$

is a bijection,[10] i.e. if and only if

$$\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}](-)^{\mathrm{t}}: \mathbf{Boole}[\mathscr{P}(\underline{R}), \mathbf{2}] \to \mathbf{SemiL}[R, |\mathbf{2}|]$$

is a bijection. Now an element of $\mathbf{Boole}[\mathscr{P}(\underline{R}), \mathbf{2}]$ is the characteristic function χ_a of a (principal, because $\underline{R}$ is finite) ultrafilter, the ultrafilter corresponding to point $a \in \underline{R}$. Similarly, an element of $\mathbf{SemiL}[R, |\mathbf{2}|]$ is the characteristic function $\chi_{[a]}$ of a (principal) filter of R, the filter generated by a. On the other hand, the function $\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}](-)^{\mathrm{t}}$ maps χ_a onto $(\downarrow^{\mathrm{T}}\chi_a)^{\mathrm{t}} = \downarrow|\chi_a|$ and, for every $b \in \underline{R}$, $|\chi_a|(\downarrow(b)) = \top$ iff there exists $c \leqslant b$ such that $c = a$ iff $a \leqslant b$. That is, the function $\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}(-)^{\mathrm{t}}$ maps χ_a into $\chi_{[a]}$ and so it is obviously bijective. $\square$

Second Proof. Notice that for every $a \in \underline{R}$,

$$\{a\} = \downarrow(a) \cap \bigcap_{x \in (\underline{R} \setminus \uparrow a)} (\underline{R} \setminus \downarrow(x)),$$

where $\uparrow a = \{y \mid a \leqslant y\}$: in fact, for every $b \in \underline{R}$, $b \in \downarrow(a) \cap \bigcap_{x \in (\underline{R} \setminus \uparrow a)} (\underline{R} \setminus \downarrow(x))$ iff $b \leqslant a \,\&\, \forall x \in \underline{R}(a \nleqslant x \Rightarrow b \nleqslant x)$, i.e. iff $b \leqslant a \,\&\, a \leqslant b$, that is iff $b = a$. Now for every Boolean algebra B and for every semilattice morphism $k: R \to |B|$, k^{T} should be the unique Boolean morphism such that the triangle

$$\begin{tikzcd}
 & R \arrow[dl, "\downarrow"'] \arrow[dr, "k"] & \\
|\mathscr{P}(R)| \arrow[rr, "|k^{\mathrm{T}}|"'] & & |B|
\end{tikzcd}$$

commutes, hence for every $S \subseteq \underline{R}$, we *must* have

$$k^{\mathrm{T}}(S) = \bigvee_{a \in S} \left(k(a) \wedge \bigwedge_{x \in (\underline{R} \setminus \uparrow a)} \neg k(x) \right). \tag{5}$$

(to understand properly the formula, recall that in a lattice the meet (dually, the join) operation applied to the empty set of indices gives $\top$ (dually, $\bot$) as a result).

---

[10] We recall that $\mathbf{Boole}[-, \mathbf{2}]$: $\mathbf{Boole}^{op} \to \mathbf{Set}$ is the contravariant functor associating with a Boolean algebra B the set of Boolean morphisms $B \to \mathbf{2}$ (i.e. equivalently, the set of the ultrafilters of B) and with a Boolean morphism $\mu: C \to B$ the "composition with μ" function. Proving that $\mathbf{Boole}[\mu, \mathbf{2}]$ is bijective iff μ is an isomorphism, requires in general a noncompletely trivial argument based on the ultrafilter theorem, but in our case we may simply argue as follows: assume that $\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}]$ is bijective and notice that $\downarrow^{\mathrm{T}}\sigma_{\mathscr{P}(R)} = \sigma_{P(\underline{R})}(\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}])^{-1}$, where the σ's are the Stone embeddings into the power sets of the sets of the ultrafilters (this equation may be easily checked directly, but follows from the fact that the Stone embedding is the unity of an adjointness, hence it is a natural transformation). Now since $\underline{R}$ is finite, $\sigma_{\mathscr{P}(\underline{R})}$ is an isomorphism and so is $\sigma_{P(R)}$ ($P(R)$ happens to be finite too, because the power set of the set of its ultrafilters is in bijective correspondence, trough $(\mathbf{Boole}[\downarrow^{\mathrm{T}}, \mathbf{2}])^{-1}$, with the power set of the finite set $\mathbf{Boole}[\mathscr{P}(\underline{R}), \mathbf{2}]$). Hence $\downarrow^{\mathrm{T}}$ is an isomorphism.

Preservation of $\bot$ and $\bigvee$ by k^{T} is immediate; for preservation of $\bigwedge$ it is sufficient to apply the distributive laws and to observe that

$$k(a_1) \wedge \bigwedge_{x_1 \in (\underline{R} \setminus \uparrow a_1)} \neg k(x_1) \wedge k(a_2) \wedge \bigwedge_{x_2 \in (\underline{R} \setminus \uparrow a_2)} \neg k(x_2)$$

is equal to $\bot$ if $a_1 \neq a_2$. Preservation of $\top$ is a little more difficult; for any $S \subseteq \underline{R}$, put

$$S^* = \bigwedge_{x \in S} k(x) \wedge \bigwedge_{x \in (\underline{R} \setminus S)} \neg k(x).$$

By the lemma below, $\bigvee_{S \subseteq \underline{R}} S^* = \top$, hence in order to show that $k^{\mathrm{T}}(\underline{R}) = \top$, it is sufficient to prove that $S^* = \bot$ in case S is not of the form $\uparrow a$ for some $a \in \underline{R}$. Otherwise said, we must prove that if $S^* \neq \bot$, then S is a filter of R (recall that R is finite). This is easily done by using the fact that k is a semilattice morphism. Preservation of complement is a consequence of preservation of $\bot$, $\vee$, $\top$, $\wedge$, as usual with Boolean algebras. Finally, to show that $\downarrow |k^{\mathrm{T}}| = k$ take any $y \in \underline{R}$ and observe that $k(y)$ is equal to $k(y) \wedge k^{\mathrm{T}}(\underline{R})$, that is to

$$\bigvee_{a \in \underline{R}} \left(k(y) \wedge k(a) \wedge \bigwedge_{x \in (\underline{R} \setminus \uparrow a)} \neg k(x) \right).$$

Again for $a \nleqslant y$ the related meet expression is zero, so the whole expression is equal to $k^{\mathrm{T}}(\downarrow(y))$. $\square$

Lemma 2.2. *Let B be a Boolean algebra, $\underline{R}$ be a finite set and $h: \underline{R} \to \underline{B}$ be a function. Then the following equation holds*

$$\top = \bigvee_{S \subseteq \underline{R}} \left(\bigwedge_{a \in S} h(a) \wedge \bigwedge_{a \in (\underline{R} \setminus S)} \neg h(a) \right).$$

Proof. By induction on the cardinality of $\underline{R}$. $\square$

From now on, we assume that the value of the functor P on a finite semilattice R is exactly $\mathscr{P}(\underline{R})$ and that the inverse transpose of a morphism ξ: $P(R) \to B$ is the composite morphism $\downarrow |\xi|$, that is for $a \in \underline{R}$,

$$\xi^{\mathrm{t}}(a) = \xi(\{y \mid y \leqslant a\}). \tag{6}$$

The value of the functor P at a semilattice morphism h: $R_1 \to R_2$ for finite R_1, R_2 can also be easily computed by the standard procedure: it is the unique Boolean morphism such that the square

$$\begin{array}{ccc} R_1 & \xrightarrow{h} & R_2 \\ \downarrow{\scriptstyle\downarrow} & & \downarrow{\scriptstyle\downarrow} \\ |P(R_1)| & \longrightarrow & |P(R_2)| \end{array}$$

commutes, i.e. it is $(h\downarrow)^{\mathrm{T}}$. So, by (5), for $S \subset \underline{R}_1$, we have that

$$P(h)(S) = \bigcup_{a\in S}\left(\{x \mid x \leqslant h(a)\} \cap \bigcap_{y\notin \uparrow a} \{x \mid x \nleqslant h(y)\}\right).$$

Now given $b \in \underline{R}_2$, b belongs to $P(h)(S)$ iff

$$\exists a \in S(b \leqslant h(a)\,\&\,\forall y(a \nleqslant y \Rightarrow b \nleqslant h(y)))$$

that is, iff

$$\exists a \in S(h^*(b) \leqslant a\,\&\,\forall y(h^*(b) \leqslant y \Rightarrow a \leqslant y)),$$

where h^* is the left adjoint to h.[11] This means that $b \in P(h)(S)$ iff $\exists a \in S(h^*(b) \leqslant a \& a \leqslant h^*(b))$ i.e. iff $h^*(b) \in S$. So $P(h)$ *is the inverse image along* h^*, *the left adjoint to* h.

To perform our main task, we recall a standard information on coproducts of finite Boolean algebras, i.e. that given two finite sets X and Y, $\mathscr{P}(X \times Y)$ is the coproduct in **Boole** of $\mathscr{P}(X)$ and $\mathscr{P}(Y)$, with the inverse images along the two projections as canonical injections. Notice that, given a Boolean algebra C and two imaps μ: $\mathscr{P}(X) \to C$, v: $\mathscr{P}(Y) \to C$, the map $[\mu, v]$ is so determined (for $S \subseteq X \times Y$)

$$[\mu, v](S) = \bigwedge_{\langle x,\, y\rangle \in S} (\mu(\{x\}) \wedge v(\{y\})). \tag{7}$$

Having this in mind, it is not difficult to compute the graded modal algebra $Gr(B)$ of Proposition 1.2 for a finite Boolean algebra B. We use formulas (1) and (2) of the previous section. Such formulas involve the endofunctor $T = | - |P$ and coproducts. As finite Boolean algebras are dual to finite sets, coproducts are turned into products in the dual category. That is, for finite $B = \mathscr{P}(L)$, formulas (1) and (2) identify a combinatorial construction on L: we introduce it and then show how it can be deduced.

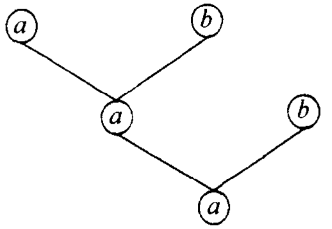
Given any set L (the set of labels), we define inductively the set T_i^L, for $i \in N$, called the sets of *L-labelled commutative idempotent trees* (briefly, *trees*) *of degree i*:

$$T_0^L = L, \qquad T_{i+1}^L = L \times \mathscr{P}(T_i^L).$$

Notice that equal immediate subtrees are identified, this is why, in defining T_{i+1}^L, we used the power set of T_i^L and not for instance the set of lists of elements of T_i^L. A subset f of T_i^L will be called a (*L-labelled commutative idempotent*) *forest of degree i*. Notice that any $t \in T_i^L$ may be represented in a graphical way as a labelled tree

[11] Given posets $\langle P, \leqslant\rangle$, $\langle Q, \leqslant\rangle$ and order-preserving functions h: $\langle P, \leqslant\rangle \to \langle Q, \leqslant\rangle$, h^*: $\langle Q, \leqslant\rangle \to \langle P, \leqslant\rangle$, we have that h^* is left adjoint to h iff for every $a \in P$, $b \in Q$, $b \leqslant h(a) \Leftrightarrow h^*(b) \leqslant a$. According to the adjoint functor theorem, we have that h: $\langle P, \leqslant\rangle \to \langle Q, \leqslant\rangle$ has a left adjoint in case $\langle P, \leqslant\rangle$ is complete and h preserves meets (this is our case, because we are considering finite semilattices). Moreover, it is not difficult to see that the left adjoint h^* is given by the formula $h^*(b) = \bigwedge_{b \leqslant h(a)} a$.

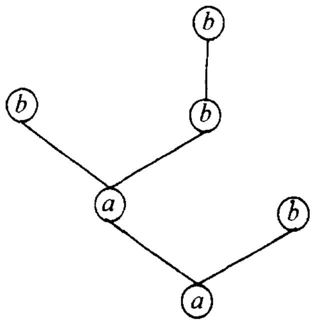
whose leaves have height at most i, e.g. for $a, b \in L$, $\langle a, \{\langle a, \{a, b\}\rangle, \langle b, \emptyset\rangle\}\rangle$ is represented as



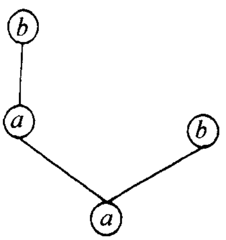
For $i > 0$, each $t \in T_i^L$ decomposes uniquely in a root-label to be noted t_R and in a forest t_S of degree $i - 1$, the forest of its *immediate subtrees* (or, simply, *subtrees*). We thus have relations R_i: $T_{i+1}^L \to T_i^L$ given, in our actual notation, by $tR_i u$ iff $u \in t_S$. We now define the *leaf-cuts* functions λ_i: $T_{i+1}^L \to T_i^L$ by

$$\lambda_0(t) = t_R, \qquad \lambda_{i+1}(t) = \langle t_R, \exists_{\lambda_i}(t_S)\rangle.$$

The function λ_i has an obvious graphic interpretation (it removes the leaves of height $i + 1$), notice however that, due to the fact that our trees are idempotent, applying leaf-cuts may considerably "reduce" the shape (the same observation applies also to other combinatorial operations to be introduced during the paper), for example if we apply λ_2 to the following tree



the result is



Notice that more complicated chains of simplifications (needing more than one step) are possible: they are all implicit in the above definition of leaf-cut.

Turning to formulas (1) and (2) of the previous section in the case of the modal algebra freely generated by the Boolean algebra $B = \mathscr{P}(L)$ (for finite L), let us prove that $B_i = \mathscr{P}(T_i^L)$: in fact $B_0 = \mathscr{P}(L) = \mathscr{P}(T_0^L)$ and

$$B_{i+1} = \mathscr{P}(L) + P(|B_i|) = \mathscr{P}(L) + \mathscr{P}\mathscr{P}(T_i^L) = \mathscr{P}(L \times \mathscr{P}(T_i^L)) = \mathscr{P}(T_{i+1}^L).$$

The ε_i's functions are the inverse images along the leaf-cuts: in fact $\varepsilon_0 = \iota_1 = \lambda_0^{-1}$, where λ_0, the 0th leaf-cut, is the first projection $L \times \mathscr{P}(L) \to L$; moreover,

$$\varepsilon_{i+1} = 1_B + P(|\varepsilon_i|) = 1_B + (\exists_{\lambda_i})^{-1} = (1_L \times \exists_{\lambda_i})^{-1} = \lambda_{i+1}^{-1},$$

because $P(|\lambda_i^{-1}|)$ is the inverse image along the left adjoint to the inverse image along λ_i, that is, it is the inverse image along the direct image along λ_i. Finally, the hemimorphisms $\Box_i = p_i^{\iota}$[12] are determined by the immediate subtree relations. In fact they are the inverse transpose of the second injections ι_2: $\mathscr{P}(\mathscr{P}(T_i^L)) \to \mathscr{P}(L) + \mathscr{P}(\mathscr{P}(T_i^L)) = \mathscr{P}(L \times \mathscr{P}(T_i^L))$; hence as ι_2 is the inverse image along the function that associates with a tree t of forest t_S, ι_2^{ι} (see formula (6)) associates with a forest f, the forest $\iota_2(\{g \mid g \subseteq f\}) = \{t \mid t_S \subseteq f\} = \{t \mid \forall u (u \in t_S \Rightarrow u \in f)\}$. This is a *root-addition* operation that we call ρ_i^+:

$$\rho_i^+(f) = \{t \mid \forall u (u \in t_S \Rightarrow u \in f)\}.$$

When applied to a forest of degree i, ρ_i^+ is the forest of degree $i+1$ that contains all the trees obtained by attaching a label at the bottom of a subforest of f. We have so proved the following result.

Theorem 2.3. *The graded modal algebra freely generated by a finite Boolean algebra* $B = \mathscr{P}(L)$ *is* $\mathscr{P}_{gr}(T^L) =_{df} \langle \{\mathscr{P}(T_i^L)\}_i, \{\lambda_i^{-1}\}_i, \{\rho_i^+\}_i \rangle$ *and the modal algebra freely generated by it is* $\langle \varinjlim(\mathscr{P}_{gr}(T^L)), \Box \rangle$.

The modal operator $\Box$ in $\varinjlim(\mathscr{P}_{gr}(T^L))$ is defined on the representative elements of equivalence classes:

$$\Box([f, i]) = [\rho_i^+(f), i+1].$$

This fact comes from formula (4) of the previous section.[13]

By the above theorem, given a graded modal algebra $\mathscr{B}$ and a morphism μ_0: $B = \mathscr{P}(L) \to \mathscr{B}_0$, there exists a unique graded morphism μ: $\mathscr{P}_{gr}(T^L) \to \mathscr{B}$, whose 0th component is μ_0. The recursive definition of $\mu = \{\mu_i\}_i$ is the following (see formula (3) in Section 1 and recall that $\Box_i^{\mathrm{T}} = p_i$)

$$\mu_{i+1} = [\mu_0 \varepsilon_0 \cdots \varepsilon_i, (|\mu_i| \Box_i)^{\mathrm{T}}].$$

---

[12] Formulas (2) give the arrows p_i needed to build a graded T-objects. The hemimorphisms of graded modal algebras are obtained by inverse transposition of such p_i.

[13] In fact, according to that formula and keeping the notation of the previous section, the modal operator in the free algebra is $[P|\eta_k| : p_k \eta_{k+1}]_k^{\iota}$ which is equal to $[|\eta_k| : p_k^{\iota} |\eta_{k+1}|]_k$ (hence to $[|\eta_k| : \rho_k^+ |\eta_{k+1}|]_k$, as claimed in the text): to see this, notice that for every i, $|\eta_i| [P|\eta_k| : p_k \eta_{k+1}]_k^{\iota} = (P|\eta_i| [P|\eta_k| : p_k \eta_{k+1}]_k)^{\iota} = p_i^{\iota} |\eta_{i+1}|$.

We want to compute the value of this function at a forest f of degree $i+1$, because the expressions that we get in this way will help in understanding the linguistic meaning of our forests. According to the above formula (7) for unique maps from coproducts of finite Boolean algebras, we have that (let's write $\varepsilon_{0,i+1}$ instead of $\varepsilon_0\varepsilon_1 \dots \varepsilon_i$):

$$\mu_{i+1}(f) = \bigvee_{t\in f} (\varepsilon_{0,i+1}(\mu_0(\{t_R\})) \wedge (|\mu_i|\Box_i)^{\mathrm{T}}(\{t_S\})).$$

On the other hand, for every forest g of degree i, by formula (5) for transposition, we have that

$$(|\mu_i|\Box_i)^{\mathrm{T}}(\{g\}) = \Box_i(\mu_i(g)) \wedge \bigwedge_{x\notin \uparrow g} \neg\Box_i(\mu_i(x)).$$

This expression may be simplified, because

$$\bigwedge_{x\notin\uparrow g} \neg\Box_i(\mu_i(x)) = \bigwedge_{u\in g} \neg\Box_i(\mu_i(T_i^L\setminus\{u\})),$$

as every $x\notin\uparrow g$ is smaller than a forest which is the complement of a singleton $\{u\}$, for suitable $u\in g$. We thus obtain (define $\Diamond_i$ as usual as $\neg\Box_i\neg$):

$$(|\mu_i|\Box_i)^{\mathrm{T}}(\{g\}) = \Box_i(\mu_i(g)) \wedge \bigwedge_{u\in g} \Diamond_i(\mu_i(\{u\}))$$

or, again

$$(|\mu_i|\Box_i)^{\mathrm{T}}(\{g\}) = \bigwedge_{u\notin g} \neg\Diamond_i(\mu_i(\{u\})) \wedge \bigwedge_{u\in g} \Diamond_i(\mu_i(\{u\})).$$

Coming back to our original problem (computing $\mu_{i+1}(f)$), we have that

$$\mu_{i+1}(f) = \bigvee_{t\in f}\left(\varepsilon_{0,i+1}(\mu_0(\{t_R\})) \wedge \bigwedge_{u\notin t_S} \neg\Diamond_i(\mu_i(\{u\})) \wedge \bigwedge_{u\in t_S} \Diamond_i(\mu_i(\{u\}))\right). \tag{8}$$

We recall that the forgetful functor

$\underline{(-)}$: **Boole** $\to$ **Set**

associating with a Boolean algebra its carrier set, does have a left adjoint. The value of this left adjoint on a finite set G is $\mathscr{P}(\mathscr{P}(G))$, the canonical embedding $\eta_G\colon G \to \underline{\mathscr{P}(\mathscr{P}(G))}$ is the map associating with any $p\in G$ the subset of $\mathscr{P}(G)$ given by

$$\eta_G(p) = \{S \subseteq G \mid p\in S\} \tag{9}$$

and the transpose of a map $k\colon G\to \underline{B}$ is the Boolean morphism whose value at any $S\subseteq\mathscr{P}(G)$ is

$$\bigvee_{S\in\mathscr{S}}\left(\bigwedge_{p\in S} k(p) \wedge \bigwedge_{p\in(G\setminus S)} \neg k(p)\right). \tag{10}$$

We have the following immediate consequence of Theorem 2.3:

Theorem 2.4. *The graded modal algebra freely generated by a finite set G is $\mathscr{P}_{gr}(T^{\mathscr{P}(G)})$ and the modal algebra freely generated by it is $\langle \varinjlim(\mathscr{P}_{gr}(T^{\mathscr{P}(G)})), \Box\rangle$.*

We study a little more carefully the composite adjointness between **Gma** and **Set**. Notice that, given a finite set G, the canonical embedding of G into $\overline{(\mathscr{P}_{gr}(T^{\mathscr{P}(G)}))_0}$ is still given by formula (9). We compute also the transpose of a set-theoretic function k: $G \to \underline{\mathscr{B}_0}$, where $\mathscr{B}$ is a graded modal algebra. By formulas (10) and (8), the transpose of k is the graded morphism $\mu = \{\mu_i\}_i$: $\mathscr{P}_{gr}(T^{\mathscr{P}(G)}) \to \mathscr{B}$ whose value at a forest f of suitable degree is recursively defined as follows:

$$\mu_0(f) = \bigvee_{a \in f}\left(\bigwedge_{p \in a} k(p) \wedge \bigwedge_{p \notin a} \neg k(p)\right);$$

$$\mu_{i+1}(f) = \bigvee_{t \in f}\left(\bigwedge_{p \in t_R} \varepsilon_{0,i+1}(k(p)) \wedge \bigwedge_{p \notin t_R} \neg \varepsilon_{0,i+1}(k(p)) \right.$$
$$\left. \wedge \bigwedge_{u \notin t_S} \neg \Diamond_i(\mu_i(\{u\})) \wedge \bigwedge_{u \in t_S} \Diamond_i(\mu_i(\{u\}))\right).$$

Suppose now that $\mathscr{B}$ is the graded Lindenbaum algebra mentioned in the previous section. Then the above formulas show how to associate with a forest f of degree n a formula $\Phi^n(f)$ of modal degree n:

$$\Phi_0(f) = \bigvee_{a \in f}\left(\bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \neg p\right),$$

$$\Phi_{n+1}(f) = \bigvee_{t \in f}\left(\bigwedge_{p \in t_R} p \wedge \bigwedge_{p \notin t_R} \neg p \wedge \bigwedge_{u \in t_S} \Diamond \Phi^n(\{u\}) \wedge \bigwedge_{u \notin t_S} \neg \Diamond \Phi^n(\{u\})\right). \tag{11}$$

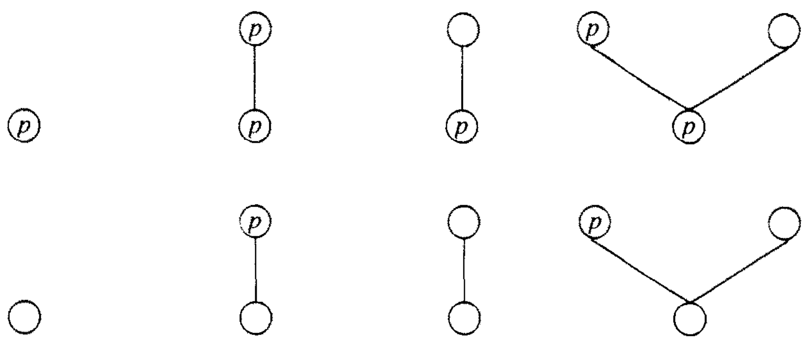
We so found the *normal forms* of [7]. Conversely, from the proof of the fact that the graded Lindenbaum algebra of Section 1 is also the free graded modal algebra, we get an inverse procedure in order to associate with a formula α^n of modal degree n a forest $\varphi^n(\alpha)$ of degree n:

$$\begin{aligned}
\varphi^0(\bot) &= \emptyset,\\
\varphi(\top) &= T_0^{\mathscr{P}(G)},\\
\varphi^0(p) &= \{a \subseteq G \mid p \in a\},\\
\varphi^{\max(n_1,n_2)}(\alpha_1^{n_1} \vee \alpha_2^{n_2}) &= \lambda^{-1}_{\max(n_1,n_2),n_1}(\varphi^{n_1}(\alpha_1)) \cup \lambda^{-1}_{\max(n_1,n_2),n_2}(\varphi^{n_2}(\alpha_2)),\\
\varphi^{\max(n_1,n_2)}(\alpha_1^{n_1} \wedge \alpha_2^{n_2}) &= \lambda^{-1}_{\max(n_1,n_2),n_1}(\varphi^{n_1}(\alpha_1)) \cap \lambda^{-1}_{\max(n_1,n_2),n_2}(\varphi^{n_2}(\alpha_2)),\\
\varphi^n(\neg \alpha^n) &= T_n^{\mathscr{P}(G)} \setminus \varphi^n(\alpha^n),\\
\varphi^{n+1}(\Box \alpha^n) &= \rho_n^+(\varphi^n(\alpha^n)).
\end{aligned} \tag{12}$$

We used the following convention: for i, $j \in N$, $i \leqslant j$, the iterated leaf-cut $\lambda_{j-1} \ldots \lambda_{i+1}\lambda_i$ is indicated with $\lambda_{j,i}$ (in case $i = j$, $\lambda_{j,i}$ is the identity function). It is useful to define also $\varphi^n(\alpha)$ for $n \geqslant d(\alpha) = m$ as $\lambda_{n,m}^{-1}(\varphi^m(\alpha))$.

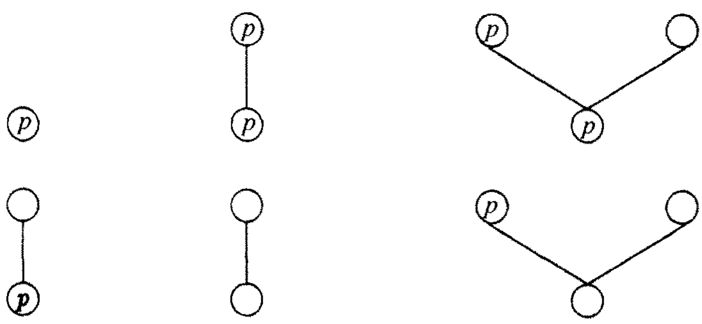
The two above procedures are inverse (they describe an algebraic isomorphism), which means that for every n and for every forest f of degree n, f is equal to $\varphi^n(\Phi^n(f))$ and moreover that, for every n and for every formula α of modal degree n, α and $\Phi^n(\varphi^n(\alpha))$ are provably equivalent in K. A third fact coming directly from the construction of free algebras as colimits, is the re-identification of normal forms through different degrees. This means that $\vdash \Phi^n(f) \leftrightarrow \Phi^{n+1}(g)$ iff $\lambda_n^{-1}(f) = g$ for every f, g of degrees n and $n + 1$, respectively.

Let us give a detailed example. If $G = \{p\}$, the set of labels $\mathscr{P}(G)$ contains exactly two elements, namely $\{p\}$ and $\emptyset$. There are eight trees of degree 1, which are the following ones:



Using (12), we compute the forest corresponding to the formula $\neg \Box p \vee p$ of modal degree 1. The forest $\varphi^0(p)$ is equal to the singleton $\{ p \}$, whereas the forest $\varphi^1(\Box p)$ consists of the following four trees

Consequently, the forest of degree 1 corresponding to the formula $\neg \Box p \vee p$ contains the six trees indicated in the picture below



Using (11), we obtain that

$$\begin{aligned}\Phi^1(\varphi^1(\neg\Box p \vee p)) = {} & (p \wedge \neg\Diamond p \wedge \neg\Diamond\neg p) \vee (p \wedge \Diamond p \wedge \neg\Diamond\neg p)\\ & \vee (p \wedge \Diamond p \wedge \Diamond\neg p) \vee (p \wedge \Diamond\neg p \wedge \neg\Diamond p)\\ & \vee (\neg p \wedge \Diamond\neg p \wedge \neg\Diamond p) \vee (\neg p \wedge \Diamond p \wedge \Diamond\neg p),\end{aligned}$$

which is in fact provably equivalent to $\neg\Box p \vee p$.

Notice that the functions λ_i are all surjective, in fact they have sections (i.e. left inverses) σ_i: $T_i^L \to T_{i+1}^L$ given by the inductive definitions:

$$\sigma_0(t) = \langle t, \emptyset \rangle, \qquad \sigma_{i+1}(t) = \langle t_R, \exists_{\sigma_i}(t_S) \rangle.$$

This fact implies that the transition Boolean morphisms $\varepsilon_i = \lambda_i^{-1}$ in the graded Lindenbaum algebra are all injective and, as shown in the previous section, *this is a proof* of the fact that in the syntactic calculus of the modal system K a formula of modal degree n is provable if and only if it admits a proof whose modal degree does not exceed n.

The fact that the leaf-cut functions are surjective has another important (strictly related) consequence, namely that *drawing the forest corresponding to a formula is a descision procedure* for the modal system K. In fact, a formula α^n is a theorem precisely iff its equivalence class is equal to $\top$ in the Lindenbaum algebra, i.e. iff the equivalence class of $\varphi^n(\alpha^n)$ is equal to the equivalence class of the total forest of degree n: however, as leaf-cuts are surjective, two forests of the same degree are in the same equivalence class iff they are equal. Notice that, if leaf-cuts were not surjective, drawing the forest corresponding to a formula and checking whether it is the total forest of the modal degree of the formula would not be sufficient to decidability, because in case the answer were negative, we would have to apply inverse image along leaf-cuts and check again whether the forest becomes total. In this case our procedure would be only a semi-decision procedure, exactly as it happens with the enumeration of theorems in the syntactic calculus. This discussion is not merely theoretical: we shall see in Section 5 that it is in general possible to apply the "graded version" of correspondence theory to describe free algebras in the variety corresponding to a given modal logic by imposing special conditions on trees; however, the leaf-cut functions might be not surjective anymore on the sets of the selected trees and so decidability might be lost [25] (even if the specific modal axioms are effectively given and the related conditions on trees are consequently effectively identified). More: choosing different equivalent axiomatizations for the same system may produce nonisomorphic free graded modal algebras, with surjective leaf-cuts in one case but not in the other (indeed this happens for very simple logics, like $S4$). In fact the existence of a recursive axiomatization producing a free graded modal algebra with surjective leaf-cuts is equivalent to the decidability of the system (see Section 5).

There is an interesting well-known *algebraic* result [3] on finitely generated free modal algebras, namely the fact that they are *atomic*. It is indeed easily seen that atoms are precisely the equivalence classes of singleton forests of the kind $\{\sigma_i(t)\}$ (these are the only singleton forests whose inverse image along a leaf-cut is again a singleton).

We recall some basic facts. Given a Kripke frame $\langle W, R\rangle$, we can build a modal algebra $\langle \mathscr{P}(W), \square\rangle$ by defining the endo-hemimorphism $\square$ as follows (for $S \subseteq W$):

$$\square S = \{w \in W \mid \forall v \in W (wRv \Rightarrow v \in S)\}.$$

Conversely, with any modal algebra $M = \langle B, \square\rangle$ we may associate a Kripke frame $\langle SpecM, R\rangle$ as follows:

- $SpecM$ is the set of the ultrafilters of M (seen as a Boolean algebra);
- R is defined so, for $u, v \in SpecM$

$$uRv \Leftrightarrow \forall x \in \underline{B}\ (\square x \in u \Rightarrow x \in v).$$

Given a set G, the *canonical frame* [13] (for the language having G as the set of propositional letters) is just the Kripke frame associated with the free algebra $\mathscr{F}(G)$ with G as the set of free generators. From Theorem 2.4, it is rather easy to find the following description for canonical frames in case G is finite: first of all, notice that an ultrafilter u of $\mathscr{F}(G)$ may be identified with a succession of trees

$$u = \{u_n \in T_n^{\mathscr{P}(G)} \mid n \in N\}$$

such that for all $n \in N$, $\lambda_n(u_{n+1}) = u_n$ (this identification can be easily seen directly, but follows also from the fact that taking ultrafilters is a contravariant representable functor, hence changes colimits into limits). In this way, Stone embedding associates with an equivalence class of forests $[f, n]$ the set of the u such that $u_n \in f$. The canonical relation in $Spec\ \mathscr{F}(G)$ is consequently determined as follows:

$$uRv \Leftrightarrow \forall n \in N,\quad \forall f \subseteq T_n^{\mathscr{P}(G)}$$

$$u_{n+1} \in \{t \in T_{n+1}^{\mathscr{P}(G)} \mid t_S \subseteq f\} \Rightarrow v_n \in f.$$

The quantifier "for all f" can be specialized to $f = (u_{n+1})_S$, without losing in generality, hence we obtain the following corollary.

Corollary 2.5. *The canonical frame $\langle Spec\ \mathscr{F}(G), R\rangle$, for finite G, consists of the set of the successions of trees $u = \{u_n \in T_n^{\mathscr{P}(G)}\}_n$ such that for every $n \in N$, $u_n = \lambda_n(u_{n+1})$. The canonical relation R contains exactly the pairs $\langle u, v\rangle$ such that for every $n \in N$, v_n is an immediate subtree of u_{n+1}.*

Notice that the ultrafilters corresponding to atoms are the "definitely constant" successions of trees. Analogous characterizations can be obtained for free algebras with infinitely many generators and their related canonical frames: in fact, there is a standard description of them in terms of (co)limits and the inclusion morphisms needed $\mathscr{F}(\iota)$: $\mathscr{F}(G_1) \rightarrow \mathscr{F}(G_2)$ (for finite G_1 and G_2 with $G_1 \subseteq G_2$) are simply inverse images along label restrictions (see Section 4).

3. Other examples

One of the advantages of the categorical context of Section 1 lies in the fact that we have a general procedure that can be applied in various situations. We summarize here the results that can be easily obtained for examples I–IV of Section 1 (only the part concerning example IV will be used in the sequel).

Example I. This example deals with the Boolean algebras with endofunctions, which are T-objects for the endofunctor that associates with a Boolean algebra the Boolean algebra freely generated by its underlying set. Such an endofunctor preserves chain colimits, hence we can apply all the results from Section 1. Again, starting with a finite Boolean algebra $B = \mathscr{P}(L)$, formulas (1) and (2) describe a combinatorial construction on L, the construction of *neighborhood trees* N_i^L:

$$N_0^L = L, \qquad N_{i+1}^L = L \times \mathscr{P}(\mathscr{P}(N_i^L)).$$

Such trees can be visualized (up to a certain extent), what seems to be harder to understand graphically is the action of the transition maps v_i (playing the role of the leaf-cuts λ_i in the case of modal algebras). They are defined as follows:

$$v_0(\langle a, \mathscr{S}\rangle) = a, \qquad v_{i+1}(\langle a, \mathscr{S}\rangle) = \langle a, \{u \mid v_i^{-1}(u) \in \mathscr{S}\}\rangle.$$

Thus the double inverse image enters in the definition of $v_{i+1} = 1_L \times (v_i^{-1})^{-1}$. This has an unexpected consequence.

Proposition 3.1. *The Boolean algebra with an endofunction freely generated by a finite Boolean algebra $\mathscr{P}(L)$ ($L \neq \emptyset$) is atomless.*

Proof. Recall that a set-theoretic function $f: X \to Y$ is surjective (injective) iff inverse image along it is injective (surjective). Moreover, if f is injective and not bijective, then every subset of X has more than one preimage along f^{-1}. This shows that each neighborhood tree u of height i can be written as $v_i(t)$ for *many* t of height $i + 1$. Now the result follows, because the mentioned free Boolean algebra with endofunction is the colimit of the chain diagram given by the $\mathscr{P}(N_i^L)$ and the v_i^{-1}. $\square$

Normal forms can be deduced from formula (3) of Section 1; they are specified once we know how to associate a formula Φ^i with each neighborhood forest $f \subseteq N_i^{\mathscr{P}(G)}$:

$$\Phi^0(f) = \bigvee_{a \in f} \left(\bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \neg p \right),$$

$$\Phi^{i+1}(f) = \bigvee_{t \in f} \left(\bigwedge_{p \in t_R} p \wedge \bigwedge_{p \notin t_R} \neg p \wedge \bigwedge_{u \in t_S} \Box \Phi^i(u) \wedge \bigwedge_{u \notin t_S} \neg \Box \Phi^i(u) \right).$$

We may try to interpret all this as follows: normal forms having only one disjunct stand for approximate descriptions of a point in a model. In the case of the previous section this description was very simple: we specified the "true" root-formula of degree zero and we specified the set of formulas of lower degree "true in an accessible point". Here the strategy is different: we again specify the true root-formula, but then we use formulas of the lower degrees as formal neighborhoods and we specify which formal neighborhoods are neighborhoods of our imaginary point.

Example II. Here we have Boolean algebras with finitely many hemimorphisms $\Box_1, \dots, \Box_n$. To check the chain colimit preservation property it is sufficient to use two preliminary results of general nature: (a) if a n-ary functor $F\colon \mathbf{C}\times \cdots \times \mathbf{C} \to \mathbf{D}$ preserves chain colimits in each variable separately, then the diagonalized functor ΔF: $\mathbf{C} \to \mathbf{C}\times \cdots \times \mathbf{C} \to \mathbf{D}$ preserves chain colimits; (b) the "coproduct of n-copies" n-ary functor preserves chain colimits in each variable separately.

This time formulas (1) and (2) give us, as expected, *multiple trees*, i.e. trees which are labelled also in the edges by the set $\{\Box_1, \dots, \Box_n\}$:

$$M_0^L = L, \qquad M_{i+1}^L = L \times (\mathscr{P}(N_i^L))^n.$$

Transition maps are the multiple versions of leaf-cuts:

$$\mu_0(\langle a, S_1, \dots, S_n\rangle) = a,$$

$$\mu_{i+1}(\langle a, S_1, \dots, S_n\rangle) = \langle a, \exists_{\mu_i}(S_1), \dots, \exists_{\mu_i}(S_n)\rangle.$$

Normal forms are the obvious ones:

$$\Phi^0(f) = \bigvee_{a\in f}\Big(\bigwedge_{p\in a} p \wedge \bigwedge_{p\notin a}\neg p\Big),$$

$$\Phi^{i+1}(f) = \bigvee_{t\in f}\Big(\bigwedge_{p\in t_R} p \wedge \bigwedge_{p\notin t_R}\neg p,$$

$$\wedge \bigwedge_{k=1}^{n}\Big(\bigwedge_{u\in t_{S_k}} \Diamond_k\Phi^i(\{u\}) \wedge \bigwedge_{u\notin t_{S_k}}\neg\Diamond_k\Phi^i(\{u\})\Big)\Big).$$

Example III. Here we have Boolean algebras with binormal binary modal operators. To check chain colimits preservation property, one simply has to show that the tensor product bifunctor preserves chain colimits in each variable separately. But this is obvious because for each semilattice R, the functor $R\otimes(-)$ has a right adjoint, the internal hom functor: otherwise said, it is possible to define the semilattice of linear maps (as in linear algebra) and to show the existence of a natural bijection between linear maps $R\otimes S \to T$ and $S \to T^R$. Another useful information is that the free semilattice $\mathscr{S}(X)$ on a finite set X is $\langle \mathscr{P}(X), \supseteq\rangle$ and consequently that $\mathscr{S}(X)\otimes\mathscr{S}(Y)$ is $\mathscr{S}(X\times Y) = \langle \mathscr{P}(X\times Y), \supseteq\rangle$. Formulas (1) and (2) now give *relational trees*:

$$R_0^L = L, \qquad R_{i+1}^L = L \times \mathscr{P}(R_i^L \times R_i^L),$$

Relational leaf-cuts are so defined:

$$\rho_0(\langle a, S\rangle) = a, \qquad \rho_{i+1}(\langle a, S\rangle) = \langle a, \exists_{\rho_i\times\rho_i}(S)\rangle.$$

Normal forms are the following ones:

$$\Phi^0(f) = \bigvee_{a \in f} \Big(\bigwedge_{p \in a} p \wedge \bigwedge_{p \notin a} \neg p \Big),$$

$$\Phi^{i+1}(f) = \bigvee_{t \in f} \Big(\bigwedge_{p \in t_R} p \wedge \bigwedge_{p \notin t_R} \neg p \wedge \bigwedge_{\langle u_1, u_2 \rangle \in t_S} \langle \Phi^i(\{u_1\}), \Phi^i(\{u_2\}) \rangle$$

$$\wedge \bigwedge_{\langle u_1, u_2 \rangle \notin t_S} \neg \langle \Phi^i(\{u_1\}), \Phi^i(\{u_2\}) \rangle \Big),$$

where $\langle x, y \rangle = \neg [\neg x, \neg y]$.

Example IV. Here we take into consideration Kripke frames and open morphisms, which are co-objects for the covariant power set functor. The general theory is now dualized, hence we should check the preservation of chain limits. The condition is false and indeed the co-free Kripke frame generated by a set does not exist. In fact, an argument due to Lambek shows that an initial object in a T-object category must be a fixed point for T (i.e. if $\langle X, x \rangle$ is initial in **TObj**, then x is an isomorphism). The same holds for co-T-objects with terminals. The covariant power set functor, however, cannot have fixed points (for Cantor's theorem on cardinality) and as **Set** does have a terminal object (which should be preserved by an hypothetic right adjoint), the outcome is that the right adjoint we are interested in, does not exist.

However, the general theory of Section 1 is not useless: we cannot apply Proposition 1.3 but Propositions 1.1 and 1.2 are still valid. We get a new semantic for modal logic (a graded version of Kripke semantics) which has some independent interest because all logics are complete with respect to it. What is a graded co-T-object? It is a *graded Kripke frame* $W = \langle \{W_i\}_i, \{\lambda_i\}_i, \{R_i\}_i \rangle$, that is a succession of sets W_0, W_1, $W_2, \ldots$ endowed with functions $\{\lambda_i\colon W_{i+1} \to W_i\}_i$ and with relations $\{R_i\colon W_{i+1} \to W_i\}_i$ such that the following squares of relations commute:[14]

$$\begin{array}{ccc} W_{i+2} & \xrightarrow{\lambda_{i+1}} & W_{i+1} \\ {\scriptstyle R_{i+1}}\downarrow & & \downarrow{\scriptstyle R_i} \\ W_{i+1} & \xrightarrow[\lambda_i]{} & W_i \end{array}$$

Equivalently, this means that the following two conditions are satisfied (for every $u \in W_i$, $v \in W_{i+1}$, $w \in W_{i+2}$):

$(gr\,1)$ $\quad wR_{i+1}v \;\Rightarrow\; \lambda_{i+1}(w) R_i \lambda_i(v)$,

$(gr\,2)$ $\quad \lambda_{i+1}(w) R_i u \;\Rightarrow\; \exists z \in W_{i+1}(wR_{i+1}z \,\&\, \lambda_i(z) = u)$.

[14] This definition is actually the general one of Section 1, because of the existence of a bijection between relations on a set X and functions $X \to \mathscr{P}(X)$.

With any graded Kripke frame W one can easily associate a graded modal algebra $\mathscr{P}_{gr}(W)$: for every $i \in N$, $\mathscr{P}_{gr}(W)_i$ is the power set Boolean algebra $\mathscr{P}(W_i)$, ε_i is the inverse image morphism λ_i^{-1}: $\mathscr{P}(W_i) \to \mathscr{P}(W_{i+1})$, finally the hemimorphisms $\Box_i$: $\mathscr{P}(W_i) \to \mathscr{P}(W_{i+1})$ are defined by (let $S \subseteq W_i$)

$$\Box_i(S) = \{w \in W_{i+1} \mid \forall v \in W_i(wR_iv \ \Rightarrow\ v \in S)\}.$$

The graded modal algebra condition (Gr) for $\mathscr{P}_{gr}(W)$ is easily seen to be equivalent to the conjunctions of the two graded Kripke frame conditions $(gr\,1)$ and $(gr\,2)$ for W.

Formulas (1) and (2) of Section 1, once dualized and applied to our case, simply describe, given any set L, the L-labelled commutative idempotent trees and the leaf-cuts. Thus Proposition 1.1 simply says that $T^L = \langle\{T_i^L\}_i, \{\lambda_i\}_i, \{R_i\}_i\rangle$ (where the R_i's are the immediate subtree relations) is a graded Kripke frame.

The notion of a *graded morphism* g: $W \to V$ between two graded Kripke frames $W = \langle\{W_i\}_i, \{\lambda_i\}_i, \{R_i\}_i\rangle$ and $V = \langle\{V_i\}_i, \{\lambda_i\}_i, \{R_i\}_i\rangle$ also come directly from the general theory of Section 1: it is a collection of functions $\{g_i$: $W_i \to V_i\}$ such that the following squares commute:

$$\begin{array}{ccc} W_{i+1} & \xrightarrow{g_{i+1}} & V_{i+1} \\ {\scriptstyle\lambda_i}\downarrow & & \downarrow{\scriptstyle\lambda_i} \\ W_i & \xrightarrow[g_i]{} & V_i \end{array} \qquad\qquad \begin{array}{ccc} W_{i+1} & \xrightarrow{g_{i+1}} & V_{i+1} \\ {\scriptstyle R_i}\downarrow & & \downarrow{\scriptstyle R_i} \\ W_i & \xrightarrow[g_i]{} & V_i \end{array}$$

Proposition 1.2 can be restated as follows.

Proposition 3.2. *Let L be a set, W a graded Kripke frame and g_0: $W_0 \to L$ a set-theoretic function; then there exist a unique graded morphism g: $W \to T^L$, whose 0-th component is precisely g_0.*

The definition of the g_i $(i > 1)$ is given by the (dualized) formula (3) of Section 1 and is the following one:

$$g_{i+1}(w) = \langle g_0(\lambda_{i+1,0}(w)), \exists_{g_i}(R_i(w))\rangle, \tag{13}$$

where $R_i(w)) =_{df} \hat{R}_i(w) = \{v \mid wR_iv\}$ and (as usual) $\lambda_{i+1,0}(w)$ means $\lambda_0\lambda_1, \ldots \lambda_i(w)$.

4. Segment-by-label replacements

In this section we characterize the uniform substitution, that is, we deal with the following question: what happens if we rewrite a normal form as a normal form after applying to it a substitution operation? The answer will be in terms of a new basic operation on trees. We perform our analysis only in the case of modal system K, although we use a general method.

Given a graded Kripke frame $W = (\{W_i\}_i, \{\lambda_i\}_i, \{R_i\}_i)$, let us indicate with $W_{\uparrow k}$ (for any given natural number k) the graded Kripke frame $\langle\{W_{k+i}\}_i, \{\lambda_{k+i}\}_i, \{R_{k+i}\}_i\rangle$. Similarly, if $h = \{h_i: W_i \to V_i\}_i$ is a graded Kripke frame morphism, by $h_{\uparrow k}$: $W_{\uparrow k} \to V_{\uparrow k}$, we indicate the graded Kripke frame morphism whose i-th components is h_{k+i}.

Given two sets of labels L, M, a *segmented-by-label replacement* of degree k is a morphism

$$s: T^L_{\uparrow k} \to T^M$$

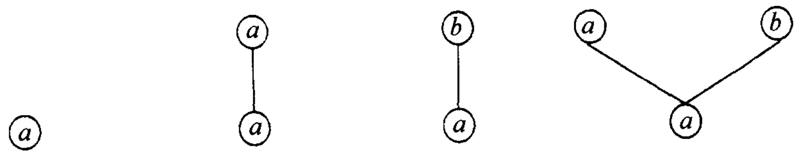
in the category **GrK** of graded Kripke frames. From Proposition 3.2. of Section 3, we know that there is a bijection

$$\mathbf{Set}[T^L_k, M] \simeq \mathbf{GrK}[T^L_{\uparrow k}, T^M]$$

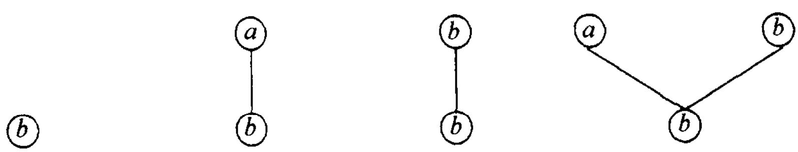
associating with s_0: $T^L_k \to M$, the segment-by-label replacement s, whose ith component s_i: $T^L_{k+i} \to T^M_i$, according to formula (13) in Section 3, is given, for $t \in T^L_{k+i}$ $(i \geq 1)$, by

$$s_i(t) = \langle s_0(\lambda_{k+i,k}(t)), \exists_{s_{i-1}}(t_S)\rangle.$$

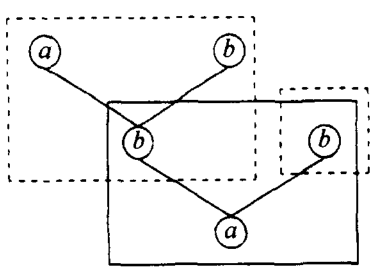
The geometric meaning of the segment-by-label replacement determined by s_0 is explained in the following example. Let L contain two labels, say a and b, and let M contain three labels, say c, d and e. Suppose $k = 1$ and that s_0 associates c with the following four trees:



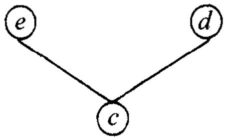
and d, e, respectively, with the first and the last two trees in the remaining list.



Then s_1 associates with the following tree:



the tree



Notice that, once again, leaf-cuts eliminate only the leaves of maximum height and that hidden simplifications are possible at each step because of idempotency of trees. We already know an example of segment-by-label replacement: for every set of labels L and for every $k \in N$, $\lambda^k = \{\lambda_{k+i,i}\}_i$ is a graded Kripke frame morphism $T^L_{\uparrow k} \to T^L$, hence it is a segment-by-label replacement. Another example is the re-labelling operation: given any map h_0: $L \to M$, the induced segment-by-label replacement h: $T^L \to T^M$ is the operation of replacing in L-trees each label a by $h_0(a)$ thus getting an M-tree.

We study a little more carefully the category of finitely generated free modal algebras; let us use $\underline{m} = \{p_1, \ldots, p_m\}$ as canonical finite sets of free generators. The free modal algebras on them $\mathcal{F}(\underline{m})$ are described in Theorem 2.4; a morphism $\mathcal{F}(\underline{m}) \to \mathcal{F}(\underline{n})$ is uniquely determined by the choice of the images of the free generators, hence we write it as $\langle [f_1, k], \ldots, [f_m, k]\rangle^{\mathcal{F}}$ meaning that the m-tuple of equivalence classes of forests $\langle [f_1, k], \ldots, [f_m, k]\rangle$ with labels in $\mathcal{P}(\underline{n})$ are the images of the free generators. On the other hand, we recall that the underlying set of $\mathcal{F}(\underline{n})$ is the chain colimit in the category of sets of the chain diagram $\{\lambda_i^{-1}$: $\mathcal{P}(T_i^{\mathcal{P}(\underline{n})}) \to \mathcal{P}(T_{i+1}^{\mathcal{P}(\underline{n})})\}_i$. Forgetting the functions λ_i^{-1} in the notation for simplicity, we indicate the underlying set of $\mathcal{F}(\underline{n})$ as $\varinjlim_i \mathcal{P}(T_i^{\mathcal{P}(\underline{n})})$. We have the following bijections:

$$\begin{aligned}
\mathbf{Ma}\,[\mathcal{F}(\underline{m}), \mathcal{F}(\underline{n}) &\simeq \mathbf{Set}\,[\underline{m}, \varinjlim_i \mathcal{P}(T_i^{\mathcal{P}(\underline{n})})] \simeq (\varinjlim_i \mathcal{P}(T_i^{\mathcal{P}(\underline{n})}))^m \\
&\simeq \varinjlim_i (\mathcal{P}(T_i^{\mathcal{P}(\underline{n})})^m) \qquad \simeq \varinjlim_i \mathbf{Set}\,[\underline{m}, \mathcal{P}(T_i^{\mathcal{P}(\underline{n})})] \\
&\simeq \varinjlim_i \mathbf{Set}\,[T_i^{\mathcal{P}(\underline{n})}, \mathcal{P}(\underline{m})] \simeq \varinjlim_i \mathbf{GrK}\,[T_{\uparrow i}^{\mathcal{P}(\underline{n})}, T^{\mathcal{P}(\underline{m})}],
\end{aligned}$$

where we use the fact that finite limits and filtered colimits commute in the category of sets, the fact that the power set functor is self-adjoint and Proposition 3.2.[15]

[15] We give further explanations. As to the first fact, we used a very particular case (to be easily checked directly) of a general important property (see [17]): given a chain diagram in **Set**, say $\{h_i$: $X_i \to X_{i+1}\}$, and given a natural number, say m, the two sets $(\varinjlim_i X_i)^m$ and $\varinjlim_i (X_i^m)$ are in bijective correspondence.

As to the second, notice that the functor $\mathcal{P}$: $\mathbf{Set}^{op} \to \mathbf{Set}$ associating with a set its power set and with a function its inverse image functions, is self-adjoint because given two sets X and Y, there is a natural bijection $(-)^{\text{t}}$: $\mathbf{Set}[X, \mathcal{P}(Y)] \to \mathbf{Set}[Y, \mathcal{P}(X)]$, defined as follows, for $h: X \to \mathcal{P}(Y)$ and $y \in Y$: $h^{\text{t}}(y) = \{x \in X \mid y \in h(x)\}$.

Computing carefully step by step the above bijections, one realializes that we established the following fact. Fix n and m and consider the set of pairs $\langle s, i\rangle$, where i is a natural number and $s \in \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}(\underline{m})}]$ is a segment-by-label replacement of degree i and introduce in the set of such pairs the following equivalence relation:

$$\langle s, i\rangle \approx \langle r, j\rangle \iff \exists k \geqslant i, j, \qquad (\lambda^{k-1})_{\uparrow i}\, s = (\lambda^{k-j})_{\uparrow j}\, r.$$

The quotient set is $\varinjlim_i \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}(\underline{m})}]$ and we established that the (well-defined) function

$$F([s, i]) = \langle [\{t \mid p_1 \in s_0(t)\}, i], \ldots, [\{t \mid p_m \in s_0(t)\}, i]\rangle^{\mathscr{F}} \tag{14}$$

is a bijection onto $\mathbf{Ma}\,[\mathscr{F}(\underline{m}), \mathscr{F}(\underline{n})]$.

There is another interesting description of the bijection F. Give a segment-by-label replacement $s \in \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}(\underline{m})}]$ of degree i, it obviously induces a graded modal algebras morphism[16]

$$s^{-1}\colon \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{m})}) \to \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})}_{\uparrow i})$$

and hence, by Propoistion 1.3, a modal algebras morphism[17]

$$\varinjlim s^{-1}\colon \mathscr{F}(\underline{m}) \to \varinjlim \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})}_{\uparrow i}).$$

Notice that the naturality is important in the case we are interested in: if we consider the chain diagrams

$$\cdots \to \mathbf{Set}\,[\underline{m}, \mathscr{P}(T^{\mathscr{P}(\underline{n})}_{i})] \xrightarrow{(-)\lambda_i^{-1}} \mathbf{Set}\,[\underline{m}, \mathscr{P}(T^{\mathscr{P}(\underline{n})}_{i+1})] \to \cdots$$

and

$$\cdots \to \mathbf{Set}\,[T^{\mathscr{P}(\underline{n})}_{i}, \mathscr{P}(\underline{m})] \xrightarrow{\lambda_i(-)} \mathbf{Set}\,[T^{\mathscr{P}(\underline{n})}_{i+1}, \mathscr{P}(\underline{m})] \to \cdots$$

then the above bijections $(-)^t$ form "vertical" isomorphisms between then making all the related squares to commute. This fact easily implies that the two chain colimit are isomorphic. A similar considerations applies to the last passage in the text: this time the natural bijection $\mathbf{Set}\,[T^{\mathscr{P}(\underline{n})}_{i}, \mathscr{P}(\underline{m})] \simeq \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}(\underline{m})}]$ comes from the adjointness of Proposition 3.2. The transposition simply consists in expanding a function $T^{\mathscr{P}(\underline{n})}_{i} \to \mathscr{P}(\underline{m})$ to a graded morphism $T^{\mathscr{P}(\underline{n})}_{\uparrow i} \to T^{\mathscr{P}(\underline{m})}$. In this way we again get vertical natural bijections showing the further isomorphism with the colimit of the chain diagram:

$$\cdots \to \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}}(\underline{m})] \xrightarrow{\lambda^1_{\uparrow i}(-)} \mathbf{GrK}\,[T^{\mathscr{P}(\underline{n})}_{\uparrow i+1}, T^{\mathscr{P}(\underline{m})}] \to \cdots$$

(recall from the above definitions that $(\lambda^1)_{\uparrow i}$ is $\{\lambda_k\}_{k \geqslant i}$, hence the value at it of the left adjoint 0th component functor $\mathbf{GrK} \to \mathbf{Set}$ is λ_i).

[16] This is general: any graded Kripke frame morphism $s = \{s_i\}_i\colon W \to V$ gives rise to a graded modal algebras morphism $s^{-1} = \{s_i^{-1}\}_i\colon \mathscr{P}_{gr}(V) \to \mathscr{P}_{gr}(W)$.

[17] Given a graded modal algebra $\mathscr{B}$, we write here $\varinjlim \mathscr{B}$ instead of $\langle \varinjlim \mathscr{B}, \Box\rangle$. In this way, we can call directly $\varinjlim$ the left adjoint to the constant functor $\overrightarrow{Con}$: $\mathbf{Ma} \to \mathbf{Gma}$.

Now it is easily seen that

$$\varinjlim(\lambda^i)^{-1}\colon \mathscr{F}(\underline{n}) = \varinjlim \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})}) \to \varinjlim \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})}_{\uparrow i})$$

is an isomorphism. So composing with its inverse[18] $(\varinjlim(\lambda^i)^{-1})^{-1}$ we finally get a modal morphism

$$(\varinjlim s^{-1})(\varinjlim(\lambda^i)^{-1})^{-1}\colon \mathscr{F}(\underline{m}) \to \mathscr{F}(\underline{n}).$$

Computing what we obtained, we have that

$$F([s, i]) = (\varinjlim s^{-1})(\varinjlim(\lambda^i)^{-1})^{-1} \tag{15}$$

(to prove this this it is sufficient to show the coincidence of the values at the free generators, i.e. that for every $k = 1, \ldots, m$, $(\varinjlim(\lambda^i)^{-1})^{-1}(\varinjlim s^{-1}([\{a \in T_0^{\mathscr{P}(\underline{m})} \mid p_k \in a\}, 0])) = [\{t \mid p_k \in s_0(t)\}, i]$).

Suppose now we are given two segment-by-label replacements

$$s\colon T^{\mathscr{P}(\underline{p})}_{\uparrow i} \to T^{\mathscr{P}(\underline{n})} \quad \text{and} \quad r\ T^{\mathscr{P}(\underline{n})}_{\uparrow j} \to T^{\mathscr{P}(\underline{m})}$$

of degrees i and j and let us try to compute $F([r, j])F([s, i])$. According to (15) we get that

$$F([r, j])F([s, i]) = (\varinjlim r^{-1})(\varinjlim(\lambda^j)^{-1})^{-1}(\varinjlim s^{-1})(\varinjlim(\lambda^i)^{-1})^{-1}.$$

Now observe that from the commutativity of the square

$$\begin{array}{ccc} T^{\mathscr{P}(\underline{p})}_{\uparrow(j+i)} & \xrightarrow{(\lambda^j)_{\uparrow i}} & T^{\mathscr{P}(\underline{p})}_{\uparrow i} \\ {\scriptstyle s_{\uparrow j}}\downarrow & & \downarrow{\scriptstyle s} \\ T^{\mathscr{P}(\underline{n})}_{\uparrow j} & \xrightarrow[\lambda^j]{} & T^{\mathscr{P}(\underline{n})} \end{array}$$

i.e. from the equation

$$s_{\uparrow j}\lambda^j = (\lambda^j)_{\uparrow i}s$$

(holding ultimately because s is a graded Kripke frame morphism), we may pass to

$$(\varinjlim(\lambda^j)^{-1})(\varinjlim s_{\uparrow j}^{-1}) = (\varinjlim s^{-1})(\varinjlim(\lambda^j)_{\uparrow i}^{-1}),$$

i.e. after using the inverses of the two isomorphisms involved, to

$$(\varinjlim s_{\uparrow j}^{-1})(\varinjlim(\lambda^j)_{\uparrow j}^{-1})^{-1} = (\varinjlim(\lambda^j)^{-1})^{-1}(\varinjlim s^{-1}).$$

---

[18] Notice that we use the same symbol $(-)^{-1}$ to denote the inverse of a bijective map and the inverse image function on power sets. The effect of the isomorphism $(\varinjlim(\lambda^i)^{-1})^{-1}$ is simply that of mapping the equivalence class of, say f, in $\varinjlim \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})}_{\uparrow i})$ onto the equivalence class of f in $\varinjlim \mathscr{P}_{gr}(T^{\mathscr{P}(\underline{n})})$.

Consequently,

$$\begin{aligned} F([r,j])F([s,i]) &= (\varinjlim r^{-1})(\varinjlim s_{\uparrow j}^{-1})(\varinjlim (\lambda^j)_{\uparrow i}^{-1})^{-1}(\varinjlim (\lambda^i)^{-1})^{-1} \\ &= (\varinjlim s_{\uparrow j} r)^{-1})(\varinjlim (\lambda^j_{\uparrow i}\lambda^i)^{-1})^{-1} \\ &= (\varinjlim s_{\uparrow j} r)^{-1}(\varinjlim (\lambda^{i+j})^{-1})^{-1} \\ &= F([s_{\uparrow j} r, i+j]). \end{aligned}$$

Our results allow us to introduce a category **Em** such that the bijections found in (15) extend to an equivalence of categories between $\mathbf{Em}^{op}$ and the category of finitely generated free modal algebras. Objects of **Em** are the sets $\underline{n}$, $\underline{m}$, ... (you can identify them with natural numbers, if you like) and the sets of arrows $\mathbf{Em}[\underline{n}, \underline{m}]$ are the sets $\varinjlim_i \mathbf{GrK}[T^{\mathscr{P}(\underline{n})}_{\uparrow i}, T^{\mathscr{P}(\underline{m})}]$. Composition of $[s,i] \in \mathbf{Em}[\underline{p}, \underline{n}]$ and $[r,j] \in \mathbf{Em}[\underline{n}, \underline{m}]$ is defined as $[s_{\uparrow j} r, i+j]$ (we do not need to check that it is well-defined or associative or unitary, it follows from the facts that $F([r,j])F([s,i]) = F([s_{\uparrow j} r, i+j])$ and that F is bijective). We already know that F is a full and faithful (i.e bijective on arrows) functor; the fact that it is essentially surjective is trivial (every finitely generated free modal algebra is isomorphic to $\mathscr{F}(\underline{n})$ for some $\underline{n}$), hence it is an equivalence of categories (see [17]).

Theorem 4.1. *The opposite category of* **Em** *is equivalent to the category of finitely generated free modal algebras.*

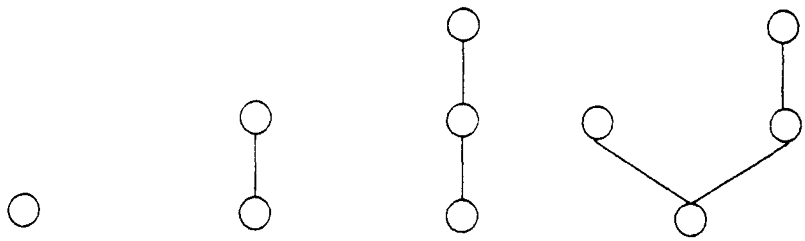
In the context of functorial semantics [15], Theorem 4.1 says that **Em** *is the equational theory of modal algebras in invariant sense.* Theorem 4.1 describes the combinatorial mechanism of substitution. Suppose we are given a modal formula $\alpha(p_1, \dots, p_n)$ of modal degree $\leqslant k$ and modal formulas $\beta_1(p_1, \dots, p_m), \dots, \beta_n(p_1, \dots, p_m)$ of modal degrees $\leqslant j$. We want to compute the forest $\varphi^{k+j}(\alpha(\beta_1/p_1, \dots, \beta_n/p_n))$, in terms of the forests $f = \varphi^k(\alpha)$ and $g_1 = \varphi^j(\beta_1), \dots, g_n = \varphi^j(\beta_n)$. As we know, the equivalence class of the forest we are looking for is equal to $\langle [g_1,j], \dots, [g_n, j] \rangle^{\mathscr{F}}([f,k])$. We now can find a segment-by-label replacement $s_{\beta_1,\dots,\beta_n}\colon T^{\mathscr{P}(\underline{m})}_{\uparrow j} \to T^{\mathscr{P}(\underline{n})}$ of degree j, such that $F([s_{\beta_1,\dots,\beta_n}, j])$ is equal to $\langle [g_1,j], \dots, [g_n,j] \rangle^{\mathscr{F}}$ (such pair $\langle s_{\beta_1,\dots,\beta_n}, j \rangle$ is unique, up to equivalence of segment-by-label replacements, because F is bijective). In fact, $s_{\beta_1,\dots,\beta_n}$ is easily determined, according to (14), as follows (recall that it is sufficient to determine its 0th component):

$$(*) \quad (s_{\beta_1,\dots,\beta_n})_0(t) = \{p_i \in \underline{n} \mid t \in g_i\},$$

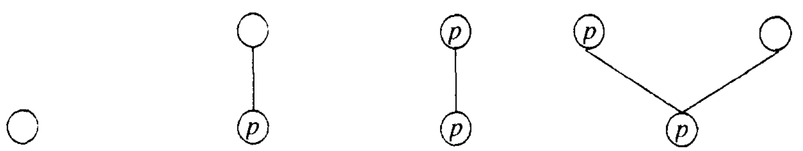
We now use the other formula (15) for $F([s_{\beta_1,\dots,\beta_n}, i])$ and obtain that the forest $\varphi^{k+j}(\alpha(\beta_1/p_1, \dots, \beta_n/p_n))$ is *the inverse image of* $\varphi^k(\alpha)$ *along the kth component of the segment-by-label replacement* $s_{\beta_1,\dots,\beta_n}$.

We give an example. Suppose we substitute $\Diamond\top$ for p in $\neg\Box p \vee p$ (notice that the result is provably equivalent to $\Diamond\top$). As $\Diamond\top$ corresponds to the singleton-forest $\{\langle\emptyset, \{\emptyset\}\rangle\}$, our $(s_{\Diamond\top})_0$ associates with the two trees of level 1

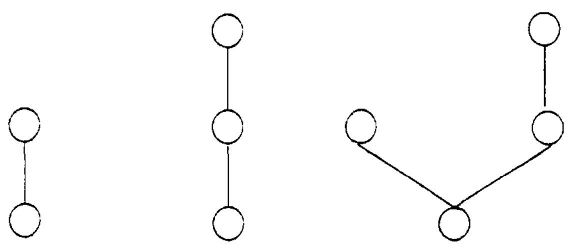
the labels $\{p\}$ and $\emptyset$, respectively. Hence $(s_{\Diamond\top})_1$ associates with the four trees of level 2



the trees



respectively. Taking the inverse image of $\varphi^1(\neg\Box p \vee p)$ (see Section 2 for the corresponding picture), we get the forest



which is in fact the forest $\varphi^2(\Diamond\top\rangle = \varphi^2((\neg\Box p \vee p)(\Diamond\top/p))$.

The following further example will be useful in next sections. Suppose that $n \leqslant m$, then the free algebra functor applied to the inclusion $\underline{n} \subseteq \underline{m}$ give rise to a "cilindrification" morphism from $\mathscr{F}(\underline{n})$ into $\mathscr{F}(\underline{m})$ (it is the morphism corresponding to the trivial substitution of p_i for p_i, for every $i = 1, \dots, n$). This morphism acts as the inverse image along the components of the "restriction to $\underline{n}$" segment-by-label replacement. This segment-by-label replacement intersects all the labels with $\underline{n}$ (that is, deletes $p_{n+1}, \dots, p_m$ everywhere). Formally, it is defined as follows:

$$a_{|\underline{n}} = a \cap \underline{n}, \qquad t_{|\underline{n}} = \langle t_R \cap \underline{n}, \{u_{|\underline{n}} \mid u \in t_S\}\rangle.$$

For a forest f (of any degree k) we indicate with $f^{|\underline{m}}$ the *extension of f to $\underline{m}$*, i.e. the forest $\{t \in T_k^{\wp(\underline{m})} | t_{|\underline{n}} \in f\}$. We so have that the cilindrification morphism maps $[f, k]$ onto $[f^{|\underline{m}}, k]$.

5. The lattice of modal logics

A *modal logic* is a variety of modal algebras or, equivalently, a set of formulas in the propositional modal language (with countably many propositional letters) which is closed under modus ponens, necessitation and uniform substitution. Set-theoretic inclusion endows the set of modal logics with a *complete lattice* structure.

Clearly, each modal logic has its own normal forms: these are obtained from the normal forms of K by "deleting trees that become inconsistent". Otherwise said, we can associate with a logic L a *collection of forests* (where, by a collection of forests, we mean an element $f \in \prod_{n,k} T_k^{\wp(\underline{n})}$): for all n, $k \geqslant 0$, f_k^n is the forest corresponding to the smallest (with respect to provability in K) formula of degree k in the letters $p_1, \dots, p_n$ that is provable in the logic L. L itself is completely determined by its associated collection of forests. The natural question is: which collections of forests are collections coming from a logic? The answer is not difficult, using the results of the previous section on uniform substitution.

Theorem 5.1. *Take the collections of forests f satisfying conditions* (I)–(III) *below and endow them with a partial ordering by using reverse componentwise inclusion (i.e. put $f \leqslant g$ iff for all $n, k \in N$, $f_k^n \supseteq g_k^n$); then you get a lattice $\langle L, \leqslant \rangle$ which is isomorphic to the lattice of modal logics.*

(I) *For every $n, k \in N$, $\exists_{\lambda_k}(f_{k+1}^n) \supseteq f_k^n$;*

(II) *for every $n, k \in N$, $\rho_k^-(f_{k+1}^n) \subseteq f_k^n$, where ρ_k^{-1} is the root-cut operation, i.e. $\rho_k^-(f_{k+1}^n) = \{t \in T_k^{\wp(\underline{n})} | \exists u \in f_{k+1}^n (t \in u_S)\}$;*

(III) *for every m, n, k, $i \in N$ and for every segment-by-label replacement s: $T_{\uparrow i}^{\wp(\underline{m})} \to T^{\wp(\underline{n})}$, $\exists_{s_k}(f_{k+i}^m) \subseteq f_k^n$.*

Proof. We make some preliminary remarks. As ρ_k^- is left adjoint to the root-addition operation ρ_k^+, condition (II) may be equivalently stated by asking that $f_{k+1}^n \subseteq \rho_k^+(f_k^n)$; the same observation applies to condition (III), where inverse image may be used instead of direct image. Notice also that, as leaf-cuts are segment-by-label replacements, we may use (I) and (III) in order to get $\exists_{\lambda_k}(f_{k+1}^n) = f_k^n$ for every $n, k \in N$. The situation is similar for label-restrictions: if $n \leqslant m$, then for every k, $f_k^n = (f_k^m)_{|\underline{n}}$, where $(-)_{|\underline{n}}$ is the direct image along the label-restriction from $\underline{m}$ to $\underline{n}$. This is due to the fact that the segment-by-label-replacement given by label-restriction from $\underline{m}$ to $\underline{n}$

has a section (the inclusion $\iota = \{\iota_k\}_k$)[19] which is also a segment-by-label replacement. $f_k^n = (f_k^m)_{|\underline{n}}$ is shown as follows: one half is an instance of (III), moreover $\exists_{\iota_k}(f_k^n) \subseteq f_k^m$ is also an instance of (III) and if we apply to it direct image along label-restriction we get exactly $f_k^n \subseteq (f_k^m)_{|\underline{n}}$.

We associate with a collection f satisfying conditions (I)–(III) the set of formulas (which will be proved to be a logic):

$$L_f = \{\alpha(p_1, \ldots, p_n) \mid d(\alpha) \leqslant k \ \& \ \varphi^k(\alpha) \supseteq f_k^n\}.$$

The above remarks show that, although a formula can be represented in different ways (there are infinitely many k larger than its modal degree and there are infinitely many n such that the propositional letters of α are included in $p_1, \ldots, p_n$), these different representations do not have any relevance concerning its membership to the set L_f just defined. For instance, for α of degree less or equal to k, $f_{k+1}^n \subseteq \varphi^{k+1}(\alpha)$ iff $f_{k+1}^n \subseteq \lambda_k^{-1}\varphi^k(\alpha)$ iff $\exists_{\lambda_k}(f_{k+1}^n) \subseteq \varphi^k(\alpha)$ iff $f_k^n \subseteq \varphi^k(\alpha)$ (because $\exists_{\lambda_k}(f_{k+1}^n) = f_k^n$). Thus, it is obvious that L_f is closed under modus ponens. Closure under necessitation and uniform substitution is due to conditions (II) and (III) (recall that necessitation is root-addition and uniform substitution is inverse image along segment-by-label replacements).

Conversely, given a logic L, we associate with it the collection $l = \{l_k^n\}$ given by

$$l_k^n = \bigcap \varphi^k(\alpha),$$

where the above intersection extends to all formulas $\alpha \in L$ having modal degree less or equal to k and containing at most $p_1, \ldots, p_n$ as propositional letters. Conditions (II) and (III) follow from closure of L under necessitation and uniform substitution. To prove condition (I), notice that (by adjointness between direct and inverse image) for every n, k, $l_{k+1}^n \subseteq \lambda_k^{-1}\exists_{\lambda_k}(l_{k+1}^n)$; however, $\Phi^{k+1}(l_{k+1}^n) \in L$ and L is closed under modus ponens, hence $\Phi^{k+1}(\lambda_k^{-1}\exists_{\lambda_k}(l_{k+1}^n)) \in L$, that is $\Phi^k(\exists_{\lambda_k}(l_{k+1}^n)) \in L$, which means $l_k^n \subseteq \exists_{\lambda_k}(l_{k+1}^n)$ by the definition of l_k^n.

The remaining verification that the two passages are an order (hence a lattice) isomorphism is straightforward.[20] □

Conditions (I)–(III) may be summarized, using a transparent terminology, by saying that a modal logic is a collection of forests which is up-invariant with respect to direct images along leaf-cuts, down-invariant with respect to root-cuts and down-invariant with respect to direct images along segment-by-label replacements. The above theorem might be used in order to produce purely combinatorial proofs of properties of the lattice of modal logics. We give here a very simple example: as it is

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[19] The terms of substitution, this section corresponds to the operation of replacing identically $p_1, \ldots, p_n$ and the exceeding propositional letters $p_{n+1}, \ldots, p_m$ by $\perp$.

[20] There is another slightly more conceptual way of proving Theorem 5.1: it consists in computing the quotient categories of **Em** that still are algebraic theoreies in the sense of Lawvere functorial semantics (see [15]).

well-known [18] this lattice has two maximal elements, corresponding to the collections of forests given by the trees t such that $t_S = \emptyset$[21] (the logic axiomatized by $\Box\bot$) and by the thin trees with identical label (the logic axiomatized by $p \leftrightarrow \Box p$). A proof from conditions (I)–(III) that these elements are the maximal ones is rather instructive and works as follows: given a collection f, either all trees in the forests f^n_{k+1} have only leaves of maximum height of there is some f^n_{k+1} containing a tree having a leaf of height less than $k+1$. In the former case condition (III) applied to the segment-by-label replacement that identifies all labels with a preassigned one, shows that all thin trees with equal label belong to the members of the collection of forests. In the second case, after applying leaf and roots-cuts (see conditions (II) and (III)) sufficiently many times, we realize that for some n, f^n_1 contains a tree that consists only of its root. By (III) all such trees belong to f^m_1 for all m and so, by (I), they belong to all the members of the collection. When compared with the traditional proof in term of generalized frames, the above proof, which is of syntactic nature despite the apparency, shows that only substitutions of modal degree zero (i.e. label renamings) are needed.

A natural question that arises at this point is the following: how to identify the collection of forests corresponding to some given logic? This operation is not easy and in general cannot be performed in an effective way, because the recursive identification of the normal forms of a logic means exactly its decidability (if we know the normal forms, computing the forest of a formula is a decision procedure and conversely, if the logic is decidable, one can effectively compute for every n, k the "minimum" provable formula of degree $\leqslant k$ with $\leqslant n$ propositional letters). However, we give here a procedure that works in many concrete cases and is not too far from reaching the goal. We need some preliminary investigation of standard algebraic nature.

Given a graded modal algebra $\mathscr{B} = \langle \{B_i\}_i, \{\varepsilon_i\}_i, \{\Box_i\rangle_i$, a *graded filter* on it is a succession $F = \{F_i\}_i$ of filters of the B_i with the additional property that if $x \in F_i$ then $\varepsilon_i(x)$ and $\Box_i x$ are both in F_{i+1}. These filters have the expected properties: if μ: $\mathscr{B} \to \mathscr{A}$ is a graded morphism, then $Ker(\mu) = \{Ker(\mu_i)\}_i$ is a graded filter (the kernel of a Boolean morphism is, as usual, the set of elements mapped onto $\top$). Moreover, if F is a graded filter of $\mathscr{B}$, then $\mathscr{B}/F$ is a graded modal algebra, where $(\mathscr{B}/F)_i$, is B_i/F_i and all the operations (included the ε_i and the $\Box_i$) are defined using the representative elements of the equivalence classes. The family of canonical projections q: $\mathscr{B} \to \mathscr{B}/F$ is a graded morphism. Moreover, the customary universal property of quotients holds: for every $\mathscr{A}$, μ: $\mathscr{B} \to \mathscr{A}$, if $F \subseteq Ker(\mu)$, then there exists a unique $\hat{\mu}$: $\mathscr{B}/F \to \mathscr{A}$ such that $q\hat{\mu} = \mu$. Notice also that for a finite graded Kripke frame $W = \langle \{W_i\}_i, \{\lambda_i\}_i, \{R_i\}_i \rangle$ (a graded Kripke frame W is said to be finite iff all the sets W_i are finite), a graded filter $F = \{F_i\}_i$ of its corresponding graded modal algebra $\mathscr{P}_{gr}(W)$ determines by duality, a *graded subframe* of W. This means a collection of subsets $\{S_i \subseteq W_i\}_i$ (in our case S_i is $\bigcap_{X \in F_i} X$) such that for $u \in S_{i+1}$, $\lambda_i(u) \in S_i$ and $R_i(u) \subseteq S_i$.

[21] All consistent logics correspond to collections of forests f such that $f^n_0 = T_0^{\mathscr{P}(n)}$. This is a consequence of the fact that there are no nontrivial varieties of Boolean algebras, but can also be easily established using (III).

Suppose now we are given a set of equations E defining a variety of modal algebras. These equations can be identified with formulas in the propositional modal language with countably many propositional letters. We want to give a meaning to the sentence $\mathscr{B} \models E$, where $\mathscr{B} = \langle \{B_i\}_i, \{\varepsilon_i\}_i, \{\Box_i\}_i \rangle$ is a graded modal algebra. An *evaluation* $v = \langle N, e \rangle$ is a pair of functions that associate with a formula α (in the above-mentioned modal propositional language with countably many propositional letters) a natural number $N(\alpha)$ and an element $e(\alpha)$ of $B_{N(\alpha)}$ in such a way that the following conditions are satisfied:

$$
\begin{aligned}
e(\top) &= \top_{B_{N(\top)}},\\
e(\bot) &= \bot_{B_{N(\top)}},\\
N(\neg\alpha) &= N(\alpha),\\
e(\neg\alpha) &= \neg e(\alpha),\\
N(\alpha \vee \beta) &= \max(N(\alpha), N(\beta)),\\
e(\alpha \vee \beta) &= \varepsilon_{N(\alpha), N(\alpha \vee \beta)}(e(\alpha)) \vee \varepsilon_{N(\beta), N(\alpha \vee \beta)}(e(\beta)),\\
N(\alpha \wedge \beta) &= \max(N(\alpha), N(\beta)),\\
e(\alpha \wedge \beta) &= \varepsilon_{N(\alpha), N(\alpha \wedge \beta)}(e(\alpha)) \wedge \varepsilon_{N(\beta), N(\alpha \wedge \beta)}(e(\beta)),\\
N(\Box\alpha) &= N(\alpha) + 1,\\
e(\Box\alpha) &= \Box_{N(\alpha)} e(\alpha).
\end{aligned}
$$

Notice that an evaluation is entirely determined by its value on atomic formulas. Moreover, if $v = \langle N, e \rangle$ is an evaluation, so is $v' = \langle N', e' \rangle$, where for some fixed k, $N'(\alpha) = N(\alpha) + k$ and $e'(\alpha) = \varepsilon_{N(\alpha), N'(\alpha)} e(\alpha)$ (this fact will be used in the sequel without explicit mention).

Now $\mathscr{B} \models E$ is defined as follows: for all $\alpha \in E$, for all evaluations $v = \langle N, e \rangle$, $e(\alpha) = \top$ (here $\top$ is obviously the unit of $B_{N(\alpha)}$). Notice that if $\mathscr{B} \models E$ then $\varinjlim \mathscr{B}$ belongs to the variety of modal algebras defined by E. Moreover, if a modal algebra $\langle B, \Box \rangle$ is in such variety, then the constant graded modal algebra $Con(\langle B, \Box \rangle)$ satisfies the equations E in the sense of the above definition.

Given any graded modal algebra $\mathscr{B}$ and any set of equations E, the following F^E is a graded filter of $\mathscr{B}$:

$$
\begin{aligned}
F_i^E = \{x \in B_i \mid \exists n \geqslant 0,\ \exists \alpha_1, \ldots, \alpha_n \in E,\ \exists v_1, \ldots, v_n\\
(N_1(\alpha_1), \ldots, N_n(\alpha_n) \leqslant i \,\&\, \Box_{N_1(\alpha_1), i}(e_1(\alpha_1)) \wedge \cdots \wedge \Box_{N_n(\alpha_n), i}(e_n(\alpha_n)) \leqslant x)\}
\end{aligned}
$$

(where, e.g. $\Box_{N_n(\alpha_1), i}(e_1(\alpha_{n+1})) = \Box_{i-1} \cdots \Box_{N_1(\alpha_1)+1} \Box_{N_1(\alpha_1)} e_1(\alpha_1)$). It turns out easily that $\mathscr{B}/F^E \models E$ and that, because of the universl property of quotients, this construction is universal in the sense that graded morphisms from $\mathscr{B}$ into a graded modal algebra satisfying equations E are in bijective correspondence (through composition with the canonical projection) with graded morphisms with domain $\mathscr{B}/F^E$ and the same codomain. Putting together the various results reached so far we can conclude that

Theorem 5.2. *The free modal algebra satisfying the equations E generated by the finite Boolean algebra* $\mathscr{P}(L)$ *is* $\varinjlim(\mathscr{P}(T^L)/F^E)$.

Proof. This is easily seen as follows: let us indicate with $\mathbf{Ma}^E$ and $\mathbf{Gma}^E$ the full subcategories of **Ma** and **Gma** satisfying the equations E (in the former case satisfaction is intended in its standard meaning, in the latter in the meaning just explained). We are interested in computing the left adjoint to the following composition of functors:

$$\mathbf{Ma}^E \xrightarrow{\text{incl}} \mathbf{Ma} \xrightarrow{Con} \mathbf{Gma} \xrightarrow{(-)_0} \mathbf{Boole}.$$

As $\langle B, \Box\rangle \models E$ implies $Con(\langle B, \Box\rangle) \models E$ this composition is the same as

$$\mathbf{Ma}^E \xrightarrow{Con} \mathbf{Gma}^E \xrightarrow{\text{incl}} \mathbf{Gma} \xrightarrow{(-)_0} \mathbf{Boole}.$$

The left adjoint to the restriction of Con is the restriction of $\varinjlim$ because $\mathscr{B} \models E$ implies $\varinjlim \mathscr{B} \models E$ and the left adjoint to the inclusion $\mathbf{Gma}^E \to \mathbf{Gma}$ is the quotient construction given above. $\Box$

To make the above result more useful, we have to explain the meaning of the definition of F^E in finite graded Kripke frames. Given a finite graded Kripke frame $W = \langle \{W_i\}_i, \{\lambda_i\}_i, \{R_i\}_i\rangle$, a formula α and an element $u \in W_i$, we write $u \models^i \alpha$ in order to mean that for all evaluations $v = \langle N, e\rangle$ on the graded modal algebra $\mathscr{P}_{gr}(W)$ such that $N(\alpha) = i$ we have that $u \in e(\alpha)$. Similarly, $u \models^i E$ will mean $u \models^i \alpha$ for all $\alpha \in E$. Let us put

$$R^*(u) = \{v \mid \exists j \leqslant i \exists z_0, \ldots, z_j (v \in W_{i-j} \,\&\, z_0 = u$$
$$\&\, z_j = v \,\&\, z_0 R_{i-1} z_1 R_{i-2} z_2 \ldots z_{j-1} R_{i-j} z_j)\}$$

(it is intended that we take the W_i disjoint and that u itself belongs to $R^*(u)$). The definition of $\mathscr{P}_{gr}(W)/F^E$ determines, by duality a graded Kripke subframe of W. In fact $\mathscr{P}_{gr}(W)/F^E$ is isomorphic to $\mathscr{P}_{gr}(W^E)$ where

$$u \in (W^E)_i \text{ iff } \forall k \leqslant i, \forall v \in W_k \; (v \in R^*(u) \;\Rightarrow\; v \models^k E).$$

To prove it, check that $u \in (W^E)_i$ iff $\forall \alpha \in E$, $\forall k \leqslant i$ $u \models^i \Box^{i-k}\alpha$ iff $\forall x \in F_i^E$ $u \in x$: this means that $W_i^E = \bigcap_{x \in F_i^E} x$, i.e. that W_i^E is the generator of the filter F_i^E.

This is what we can do in general, however, in many concrete cases, the *graded version of correspondence theory* [27] may be useful in order to make the condition "$v \models^k E$" elementary.

Let us take the example of the modal system $S4$, whose standard axiomatization is the following

$$\Box p \to p, \qquad \Box p \to \Box\Box p.$$

Now it is easily seen that

$$v \models^{k+1} \Box p \to p \quad \text{iff } vR_k\lambda_k(v);$$

$$v \models^{k+2} \Box p \to \Box\Box p \quad \text{iff } \forall z, w \ (vR_{k+1}z \,\&\, zR_k w \ \Rightarrow \ \lambda_{k+1}(v)R_k w)$$

(notice that the relation $v \models^k \alpha$ is always true if k is less than the modal degree of α). In terms of trees on a finite set L, we can define $(T^L)^{S4}$ to be the graded Kripke frame consisting for each n of the set of reflexive and transitive trees of height n (the λ_i and the R_i are the restricted leaf-cuts functions and immediate subtree relations),[22] where

a tree $t \in T^L_{k+1}$ is *reflexive* iff $\lambda_k(t) \in t_S$ and (this is a recursive definition) all trees in t_S are reflexive (all trees of height zero are reflexive);

a tree $t \in T^L_{k+2}$ is *transitive* iff for all $z \in t_S$ and $w \in z_S$, $w \in (\lambda_{k+1}(t))_S$ and all trees in t_S are transitive (all trees of height 0 or 1 are transitive).

It follows that $\varinjlim \mathscr{P}_{gr}(T^L)^{S4}$ is the free $S4$-algebra generated by the finite Boolean algebra $\mathscr{P}(L)$.. However, *we have not found* the normal forms for $S4$ in this way. In fact our procedure produces a collection of forests (namely, in our case, the reflexive and transitive trees) that satisfies only conditions (II) and (III) of Theorem 5.1, but not necessarily condition (I). Condition 1 says that "no tree disappears in the colimit": in the present situation this would mean that for every reflexive and transitive tree t of degree, say n, there exists a reflexive and transitive tree t' of degree $n+1$ such that $\lambda_n(t') = t$. This is not true, as easy counterexamples show.

How to make the relevant corrections? We point out that the above description of the free algebra is not satisfactory, for instance because it does not give any decision procedure: if $\varphi^k(\alpha)$ does not contain all the reflexive and transitive trees of height k, this does not mean that α is not a theorem of $S4$, because the exceeding trees might disappear in the colimit. A possible solution would be that of trying to describe effectively the vanishing trees, another solution that we are following here consists in *changing the axiomatization* of $S4$. This will not change the corresponding variety of modal algebras, but will change the full subcategory of the graded modal algebras that satisfy the new equations. In this way we can hope that the free objects in this more restricted category will correspond to graded subframes having surjective leaf-cuts. The new axiomatization will have the property that in order to prove a formula

[22] It is a consequence of our approach, e.g. that a subtree of a reflexive tree is reflexive and that the leaf-cut of a reflexive tree is a reflexive tree. This is due to the fact that F^L is a graded filter, hence determines a graded subframe.

α proofs of modal degree less or equal to the degree of α will be sufficient. We replace $\Box p \to \Box\Box p$ by

$$\Box(\Box p \to q) \to \Box(\Box p \to \Box q).$$

Using graded correspondence, we get the notion of strongly transitive tree, where:

a tree $t \in T^L_{k+2}$ is *strongly transitive* iff all trees in t_S are strongly transitive and for all $z \in t_S$ and $w \in z_S$, there exists $z' \in t_S$ such that $\lambda_k(z') = w$ and $z'_S \subseteq z_S$ (all trees of height 0 or 1 are strongly transitive).

In this way we obtained Fine's normal forms [7] for $S4$. In fact condition (I) of Theorem 5.1 for reflexive and strongly transitive trees can be proved as follows (simultaneously, we give also a proof by normal forms of a classical theorem [5]):

Theorem 5.3. *Leaf-cuts restricted to reflexive and strongly transitive trees are surjective. Moreover, the S4 modal algebra freely generated by a finite Boolean algebra is atomic.*

Proof. Let $(T^L)^{S4*}$ be the graded Kripke frame of reflexive and strongly transitive trees with labels in L (we recall that $\varinjlim \mathscr{P}_{gr}((T^L)^{S4*})$ is the $S4$ modal algebra freely generated by the finite Boolean algebra $\mathscr{P}(L)$). For the purpose of this proof, all trees will be automatically considered reflexive and strongly transitive. Let us introduce in the sets $(T^L)^{S4*}_k$ a preorder relation $\leqslant_k$ by putting:

$$a \leqslant_0 b \quad \forall a, b \in L; \qquad t \leqslant_{k+1} u \quad \text{iff } t_S \subseteq u_S.$$

We use the notation $\downarrow v$, for $v \in (T^L)^{S4*}_k$, in order to indicate the set $\{z \in (T^L)^{S4*}_k \mid z \leqslant_k v\}$. We collect some preliminary facts.

(i) *For all* $v \in (T^L)^{S4*}_{k+1}$ *and* $u \in v_S$, $u \leqslant_k \lambda_k(v)$. This is trivial for $k = 0$ and for $k = j + 1$ it amounts to show that $u_S \subseteq \exists_{\lambda_j}(v_S)$, which follows from transitivity.

(ii) *For all* $v \in (T^L)^{S4*}_k$, $t \in (T^L)^{S4*}_{k+1}$ *such that* $v \in t_S$, $v_t =_{df} \langle v_R, (\downarrow v) \cap t_S \rangle$ *is a reflexive and strongly transitive tree of height* $k + 1$ *such that* $\lambda_k(v_t) = v$ (we assume $v_R = v$ for trees of height 0). Again this is trivial for $k = 0$. For $k = j + 1$, showing that $\lambda_{j+1}(v_t) = v$ amounts to show that $\exists_{\lambda_j}(\downarrow v \cap t_S) = v_S$. One inclusion follows from reflexivity, the other one from strong transitivity. The reflexivity of v_t is immediate and its strong transitivity follows from the strong transitivity of t.

(iii) *For all* $t \in (T^L_{k+1})^{S4*}$, $\eta_{k+1}(t) =_{df} \langle t_R, \{v_t\}_{v \in t_S} \rangle$ *is a reflexive and strongly transitive tree of height* $k + 2$. Reflexivity follows from $t = (\lambda_k(t))_t$ which is a consequence of (i). To show strong transitivity, suppose that $k \geqslant 1$, that $u \in (\eta_{k+1}(t))_S$ and $z \in u_S$. It follows that $u = v_t$ for some $v \in t_S$. But then $z \in (\downarrow v \cap t_S)$, consequently z_t is the element of $(\eta_{k+1}(t))_S$ such that $\lambda_k(z_t) = z$ and $(z_t)_S \subseteq (v_t)_S = u_S$ (recall that $\downarrow z \subseteq \downarrow v$ as $z \leqslant_{k-1} v$).

It is now obvious that restricted leaf-cults are surjective, in fact they have the η_{k+1} as sections (λ_0 has the trivial section $\eta_0(a) = \langle a, \{a\}\rangle$).[23]

For the result of atomicity we adopt an argument similar to the argument we used for the modal system K in Section 2. A reflexive and strongly transitive tree v of height $k+2$ is said to be *reduced* iff for all $z \in v_S$ and $w \in z_S$, there exists a unique $z' \in v_S$ such that $\lambda_k(z') = w$. It follows from the definition of the η_{k+1} that for every t, $\eta_{k+1}(t)$ is reduced.

(iv) *Suppose that* $t \in (T^L)^{S4*}_{k+2}$ *is reduced and that* $z \in (T^L)^{S4*}_{k+3}$ *is such that* $\lambda_{k+2}(z) = t$; *then* $z = \eta_{k+2}(t)$. This means that $z_S = \{v_t\}_{v \in t_S}$. The inclusion $\supseteq$ easily follows from the inclusion $\subseteq$ and the hypothesis $\lambda_{k+2}(z) = t$. Moreover, to show the inclusion $\subseteq$ is sufficient to prove that for every $x \in z_S$, $x = (\lambda_{k+1}(x))_t$, i.e. that $x_S = (\downarrow\lambda_{k+1}(x)) \cap t_S$. Take $y \in x_S$; $y \leqslant_{k+1} \lambda_{k+1}(x)$ follows from (i) and $y \in t_S$ from the transitivity of z and the hypothesis $\lambda_{k+2}(z) = t$. Now, conversely, suppose that $y \leqslant_{k+1} \lambda_{k+1}(x)$ and $y \in t_S$. The former fact, together with reflexivity, implies $\lambda_k(y) \in (\lambda_{k+1}(x))_S$. Hence, there is a $y' \in x_S$ such that $\lambda_k(y') = \lambda_k(y)$. As the inclusion $x_S \subseteq (\downarrow\lambda_{k+1}(x)) \cap t_S$ has been shown above, $y' \in t_S$ and so $y = y'$ because t is reduced, y is reflexive and $y \in t_S$. Consequently, $y \in x_S$, as desired.

From (iv) it follows that the equivalence classes of the singleton forests $\{\eta_{k+1}(t)\}$ are all the atoms (recall that in the colimit every forest is identified with its inverse images along leaf-cuts, hence every atom is identified with a singleton forest of the specified kind). Now take a nonempty forest f of height $k+1$: it does contain an element t, hence the atom $[\{\eta_{k+1}(t)\}, k+2]$ is smaller than the equivalence class of f in the free $S4$ modal algebra. $\square$

The above effective enumeration of the atoms, when compared with the traditional one [4, 24] looks much more abstract because the relationship with finite Kripke models has not been exploited. On the other hand, the present approach gives other informations on the syntactic side (e.g. normal forms, properties of the different axiomatizations) and *does not depend* on finite model property.

We conclude the section by giving some information on the intuitionistic logic. As intuitionistic implication is a binary connective that preserves conjunctions in the second argument and changes disjunctions into conjunctions in the first, Heyting algebras can be represented as T-objects satisfying additional equations (the situation is analogous to example III in Section 1, we simply have to pass to the opposite lattice in the first argument before taking the tensor product). The problem of forcing the additional equations seems, however, not to be easy and a more direct approach, still in the spirit of this paper, is probably preferable. This is done in [8], where a simplified version of the construction of [26] is presented. Normal forms for intuitionistic logic turn out to be very similar to normal forms for $S4$, *from a geometric point of view* (see [8]): indeed they correspond exactly to reflexive and strongly transitive trees with

[23] We point out that there are also other sections, e.g. $\eta'_k(t) =_{df} \langle t_R, \downarrow t\rangle$.

labels in $\mathscr{P}(n)$ satisfying the additional requirement that labels are *increasing* subsets (this condition is formally expressed by asking that $v \in t_S$ implies $t_R \subseteq v_R$).

6. Propositional quantifiers and tense operators

The mathematical experience with free algebras shows that they are usually very rich, in the sense that they contain additional interesting opperators. For instance, finitely generated free Heyting algebras have also a dual-Heyting structure (see [8] and [10]), moreover, the cilindrification morphisms among them preserve this dual structure and have both left and right adjoints (see [20, 10]). From a logical point of view, these facts mean that richer logics can be syntactically interpreted in propositional intuitionistic logic. A strong form of interpolation theorem follows immediately. It is worth noticing that some recent papers [11, 23] applied normal forms in connection with analogous questions related to interpolation for the modal logic of provability. We deal here with the basic modal system K: the case of this system is much easier (because we do have to increase the degree in order to define propositional quantifiers. On the other hand, in this simple case, the proofs below illustrate in a neat and clean way the phenomena leading to such richness of free algebras.

Higher-order propositional modal calculus K^2 is introduced as follows: we take a countable set of propositional variables G and we allow formulas to be built using also the existential quantifier $\exists$. We use the notation $\alpha(p_1, \dots, p_n)$ or simply $\alpha(\underline{p})$, to mean that the free variables of α are among $p_1, \dots, p_n$. The notation $\alpha(\beta_1/p_1, \dots, \beta_n/p_n)$, or simply $\alpha(\underline{\beta}/\underline{p})$, is used for substitution. Substitution is defined inductively in the obvious way, however, for quantified formulas, in order to avoid well-known troubles, we put $(\exists p\alpha(\underline{p}, p))(\underline{\beta}/\underline{p}) = \exists q(\alpha(\underline{\beta}/\underline{p}, q/p))$, where q is a variable not occurring in $\underline{\beta}$ (q may be chosen according to some purely conventional criterion, for instance an alphabetic criterion). As axioms and rules, we have customary axioms and rules for K and, in addition, the following ones:

$$\vdash_{K^2} \alpha(\underline{p}, p) \to \beta(\underline{p}) \ \Rightarrow\ \vdash_{K^2} \exists p\alpha(\underline{p}, p) \to \beta(\underline{p})$$

$$\vdash_{K^2} \alpha(\underline{p}, \beta/p) \to \exists p\alpha(\underline{p}, p).$$

We look for a translation τ associating with every formula $\alpha(\underline{p})$ in K^2 a formula $\tau(\alpha)(\underline{p})$ in K (notice that the set of free variables in the formula cannot increase), satisfying the following conditions (we use $=$ for "provable equivalence", i.e. identity in Lindenbaum algebras):

$$\tau(p) = p,$$

$$\tau(\top) = \top,$$

$$\tau(\bot) = \bot,$$

$$
\begin{aligned}
&\tau(\alpha_1 \wedge \alpha_2) = (\alpha_1) \wedge (\tau\alpha_2),\\
&\tau(\alpha_1 \vee \alpha_2) = \tau(\alpha_1) \vee (\tau\alpha_2),\\
&\tau(\Box\alpha) \quad = \Box\tau(\alpha),\\
&\tau(\neg\alpha) \quad = \neg\tau(\alpha),\\
&\tau(\alpha(\underline{\beta}/\underline{p})) = \tau(\alpha)(\tau(\underline{\beta})/\underline{p}),\\
&\vdash_{K^2}\alpha \quad \Rightarrow \vdash\tau(\alpha),\\
&\tau(\exists p\alpha) \quad = \tau(\exists p\tau(\alpha)).
\end{aligned}
\tag{16}
$$

These conditions mean that τ translates identically (up to provability) the Boolean connectives, the box operator and the substitution; moreover, traslation preserves provability and finally translation of quantified formulas $\exists p\alpha$ is made in an uniform way, in the sense that it can always be reduced to the traslation of a formula, namely $\exists p\tau(\alpha)$, that contains a single occurrence of a quantifier. The traslation, satisfying all the above requirements is unique up to probability, provided that it exists. In fact, given two such translations τ_1 and τ_2, it is possible to show that for every formula α in the language of K^2, $\vdash\tau_1(\alpha) \to \tau_2(\alpha)$: this is established by an easy induction on α, in the case $\alpha = \exists p\beta$ we argue as follows. By induction hypothesis, $\vdash\tau_1(\beta) \to \tau_2(\beta)$, moreover, as $\beta \to \exists p\beta$ is provable in K^2, we have that $\vdash\tau_1(\beta) \to \tau_2(\alpha)$. However, provability in K implies provability in K^2 (for formulas in the language of K, i.e. for quantifier-free formulas), hence as $\tau_2(\alpha)$ does not contain p free (translation does not add new free variables), $\vdash_{K^2}\exists p\tau_1(\beta) \to \tau_2(\alpha)$. This implies $\vdash\tau_1(\alpha) \to \tau_2(\alpha)$ for preservation of provability, uniformity and the fact that translation is provably identical for quantifier-free formulas.

We formulate algebraically and solve by our geometric representation of normal forms the problem of finding the above translation. By the definition of free algebra, given two finite subsets of the countable set G, say $\underline{p} = p_1, \dots, p_n$ and $\underline{q} = q_1, \dots, q_m$, it is clear that a morphism $\mathscr{F}(\underline{p}) \to \mathscr{F}(\underline{q})$ is uniquely determined by the choice of n equivalence classes of formulas in the variables $\underline{q}$; we indicate the morphism corresponding to $[\underline{\beta}]$ as $[\underline{\beta}]^{\mathscr{F}}$. Notice that this morphism is precisely substitution, i.e. for $[\alpha] \in \mathscr{F}(\underline{p})$, $[\underline{\beta}]^{\mathscr{F}}([\alpha]) = [\alpha(\underline{\beta}/\underline{p})]$. Using this notation, the morphism corresponding to the inclusion $\underline{p} \subseteq \underline{p}, p$, i.e. corresponding to the "cilindrification" substitution, is $[\underline{p}]^{\mathscr{F}}$. We prove that the existence of the translation is equivalent to the fact that these morphisms have left adjoints (to be called $\exists p$) satisfying *Beck conditions* [16], i.e. such that the following squares commute, for $\underline{\beta} \in \mathscr{F}(\underline{p})$:

$$
\begin{tikzcd}
\mathscr{F}(\underline{p}, p) \arrow[r, "\exists_p"] \arrow[d, "{[\underline{\beta}, q]^{\mathscr{F}}}"'] & \mathscr{F}(\underline{p}) \arrow[d, "{[\underline{\beta}]^{\mathscr{F}}}"] \\
\mathscr{F}(\underline{q}, q) \arrow[r, "\exists_q"'] & \mathscr{F}(\underline{q})
\end{tikzcd}
$$

(notice that it is implicit in our notation that q does not appear in $\underline{q}$). The commutativity of the above square means that for every formula $[\alpha(\underline{p}, p)]$ the equation

$$(B)\quad [\underline{\beta}(\underline{q})]^{\mathscr{F}}(\exists p[\alpha(\underline{p}, p)]) = \exists q[\alpha(\underline{\beta}/\underline{p}, q/p)]$$

is satisfied; this equation corresponds in the inductive definition of substitution to the case of quantified formulas (the other inductive cases are hidden in the fact that $[\underline{\beta}]^{\mathscr{F}}$ is a morphism between modal algebras). Although this is clear from the point of view of the categorical analysis of logic [16], we formally prove that the traslation exists iff the cilindrification morphisms $[\underline{p}]^{\mathscr{F}}$ have a left adjoint satisfying the Beck conditions.

Suppose that the translation actually exists. Define the value of the left adjoint to $[\underline{p}]^{\mathscr{F}} : \mathscr{F}(\underline{p}) \to \mathscr{F}(\underline{p}, p)$ at $[\alpha(\underline{p}, p)]$ as an element of the equivalence class of $[\tau(\exists p\alpha)]$. The definition is correct because provability in K implies provability in K^2 (for formulas in the language of K). We prove the adjointness relation, i.e. that for $[\beta] \in \mathscr{F}(\underline{p})$, we have

$$[\tau(\exists p\alpha)] \leqslant [\beta] \quad \text{iff} \quad [\alpha] \leqslant [\beta]$$

(recall that $[\underline{p}]^{\mathscr{F}}$ associates with $[\beta]$, $[\beta]$ itself seen as an element of $\mathscr{F}(\underline{p}, p)$). This amounts to prove that $\vdash \tau(\exists p\alpha) \to \beta$ iff $\vdash \alpha \to \beta$. The left-to-right side follows from the facts that $\vdash_{K^2} \alpha \to \exists p\alpha$, that translation preserves provability and acts identically on quantifier-free formulas. The right-to-left side is obtained by similar passages and by the inference rule for introduction of $\exists$. Finally, the Beck condition requires that for every $\underline{q}$ and for every (quantifier-free) formula $\underline{\beta}(\underline{q})$ we have that

$$\vdash \tau(\exists p\alpha)(\underline{\beta}/\underline{p}) \leftrightarrow \tau(\exists q\alpha(\underline{\beta}/\underline{p}, q/p)),$$

where $q \notin \underline{q}$. This follows from the property of preservation of substitution of τ (and from the fact that alphabetic variants are provably equivalent, in case q is not the conventional choice for this substitution).

Suppose now that the cilindrification morphisms have left adjoints satisfying the Beck conditions. To find τ use some of the above clauses (16) as an inductive definition and to define $\tau(\exists p\alpha(\underline{p}, p))$ (where α contains exactly $\underline{p}$ as free variables other than p) take an element of the equivalence class of the value of the left adjoint $\exists p$ to $[\underline{p}]^{\mathscr{F}}$: $\mathscr{F}(\underline{p}) \to \mathscr{F}(\underline{p}, p)$ at $[\tau(\alpha)]$. Here we need a preliminary remark, due to the fact that in the traditional logical calculi Beck condition is hidden everywhere, also in notation. When we write $\alpha(\underline{p}, p)$ we express the fact that α has free variables *among* $\underline{p}, p$. Suppose that $\underline{p} = \underline{q}, \underline{r}$ and that $\underline{q}$ are exactly the variables other than p occurring free in α. We defined $[\tau(\exists p\alpha)]$ as $\exists p[\tau(\alpha(\underline{q}, p))]$ where $\exists p$ is the left adjoint to $[\underline{q}]^{\mathscr{F}}$: $\mathscr{F}(\underline{q}) \to \mathscr{F}(\underline{q}, p)$. So $\exists p[\tau(\alpha(\underline{q}, p))]$ is an equivalence class of formulas in $\mathscr{F}(\underline{q})$, hence it can be seen also as an equivalence class of formulas in $\mathscr{F}(\underline{p}) = \mathscr{F}(\underline{q}, \underline{r})$: to do this it is sufficient to pass to $[\underline{q}]^{\mathscr{F}}(\exists p[\tau(\alpha(\underline{q}, p))])$. On the other hand, $[\tau(\alpha)]$ can be considered also as an element of $\mathscr{F}(\underline{q}, \underline{r}, p) = \mathscr{F}(\underline{p}, p)$ and it makes sense to consider

$\exists p[\tau(\alpha)]$, where this time $\exists p$ is the left adjoint to $[\underline{p}]^{\mathscr{F}}: \mathscr{F}(\underline{p}) \to \mathscr{F}(\underline{p}, p)$. The point is that the value that we get in this way is exactly $[\underline{q}]^{\mathscr{F}}(\exists p[\tau(\alpha(\underline{q}, p))])$, because of Beck condition applied to the square:

$$\begin{array}{ccc} \mathscr{F}(\underline{q}, p) & \xrightarrow{\exists_p} & \mathscr{F}(\underline{q}) \\ {\scriptstyle [\underline{q}, p]^{\mathscr{F}}}\downarrow & & \downarrow{\scriptstyle [\underline{q}]^{\mathscr{F}}} \\ \mathscr{F}(\underline{p}, p) & \xrightarrow[\exists_p]{} & \mathscr{F}(\underline{p}) \end{array}$$

The moral of all this is that the equivalence class of formulas in $\mathscr{F}(\underline{p})$ corresponding to $\tau(\exists p\alpha(\underline{p}, p)$ is the value of the left adjoint to $[\underline{p}]^{\mathscr{F}}: \mathscr{F}(\underline{p}) \to \mathscr{F}(\underline{p}, p)$ computed at $[\tau(\alpha(\underline{p}, p))]$, no matter if we defined it by disregarding some irrelevant free variables.

Preservation of substitution is established by induction: for instantance, in the inductive step for the existential quantifier, we have to show that

$$[\tau(\exists p\gamma)(\tau(\underline{\beta})/\underline{p})] = [\tau(\exists q\gamma(\underline{\beta}/\underline{p}, q/p))].$$

However, $[\tau(\exists p\gamma)(\tau(\underline{\beta})/\underline{p})] = [\tau(\underline{\beta})]^{\mathscr{F}}(\exists p[\tau(\gamma)])$ which is equal to (by Beck condition) $\exists q([\tau(\underline{\beta}), q]^{\mathscr{F}}([\tau(\gamma)]))$, i.e. to $\exists q[\tau(\gamma)(\tau(\underline{\beta})/\underline{p}, \tau(q)/p)]$. Now it is sufficient to apply the induction hypothesis.

The uniformity condition is trivially satisfied: applying τ twice is the same thing as applying it once (because it acts identically on quantifier-free formulas).

We finally show the preservation of provability, by induction on the proof of $\vdash_{K^2}\alpha$. We examine only the relevant cases, i.e. the cases corresponding to the specific axiom and to the specific rule for K^2. It is trivially seen that if $[\tau(\alpha(\underline{p}, p))] \leqslant [\tau(\beta(\underline{p}))]$ then $[\tau(\exists p(\alpha(\underline{p}, p)))] \leqslant [\tau(\beta(\underline{p}))]$.[24] Showing that $[\tau(\alpha(\underline{p}, \beta(\underline{p})/p))] \leqslant \exists p[\tau(\alpha(\underline{p}, p))]$ is only a little more difficult, but standard (notice that we may freely suppose that β contains free variables among the $\underline{p}$'s, because we can evidentiate as many variables as we like). By the adjointness conditions, $[\tau(\alpha)] \leqslant [\underline{p}]^{\mathscr{F}}(\exists p[\tau(\alpha)])$ and so, as $[\underline{p}, \tau(\beta(\underline{p}))]^{\mathscr{F}}$ is a morphism, $[\underline{p}, \tau(\beta(\underline{p}))]^{\mathscr{F}}([\tau(\alpha)]) \leqslant [\underline{p}, \tau(\beta(\underline{p}))]^{\mathscr{F}}[\underline{p}]^{\mathscr{F}}(\exists p[\tau(\alpha)])$.

[24] Notice that the latter inequation holds in $\mathscr{F}(\underline{p})$, whereas the former holds in $\mathscr{F}(\underline{p}, p)$. We recall that, in any case, an inequation in a Lindenbaum algebra is equivalent to provability of implication in the whole calculus, that is to what is important for us (in our case, for instance, instead of proving "$\vdash\tau(\underline{p}, p)) \to \tau(\beta(\underline{p}))$ implies $\vdash\tau(\exists p(\alpha(\underline{p}, p))) \to \tau(\beta(\underline{p}))$", we show the corresponding algebraic statement indicated in the text). To understand the passage, recall that $\tau(\beta)$ does not contain p, hence $[\tau(\beta)]$ seen as an element of $\mathscr{F}(\underline{p}, p))$ is equal to $[\underline{p}]^{\mathscr{F}}([\tau(\beta)])$ (where this time $[\tau(\beta)]$ is seen as an element of $\mathscr{F}(\underline{p})$). This observation allows to introduce the left adjoint and to pass from $[\tau(\alpha(\underline{p}, p))] \leqslant_{\mathscr{F}(\underline{p}, p)} [\tau(\beta(\underline{p}))]$ to $[\tau(\exists p(\alpha(\underline{p}, p)))] \leqslant_{\mathscr{F}(\underline{p})} [\tau(\beta(\underline{p}))]$.

This is equivalent to $[\tau(\alpha)(\underline{p}/\underline{p}, \tau(\beta)/p)] \leqslant \exists p[\tau(\alpha)]$,[25] i.e. (by preservation of substitution by τ, a fact we have already established) to $[\tau(\alpha(\underline{p}, \beta(\underline{p})/p))] \leqslant \exists p[\tau(\alpha(\underline{p}, p))]$.

We can finally pass to the geometric point of view and show what we need for the existence of the translation.

Theorem 6.1. *The cilindrification morphisms have a left adjoint satisfying the Beck condition.*

Proof. Cilindrification morphisms act on forests by inverse image along label restrictions (see Section 4), that is they associate with a forest f of any degree with labels in $\mathscr{P}(\underline{p})$, the set of trees of the same degree with labels in $\mathscr{P}(\underline{p}, p)$, such that the tree obtained from them by deleting p everywhere in the labels, belongs to f. As the left adjoint of inverse image is direct image, we define, for $f \subseteq T_n^{\mathscr{P}(\underline{p}, p)}$,

$$\exists p[f, n] = [f_{|\underline{p}}, n],$$

where $f_{|\underline{p}} = \{t_{|\underline{p}}/t \in f\}$. We only have to verify that the definition is correct (i.e. does not depend on the representative elements of the equivalence classes) and that Beck condition is satisfied.

As to *the correctedness of the definition*, what we have to show is the following fact (we can limit ourselves to simple leaf-cuts, for iterated leaf-cuts the same argument can be repeatedly applied):

$$\lambda_n^{-1}(f_{|\underline{p}}) = (\lambda_n^{-1}(f))_{|\underline{p}}.$$

Now, in general, given a commutative square of sets and functions

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle l} \\ Z & \xrightarrow[m]{} & T \end{array}$$

the "Beck condition" for it (i.e. the condition "for every $S \subseteq Z$, $l^{-1}(\exists_m(S)) = \exists_h(k^{-1}(S))$") is easily seen to be equivalent to the fact that the square is a weak pullback of sets (i.e. to the fact that for every $y \in Y$, $z \in Z$, $l(y) = m(z)$ implies that there exists a (not necessarily unique) $x \in X$ such that $h(x) = y$ and $k(x) = z$). This

[25] In general, $[\underline{\gamma}]^{\#}[\underline{\delta}(\underline{q})]^{\#}$ is equal to $[\underline{\delta}(\underline{\gamma}/\underline{q})]^{\#}$ (this follows from the definition: e.g. $[\underline{\gamma}]^{\#}$ is the unique modal morphism that, applied to any $[\vartheta(\underline{q})]$ gives $[\vartheta(\underline{\gamma}/\underline{q})]$ as a result, etc.). In our case, $[\underline{p}, \tau(\beta(\underline{p}))]^{\#}[\underline{p}]^{\#}$ gives as a result the morphism corresponding to the equivalence classes of the formulas obtained by taking the substitution of $\underline{p}$ for $\underline{p}$ in $\underline{p}$, hence it gives the identity morphism as a result.

observation reduces our problems to that of showing that the following square is a weak pullback:

$$\begin{tikzcd}
T_{n+1}^{\mathscr{P}(\underline{p},p)} \arrow[r, "(-)_{|\underline{p}}"] \arrow[d, "\lambda_n"'] & T_{n+1}^{\mathscr{P}(\underline{p})} \arrow[d, "\lambda_n"] \\
T_{n}^{\mathscr{P}(\underline{p},p)} \arrow[r, "(-)_{|\underline{p}}"'] & T_{n}^{\mathscr{P}(\underline{p})}
\end{tikzcd}$$

So let us take $u \in T_{n+1}^{\mathscr{P}(\underline{p},p)}$ and $v \in T_{n}^{\mathscr{P}(\underline{p},p)}$ such that $\lambda_n(u) = v_{|\underline{p}}$ and let us look for a $t \in T_{n+1}^{\mathscr{P}(\underline{p},p)}$ such that $t_{|\underline{p}} = u$ and $\lambda_n(t) = v$. We argue by induction on n. For $n = 0$, put $t = \langle v, u_S \rangle$. For $n = k + 1$, notice that our hypothesis entails that $v_R \cap \underline{p} = u_R$ and that (let $u_S = \{u_1, \ldots, u_m\}$ and $v_S = \{v_1, \ldots, v_h\}$) $\{\lambda_k(u_1), \ldots, \lambda_k(u_m)\} = \{(v_1)_{|\underline{p}}, \ldots, (v_h)_{|\underline{p}}\}$. By induction hypothesis, there exists, for every i, j ($1 \leqslant i \leqslant h$, $1 \leqslant j \leqslant m$) such that $(v_i)_{|\underline{p}} = \lambda_k(u_j)$, a tree $t_{i,j} \in T_{k+1}^{\mathscr{P}(\underline{p},p)}$ such that $\lambda_{k+1}(t_{ij}) = v_i$ and $(t_{ij})_{|\underline{p}} = u_j$. We put $t = \langle v_R, \{t_{ij}\} \rangle$, thus getting $\lambda_{k+2}(t) = v$ and $t_{|\underline{p}} = u$, as required.

The examination of *the Beck condition*[26] leads to a problem similar to the above one. We have to prove the commutativity of the following squares:

$$\begin{tikzcd}
\mathscr{F}(\underline{p}, p) \arrow[r, "\exists_p"] \arrow[d, "{[\underline{\beta}, q]^{\mathscr{F}}}"'] & \mathscr{F}(\underline{p}) \arrow[d, "{[\underline{\beta}]^{\mathscr{F}}}"] \\
\mathscr{F}(\underline{q}, q) \arrow[r, "\exists_q"'] & \mathscr{F}(\underline{q})
\end{tikzcd}$$

where we may freely assume that $\underline{p} = p_1, \ldots, p_n$, $\underline{q} = p_1, \ldots, p_m, p = p_{n+1}, q = p_{m+1}$. Suppose also that the formulas $\underline{\beta} = \beta_1, \ldots, \beta_n$ (containing at most the propositional letters $\underline{q}$) have modal degree less or equal to k. Translating everything into the combinatorial framework, our aim consists in proving that for every natural number i and for every forest $f \subseteq T_i^{\mathscr{P}(\underline{p},p)}$

$$(s_{\underline{\beta}})_i^{-1}(f_{|\underline{p}}) = ((s_{\underline{\beta},q})_i^{-1}(f))_{|\underline{q}}.$$

We recall that (see (*) in Section 4)

$$(s_{\underline{\beta}})_0(t) = \{p_i \mid t \in \varphi^k(\beta_i)\}.$$

[26] We cannot completely dispense with checking the Beck condition if we want the claimed translation from K^2 into K. In fact, the existence of such a translation implies (see below) a strong form of interpolation theorem. On the other hand, there are many locally finite vcarieties of modal algebras in which interpolation fails [19]. In such varieties, all morphisms between finitely generated free algebras have adjoint, because such algebras are finite. Consequently, it is just Beck condition that does not hold in these cases (the reader can see by himself that Beck condition is really used in the below proof of existence of interpolants: it is sufficient to give each passage its algebraic meaning).

Analogously (notice that in computing $s_{\underline{\beta},q}$ we consider the $\underline{\beta}$ as formulas in $\underline{q}$, q hence the $\varphi^k(\beta_i)$ must be extended by taking inverse images along label restrictions):

$$(s_{\underline{\beta},q})_0(u) = \{p_i | (i \leqslant n \& u_{|\underline{q}} \in \varphi^k(\beta_i)) \text{ or } (i = n+1 \& q \in u_R)\}$$

$$= (s_{\underline{\beta}})_0(u_{|\underline{q}}) \cup (u_R)_{\downarrow p},$$

where we indicated directly with $(u_R)_{\downarrow p}$ the operation of considering the root u_R of u (we assume that $u_R = u$ in case $k = 0$) and taking $\{p\}$ if $q \in u_R$ or $\emptyset$ in case $q \notin u_R$. So we must simply show that the following commutative squares are weak-pullbacks (for every $i \in N$):

$$\begin{array}{ccc} T_{k+i}^{\mathscr{P}(\underline{q},q)} & \xrightarrow{(s_{\underline{\beta},q})_i} & T_i^{\mathscr{P}(\underline{p},p)} \\ \Big\downarrow{\scriptstyle (-)_{|\underline{q}}} & & \Big\downarrow{\scriptstyle (-)_{|\underline{p}}} \\ T_{k+i}^{\mathscr{P}(\underline{q})} & \xrightarrow[(s_{\underline{\beta}})_i]{} & T_i^{\mathscr{P}(\underline{p})} \end{array}$$

Theorem 6.1 will be completely proved once we have shown that given $t \in T_i^{\mathscr{P}(\underline{p},p)}$ and $v \in T_{k+i}^{\mathscr{P}(\underline{p})}$ such that $t_{|\underline{p}} = (s_{\underline{\beta}})_i(v)$, there exists $u \in T_{k+i}^{\mathscr{P}(\underline{q},q)}$ such that $u_{|\underline{q}} = v$ and $(s_{\underline{\beta},q})_i(u) = t$.We argue by induction on i.

If $i = 0$, suppose that $t_{|\underline{q}} = (s_{\underline{\beta}})_0(v)$ and take as u the tree that one gets from v by adding to its root $t_{\downarrow q}$ (i.e. q if $p \in t$, nothing otherwise). Clearly $u_{|\underline{q}} = v$; moreover $(s_{\underline{\beta},q})_0(u) = (s_{\underline{\beta}})_0(u_{|\underline{q}}) \cup (u_R)_{\downarrow p} = t_{|\underline{p}} \cup t_{|p} = t$.

If $i = j+1$, suppose that $t_{|\underline{p}} = (s_{\underline{\beta}})_{j+1}(v)$. This means that $(t_R)_{|\underline{p}} = (s_{\underline{\beta}})_0(\lambda_{k+i,i}(v))$ and that $(t_S)_{|\underline{p}} = \exists_{(s_{\underline{\beta}})_j}(v_S)$. By induction hypothesis, for every $t_s \in t_S$ and for every $v_r \in v_S$ such that $(t_s)_{|\underline{p}} = (s_{\underline{\beta}})_j(v_r)$, there exists $u_{r,s}$ such that $(u_{r,s})_{|\underline{q}} = v_r$ and $(s_{\underline{\beta},q})_j(u_{r,s}) = t_s$. Let us put $u = \langle v_R \cup (t_R)_{\downarrow q}, \{u_{r,s}\}\rangle$ (with the usual convention for $(t_R)_{\downarrow q}$). Clearly $u_{|\underline{q}} = v$. Moreover $(s_{\underline{\beta},q})_i(u) = \langle (s_{\underline{\beta},q})_0(\lambda_{k+i,i}(u)), t_S\rangle$. We show that $(s_{\underline{\beta},q})_0(\lambda_{k+i,i}(u)) = t_R$. In fact,

$$(s_{\underline{\beta},q})_0(\lambda_{k+i,i}(u)) = (s_{\underline{\beta}})_0((\lambda_{k+i,i}(u))_{|\underline{q}}) \cup ((\lambda_{k+i,i}(u))_R)_{\downarrow p}$$

$$= (s_{\underline{\beta}})_0(\lambda_{k+i,i}(u_{|\underline{q}})) \cup (t_R)_{|p}$$

$$= (s_{\underline{\beta}})_0(\lambda_{k+i,i}(v)) \cup (t_R)_{|p}$$

$$= (t_R)_{|\underline{p}} \cup (t_R)_{|p} = t_R,$$

where we used the fact that label restrictions are graded Kripke frames morphisms and the fact (already proved) that $u_{|\underline{q}} = v$. $\square$

Theorem 6.1 shows that there exists a translation from K^2 into K with the properties (16): we express this fact by saing that *K is a syntactic model of K^2*. To translate a formula of the kind $\exists p\alpha(\underline{p}, p)$, take the translation of α which is inductively given, *transform it into a forest with labels in $\mathscr{P}(\underline{p}, p)$*, delete p wherever it appears in the labels and finally go back to a formula.

Theorem 6.1 can be easily improved by showing that the translation extends to K^2 with the Barcan formula, but:

Theorem 6.2. *K is a not a syntactic model of K^2 with equality.*[27]

Proof. We show that there cannot be any propositional formula in two variables that can be the translation of $p_1 = p_2$. Suppose the contrary, i.e. that there exists such a formula. It will have modal degree say n and it will correspond to a forest $e \subseteq T_n^{\mathscr{P}(\{p_1, p_2\})}$. Let us put $e_k = \lambda_{k,n}^{-1}(e)$ for $k \geqslant n$. The translation of $p = p$ must be a theorem in K, this means that for all $k \geqslant n$ and for all $t \in T_k^{\mathscr{P}(\{p\})}$

$$\delta_k(t) \in e_k,$$

where δ is the morphism corresponding to the diagonal substitution of p both for p_1 and p_2 (from the results of Section 4, it is clear that δ_k is the label renaming operation that changes the empty label into itself and the label $\{p\}$ into $\{p_1, p_2\}$). By the substitutivity axiom, the translation of the formulas $(p_1 = p_2 \wedge \alpha(p_1/p)) \rightarrow \alpha(p_2/p)$ are theorems of K. Fix now $k \geqslant n$ and $t \in T_k^{\mathscr{P}(\{p_1, p_2\})}$; let us indicate with $(t_{|p_i})_{\downarrow p}$ (for $i = 1, 2$), the tree obtained from t by restricting it to p_i and by renaming p_i as p. If we take as $\alpha(p)$ the formula corresponding to the singleton forest $\{(t_{|p_1})_{\downarrow p}\}$, we get that

$$t \in e_k \;\Rightarrow\; (t_{|p_1})_{\downarrow p} = (t_{|p_2})_{\downarrow p}.$$

The two conditions we found are contradictory: for instance, according to the former any thin tree with empty labels of height $k \geqslant n$ should be in e_k. However, if we attach the label $\{p_1\}$ to it at the top, we get a tree in $e_{k+1} = \lambda_k^{-1}(e_k)$ which clearly does not satisfy the latter condition. $\square$

Theorem 6.1 can be used in order to give a combinatorial procedure enumerating all the Craig's interpolants (up to provable equivalence) of two given modal formulas and determining whether they are finitely many or not.

Given two modal formulas $\alpha(\underline{r}, \underline{p})$ and $\beta(\underline{r}, \underline{q})$ sharing the propositional letters $\underline{r}$ and such that

$$\alpha(\underline{r}, \underline{p}) \vdash \beta(\underline{r}, \underline{q}) \tag{17}$$

[27] Equivalently [16], this means that not all the substitution morphisms (in particular, not the diagonal morphism) have a left adjoint.

(we in general write $\gamma_1, \dots, \gamma_n \vdash \gamma$ for $\vdash \gamma_1 \wedge \cdots \wedge \gamma_n \to \gamma$), we say that the formula $\gamma(\underline{r})$ is an *interpolant* for them in case we have that

$$\alpha(\underline{r}, \underline{p}) \vdash \gamma(\underline{r}) \quad \text{and} \quad \gamma(\underline{r}) \vdash \beta(\underline{r}, \underline{q}).$$

Craig's interpolation theorem says that at least one interpolant always exists. We use here a syntactic idea applied in [20] to the case of intuitionistic logic: the idea is a simple corollary of the definability of propositional quantifiers. For a modal formula α (in the language of K), let us indicate $\tau(\exists p\alpha)$ directly with $\exists p\alpha$. We know from Theorem 6.1 that we may freely use in K the second-order syntax of K^2, for instance the following four rules are at our disposal:

$$\alpha(\underline{p}, p) \vdash \beta(\underline{p}) \Leftrightarrow \exists p\alpha(\underline{p}, p) \vdash \beta(\underline{p}),$$

$$\beta(\underline{p}) \vdash \alpha(\underline{p}, p) \Leftrightarrow \beta(\underline{p}) \vdash \forall p\alpha(\underline{p}, p)$$

(where $\forall p$ is defined as usual as $\neg\exists p\neg$). We so have that $\gamma(\underline{r})$ is an interpolant of two formulas $\alpha(\underline{r}, \underline{p})$ and $\beta(\underline{r}, \underline{q})$ satisfying (17) iff

$$\exists \underline{p}\alpha(\underline{r}, \underline{p}) \vdash \gamma(\underline{r})$$

and

$$\gamma(\underline{r}) \vdash \forall \underline{q}\beta(\underline{r}, \underline{q}).$$

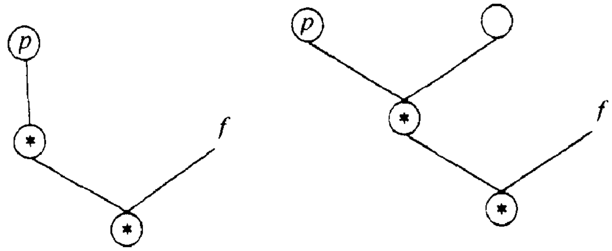
Thus, as

$$\exists \underline{p}\alpha(\underline{r}, \underline{p}) \vdash \forall \underline{q}\beta(\underline{r}, \underline{q})$$

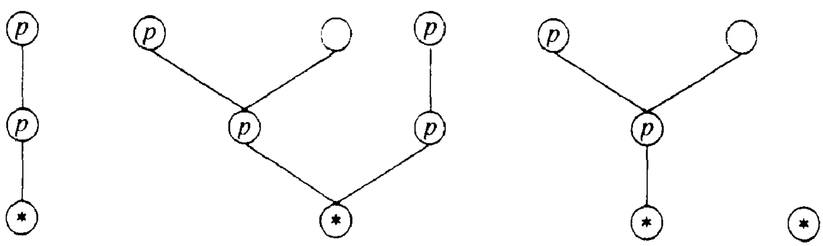
is easily deduced from (17), we have that there always exist a *minimum interpolant* $\exists \underline{p}\alpha(\underline{r}, \underline{p})$ and a *maximum interpolant* $\forall \underline{q}\beta(\underline{r}, \underline{q})$. The following is consequently a *procedure* in order to determine, given two modal formulas $\alpha(\underline{r}, \underline{p})$ and $\beta(\underline{r}, \underline{p})$ satisfying (17), all their interpolants (up to provable equivalence) of modal degree n (where n is any given natural number greater or equal to the maximum of the modal degrees of $\alpha(\underline{r}, \underline{p})$ and $\beta(\underline{r}, \underline{q})$):

- transform α and β into forests by computing $\varphi^n(\alpha)$ and $\varphi^n(\beta)$;
- delete the $\underline{p}$'s wherever they appear in the labels of $\varphi^n(\alpha)$: in this way we have computed the forest $\varphi^n(\exists \underline{p}\alpha)$;
- perform the dual operation on the forest $\varphi^n(\beta)$ (i.e. take the trees $t \in T_n^{\wp(\underline{r})}$, such that for all $u \in T_n^{\wp(\underline{r}, \underline{q})}$, $u_{|\underline{r}} = t$ implies that $u \in \varphi^n(\beta)$): in this way we have computed the forest $\varphi^n(\forall \underline{q}\beta)$;
- the interpolants of modal degree n are exactly (up to provable equivalence) the formulas $\Phi^n(g)$, where $g \subseteq T_n^{\wp(\underline{r})}$ is a forest such that $\varphi^n(\exists \underline{p}\alpha) \subseteq g \subseteq \varphi^n(\forall \underline{q}\beta)$.

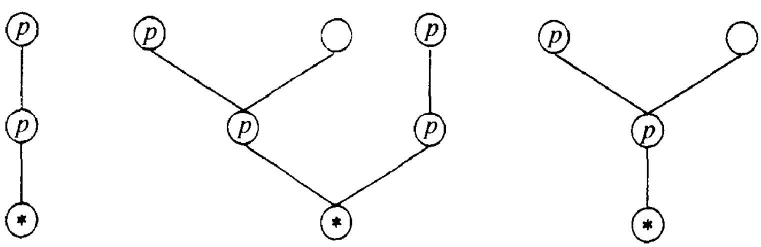
We give here an example of the above procedure. Let us compute the interpolants of modal degree 2 of $\Box(p \wedge \Diamond p) \wedge \Diamond\Diamond p$ and $\Box\Box q \to \Diamond\Diamond q$. The forest $\varphi^2(\Diamond\Diamond p)$ contains the following trees



where each occurrence of $*$ denotes an arbitrary label and f an arbitrary forest of degree 1 with labels in the power-set of $\{p\}$. On the other hand, the forest $\varphi^2(\Box(p \wedge \Diamond p))$ consists only of the following four kinds of trees:

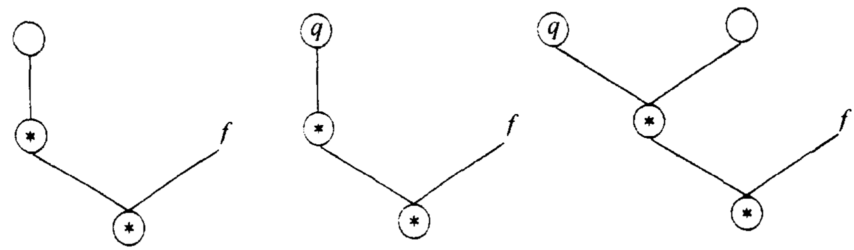


Consequently, $\varphi^2(\Box(p \wedge \Diamond p) \wedge \Diamond\Diamond p)$ is the forest so represented

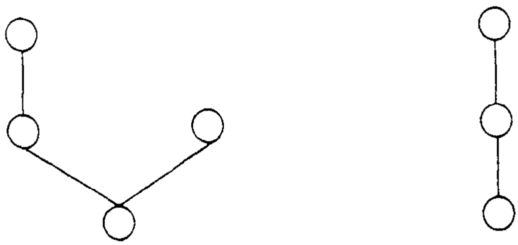


Deleting p everywhere, we obtain that $\varphi^2(\exists p(\Box(p \wedge \Diamond p) \wedge \Diamond\Diamond p))$ contains only the following tree:

The forest $\varphi^2(\Box\Box q \to \Diamond\Diamond q)$ is the following (notice that $\Box\Box q \to \Diamond\Diamond q$ is provably equivalent to $\Diamond\Diamond\neg q \vee \Diamond\Diamond q$ and that $\varphi^2(\Diamond\Diamond p)$ has been determined above)



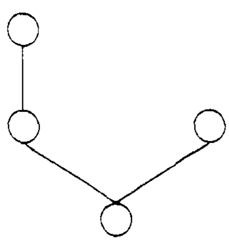
Computing $\varphi^2(\forall q(\Box\Box q \to \Diamond\Diamond q))$ yields



We have so concluded that there are only two interpolants, namely the following (recall that in using formulas (11) of Section 2, we have to put $\top$ whenever the set of indexes of a conjunction is empty):

$$\Diamond\Diamond\top \wedge \neg\Diamond\neg\Diamond\top, \qquad (\Diamond\Diamond\top \wedge \neg\Diamond\neg\Diamond\top) \vee (\Diamond\Diamond\top \wedge \Diamond\neg\Diamond\top).$$

To find interpolants of higher modal degrees one simply takes the inverse image along leaf-cuts of the whole situation. In our case, for instance, as there are seven trees of degree 3 whose leaf-cut is equal to the tree,



we can conclude that there are $2^7 = 128$ interpolants of modal degree 3.

Some facts are quite evident from the above analysis: for two formulas (of modal degrees less or equal to $n + 1$) $\alpha(\underline{r}, \underline{p})$ and $\beta(\underline{r}, \underline{q})$ satisfying (17), the total number of interpolants of any degree is *finite* (up to provable equivalence) iff the forest $\varphi^{n+1}(\exists \underline{p}\alpha)\backslash\varphi^{n+1}(\forall \underline{q}\beta)$ is either empty or contains only *atomic* trees, i.e. trees of the kind $\sigma_n(\overline{t})$ (which are exactly the trees of degree $n + 1$ having the unique extension property, see Section 2). This observation clearly shows also that if the interpolants are finitely many, their number is *bounded by the cardinality of the set of forests of degree n* with labels in $\mathscr{P}(\underline{r})$. For formulas of modal degree zero, we have that interpolants are infinitely many or just only one (this is a general fact: if there is only one interpolant in any degree, then increasing the degree one cannot find new interpolants).

We conclude the section by showing another syntactic interpretation of a stronger logic. We recall that the bimodal system K_t contains in the language two modal connectives F and $\Box$ (the customary notation in the literature is F for the "possible-in-the-future" operator F and H for the "necessary-in-the-past" operator $\Box$); as axioms we have tautologies and as inference rules we have modus ponens and the two additional rules (this axiomatization is equivalent to the standard one)

$$\vdash_{K_t} F\alpha \to \beta \;\Rightarrow\; \vdash_{K_t} \alpha \to \Box\beta,$$

$$\vdash_{K_t} \alpha \to \Box\beta \;\Rightarrow\; \vdash_{K_t} F\alpha \to \beta.$$

From an algebraic point of view, we have *tense algebras*, which are Boolean algebras endowed with a pair of adjoint functors. This means that a tense algebra is a structure

$$T = \langle \underline{T},\ \wedge,\ \top,\ \vee,\ \bot, \neg, F, \Box\rangle,$$

where $\langle \underline{T},\ \wedge,\ \top,\ \vee,\ \bot, \neg\rangle$ is a Boolean algebra and F and $\Box$ are (order-preserving) unary operators such that for every $x, y \in \underline{T}$

$(A)\ Fx \leqslant y \;\Leftrightarrow\; x \leqslant \Box y.$

We shall show that tense logic can be interpreted in modal logic, in the same sense as we meant above for K^2. That is, we prove that, given a set of propositional letters G, there exists a translation τ associating with a tense formula α (i.e. a formula containing Boolean connectives and F, $\Box$) a modal formula $\tau(\alpha)$ (i.e. a formula containing Boolean connectives and $\Box$), in such a way that, up to provable equivalence, translation (which is supposed to preserve provability) acts identically on Boolean connectives and on $\Box$.[28] We leave the reader to verify that, in presence of the uniformity condition $\tau(F\alpha) = \tau(F\tau(\alpha))$ such translation is unique and that its existence is equivalent, from the algebraic point of view, to the existence of the left adjoint to $\Box$ in the modal algebra freely generated by G. As all free modal algebras are colimit of finitely generated free ones, it is sufficient to show the result for finite G and to show that the left adjoint so found is preserved by the cjlindrification morphisms (in this way it is automatically proved that the left adjoint exists in the colimit too).

Given any forest f of degree $n+1$ with labels in the power set of G, define its *root-cut* $\rho^{-}(f)$ as the forest of degree n (in the same set of labels) obtained by deleting all the root-labels of the trees in f, that is

$$\rho\ \ (f) = \{u \in T_n^{\wp(G)} \mid \exists t \in f(u \in t_S)\}.$$

[28] Notice that Thomason's [25] reduction of tense logic to modal logic does not fulfills these requirements and so is different from our τ.

We recall that every element in $\mathscr{F}(G)$ may be represented as a forest of degree at least 1 (for, we may always apply λ_0^{-1} to any forest of degree 0 and get an equivalent forest of degree 1). We simply put

$$F[f, n+1] = [\rho^-(f), n].$$

There are three things to be checked, namely the fact that te definition is correct, the adjointness relation (A) and the preservation of the tense operator by the cilindrification morphisms.

As to *the correctedness of the definition*, we show that for every $n \geqslant m$ and for every forest $f \subseteq T_{m+1}^{\mathscr{P}(G)}$, the following equation holds:

$$\rho^-(\lambda_{n+1,m+1}^{-1}(f)) = \lambda_{n,m}^{-1}(\rho^-(f)).$$

This means that for every tree $w \in T_n^{\mathscr{P}(G)}$,

$$\exists t(\lambda_{n+1,m+1}(t) \in f \ \&\ w \in t_S) \iff \exists v \in f(\lambda_{n,m}(w) \in v_S).$$

The left-to-right side is trivial, for the other one the appropriate choice for t is $\langle v_R, \exists_{\sigma_{m,n}}(v_S) \cup \{w\}\rangle$ (see Section 2 for the definition of the σ_i).

As to *the adjointness relation*, we have to show that

$$F[f, n+1] \leqslant [g, m] \quad \text{iff} \quad [f, n+1] \leqslant \Box[g, m].$$

We may assume that $m = n$, so we are simply asked to show that

$$\{u \in T_n^{\mathscr{P}(G)} \mid \exists t \in f(u \in t_S)\} \subseteq g \quad \text{iff} \quad f \subseteq \{t \in \mathscr{P}_{n+1}^{\mathscr{P}(G)} \mid t_S \subseteq g\},$$

which is in fact trivially true.

Finally, we show *the preservation of F by the cilindrification morphisms.*[29] This means that given $m \geqslant 0$, given two finite sets G_1, G_0 such that $G_1 \subseteq G_0$ and given a forest $f \subseteq T_{m+1}^{\mathscr{P}(G)}$, the following equations holds:

$$\rho^-(f^{|G_0}) = (\rho^-(f))|^{G_0},$$

where $(-)^{|G_0}$ is the inverse image along label restriction from G_0 to G_1. In other words, we must prove that for every tree $u \in T_m^{\mathscr{P}(G_0)}$,

$$\exists t(t_{|G_0} \in f \ \&\ u \in t_S) \iff \exists v \in f(u_{|G_0} \in v_S).$$

The left-to-right side is trivial, for the other one put $t = \langle v_R, v_S \cup \{u\}\rangle$.

We have so established the following result:

Theorem 6.3. *Free modal algebras are tense algebras, in the sense that their box operators have a left ajoint.*

[29] According to the analysis of [9], this is equivalent to the Barcan formula for $\Box$, which in fact holds as we noticed above.

Theorem 6.3 guarantees the existence of a translation with the above characteristics: to translate $F\alpha$, transform $\tau(\alpha)$ into a forest of degree at least one, remove its root-labels and go back to a formula.[30]

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Appendix

We give here some information on the categorical background used in the paper. What follows might be formally sufficient to understand the paper, however, this appendix is conceived only as a guide in order to orient the readers which are suggested to consult a textbook for more examples, proofs and motivations.

A *category* $\mathbf{C}$ consists of two classes, the class of the objects O and the class of the arrows A of $\mathbf{C}$. To each arrow are associated two objects, its domain and its codomain. We usually write $k: B \to C$ to mean that $k \in A$ and that B is the domain of k and C its codomain. $\mathbf{C}[B, C]$ denotes the totality of arrows of domain B and codomain C. To each object $B \in O$ an arrow 1_B of domain B and codomain B is assigned. A partial operation of composition $\mathbf{C}[B, C] \times \mathbf{C}[C, D] \to \mathbf{C}[B, D]$ is also given and is required to satisfy the following identity and associativity axioms:

$$k1_C = k, \qquad 1_B k = k, \qquad k(lm) = (kl)m$$

for every $k: B \to C$, $l: C \to D$, $m: D \to E$. This completes the definition of a category. Examples are easily found: sets and functions, Boolean algebras and Boolean morphisms, etc. Notice also that a single preordered set is a category ($\mathbf{C}[B, C]$ is in this case either a singleton or empty, depending on the relation $B \leqslant C$ to hold or not).

The notion of *functor* is the notion of morphism between categories $F: \mathbf{C} \to \mathbf{D}$. It is a correspondence that associates with each object B of $\mathbf{C}$ an object $F(B)$ of $\mathbf{D}$ and with an arrow $k \in \mathbf{C}[B, C]$ an arrow $F(k) \in \mathbf{D}[F(B), F(C)]$ in such a way that identity and composition are preserved, i.e.:

$$F(1_B) = 1_{F(B)}, \qquad F(kl) = F(k)F(l)$$

for every B and for every pair of composable arrows k, l. If $\mathbf{C}$ and $\mathbf{D}$ are both preordered sets, then the notion of functor reduces to the notion of order-preserving map.

[30] Notice that the translation τ we have identified does not reflect provability, in the sense that provability of $\tau(\alpha)$ does not imply provability of α in K_t: this is easily seen from the fact that Fp is translated into a formula provably equivalent to $\top$. Moreover, the tense operator F is not preserved by all substitution morphisms.

Given two categories $\mathbf{C}$, $\mathbf{D}$ and two functors $|-|: \mathbf{C} \to \mathbf{D}$, $P: \mathbf{D} \to \mathbf{C}$, we say that P is left adjoint to $|-|$ iff for every pair B, R of objects of $\mathbf{C}$ and $\mathbf{D}$, respectively, there is a "transposition" bijection

$$(-)^{\mathrm{T}}: \mathbf{D}[R, |B|] \to \mathbf{C}[P(R), B]$$

(depending of course on R and B, but we usualy omit such subscripts) satisfying the following naturality conditions

$$(h|\mu|)^{\mathrm{T}} = h^{\mathrm{T}}\mu, \qquad (kh)^{\mathrm{T}} = P(k)h^{\mathrm{T}}$$

for $h: R \to |B|$ and $k: S \to R$ in $\mathbf{D}$ and for $\mu: B \to C$ in $\mathbf{C}$. The inverse transposition

$$(-)^{\mathrm{t}}: \mathbf{C}[P(R), B] \to \mathbf{D}[R, |B|],$$

consequently (better, equivalently) satisfies the corresponding naturality conditions

$$\varphi^{\mathrm{t}}|\mu| = (\varphi\mu)^{\mathrm{t}}, \qquad k\varphi^{\mathrm{t}} = (P(k)\varphi)^{\mathrm{t}}$$

for $\varphi: P(R) \to B$ and $\mu: B \to C$ in $\mathbf{C}$ and for $k: S \to R$ in $\mathbf{D}$. The naturality conditions may be also expressed as the commutativity of squares like:

$$\begin{tikzcd} \mathbf{C}[P(R), B] \arrow[r, "P(k)(-)"] \arrow[d, "(-)^{\mathrm{t}}"'] & \mathbf{C}[P(S), B] \arrow[d, "(-)^{\mathrm{t}}"] \\ \mathbf{D}[R, |B|] \arrow[r, "k(-)"'] & \mathbf{D}[S, |B|] \end{tikzcd}$$

where $P(k)(-)$ means left composition with $P(k)$ and similarly $k(-)$ means left composition with k (we used this squares formulation in Section 4).

If both $\mathbf{C}$ and $\mathbf{D}$ are preordered sets, we may ignore of course the naturality conditions and the definition of adjoint functors simply requires that

$$P(R) \leqslant B \text{ iff } R \leqslant |B|$$

for every $R \in \mathbf{D}$ and $B \in \mathbf{C}$.

Notice that the left adjoint to a functor $|-|: \mathbf{C} \to \mathbf{D}$ is unique up to isomorphism (provided that it exists). Moreover, given four functors $|-|_1: \mathbf{C}_1 \to \mathbf{C}_2$, $P_1: \mathbf{C}_2 \to \mathbf{C}_1$, $|-|_2: \mathbf{C}_2 \to \mathbf{C}_3$ and $P_2: \mathbf{C}_3 \to \mathbf{C}_2$, such that P_1 is left adjoint to $|-|_1$ and P_2 is left adjoint to $|-|_2$, we have that P_2P_1 is left adjoint to the composite functor $|-|_1|-|_2$.

There are some general theorems for existence of left adjoints: we leave the reader to consult a textbook for the precise formulations (during the paper we used once such theorems, but not in a way that may produce difficulties in understanding the constructions: the real point is to give explicit descriptions of such adjoints, not merely to prove that they exist).

Another important notion, strictly related to adjoint functors, is the notion of universal pair. Given two categories $\mathbf{C}$, $\mathbf{D}$, a functor $|-|: \mathbf{C} \to \mathbf{D}$ and an object R in $\mathbf{D}$, we say that a pair $\langle P(R), \eta_R \rangle$ (where $P(R)$ is an object of $\mathbf{C}$ and $\eta_R \in \mathbf{D}[R, |P(R)|]$) is universal to the functor $|-|$ iff for every object B in $\mathbf{C}$ and for every arrow $h: R \to |B|$, there is a unique arrow h^{T}: $P(R) \to B$ in $\mathbf{C}$ such that $\eta_R|h^{\mathrm{T}}| = h$. Notice that the concept of universal pair is nothing but the abstract version (i.e. referred to an arbitrary functor $|-|$) of the familiar concept of free algebra.

Adjoint functors and universal pairs are related in the following way: if the functor $|-|$: $\mathbf{C} \to \mathbf{D}$ does have a left adjoint P: $\mathbf{D} \to \mathbf{C}$, then for every R in $\mathbf{D}$, the pair $\langle P(R), 1^{\mathrm{t}}_{P(R)} \rangle$ is universal to the functor $|-|$. Conversely, if for every object R is $\mathbf{D}$ there exists a universal pair $\langle P(R), \eta_R \rangle$ to the functor $|-|$: $\mathbf{C} \to \mathbf{D}$, then $|-|$ turns out to have a left adjoint P: its value on the objects and the transposition bijection are directly given by the definition of universal pair, whereas the value of P at any arrow k: $S \to R$ in $\mathbf{D}$ is $(k\eta_R)^{\mathrm{T}}$.

The following fact is often implicitly used in the paper: suppose that we are given two functors F: $\mathbf{C} \to \mathbf{D}$, G: $\mathbf{D} \to \mathbf{E}$, an object E in $\mathbf{E}$, a universal pair $\langle G^*(E), \eta_E \rangle$ from E to G and an universal pair $\langle F^*(G^*(E)), \eta_{G^*(E)} \rangle$ from $G^*(E)$ to F. Then $\langle F^*(G^*(E)), \eta_E G(\eta_{G^*(E)}) \rangle$ is universal from E to the composite functor FG.

We recall the definition of *coproduct* of two objects B_1 and B_2 in a category $\mathbf{C}$: it is an object $B_1 + B_2$ of $\mathbf{C}$ endowed with two arrows ι_1: $B_1 \to B_1 + B_2$ and ι_2: $B_2 \to B_1 + B_2$, such that for every object C and arrows μ_1: $B_1 \to C$ and μ_2: $B_2 \to C$, there exists a unique arrow $[\mu_1, \mu_2]$: $B_1 + B_2 \to C$ such that $\iota_1[\mu_1, \mu_2] = \mu_1$ and $\iota_2[\mu_1, \mu_2] = \mu_2$. For arrows ξ_1: $B_1 \to C_1$ and ξ_2: $B_2 \to C_2$, $\xi_1 + \xi_2$ stands for $[\xi_1 \iota_1, \xi_2 \iota_2]$. The following equations may be easily deduced from the above definitions:

$$[\mu_1, \mu_2]\nu = [\mu_1\nu, \mu_2\nu], \qquad \iota_1(\xi_1 + \xi_2) = \xi_1\iota_1, \qquad \iota_2(\xi_1 + \xi_2) = \xi_2\iota_2.$$

Coproducts exist in any category which is a variety of algebras: given two algebras B_1 and B_2, we can represent them as quotient of free algebras $\mathscr{F}(X_1)/C_1$ and $\mathscr{F}(X_1)/C_2$ (with X_1 and X_2 disjoints and C_1, C_2 congruences). To build the coproduct, take $\mathscr{F}(X_1 \cup X_2)/C$ where C is the congruence generated by $C_1 \cup C_2$.

Coproducts are special cases of colimits. We do not give here the general definition of colimit but only the special case of chain colimit used in Section 1. A *chain diagram* in $\mathbf{C}$ is a succession of objects $B_0, B_1, \ldots$ equipped with morphisms ε_i: $B_i \to B_{i+1}$. A *cone* for a chain diagram as above consists of an object L and of "injection" morphisms η_i: $B_i \to L$ such that for every $i \in N$ the following triangles commute:

$$\begin{tikzcd}
B_i \arrow[dr, "\varepsilon_i"] \arrow[dd, "\eta_i"'] & \\
 & B_{i+1} \arrow[dl, "\eta_{i+1}"] \\
L &
\end{tikzcd}$$

A *chain colimit* (inductive or direct limit in the standard algebraic terminology) for a chain diagram is a universal cone for it, that is a cone as above such that for every other cone $\langle L^*, \{\eta_i^*\}_i\rangle$ there exists a unique morphism $\eta^*: L \to L^*$ such that for every $i \in N$, $\eta_i \eta^* = \eta_i^*$. Chain colimits are computed in varieties of algebras by means of equivalence classes: given a chain diagram $\langle\{B_i\}_i, \{\varepsilon_i\}_i\rangle$, one takes the disjoint union of the carrier sets $\underline{B}_i$'s and introduces in it the equivalence relation:

$$\langle a, i\rangle \approx \langle b, j\rangle \quad \text{iff}\ \exists k \geqslant i, j(\varepsilon_{i,k}(\langle a, i\rangle) = \varepsilon_{j,k}(\langle b, j\rangle)).$$

The operations are easily defined on the representative elements of the equivalence classes and the injection maps η_i simply associates with every element its equivalence class. The unique map required by the universal property is also defined on the representative elements of the equivalence classes: given another cone $\langle L^*, \{\eta_i^*\}_i\rangle$, one is forced to define $\eta^*([a, i]) = \eta_i^*(a)$ (notice that the definition is correct because $\langle L^*, \{\eta_i^*\}_i\rangle$ is a cone). A functor $|-|$: $\mathbf{B} \to \mathbf{C}$ *preserves chain colimits* whenever if $\langle L, \{\eta_i\}_i\rangle$ is a chain colimit in $\mathbf{B}$ for the chain diagram $\langle\{B_i\}_i, \{\varepsilon_i\}_i\rangle$, then $\langle |L|, \{|\eta_i|\}_i\}$ is a chain colimit in $\mathbf{C}$ for the chain diagram $\langle\{|B_i|\}_i, \{|\varepsilon_i|\}_i\rangle$. Every left adjoint functor preserves colimits; chain colimits are preserved also by forgetful functors, i.e. by functors between categories of algebras that ignore part of the operations (see e.g. the functor $|-|$: **Boole** $\to$ **SemiL** of Section 1).

The notions of product and of chain limit are dual to the above ones: they are obtained by reversing the direction of every arrow involved in the definition (these notions are rarely used in the paper).

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