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Differential Geometry and its Applications 26 (2008) 419–433

DIFFERENTIAL GEOMETRY AND ITS APPLICATIONS

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Pseudo-Riemannian 3-manifolds with prescribed distinct constant Ricci eigenvalues

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Received 6 December 2006; received in revised form 12 March 2007

Available online 21 December 2007

Communicated by O. Kowalski

Abstract

We study three-dimensional pseudo-Riemannian manifolds having distinct constant principal Ricci curvatures. These spaces are described via a system of differential equations, and a simple characterization is proved to hold for the locally homogeneous ones. We then generalize the technique used in [O. Kowalski, F. Prüfer, On Riemannian 3-manifolds with distinct constant Ricci eigenvalues, Math. Ann. 300 (1994) 17–28] for Riemannian manifolds and construct explicitly homogeneous and non-homogeneous pseudo-Riemannian metrics in $\mathbb{R}^3$, having the prescribed principal Ricci curvatures.

MSC: 53C50; 53C20; 53C25

Keywords: Pseudo-Riemannian manifolds; Ricci eigenvalues; Curvature homogeneity

1. Introduction

A pseudo-Riemannian manifold (M, g) is *homogeneous* provided that, for any points $p, q \in M$, there exists an isometry ϕ such that $\phi(p) = q$, it is *locally homogeneous* if there is a local isometry mapping a neighborhood of p into a neighborhood of q [19]. Pseudo-Riemannian homogeneous and locally homogeneous spaces are one of the most interesting research fields in pseudo-Riemannian geometry. Recently, many authors investigated the problem of extending several results concerning homogeneous Riemannian manifolds, to the more general framework of pseudo-Riemannian geometry (in particular, to Lorentzian geometry).

A pseudo-Riemannian manifold (M, g) is *curvature homogeneous up to order k* if, for any points $p, q \in M$, there exists a linear isometry $\phi: T_pM \to T_qM$ such that $\phi^*(\nabla^i R(q)) = \nabla^i R(p)$ for all $i \leqslant k$. When $k = 0$, (M, g) is simply called a *curvature homogeneous space*. Clearly, a locally homogeneous space is curvature homogeneous of any order k. Conversely, if k is sufficiently high, curvature homogeneity up to order k implies local homogeneity. This

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[1] Supported by funds of MURST (PRIN) GNSAGA and the University of Lecce.

0926-2245/$ – see front matter
doi:10.1016/j.difgeo.2007.11.031

result was proved by Singer [22] for Riemannian manifolds. Through the equivalence theorem for G-structures due to Cartan and Sternberg [23], Singer's result can be extended to the pseudo-Riemannian case.

If $\dim M = 2$, then curvature homogeneity (up to order 0) already implies local homogeneity. However, when $\dim M \geqslant 3$, a curvature homogeneous space needs not to be locally homogeneous. Several examples are known, both in Riemannian and Lorentzian geometry, of non-homogeneous curvature homogeneous spaces. We can refer to [2] for a survey and further references for the Riemannian case. Bueken and Vanhecke [7] showed that there exist non-homogeneous Lorentzian three-spaces which are curvature homogeneous up to order one. Indeed, curvature homogeneity of order two is needed to ensure that a three-dimensional Lorentzian manifold is locally homogeneous [6] (in the Riemannian case, curvature homogeneity of order one is sufficient [21]).

Riemannian curvature homogeneous three-spaces have been extensively studied. A three-dimensional Riemannian manifold (M, g) is curvature homogeneous if and only if its principal Ricci curvatures are constant. The problem of classifying three-dimensional manifolds with prescribed constant Ricci curvatures goes back to Bianchi [1]. In 1991, Yamato [24] gave criteria for local homogeneity and described the first examples of non-homogeneous Riemannian three-manifolds with constant distinct Ricci eigenvalues. In [13], Kowalski studied the case with two distinct constant Ricci eigenvalues, proving an existence result. Some non-trivial examples were also given, but a universal family of examples is not known. An alternative proof of the main existence theorem of [13] was given in [3]. In the case with all distinct principal Ricci curvatures, a general existence theorem of Cauchy–Kowalewski type was proved in [16], while [15] provides a universal family of examples, showing how to construct explicitly Riemannian metrics on $\mathbb{R}^3$ having the prescribed distinct constant Ricci eigenvalues. The argument used in [16] was further developed in [17], where the existence theorem was also extended by taking arbitrary distinct functions as Ricci eigenvalues. Note also that an explicit classification has been obtained under some additional geometric conditions. A survey can be found in [17].

The Lorentzian case appears more complex. In fact, while a Riemannian manifold (M, g) always admits an orthonormal frame diagonalizing its Ricci operator Q, in the Lorentzian case Q can take four different forms (called *Segre types*). Hence, prescribing the principal Ricci curvatures of a Lorentzian three-manifold *is not* equivalent to prescribe its Ricci operator, and one needs to specialize the study to the different Segre types.

The study of three-dimensional curvature homogeneous Lorentzian spaces was undertaken first by Bueken, who considered the diagonal case with two distinct Ricci eigenvalues [4], as well as a non-diagonal case [5]. Other non-diagonal examples have been investigated in [10] and [12]. Clearly, the diagonal case is trivial when all principal Ricci curvatures coincide: the space has constant sectional curvature (in particular, it is locally homogeneous).

As references above show, several authors focused on the problem of finding explicit examples of non-homogeneous curvature homogeneous pseudo-Riemannian three-manifolds. However, the problem of constructing explicit homogeneous pseudo-Riemannian metrics, having the required curvature properties (in particular, prescribed Ricci eigenvalues), is far from trivial. This problem is motivated both by the important role played by locally homogeneous spaces and by the abstractness of some known examples (for example, Lie groups only described through the form of their Lie algebra). In this paper, we shall provide a complete description of a three-dimensional pseudo-Riemannian manifold (M, g), having constant distinct principal Ricci curvatures and a diagonal Ricci operator. We shall then exhibit explicit examples of non-homogeneous and homogeneous pseudo-Riemannian metrics in $\mathbb{R}^3$, having the same (constant) principal Ricci curvatures of a given (M, g).

The paper is organized in the following way. In Section 2, we give a general description of (M, g) via a system of differential equations for the functions determining its Levi Civita connection. We also investigate the isometries among such spaces and characterize the locally homogeneous cases. In Section 3, we determine a system of partial differential equations, whose solutions permit to construct explicitly curvature homogeneous pseudo-Riemannian metrics on $\mathbb{R}^3$ with the prescribed distinct Ricci eigenvalues. In Section 4, a class of non-homogeneous solutions of the system is described. Finally, in Section 5 a family of pseudo-Riemannian locally homogeneous metrics on $\mathbb{R}^3$, having q, q_2 and q_3 as Ricci eigenvalues, is explicitly described. The triplet (q_1, q_2, q_3) is chosen in a way that covers all locally homogeneous Riemannian three-manifolds with distinct Ricci eigenvalues and most of the Lorentzian ones (with a diagonal Ricci operator). The remaining Lorentzian cases, both with diagonal and non-diagonal Ricci operator, will be treated in a forthcoming paper.

2. Pseudo-Riemannian 3-manifolds with constant Ricci eigenvalues: general description

Let (M, g) be a connected three-dimensional pseudo-Riemannian manifold. We denote by ∇ the Levi Civita connection of (M, g) and by R its curvature tensor, taken with the sign convention

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

Since $\dim M = 3$, its curvature tensor is completely determined by the Ricci tensor ϱ, defined, for any point $p \in M$ and $X, Y \in T_pM$, by

$$\varrho(X,Y)_p = \sum_{i=1}^{3} \varepsilon_i g\big(R(X, e_i)Y, e_i\big), \tag{2.1}$$

where $\{e_1, e_2, e_3\}$ is a pseudo-orthonormal basis of T_pM and $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all i.

Because of the symmetries of the curvature tensor, the Ricci tensor ϱ is symmetric [19]. Hence, the *Ricci operator* Q, defined by $g(QX, Y) = \varrho(X, Y)$, is self-adjoint. In the Riemannian case, this fact implies that there exists an orthonormal basis diagonalizing Q, while for a Lorentzian manifold four different cases can occur ([18, p. 261], [4]), and there exists a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, such that Q takes one of the following forms, called *Segre types*:

$$\text{Segre type } \{11,1\}: \begin{pmatrix} \bar a & 0 & 0 \\ 0 & \bar b & 0 \\ 0 & 0 & \bar c \end{pmatrix}, \qquad \text{Segre type } \{1z\bar z\}: \begin{pmatrix} \bar a & 0 & 0 \\ 0 & \bar b & \bar c \\ 0 & -\bar c & \bar b \end{pmatrix},$$

$$\text{Segre type } \{21\}: \begin{pmatrix} \bar a & 0 & 0 \\ 0 & \bar b & \varepsilon \\ 0 & -\varepsilon & \bar b - 2\varepsilon \end{pmatrix}, \qquad \text{Segre type } \{3\}: \begin{pmatrix} \bar b & \bar a & -\bar a \\ \bar a & \bar b & 0 \\ \bar a & 0 & \bar b \end{pmatrix}.$$

Assume now (M, g) is curvature homogeneous. Then, is easily seen that its Ricci operator Q has the same Segre type at every point $p \in M$ and that, at least locally, there exists a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ such that Q is given by one of the expressions above, where $\bar a$, $\bar b$ and $\bar c$ are constants. We shall refer to the eigenvalues of the Ricci operator as *principal Ricci curvatures* or, more briefly, *Ricci eigenvalues*. Throughout the paper, we shall deal with the diagonal case with three distinct principal Ricci curvatures, that is, we assume that, with respect to $\{e_i\}$, we have

$$q_i = \varepsilon_i \varrho_{ii}, \qquad \varrho_{ij} = 0 \quad \text{if } i \neq j, \tag{2.2}$$

and

$$q_i \neq q_j \quad \text{if } i \neq j. \tag{2.3}$$

Following [8], we now put

$$\nabla_{e_i} e_j = \sum_k \varepsilon_j b^i_{jk} e_k. \tag{2.4}$$

Clearly, the functions b^i_{jk} determine completely the Levi Civita connection, and conversely. Note that from $\nabla g = 0$ it follows at once

$$b^i_{kj} = -b^i_{jk}, \tag{2.5}$$

for all i, j, k. In particular,

$$b^i_{jj} = 0 \tag{2.6}$$

for all indices i and j. We now put

$$b^1_{12} = \alpha, \qquad b^1_{13} = \beta \qquad b^1_{23} = \gamma, \qquad b^2_{12} = \kappa, \qquad b^2_{13} = \mu, \qquad b^2_{23} = \nu,$$
$$b^3_{12} = \sigma, \qquad b^3_{13} = \tau, \qquad b^3_{23} = \psi. \tag{2.7}$$

Reversing the metric [18, p. 92] when needed and suitably rearranging e_1, e_2, e_3, we can assume $\varepsilon_i = 1$ for all i in the Riemannian case, and $\varepsilon_1 = -\varepsilon_2 = \varepsilon_3 = 1$ in the Lorentzian one. Thus, we can treat in a unified way the Riemannian and Lorentzian cases, by assuming $\varepsilon_1 = \varepsilon_3 = 1$ and $\varepsilon_2 = \varepsilon = \pm 1$. By (2.4)–(2.7) we get that the Levi Civita connection ∇ of (M, g) is completely determined by

$$\begin{aligned} &\nabla_{e_1} e_1 = \alpha e_2 + \beta e_3, && \nabla_{e_2} e_1 = \kappa e_2 + \mu e_3, && \nabla_{e_3} e_1 = \sigma e_2 + \tau e_3, \\ &\nabla_{e_1} e_2 = \varepsilon(-\alpha e_1 + \gamma e_3), && \nabla_{e_2} e_2 = \varepsilon(-\kappa e_1 + \nu e_3), && \nabla_{e_3} e_2 = \varepsilon(-\sigma e_1 + \psi e_3), \\ &\nabla_{e_1} e_3 = -\beta e_1 - \gamma e_2, && \nabla_{e_2} e_3 = -\mu e_1 - \nu e_2, && \nabla_{e_3} e_3 = -\tau e_1 - \psi e_2. \end{aligned} \tag{2.8}$$

In particular, from (2.8) we get at once

$$\begin{aligned} &[e_1, e_2] = -\varepsilon\alpha e_1 - \kappa e_2 + (\varepsilon\gamma - \mu) e_3, \\ &[e_1, e_3] = -\beta e_1 - (\gamma + \sigma) e_2 - \tau e_3, \\ &[e_2, e_3] = (\varepsilon\sigma - \mu) e_1 - \nu e_2 - \varepsilon\psi e_3. \end{aligned} \tag{2.9}$$

Note that, conversely, functions (b^i_{jk}) are completely determined by the Lie brackets of vectors e_1, e_2, e_3, since the well-known *Koszul formula* [19] yields

$$2\varepsilon_j \varepsilon_k b^i_{jk} = 2g(\nabla_{e_i} e_j, e_k) = g\big([e_i, e_j], e_k\big) - g\big([e_j, e_k], e_i\big) + g\big([e_k, e_i], e_j\big). \tag{2.10}$$

Before using (2.8) to express curvature conditions (2.2), we remark that functions $\alpha, \dots, \psi$ are not all independent. In fact, from (2.2) and (2.4) it easily follows

$$\nabla_i \varrho_{jk} = -\varepsilon_j \varepsilon_k (q_j - q_k) b^i_{jk}, \tag{2.11}$$

for all indices i, j, k (in particular, $\nabla_i \varrho_{jj} = 0$ for all i, j). Since (M, g) is curvature homogeneous, its scalar curvature $r = \operatorname{tr} \varrho$ is constant. The well-known *divergence formula* $dr = 2 \operatorname{div} \varrho$ [19] then implies

$$\nabla_1 \varrho_{12} + \nabla_3 \varrho_{23} = 0, \qquad \nabla_1 \varrho_{13} + \varepsilon \nabla_2 \varrho_{23} = 0, \qquad \varepsilon \nabla_2 \varrho_{12} + \nabla_3 \varrho_{13} = 0. \tag{2.12}$$

Using (2.11) in (2.12), since all q_i are distinct we obtain

$$\nu = -\frac{q_1 - q_3}{q_2 - q_3}\beta, \qquad \tau = -\frac{q_1 - q_2}{q_1 - q_3}\kappa, \qquad \psi = -\frac{q_1 - q_2}{q_2 - q_3}\alpha. \tag{2.13}$$

Then, putting $c = -\frac{q_1 - q_3}{q_2 - q_3}$, we have $c \neq 0, -1$ and we can rewrite (2.13) as follows:

$$\nu = c\beta, \qquad \tau = -\frac{1 + c}{c}\kappa, \qquad \psi = (1 + c)\alpha. \tag{2.14}$$

We can now compute the components of the curvature tensor with respect to $\{e_1\}$ starting from (2.8). Then, (2.1) and (2.2) provide the components of the Ricci tensor and the Ricci eigenvalues, respectively. Via standard calculations, also taking into account (2.14), we get

$$\begin{aligned} -q_1 = {} & e_1(\kappa) - e_2(\alpha) + \varepsilon\alpha^2 + \kappa^2 + c\beta^2 - \gamma\mu - \sigma(\varepsilon\gamma - \mu) \\ & - \frac{1 + c}{c} e_1(\kappa) - e_3(\beta) + \beta^2 + \frac{(1 + c)^2}{c^2}\kappa^2 + \varepsilon\big(\gamma\sigma - (1 + c)\alpha^2\big) + \mu(\gamma + \sigma), \end{aligned} \tag{2.15}$$

$$\begin{aligned} -q_2 = {} & e_1(\kappa) - e_2(\alpha) + \varepsilon\alpha^2 + \kappa^2 + c\beta^2 - \gamma\mu - \sigma(\varepsilon\gamma - \mu) \\ & + (1 + c)e_2(\alpha) - ce_3(\beta) + c^2\beta^2 + (1 + c)^2\varepsilon\alpha^2 - \frac{1 + c}{c}\kappa^2 - \mu\sigma + \gamma(\mu - \varepsilon\sigma), \end{aligned} \tag{2.16}$$

$$\begin{aligned} -q_3 = {} & -\frac{1 + c}{c} e_1(\kappa) - e_3(\beta) + \beta^2 + \frac{(1 + c)^2}{c^2}\kappa^2 + \varepsilon\big(\gamma\sigma - (1 + c)\alpha^2\big) + \mu(\gamma + \sigma) \\ & + (1 + c)e_2(\alpha) - ce_3(\beta) + c^2\beta^2 + (1 + c)^2\varepsilon\alpha^2 - \frac{1 + c}{c}\kappa^2 - \mu\sigma + \gamma(\mu - \varepsilon\sigma). \end{aligned} \tag{2.17}$$

We can express conditions (2.15)–(2.17) in a simpler form. In fact, putting

$$\lambda_1 = \frac{1}{2}(-q_1 + q_2 + q_3), \qquad \lambda_2 = \frac{1}{2}(q_1 - q_2 + q_3), \qquad \lambda_3 = \frac{1}{2}(q_1 + q_2 - q_3), \tag{2.18}$$

we have that the Ricci principal curvatures $\{q_i\}$ determine the values of $\{\lambda_i\}$, and conversely. Note also that

$$q_1 = \lambda_2 + \lambda_3, \qquad q_2 = \lambda_1 + \lambda_3, \qquad q_3 = \lambda_1 + \lambda_2$$

and so,

$$\lambda_i - \lambda_j = -(q_i - q_j), \tag{2.19}$$

for all indices i, j. From (2.18) it follows that (2.15)–(2.17) are equivalent to

$$-\lambda_3 = e_1(\kappa) - e_2(\alpha) + \varepsilon\alpha^2 + \kappa^2 + c\beta^2 - \gamma\mu - \sigma(\varepsilon\gamma - \mu), \tag{2.20}$$

$$-\lambda_2 = -\frac{1+c}{c}e_1(\kappa) - e_3(\beta) + \beta^2 + \frac{(1+c)^2}{c^2}\kappa^2 + \varepsilon\bigl(\gamma\sigma - (1+c)\alpha^2\bigr) + \mu(\gamma+\sigma), \tag{2.21}$$

$$-\lambda_1 = (1+c)e_2(\alpha) - ce_3(\beta) + c^2\beta^2 + (1+c)^2\varepsilon\alpha^2 - \frac{1+c}{c}\kappa^2 - \mu\sigma + \gamma(\mu - \varepsilon\sigma). \tag{2.22}$$

Now, again by (2.1) and (2.8) and taking into account (2.7), we easily obtain that the second equation in (2.2) is satisfied if and only if

$$e_1(\mu) - e_2(\beta) - \frac{1}{c}\kappa\mu + \varepsilon(c-1)\alpha\beta - \varepsilon\frac{1+2c}{c}\gamma\kappa = 0, \tag{2.23}$$

$$(1+c)e_1(\alpha) - e_3(\gamma) + (1+c)\beta\gamma + \frac{(1+c)(2+c)}{c}\alpha\kappa + (c-1)\beta\sigma = 0, \tag{2.24}$$

$$e_2(\sigma) - e_3(\kappa) + c\varepsilon\alpha\sigma + (2+c)\alpha\mu + (1+2c)\beta\kappa = 0. \tag{2.25}$$

Therefore, we can conclude that curvature homogeneous pseudo-Riemannian three-manifolds (M,g), having a diagonal Ricci operator (that is, of Segre type $\{11,1\}$) and (constant) principal Ricci curvatures q_1, q_2 and q_3, are characterized by Eqs. (2.20)–(2.25). By (2.8) (equivalently, (2.9)), functions $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, appearing in (2.20)–(2.25) determine the Levi Civita connection of (M,g). In this way, we proved the following

Theorem 2.1. *Let (M,g) be a three-dimensional pseudo-Riemannian manifold. (M,g) has a diagonal Ricci operator with constant principal Ricci curvatures q_1, q_2 and q_3 if and only if there exist (at least, locally) a pseudo-orthonormal frame field $\{e_1, e_2, e_3\}$ and six functions $\alpha, \beta, \gamma, \kappa, \mu, \sigma$, such that (2.9) and (2.20)–(2.25) hold.*

In order to give a complete local classification result, consider now two pseudo-Riemannian three-manifolds (M,g), (M',g') which are solutions of (2.20)–(2.25). In other words, according to Theorem 2.1, (M,g) and (M',g') are curvature homogeneous three-spaces, with diagonal Ricci operator and (constant) principal Ricci curvatures q_1, q_2 and q_3 and so, they admit (at least, locally) pseudo-orthonormal frame fields $\{e_1, e_2, e_3\}$ and $\{e'_1, e'_2, e'_3\}$ respectively, for which (2.20)–(2.25) hold. Then, we have the following

Theorem 2.2. *A differentiable mapping $f: M \to M'$ is an isometry if and only if*

$$f_*(e_i) = \varepsilon'_i e'_i,$$

where $\varepsilon'_i = \pm 1$ for all $i = 1, 2, 3$.

Proof. The "if" part is obvious. As concerns the "only if" part, suppose that f is an isometry. Then, f must preserve the eigenspaces of the Ricci operator. Since the principal Ricci curvatures are distinct, these eigenspaces are one-dimensional. Therefore, we must have $f_*(e_i) = \varepsilon'_i e'_i$, where $\varepsilon'_i = \pm 1$, for all $i = 1, 2, 3$. □

Remark 2.3. If only two Ricci eigenvalues are distinct, one of the eigenspaces of the Ricci operator is two-dimensional. Hence, there exist infinitely many different pseudo-orthonormal frames of Ricci eigenvectors, and one can choose a special pseudo-orthonormal frame, for which some connection functions b^i_{jk} vanish [4]. A similar technique was also used for the non-diagonal case treated in [5]. Theorem 2.2 shows a greater rigidity and complexity for the case with three distinct Ricci eigenvalues.

We now determine a simple criterion to recognize locally homogeneous three-spaces, among all solutions of (2.20)–(2.25). We start by showing that *pseudo-Riemannian three-manifolds, with distinct Ricci eigenvalues, are never locally symmetric*. In fact, since all q_i are distinct, by (2.11) we can conclude that (M,g) is locally symmetric (equivalently, Ricci-parallel) if and only if $b^i_{jk}=0$ for all i,j,k. But then, (2.20)–(2.25) imply that (M,g) is flat. In particular, $q_1=q_2=q_3=0$ and this can not occur.

As the author proved in [8], a three-dimensional locally homogeneous Lorentzian three-manifold is either locally symmetric or locally isometric to a Lie group, equipped with a left-invariant Lorentzian metric. The corresponding result for Riemannian manifolds was proved in [21]. Taking into account these results, we can now prove the following

Theorem 2.4. *Let (M,g) be a three-dimensional pseudo-Riemannian manifold, for which (2.9) and (2.20)–(2.25) hold. (M,g) is locally homogeneous if and only if the functions $\alpha,\beta,\gamma,\kappa,\mu,\sigma$ are constant.*

Proof. If (2.9) holds for some constants $\alpha,\beta,\gamma,\kappa,\mu,\sigma$, then by (2.10) we have that all b^i_{jk} are constant (at least, locally). Since the components of the Ricci tensor and of its derivatives of any order with respect to $\{e_i\}$ depend on b^i_{jk}, we have that (M,g) is curvature homogeneous up to any order k and so, it is locally homogeneous.

Conversely, assume now (M,g) is locally homogeneous. As we already noticed, (M,g) is not locally symmetric. Hence, the main results of [8] and [21] imply that (M,g) is locally isometric to a three-dimensional Lie group G, equipped with a left-invariant pseudo-Riemannian metric. The Lie algebra $\mathfrak{g}$ of G admits a pseudo-orthonormal basis $\{e'_1,e'_2,e'_3\}$, such that

$$\begin{aligned}
&[e'_1,e'_2]=k_1e'_1+k_2e'_2+k_3e'_3,\\
&[e'_1,e'_3]=k_4e'_1+k_5e'_2+k_6e'_3,\\
&[e'_2,e'_3]=k_7e'_1+k_8e'_2+k_9e'_3,
\end{aligned}\tag{2.26}$$

for some real constants $k_1,\dots,k_9$. The conclusion then follows comparing (2.26) with (2.9). □

3. The basic system of partial differential equations

Let (M,g) be a three-dimensional pseudo-Riemannian manifold, having a diagonal Ricci operator and constant distinct principal Ricci curvatures q_1,q_2 and q_3. Generalizing the technique used in [15] for the Riemannian case, we shall express Eqs. (2.20)–(2.25) via a system of partial differential equations for some functions of three variables, whose solutions permit to build explicitly pseudo-Riemannian metrics on $\mathbb{R}^3$ with the curvature properties of (M,g).

We fix a point $p\in M$ and consider a pseudo-orthonormal frame field $\{e_1,e_2,e_3\}$ as in Theorem 2.1. We then choose a surface S through p transversal to the lines generated by e_3, a local coordinates system (w,x) on S and a neighborhood U_p of p, sufficiently small that each $q\in U_p$ is situated on exactly one line generated by e_3 and passing through one point $\bar q\in S$.

Choose an orientation of S and define the coordinate function y in U_p as the oriented distance of the point q from S along the corresponding line, that is,

$$y(q)=\operatorname{dist}\bigl(q,\pi(q)\bigr),\tag{3.1}$$

where $\pi:U_p\to S$ is the corresponding projection. We also define

$$w(q)=w\bigl(\pi(q)\bigr),\qquad x(q)=x\bigl(\pi(q)\bigr).\tag{3.2}$$

In this way, a local coordinate system (w,x,y) is introduced in U_p. Notice that $e_3=\frac{\partial}{\partial y}$ and the coframe $\{\omega^1,\omega^2,\omega^3\}$ of $\{e_1,e_2,e_3\}$ must take the form

$$\begin{aligned}
&\omega^1=A\,dw+B\,dx,\\
&\omega^2=C\,dw+D\,dx,\\
&\omega^3=G\,dw+H\,dx+dy,
\end{aligned}\tag{3.3}$$

for some functions A,B,C,D,G,H.

Next, we introduce the connection forms on (M, g), putting

$$\omega^i_j = \sum_k \varepsilon_j b^i_{jk} \omega^k. \tag{3.4}$$

Connection forms completely determine the Levi Civita connection, because from (3.4) it follows

$$\nabla_{e_i} e_j = \sum_k \omega^k_j(e_i) e_k,$$

for all i, j. Moreover, from (2.6) we easily get

$$\omega^i_j + \varepsilon_i \varepsilon_j \omega^j_i = 0 \tag{3.5}$$

for all i, j. In particular, $\omega^i_i = 0$ for all i. The *structure equations* for ω^i_j give

$$d\omega^i + \sum_j \omega^i_j \wedge \omega^j = 0, \tag{3.6}$$

for all indices i. As concerns the curvature forms Ω^i_j, they are completely determined by the standard equations

$$-d\Omega^i_j = d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j. \tag{3.7}$$

Using (2.2) and taking into account (3.5) and (2.18), we obtain that (3.7) is equivalent to

$$\begin{aligned} &d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 = \varepsilon\lambda_3 \omega^1 \wedge \omega^2, \\ &d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 = \lambda_2 \omega^1 \wedge \omega^3, \\ &d\omega^2_3 - \omega^1_2 \wedge \omega^1_3 = \lambda_1 \omega^2 \wedge \omega^3 \end{aligned} \tag{3.8}$$

(where we used $\varepsilon_1 = \varepsilon_3 = 1$ and put $\varepsilon_2 = \varepsilon$). We now use (3.3) in (3.6). After some routine calculations, also taking into account (2.7) and (2.14), we obtain that (3.6) is equivalent to the following system of nine partial differential equations:

$$\begin{aligned} &B'_w - A'_x = \varepsilon\alpha\mathcal{D} + \beta\mathcal{E} + (\mu - \varepsilon\sigma)\mathcal{F}, \\ &A'_y = -\beta A - (\mu - \varepsilon\sigma)C, \\ &B'_y = -\beta B - (\mu - \varepsilon\sigma)D, \\ &D'_w - C'_x = \kappa\mathcal{D} + (\gamma + \sigma)\mathcal{E} + c\beta\mathcal{F}, \\ &C'_y = -(\gamma + \sigma)A - c\beta C, \\ &D'_y = -(\gamma + \sigma)B - c\beta D, \\ &H'_w - G'_x = (\mu - \varepsilon\gamma)\mathcal{D} - \frac{1+c}{c}\kappa\mathcal{E} + (1+c)\varepsilon\alpha\mathcal{F}, \\ &G'_y = \frac{1+c}{c}\kappa A - (1+c)\varepsilon\alpha C, \\ &H'_y = \frac{1+c}{c}\kappa B - (1+c)\varepsilon\alpha D, \end{aligned} \tag{3.9}$$

where $\mathcal{D}, \mathcal{E}, \mathcal{F}$ are auxiliary functions, defined by

$$\mathcal{D} = AD - BC, \qquad \mathcal{E} = AH - BG, \qquad \mathcal{F} = CH - DG. \tag{3.10}$$

Starting from the connection functions b^i_{jk} of (Mg), system (3.9) permits to determine the functions $A, \ldots, H$ and so, to give explicit pseudo-Riemannian metrics on $\mathbb{R}^3$, with the same Levi Civita connection of (M, g). Notice that, conversely, if $A, \ldots, H$ are known, then by (3.9) we can determine the connection functions b^i_{jk}.

We now use (3.3) to express curvature conditions (3.8). Via some very long but standard calculations, we obtain that (3.8) is equivalent to the following system of differential equations:

$$\begin{aligned}
&A\alpha'_x - B\alpha'_w + C\kappa'_x - D\kappa'_w + G\sigma'_x - H\sigma'_w - \mathcal{D}(U_3+\lambda_3) - \mathcal{E}V_3 - \mathcal{F}W_3 = 0,\\
&A\alpha'_y + C\kappa'_y + G\sigma'_y - \sigma'_w - AV_3 - CW_3 = 0,\\
&B\alpha'_y + D\kappa'_y + H\sigma'_y - \sigma'_x - BV_3 - DW_3 = 0,\\
&A\beta'_x - B\beta'_w + C\mu'_x - D\mu'_w - \frac{1+c}{c}G\kappa'_x + \frac{1+c}{c}H\kappa'_w - \mathcal{D}U_2 - \mathcal{E}(V_2+\lambda_2) - \mathcal{F}W_2 = 0,\\
&A\beta'_y + C\mu'_y - \frac{1+c}{c}G\kappa'_y + \frac{1+c}{c}\kappa'_w - A(V_2+\lambda_2) - CW_2 = 0,\\
&B\beta'_y + D\mu'_y - \frac{1+c}{c}H\kappa'_y + \frac{1+c}{c}\kappa'_x - B(V_2+\lambda_2) - DW_2 = 0,\\
&A\gamma'_x - B\gamma'_w + cC\beta'_x - cD\beta'_w + (1+c)G\alpha'_x - (1+c)H\alpha'_w - \mathcal{D}U_1 - \mathcal{E}V_1 - \mathcal{F}(W_1+\lambda_1) = 0,\\
&A\gamma'_y + cC\beta'_y + (1+c)G\alpha'_y - (1+c)\alpha'_w - AV_1 - C(W_1+\lambda_1) = 0,\\
&B\gamma'_y + cD\beta'_y + (1+c)H\alpha'_y - (1+c)\alpha'_x - BV_1 - D(W_1+\lambda_1) = 0,
\end{aligned}\tag{3.11}$$

where we put

$$\begin{aligned}
&U_1 = -c\varepsilon\alpha\gamma + (2+c)\alpha\mu + (c-1)\beta\kappa,\\
&V_1 = -\frac{(1+c)(2+c)}{c}\alpha\kappa + (1-c)\beta\gamma + (1+c)\beta\sigma,\\
&W_1 = (1+c)^2\varepsilon\alpha^2 + c^2\beta^2 - \frac{1+c}{c}\kappa^2 + \gamma\mu - \varepsilon\gamma\sigma - \mu\sigma,\\
&U_2 = (1-c)\varepsilon\alpha\beta + \frac{1+2c}{c}\varepsilon\gamma\kappa - \frac{1}{c}\kappa\mu,\\
&V_2 = -(1+c)\varepsilon\alpha^2 + \beta^2 + \frac{(1+c)^2}{c^2}\kappa^2 + \gamma\mu + \varepsilon\gamma\sigma + \mu\sigma,\\
&W_2 = -\frac{(1+c)(1+2c)}{c}\varepsilon\alpha\kappa + (1+c)\beta\mu + (c-1)\varepsilon\beta\sigma,\\
&U_3 = \varepsilon\alpha^2 + c\beta^2 + \kappa^2 - \gamma\mu - \varepsilon\gamma\sigma + \mu\sigma,\\
&V_3 = (2+c)\alpha\beta + \frac{1+2c}{c}\gamma\kappa - \frac{1}{c}\kappa\sigma,\\
&W_3 = (2+c)\alpha\mu + c\varepsilon\alpha\sigma + (1+2c)\beta\kappa.
\end{aligned}\tag{3.12}$$

In this way, we proved the following

Theorem 3.1. *Let A, B, C, D, G, H be smooth functions on the three variables w, x, y, satisfying partial differential equations (3.9) and (3.12). Then, (3.3) describes a curvature homogeneous pseudo-Riemannian metric g on $\mathbb{R}^3$, with diagonal Ricci operator and (constant) Ricci principal curvatures q_1, q_2, q_3.*

Remark 3.2. A similar argument has been already used in [15] for the Riemannian case. We can refer to [15] for a more detailed description of how the corresponding equations for the connection and the curvature are obtained. Occasional changes of sign with respect to corresponding formulas in [15] are due to the different curvature convention and the different choice of the connection forms ω^i_j.

4. A class of non-homogeneous solutions

We now look for a special class of solutions of the systems of differential equations (3.9) and (3.12), obtained by making some assumptions which remarkably simplify these equations. Here we simply adapt to our systems the procedure already used in [15]. For this reason, we prefer not to include the detailed explanation of the assumptions below and to refer to [15] for more information.

First of all, we suppose that *the connection functions* b^i_{jk} *are independent of the variable* y (notice that, by Theorem 2.4, this condition is satisfied even when (M,g) is locally homogeneous). Moreover, we assume that the following formulas hold:

$$\beta = 0, \qquad \kappa = 0, \qquad \sigma = -\gamma, \tag{4.1}$$

$$c\gamma\mu = \lambda_2 + (1+c)\lambda_3, \tag{4.2}$$

$$(1+c)\alpha^2 = \varepsilon\lambda_2 - \gamma^2. \tag{4.3}$$

Assuming $\gamma \neq 0$, (4.2) permits to write μ in function of γ. Moreover, by its definition, c satisfies $1+c<0$. If $\varepsilon\lambda_2 - \gamma^2 < 0$ and we also require $\alpha > 0$, then Eq. (4.3) permits to determine uniquely α in function of γ. We are now ready to state the following

Theorem 4.1. *Let* (M,g) *be a three-dimensional pseudo-Riemannian manifold with diagonal Ricci operator and constant distinct Ricci eigenvalues* q_1, q_2 *and* q_3, *and* $\alpha, \beta, \gamma, \kappa, \mu, \sigma$ *smooth functions on* $\mathbb{R}^2[w,x]$, *satisfying* (4.1), (4.2) *and* (4.3). *If*

$$\gamma > 0, \qquad \gamma'_w \neq 0, \qquad \gamma^2 > \max\left\{\varepsilon\lambda_2, \frac{(2+c)(\lambda_2 + (1+c)\lambda_3)}{c^2}\right\}, \tag{4.4}$$

then there exist six smooth functions A, B, C, D, G, H *on* $\mathbb{R}^3[w,x,y]$ *which are solutions of* (3.9) *and* (3.12) *and so, determine pseudo-Riemannian metrics on* $\mathbb{R}^3$ *with Ricci eigenvalues* q_1, q_2, q_3.

Proof. Because of (4.1)–(4.4) and taking into account (2.14), all connection functions are uniquely determined as functions of γ (in the case of α, since we assumed $\alpha > 0$).

Suppose now to choose a particular function γ, satisfying (4.4). Using (4.1)–(4.4), one can check that all curvature conditions expressed by (3.11) are satisfied by arbitrary functions A, B, G, H and by C, D uniquely determined by

$$C = -\frac{\gamma'_w}{f(\gamma)}, \qquad D = -\frac{\gamma'_x}{f(\gamma)}, \tag{4.5}$$

where we put $f(\gamma) = W_3$ (different from zero, because of the last condition in (4.4)).

To complete the proof applying Theorem 3.1, we must check that also connection conditions (3.9) are satisfied, under suitable assumptions on A, B, G, H. By (4.1)–(4.3), (3.9) reduces to

$$\begin{aligned}
&C'_y = 0, && D'_y = 0,\\
&A'_y = (\varepsilon\sigma - \mu)C, && B'_y = (\varepsilon\sigma - \mu)D,\\
&G'_y = -(1+a)\varepsilon\alpha C, && H'_y = -(1+a)\varepsilon\alpha D,\\
&B'_w - A'_x = \varepsilon\alpha\mathcal{D} + (\mu - \varepsilon\sigma)\mathcal{F}, &&\\
&D'_w - C'_x = 0, &&\\
&H'_w - G'_x = (\mu - \varepsilon\gamma)\mathcal{D} + (1+a)\varepsilon\alpha\mathcal{F}. &&
\end{aligned} \tag{4.6}$$

Equations in the first and fifth rows of (4.6) are satisfied because of (4.5). Next, we can integrate equations in the second and third rows of (4.6) and then require that their solutions A, B, G, H also satisfy the remaining equations in (4.6). In this way, we conclude that (4.6) is satisfied if and only if

$$\begin{aligned}
&A = C(\varepsilon\sigma - \mu)y + A_0(w,x), && B = D(\varepsilon\sigma - \mu)y + B_0(w,x),\\
&G = -(1+c)C\varepsilon\alpha y + G_0(w,x), && H = -(1+c)D\varepsilon\alpha y + H_0(w,x),
\end{aligned}$$

with A_0, B_0, G_0, H_0 smooth functions on $\mathbb{R}^2[w,x]$, satisfying

$$\begin{aligned}
&(A_0)'_x - (B_0)'_w = (CB_0 - DA_0)\varepsilon\alpha + (CH_0 - DG_0)(\varepsilon\sigma - \mu),\\
&(G_0)'_x - (H_0)'_w = (CB_0 - DA_0)(\varepsilon\gamma - \mu) - (CH_0 - DG_0)(1+c)\varepsilon\alpha.
\end{aligned} \tag{4.7}$$

Choosing B_0, H_0 as arbitrary C^∞-functions, (4.7) is a system of two linear first order ordinary differential equations for A_0, G_0, with w as a parameter. Then, the standard existence theorem yields that the solution (A_0, G_0) exists in the whole $\mathbb{R}^2[x, w]$ and this ends the proof. □

Remark 4.2. For any choice of the arbitrary two-variables functions B_0 and H_0, we obtain an explicit solution A, B, C, D, G, H for (3.9) and (3.11). Note that A_0 and G_0 depend each from an arbitrary function of the variable w. Hence, from the proof of Theorem 4.1 it follows that for any choice of q_1, q_2, q_3, there exists a family of curvature homogeneous pseudo-Riemannian metrics on $\mathbb{R}^3$, with Ricci eigenvalues q_i, formally depending on two functions of two variables and two more functions of one variable.

It should be noted that because of Theorem 2.4, all the solutions given in Theorem 4.1 are *non-homogeneous*, because (4.4) implies that γ is not constant. It is also worthwhile to emphasize that *two of such solutions, obtained starting by two different functions $\gamma = b^1_{23}$, are never* (*locally*) *isometric*. In fact, Theorem 2.2 implies that two locally isometric curvature homogeneous pseudo-Riemannian three-manifolds (M, g) and (M', g'), with respect to the suitable frames $\{e_i\}$ and $\{e'_i\}$, necessarily have the same connection functions (at most, up to sign).

We now end this section by exhibiting, for any choice of the triplet (q_1, q_2, q_3), an explicit pseudo-Riemannian metric on $\mathbb{R}^3$, having q_i as Ricci eigenvalues. In fact, a straightforward calculation proves the following

Corollary 4.3. *Let (q_1, q_2, q_3) be any real triplet. Consider an arbitrary one-variable function $\gamma = \gamma(w)$ satisfying* (4.4) *and $\alpha, \beta, \kappa, \mu, \sigma$ determined by* (4.1), (4.2), (4.3) *as functions of γ. Then, the following functions*

$$\begin{aligned} &A = C(\varepsilon\sigma - \mu)y + C\varepsilon\alpha x, && B = 1,\\ &C = -\frac{\gamma'_w}{f(\gamma)}, && D = 0,\\ &G = -C(1+c)\varepsilon\alpha y + C(\varepsilon\gamma - \mu)x, && H = 0 \end{aligned} \tag{4.8}$$

are solutions of (3.9) *and* (3.11). *So,* (3.3) *and* (4.8) *determine explicitly a* (*non-homogeneous*) *curvature homogeneous pseudo-Riemannian metric g on $\mathbb{R}^3$, having Ricci eigenvalues q_i.*

5. Homogeneous solutions

We shall now exhibit locally homogeneous pseudo-Riemannian metrics on $\mathbb{R}^3$, with prescribed distinct Ricci eigenvalues. As we noticed in Section 2, a locally homogeneous pseudo-Riemannian three-manifold with distinct Ricci eigenvalues (and diagonal Ricci operator) is locally isometric to a three-dimensional Lie group, equipped with a left-invariant pseudo-Riemannian metric. Three-dimensional Lie groups, admitting a left-invariant Riemannian metric, have been completely classified in [18], where their curvature has also been described. The classification in the Lorentzian case can be obtained by combining the results of [11] and [20]. A unified presentation of three-dimensional Lie groups with left-invariant Lorentzian metrics is given in [8], while [9] provides the description of their curvature.

In constructing homogeneous solutions of (3.9) and (3.11), we shall start from two distinct sets of conditions on the connection coefficients, corresponding to a locally homogeneous pseudo-Riemannian manifold locally isometric to a *unimodular* and a *non-unimodular* three-dimensional Lie group, respectively. Since the Ricci operator of a Riemannian manifold is always diagonalizable, these choices cover all possibilities in the Riemannian case. Referring to the classification given in [8] for the Lorentzian case, Lie groups having a unimodular Lie algebra of type $\mathfrak{g}_3$, as well as a non-unimodular Lie algebra either of type $\mathfrak{g}_5$ or $\mathfrak{g}_6$, are also included.

Unimodular case. We start from the following assumption:

$$\alpha = \beta = \kappa = 0, \qquad \gamma, \mu, \sigma \text{ constants.} \tag{5.1}$$

Note that by (5.1) and (2.14) we also have $\nu = \tau = \psi = 0$. Taking into account (2.9), (5.1) is then equivalent to requiring that

$$
\begin{aligned}
&[e_1, e_2] = k_3 e_3,\\
&[e_1, e_3] = -k_2 e_2,\\
&[e_2, e_3] = k_1 e_1,
\end{aligned}
\tag{5.2}
$$

where we put $k_1 = \varepsilon\sigma - \mu$, $k_2 = \gamma + \sigma$ and $k_3 = \varepsilon\gamma - \mu$. Every Riemannian three-manifold locally isometric to a unimodular Lie group admits (at least, locally) an orthonormal frame satisfying (5.2) [18]. The same property holds for Lorentzian three-manifolds locally isometric to a unimodular Lie group having a Lie algebra of type $\mathfrak{g}_3$ [8,20]. If (5.1) holds, then (3.12) reduces to

$$
\begin{aligned}
&W_1 = \gamma\mu - \varepsilon\gamma\sigma - \mu\sigma, \qquad && V_2 = \gamma\mu + \varepsilon\gamma\sigma + \mu\sigma, \qquad && U_3 = -\gamma\mu - \varepsilon\gamma\sigma + \mu\sigma,\\
&U_1 = V_1 = 0, && U_2 = W_2 = 0, && V_3 = W_3 = 0,
\end{aligned}
$$

and so, (3.11) becomes

$$
\begin{aligned}
&\mathcal{D}(U_3 + \lambda_3),\\
&\mathcal{E}(V_2 + \lambda_2) = 0, \qquad && A(V_2 + \lambda_2) = 0, \qquad && B(V_2 + \lambda_2) = 0,\\
&\mathcal{F}(W_1 + \lambda_1) = 0, && C(W_1 + \lambda_1) = 0, && D(W_1 + \lambda_1) = 0.
\end{aligned}
\tag{5.3}
$$

Next, again by (5.1), the Ricci eigenvalues are given by

$$
q_1 = -2\mu\sigma, \qquad q_2 = 2\varepsilon\gamma\sigma, \qquad q_3 = -2\gamma\mu. \tag{5.4}
$$

It is interesting to remark that when (5.1) holds, γ, μ, σ determine uniquely the Ricci eigenvalues via (5.4), and conversely. So, *prescribing the Ricci eigenvalues is now equivalent to prescribe the Levi Civita connection of* (M, g). Notice also that connection functions γ, μ, σ (equivalently, constants k_i) are not completely arbitrary, since all q_i must be distinct. In particular, if $k_i = 0$ for some i, then necessarily $k_j k_h \neq 0$ for $j \neq i \neq h$, otherwise by (5.1) and (5.4) the Ricci eigenvalues can not be all distinct.

Because of (5.4) and taking into account (2.18), it is easy to check that

$$
W_1 + \lambda_1 = V_2 + \lambda_2 = U_3 + \lambda_3 = 0.
$$

Therefore, all Eqs. (5.3) reduce to identities, that is, under the assumption (5.1), curvature conditions (3.11) are always satisfied.

We now turn our attention to the connection equations (3.9). Again by (5.1), we obtain that (3.9) reduces to

$$
\begin{aligned}
&A'_y = k_1 C, \qquad && B'_y = k_1 D,\\
&C'_y = -k_2 A, && D'_y = -k_2 B,\\
&G'_y = 0, && H'_y = 0,\\
&B'_w - A'_x = -k_1 \mathcal{F}, &&\\
&D'_w - C'_x = k_2 \mathcal{E}, &&\\
&H'_w - G'_x = -k_3 \mathcal{D}. &&
\end{aligned}
\tag{5.5}
$$

It is rather easy to find solutions of system (5.5). By the last two equations of (5.5), we get that G and H only depend on w and x. Moreover, differentiating by y the equations in the second row of (5.5) and using the equations in the fourth row, we get at once

$$
A''_{yy} = \eta A, \qquad B''_{yy} = \eta B, \tag{5.6}
$$

where we put $\eta = -k_1 k_2$. In the same way, interchanging A with C and B with D, we find

$$
C''_{yy} = \eta C, \qquad D''_{yy} = \eta D. \tag{5.7}
$$

One can now solve explicitly (5.6) and (5.7), for the different possibilities given by the sign of η, and then check when these solutions also satisfy the remaining equations of (5.5). Note that, because of (3.3), $\mathcal{D} = AD - BC \neq 0$ is a necessary and sufficient condition for linear independence of ω^i. Some explicit solutions of (5.5) are resumed in the following

Theorem 5.1. *Let γ, μ and σ be three non-zero constants such that the numbers q_1, q_2 and q_3 defined by (5.4) are distinct. Put $k_1 = \varepsilon\sigma - \mu$, $k_2 = \gamma + \sigma$, $k_3 = \varepsilon\gamma - \mu$ and $\eta = -k_1 k_2$. Then, (3.3) determines a family of (locally isometric) locally homogeneous pseudo-Riemannian metrics on $\mathbb{R}^3[w, x, y]$ with Ricci eigenvalues q_i, where functions A, B, C, D, G, H are the following:*

- *When $\eta > 0$ and $k_3 \neq 0$:*

$$\begin{aligned} &A = f\cosh(\sqrt{\eta}y), && B = \theta\sinh(\sqrt{\eta}y),\\ &C = \frac{\sqrt{\eta}}{k_1} f\sinh(\sqrt{\eta}y), && D = \frac{\sqrt{\eta}}{k_1}\theta\cosh(\sqrt{\eta}y),\\ &G = -\frac{1}{\theta\sqrt{\eta}} f'_x, && H = 0, \end{aligned} \tag{5.8}$$

  for a real constant $\theta \neq 0$ and

$$f = \begin{cases} a_1(w)\cosh(\sqrt{r}x) + a_2(w)\sinh(\sqrt{r}x) & \text{if } r > 0,\\ a_1(w)\cos(\sqrt{|r|}x) + a_2(w)\sin(\sqrt{|r|}x) & \text{if } r < 0, \end{cases} \tag{5.9}$$

  where $r = k_2 k_3 \theta^2$ and a_1, a_2 are two arbitrary one-variable functions.

- *When $\eta > 0$ and $k_3 = 0$:*

$$\begin{aligned} &A = A_0\cosh(\sqrt{\eta}y), && B = B_0\sinh(\sqrt{\eta}y),\\ &C = \frac{\sqrt{\eta}}{k_1} A_0\sinh(\sqrt{\eta}y), && D = \frac{\sqrt{\eta}}{k_1} B_0\cosh(\sqrt{\eta}y),\\ &G = 0, && H = 0, \end{aligned} \tag{5.10}$$

  where A_0, B_0, are arbitrary functions of (w, x), satisfying $A_0 B_0 \neq 0$.

- *When $\eta < 0$ and $k_3 \neq 0$:*

$$\begin{aligned} &A = f\cos(\sqrt{|\eta|}y), && B = \theta\sin(\sqrt{|\eta|}y),\\ &C = -\frac{\sqrt{|\eta|}}{k_1} f\sin(\sqrt{|\eta|}y), && D = \frac{\sqrt{|\eta|}}{k_1}\theta\cos(\sqrt{\eta}y),\\ &G = -\frac{1}{\theta\sqrt{\eta}} f'_x, && H = 0, \end{aligned} \tag{5.11}$$

  for a real constant $\theta \neq 0$ and f given by (5.9), where $r = -k_2 k_3 \theta^2$ and a_1, a_2 are two arbitrary one-variable functions.

- *When $\eta > 0$ and $k_3 = 0$:*

$$\begin{aligned} &A = A_0\cos(\sqrt{|\eta|}y), && B = B_0\sin(\sqrt{|\eta|}y),\\ &C = -\frac{\sqrt{|\eta|}}{k_1} A_0\sin(\sqrt{\eta}y), && D = \frac{\sqrt{|\eta|}}{k_1} B_0\cos(\sqrt{\eta}y),\\ &G = 0, && H = 0, \end{aligned} \tag{5.12}$$

  where A_0, B_0, are arbitrary functions of (w, x), satisfying $A_0 B_0 \neq 0$.

- *When $\eta = 0$: if $k_1 k_3 \neq 0 = k_2$, then*

$$\begin{aligned} &A = f(w, x), && B = k_1\theta y,\\ &C = 0, && D = \theta,\\ &G = -\frac{1}{k_1\theta} f'_x && H = 0, \end{aligned} \tag{5.13}$$

  for a real constant $\theta \neq 0$ and f described by (5.9), where $r = -k_1 k_3 \theta^2$ and a_1, a_2 are two arbitrary one-variable functions. The case $k_1 = 0 \neq k_2 k_3$ gives similar solutions.

In all the different cases, the corresponding pseudo-Riemannian metric is defined in the open subset of $\mathbb{R}^3$ *where* $\mathcal{D} \neq 0$.

Non-unimodular case. We now start by assuming that connection functions satisfy

$$\alpha = \beta = \varepsilon\sigma - \mu = 0, \qquad \gamma, \kappa, \mu \text{ constants.} \tag{5.14}$$

Indeed, we only need a special case of (5.14). In terms of Lie brackets, this special case corresponds to assume that there exist four constants a, b, c, d, such that

$$\begin{aligned} &[e_1, e_2] = ae_2 + be_3, \\ &[e_1, e_3] = ce_2 + de_3, \\ &[e_2, e_3] = 0, \quad \text{with } a + d \neq 0,\ \varepsilon ac + bd = 0. \end{aligned} \tag{5.15}$$

The Lie algebra of any three-dimensional non-unimodular Riemannian Lie group admits an orthonormal basis satisfying (5.15) [18]. Moreover, it is easily seen that also Lorentzian Lie groups having a non-unimodular Lie algebra of type either $\mathfrak{g}_5$ or $\mathfrak{g}_6$ satisfy (5.15), with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$: starting from the description of $\mathfrak{g}_5$ and $\mathfrak{g}_6$ given in [8], it suffices to rearrange suitably the vectors of the pseudo-orthonormal bases of these Lie algebras. Routine calculations show that if (5.15) holds, then the Ricci operator is diagonal, and the Ricci eigenvalues are given by

$$q_1 = -a^2 - d^2 - \frac{\varepsilon}{2}(b + \varepsilon c)^2, \qquad q_2 = -a(a+d) - \frac{\varepsilon}{2}(b^2 - c^2), \qquad q_3 = -d(a+d) + \frac{\varepsilon}{2}(b^2 - c^2) \tag{5.16}$$

(we can refer to [18] and [9] for more details). Proceeding as in the unimodular case, one can check that (3.12) now reduces to

$$\begin{aligned} &W_1 + \lambda_1 = 0, \qquad V_2 + \lambda_2 = 0, \qquad U_3 + \lambda_3 = 0, \\ &U_1 = V_1 = 0, \qquad U_2 = W_2 = 0, \qquad V_3 = W_3 = 0, \end{aligned}$$

and so, all curvature equations (3.11) are satisfied.

As concerns the connection equations (3.9), because of (5.15) they become

$$\begin{aligned} &A'_y = 0, && B'_y = 0, \\ &C'_y = cA, && D'_y = cB, \\ &G'_y = dA, && H'_y = dB. \\ &B'_w - A'_x = 0, && \\ &D'_w - C'_x = -a\mathcal{D} - c\mathcal{E}, && \\ &H'_w - G'_x = -b\mathcal{D} - d\mathcal{E}, && \end{aligned} \tag{5.17}$$

In order to provide some explicit solutions of (5.17), a straightforward calculation proves the following

Theorem 5.2. *Let* a, b, c, d *be four real constants satisfying* $a + d \neq 0$ *and* $\varepsilon ac + bd = 0$, *such that the numbers* q_1, q_2 *and* q_3 *defined by* (5.16) *are distinct. Then,* (3.3) *determines a family of* (*locally isometric*) *locally homogeneous pseudo-Riemannian metrics on* $\mathbb{R}^3[w, x, y]$ *with Ricci eigenvalues* q_i, *where functions* A, B, C, D, G, H *are the following*:

$$\begin{aligned} &A = f(w), && B = 0, \\ &C = cf(w)y + a\theta f(w)x + c_0(w), && D = \theta, \\ &G = df(w)y + b\theta f(w)x + g_0(w), && H = 0, \end{aligned} \tag{5.18}$$

for a real constant $\theta \neq 0$ *and three arbitrary one-variable functions* f, c_0, g_0. *The corresponding pseudo-Riemannian metric is defined in the open subset of* $\mathbb{R}^3$ *where* $f \neq 0$.

In [14], Kowalski and Nikčević found necessary and sufficient conditions for three real constants q_1, q_2, q_3 to be the principal Ricci curvatures of some three-dimensional locally homogeneous Riemannian space. Restricting ourselves to the case of three distinct real numbers q_1, q_2, q_3 and adapting the technique used in [14], we can easily determine the signatures of the Ricci forms which can occur for a locally homogeneous pseudo-Riemannian three-manifold (with diagonal Ricci operator). Hereby we briefly explain how to determine such signatures, referring to [14] as a basic model for the argument used.

In Section 2 we proved that a locally homogeneous pseudo-Riemannian three-space, with diagonal Ricci operator and distinct Ricci eigenvalues, can not be locally symmetric and so, is locally isometric to a (either unimodular or non-unimodular) Lie group, equipped with a left-invariant pseudo-Riemannian metric. Consider now an arbitrary triplet (q_1, q_2, q_3) of *distinct* real numbers. Starting from (5.4), we can conclude that $q_i \neq 0$ for all i is a necessary and sufficient condition for the existence of a pseudo-Riemannian Lie group G with Lie algebra (5.2), having q_1, q_2, q_3 as Ricci eigenvalues.

Suppose now that (just) one of q_i vanishes. In this case, a unimodular pseudo-Riemannian Lie group having q_1, q_2, q_3 as Ricci eigenvalues can not exist. On the other hand, (5.16) implies that the Ricci eigenvalues satisfy $q_2 + q_3 < 0$ and so, at least one of q_2 and q_3 is negative. In this case, it is possible to prove that there exists a three-dimensional non-unimodular Lie group G, equipped with a left-invariant Lorentzian metric g, having diagonal Ricci operator and q_1, q_2, q_3 as Ricci eigenvalues. Hence, and using Theorem 5.2 from [14] in the Riemannian case, we see that the only signature which could not occur for the Ricci form is $(+,+,0)$. So, we obtained the following

Theorem 5.3. *A locally homogeneous pseudo-Riemannian* 3*-manifold* (M, g) *with the distinct Ricci eigenvalues* q_1, q_2, q_3 (*and diagonal Ricci operator*) *exists if the Ricci form does not have the signature* $(+,+,0)$.

Theorems 5.1–5.3 now yield at once the following

Corollary 5.4. *For every prescribed triplet of distinct real numbers* q_1, q_2, q_3*, satisfying the restriction given in Theorem* 5.3*, some of the formulas* (5.8)–(5.13) *or* (5.18) *provide corresponding explicit pseudo-Riemannian metrics with the Ricci eigenvalues* q_1, q_2, q_3.

Remark 5.5. From the classification of three-dimensional Riemannian [18] and Lorentzian [11,20] Lie groups it follows that, both in the unimodular and non-unimodular cases, they are determined by a finite number of independent parameters. In Section 2 we proved that a locally homogeneous three-space, with diagonal Ricci operator and distinct Ricci eigenvalues, is locally isometric to one of such Lie groups. Hence, these locally homogeneous pseudo-Riemannian three-spaces depend, up to local isometries, on a finite number of real parameters.

In particular, as affirmed in Theorems 5.1 and 5.2, starting from a fixed set of coefficients (determining the homogeneous model via its Lie algebra), we get locally homogeneous pseudo-Riemannian metrics, with the prescribed Ricci eigenvalues, which formally depend on arbitrary functions, but are indeed locally isometric to one another.

Theorems 5.1 and 5.2 show that in most cases there exists a locally homogeneous pseudo-Riemannian metric with Ricci eigenvalues (q_1, q_2, q_3), defined on the whole of $\mathbb{R}^3[w, x, w]$. It remains open the problem whether for all prescribed (q_1, q_2, q_3), there exists a locally homogeneous pseudo-Riemannian metric which is *complete* and globally defined on $\mathbb{R}^3[w, x, y]$.

Acknowledgements

The author expresses his sincere gratitude toward Prof. O. Kowalski for several helpful comments and suggestions.

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