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An Analogue of a Problem of P. Erdös and E. Feldheim on L_p Convergence of Interpolatory Processes

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For the Lagrange interpolation polynomial $L_n(f, x)$ of degree $\leqslant n-1$, defined by

$$L_n(f, x_{kn}) = f(x_{kn}), \qquad k = 1, 2, ..., n, \tag{1.1}$$

of a given function f (based on n distinct points $-1 \leqslant x_{nn} < x_{n-1,n} < \cdots < x_{1n} \leqslant 1$), P. Erdös and E. Feldheim [6] proved the following

THEOREM A. *If $f \in C[-1, 1]$ and x_{kn}, $k = 1, 2, ..., n$, are the zeros of $T_n(x)$, the* nth *Chebyshev polynomial, then for any fixed $p > 0$,*

$$\lim_{n \to \infty} \int_{-1}^{1} |L_n[f, x] - f(x)|^p (1 - x^2)^{-1/2} \, dx = 0. \tag{1.2}$$

For a more detailed study of this kind of work we refer the reader to P. Erdös and P. Turán [7], R. Askey [1], P. G. Nevai [11], P. Vertesi [20], and A. K. Varma and P. Vertesi [19]. The corresponding study of mean convergence of the Hermite–Fejér interpolation process was recently initiated by Nevai and Vertresi [12]. In Ref. [12] weighted L_p convergence of Hermite–Fejér interpolation based on the zeros of a generalized Jacobi polynomial was investigated. The main result of Ref. [12] gives necessary and sufficient conditions for such convergence for all continuous functions.

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They mentioned that the main reason for the lack of the general theory appears to be the complicated structure of explicit representation for the Hermite–Fejér interpolating polynomial.

The object of this paper is to consider the problem of degree of approximation (in the L_p norm for any fixed $p>0$) of a given continuous function by various interpolatory (Hermite) processes based on the Tchebycheff nodes. Now we turn to describe these results.

Let

$$x_k \equiv x_{kn} = \cos\frac{(2k-1)\pi}{2n}, \qquad k=1, 2, ..., n, \tag{1.3}$$

be the zeros of $T_n(x)=\cos n\theta$, $\cos\theta = x$, the nth degree Tchebycheff polynomial of the first kind. In this case the well-known Hermite–Fejér interpolation polynomial is given by [8], [10]

$$H_n(f, x) = \sum_{k=1}^{n} f(x_k)\, h_k(x), \tag{1.4}$$

where

$$h_k(x) = (1-xx_k)\left(\frac{T_n(x)}{n(x-x_k)}\right)^2 \geqslant 0, \qquad \sum_{k=1}^{n} h_k(x) \equiv 1. \tag{1.5}$$

Next, we define a polynomial $Q_n(f, x)$ of degree $\leqslant 2n+1$ by

$$\begin{aligned} Q_n(f, x) = H_n(f, x) &+ (f(1) - H_n(f, 1))\left(\frac{1+x}{2}\frac{T_n(x)}{T_n(1)}\right)^2 \\ &+ (f(-1) - H_n(f, -1))\left(\frac{1-x}{2}\frac{T_n(x)}{T_n(-1)}\right)^2. \end{aligned} \tag{1.6}$$

It is known [3] that

$$\begin{aligned} &Q_n(f, x_k) = f(x_k), \qquad Q_n'(f, x_k) = 0, \qquad k=1, 2, ..., n, \\ &Q_n(f, \pm 1) = f(\pm 1). \end{aligned} \tag{1.7}$$

Concerning $H_n(f, x)$ and $Q_n(f, x)$ we shall prove the following

THEOREM 1. *Let $f \in C[-1, 1]$ and let $H_n(f, x)$ be as defined by* (1.4). *Then for any fixed $p>0$ we have*

$$\left(\int_{-1}^{1} |H_n(f, x) - f(x)|^p (1-x^2)^{-1/2}\, dx\right)^{1/p} \leqslant c_1 w_f(1/n), \tag{1.8}$$

and

$$\left(\int_{-1}^{1} |Q_n(f, x) - f(x)|^p (1-x^2)^{-1/2}\, dx\right)^{1/p} \leqslant c_2 w_f(1/n), \tag{1.9}$$

where c_1, c_2 *are positive absolute constants and* $w_f(\delta)$ *is the modulus of continuity of* f.

Remark. It is well known that for $f \in \text{Lip}\, 1$ the error function $\|f(x) - H_n(f, x)\|_{L^\infty}$ is of order $O(\log n/n)$ and this is best possible (see [2], [14], and [16]). On the other hand we can conclude from (1.8) that the corresponding estimate in L_p norm is $O(1/n)$.

If $f \in C^1[-1, 1]$ then it is known that a Hermite interpolation polynomial $H_n^*(f, x)$ of degree $\leqslant 2n-1$ which satisfies the conditions

$$H_n^*(f, x_k) = f(x_k), \qquad \frac{d}{dx}[H_n^*(f, x)]_{x_k} = f'(x_k), \qquad k = 1, 2, ..., n, \tag{1.10}$$

is given by

$$H_n^*(f, x) = \sum_{k=1}^{n} f(x_k)\, h_k(x) + \sum_{k=1}^{n} f'(x_k)\, \sigma_k(x), \tag{1.11}$$

where $h_k(x)$ is as in (1.5), the x_k's are as in (1.3), and $\sigma_k(x)$ is given by

$$\sigma_k(x) = (x - x_k)\, l_k^2(x), \qquad l_k(x) = \frac{T_n(x)}{(x - x_k)\, T_n'(x_k)}. \tag{1.12}$$

For the polynomial $H_n^*(f, x)$ we shall prove the following:

THEOREM 2. *Let* $f(x)$ *be defined and have a continuous derivative* $f'(x)$ *on* $[-1, 1]$. *Then for the Hermite interpolation polynomial* $H_n^*(f, x)$ *corresponding to the Tchebycheff abscissas of the first kind*

$$\left(\int_{-1}^{1} |f(x) - H_n^*(f, x)|^2 (1-x^2)^{-1/2}\, dx\right)^{1/2} \leqslant c_3 n^{-1} E_{2n-2}(f'), \tag{1.13}$$

where $E_{2n-2}(f')$ *is the best approximation to* $f'(x)$ *by polynomials of degree at most* $2n-2$ *and* c_3 *is a positive absolute constant.*

If we change the nodes of interpolation to the zeros of Tchebycheff polynomials of the second kind,

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad x = \cos\theta, \tag{1.14}$$

then it is known that for many interpolation processes (this includes Lagrange as well as Hermite–Fejér interpolation) convergence behaviour is very poor with respect to the nodes of (1.14) especially near 1 or -1. The situation is not improved for the Lagrange interpolation on these nodes even in problems of mean convergence. E. Feldheim [9] proved that for the same abscissas it is not true that ($r \geqslant 1$)

$$\lim_{n \to \infty} \int_{-1}^{1} |f(x) - L_n(f, x)|^{2r}\, dx = 0. \tag{1.15}$$

In fact the superior limit of the integrals in question may be $+\infty$ if $f(x)$ is a properly chosen continuous function. But the situation changes if we consider $Q_n^*(f, x)$, the so-called quasi-Hermite–Fejér interpolation polynomial of degree $\leqslant 2n+1$ based on the extended Tchebycheff nodes of the second kind. It is given by

$$\begin{aligned} Q_n^*(f, x) = {} & \left(\frac{1+x}{2} f(1) + \frac{1-x}{2} f(-1)\right) \frac{U_n^2(x)}{(n+1)^2} \\ & + \sum_{k=1}^{n} f(t_k)(1-x^2)(1-xt_k)\left(\frac{U_n(x)}{(n+1)(x-t_k)}\right)^2, \end{aligned} \tag{1.16}$$

where the t_k's are the zeros of $U_n(x)$.

In this case, P. Szasz [15] proved that $\lim_{n \to \infty} Q_n^*(f, x) = f(x)$ uniformly on $[-1, 1]$ provided $f \in C[-1, 1]$. Later Saxena and Mathur [21] proved that if $f \in C[-1, 1]$ we have

$$|Q_n^*(f, x) - f(x)| \leqslant \frac{c}{n} \sum_{k=1}^{n} w_f\left(\frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right). \tag{1.17}$$

Next let us denote by $R_n(f, x)$ the Hermite–Fejér interpolation polynomial of degree $\leqslant 2n+3$ satisfying the conditions

$$\begin{aligned} & R_n(f, t_k) = f(t_k), \qquad R_n'(f, t_k) = 0, \qquad k = 1, 2, ..., n \\ & R_n(f, \pm 1) = f(\pm 1), \qquad R_n'(f, \pm 1) = 0, \end{aligned} \tag{1.18}$$

where the t_k's are the zeros $U_n(x)$ given by (1.14). Concerning $R_n(f, x)$, the following pointwise estimate was obtained by Bojanic, Prasad, and Saxena [3]. It is given by ($-1 \leqslant x \leqslant 1$)

$$|R_n(f, x) - f(x)| \leqslant \frac{c_4}{n} \sum_{k=1}^{n} w\left(\frac{\sqrt{1-x^2}}{k} + \frac{1}{k^2}\right) + \frac{c_5}{n^2}, \tag{1.19}$$

where c_4, c_5 are positive absolute constants.

Concerning $Q_n^*(f, x)$ and $R_n(f, x)$ we shall prove the following:

THEOREM 3. *Let $f \in C[-1, 1]$ and let $Q_n^*(f, x)$ be the quasi-Hermite–Fejér interpolation polynomial of degree $\leqslant 2n+1$ as stated in* (1.16). *Then for any fixed $p > 0$, we have*

$$\left(\int_{-1}^{1} |Q_n^*(f, x) - f(x)|^p (1-x^2)^{-1/2}\, dx\right)^{1/p} \leqslant c_6 w_f(1/n). \tag{1.20}$$

Also, we have

$$\left(\int_{-1}^{1} |R_n(f, x) - f(x)|^p (1-x^2)^{-1/2}\, dx\right)^{1/p} \leqslant c_7 w_f(1/n), \tag{1.21}$$

where $R_n(f, x)$ is defined by (1.18).

2. PRELIMINARIES

Here we state some known results which we shall need later on. If $l_k(x)$ is given by (1.12) then for $-1 \leqslant x \leqslant 1$ it is known that

$$l_k^2(x) \leqslant 2, \qquad \sum_{k=1}^{n} l_k^2(x) \leqslant 2. \tag{2.1}$$

Also, from (1.4) it follows that for $-1 \leqslant x \leqslant 1$

$$\sum_{k=1}^{n} h_k(x) = 1, \qquad h_k(x) \geqslant 0, \qquad k = 1, 2, ..., n. \tag{2.2}$$

Next, if

$$\lambda_k(x) = \frac{U_n(x)}{(x - t_k)\, U_n'(t_k)} = \frac{(-1)^{k+1}(1 - t_k^2)\, U_n(x)}{(n+1)(x - t_k)}, \qquad k = 1, 2, ..., n, \tag{2.3}$$

then due to Erdös [5], Varma and Vertesi [19], and Varma [18] we have for $-1 \leqslant x \leqslant 1$ respectively

$$|\lambda_k(x)| \leqslant c_8, \qquad \frac{(1-x^2)^{1/2}}{(1-t_k^2)^{1/2}}\, |\lambda_k(x)| \leqslant \sqrt{2} \tag{2.4}$$

$$\sum_{k=1}^{n} \frac{(1-x^2)^2}{(1-t_k^2)^2}\, \lambda_k^2(x) \leqslant 2, \qquad \sum_{k=1}^{n} \frac{1-x^2}{1-t_k^2}\, \lambda_k^2(x) \leqslant 2, \tag{2.5}$$

where the t_k's are given by

$$t_k = \cos \frac{k\pi}{n+1}, \qquad k = 1, 2, ..., n. \tag{2.6}$$

Further, from a known theorem of S. A. Teljakovskii [17] it follows that there exists a polynomial $P_n(x)$ of degree $\leqslant n$ such that for all x, $-1 \leqslant x \leqslant 1$,

$$|f(x) - P_n(x)| \leqslant c_{11} w_f\left(\frac{\sqrt{1-x^2}}{n}\right) \tag{2.7}$$

and

$$(1-x^2)^{1/2} \, |P_n'(x)| \leqslant c_{12} n w_f(1/n). \tag{2.8}$$

Also for $-1 \leqslant x \leqslant 1$, we know that

$$|T_n(x)| \leqslant 1. \tag{2.9}$$

3. Some Lemmas

In this section we state and prove several lemmas which will be needed later on.

Lemma 3.1. *If $l_k(x)$ is as in* (1.12) *then for* $k = 1, 2, ..., n$,

$$\int_{-1}^{1} l_k(x)\, l_j(x)(1-x^2)^{-1/2}\, dx = \frac{\pi}{n}, \qquad k = j$$
$$= 0, \qquad k \neq j. \tag{3.1}$$

Proof. For the proof we refer the reader to Erdös and Turán [7].

Lemma 3.2. *If $\sigma_k(x)$ is given by* (1.12) *then*

$$\int_{-1}^{1} \sigma_k(x)\, \sigma_j(x)(1-x^2)^{-1/2}\, dx = 0, \qquad k \neq j$$
$$= \frac{\pi}{2n^3}(1-x_k^2), \qquad k = j. \tag{3.2}$$

Proof. From (1.12), (3.1), and using the orthogonal property of Tchebycheff polynomials we have

$$\begin{aligned}
&\int_{-1}^{1} \sigma_k(x)\,\sigma_j(x)(1-x^2)^{-1/2}\,dx \\
&=\int_{-1}^{1} (x-x_k)\,l_k^2(x)(x-x_j)\,l_j^2(x)(1-x^2)^{-1/2}\,dx \\
&=\frac{1}{T_n'(x_k)\,T_n'(x_j)}\int_{-1}^{1} \frac{1+T_{2n}(x)}{2}\,l_k(x)\,l_j(x)(1-x^2)^{-1/2}\,dx \\
&=\frac{1}{2T_n'(x_k)\,T_n'(x_j)}\int_{-1}^{1} l_k(x)\,l_j(x)(1-x^2)^{-1/2}\,dx \\
&=0, \qquad k\neq j.
\end{aligned}$$

Similarly, if $k=j$ then due to (3.1) and

$$|T_n'(x_k)|=\frac{n}{(1-x_k^2)^{1/2}}, \qquad k=1,2,...,n, \tag{3.3}$$

it follows that

$$\int_{-1}^{1} \sigma_k^2(x)\frac{1}{\sqrt{1-x^2}}\,dx=\frac{\pi}{2n(T_n'(x_k))^2}=\frac{\pi}{2n^3}(1-x_k^2). \tag{3.4}$$

This proves Lemma 3.2.

LEMMA 3.3. *If $\sigma_k(x)$ is given by (1.12) then*

$$\int_{-1}^{1}\left[\sum_{k=1}^{n} P_n'(x_k)\,\sigma_k(x)\right]^2 (1-x^2)^{-1/2}\,dx\leqslant c_{13}[w_f(1/n)]^2, \tag{3.5}$$

where $P_n(x)$ is the polynomial for which (2.7) and (2.8) are valid.

Proof. We have

$$\begin{aligned}
&\int_{-1}^{1}\left[\sum_{k=1}^{n} P_n'(x_k)\,\sigma_k(x)\right]^2 (1-x^2)^{-1/2}\,dx \\
&=\sum_{j=1}^{n}\sum_{k=1}^{n} P_n'(x_j)\,P_n'(x_k)\int_{-1}^{1}\sigma_k(x)\,\sigma_j(x)(1-x^2)^{-1/2}\,dx.
\end{aligned}$$

So from (3.2) and (2.8) we have

$$\int_{-1}^{1}\left[\sum_{k=1}^{n} P_n'(x_k)\,\sigma_k(x)\right]^2 (1-x^2)^{-1/2}\,dx$$
$$=\sum_{k=1}^{n} (P_n'(x_k))^2\,\frac{\pi}{2n^3}\,(1-x_k^2)\leqslant\frac{\pi}{2n^3}\,c_{12}^2 n^2 \sum_{k=1}^{n} (w_f(1/n))^2$$
$$=c_{13}(w_f(1/n))^2,$$

which yields (3.5). This proves Lemma 3.3.

In the work of Erdös and Feldheim [6] the following result played an important role. Let $l_{vn}(x)\equiv l_v(x)$, $v=1,2,...,n$, be the fundamental polynomial of Lagrange interpolation based on the zeros of $T_n(x)$. Then

$$\int_{-1}^{1} l_{v_1}(x)\,l_{v_2}(x)\cdots l_{v_{2k}}(x)(1-x^2)^{-1/2}\,dx=0,$$

where $v_1, v_2, ..., v_{2k}$ are distinct integers between 1 and n. For the L_p convergence of a quasi-Hermite–Fejér interpolation process based on the nodes $(1-x^2)\,U_n(x)$ the corresponding result is given by the following lemma:

LEMMA 3.4. *Let $v_1, v_2, ..., v_{2k}$ be distinct integers between* 1 *and n. Then we have*

$$\int_{-1}^{1} \chi_{v_1}(x)\,\chi_{v_2}(x)\cdots\chi_{v_{2k}}(x)(1-x^2)^{-1/2}\,dx=0, \tag{3.6}$$

where

$$\chi_v(x)=\frac{\sqrt{1-x^2}}{\sqrt{1-t_v^2}}\,\lambda_v(x)=\frac{\sqrt{1-x^2}}{\sqrt{1-t_v^2}}\,\frac{U_n(x)}{(t-t_v)\,U_n'(t_v)}. \tag{3.7}$$

Proof. From the earlier result [19, page 72] it follows that

$$(1-x^2)^k(U_n(x))^{2k-1}=\sum_{i=n}^{(2k-1)(n+1)+1} \mu_i\cos i\theta. \tag{3.8}$$

Since $t_{v_1}, ..., t_{v_{2k}}$ are distinct it follows that $U_n(x)/(x-t_{v_1})\cdots(x-t_{v_{2k}})$ is indeed a polynomial of degree $\leqslant n-2k$. Next we also note that

$$\chi_{v_1}(x)\cdots\chi_{v_{2k}}(x)=\alpha(v_1, v_2, ..., v_{2k})\,\frac{U_n(x)(1-x^2)^k(U_n(x))^{2k-1}}{(x-t_{v_1})\cdots(x-t_{v_{2k}})}. \tag{3.9}$$

Therefore on using the orthogonality of a Tchebycheff polynomial of the first kind it follows that

$$\int_{-1}^{1} \chi_{\nu_1}(x)\cdots\chi_{\nu_{2k}}(x)(1-x^2)^{-1/2}\,dx$$

$$\alpha(\nu_1, \nu_2\cdots\nu_{2k})\int_{-1}^{1} q_{n-2k}(x)\sum_{i=n}^{(2k-1)(n+1)+1}\mu_i T_i(x)(1-x^2)^{-1/2}\,dx=0.$$

Let us introduce the linear operator

$$L_n^*[f, x]=\sum_{k=1}^{n} f(t_k)\,\chi_k(x), \tag{3.10}$$

where $\chi_k(x)$ is defined by (3.7) and the t_k's are the zeros of $U_n(x)$ given by (2.6). On using (3.6) and some simple computation we obtain

$$\begin{aligned}\int_{-1}^{1} \chi_j(x)\,\chi_k(x)(1-x^2)^{-1/2}\,dx&=0, \qquad j\neq k\\ &=\frac{\pi}{n+1}, \qquad j=k.\end{aligned} \tag{3.11}$$

On using (3.11) we obtain

$$\int_{-1}^{1} (L_n^*[f, x])^2(1-x^2)^{-1/2}\,dx=\frac{\pi}{n+1}\sum_{k=1}^{n} f^2(t_k)\leqslant\pi\max_{-1\leqslant x\leqslant 1}|f^2(x)|. \tag{3.12}$$

Also, from (3.12) and the Cauchy–Schwarz inequality for integrals we also have

$$\int_{-1}^{1} |L_n^*[f, x]|\,(1-x^2)^{-1/2}\,dx\leqslant\pi\max_{-1\leqslant x\leqslant 1}|f(x)|. \tag{3.13}$$

4. Proof of the Theorems

For the proof of Theorem 1 and Theorem 3 we follow the method of Erdös and Feldheim [6]. It is enough to prove the theorems for even values of p only. In the case of the proof of Theorem 3 we limit for the case $p=4$. For arbitrary fixed even p the proof is similar.

Proof of Theorem 1. Let $P_n(x)$ be the polynomial of degree $\leqslant n$ for which (2.7) and (2.8) are valid. Due to (1.4) we have

$$H_n(f, x) - f(x) = H_n(f - P_n, x) + H_n(P_n, x) - P_n(x) + P_n(x) - f(x). \quad (4.1)$$

On using (2.2) and (2.7) it follows that

$$|H_n(f - P_n, x)| = \left| \sum_{k=1}^{n} (f(x_k) - P_n(x_k))\, h_k(x) \right| \leqslant w_f(1/n) \sum_{k=1}^{n} h_k(x) \leqslant w_f(1/n). \quad (4.2)$$

Thus to complete the proof of (1.8) we must prove that

$$\left(\int_{-1}^{1} |H_n(P_n, x) - P_n(x)|^{2p} (1 - x^2)^{-1/2}\, dx \right)^{1/2p} \leqslant c_{14} w_f(1/n). \quad (4.3)$$

From the uniqueness of Hermite interpolation we have

$$\begin{aligned} P_n(x) - H_n(P_n, x) &= \sum_{k=1}^{n} P_n'(x_k)\, \sigma_k(x) = \sum_{k=1}^{n} P_n'(x_k)(x - x_k)\, l_k^2(x) \\ &= \frac{1}{n} \sum_{k=1}^{n} P_n'(x_k)(-1)^{k-1}(1 - x_k^2)^{1/2} l_k(x)\, T_n(x) \\ &= T_n(x) \sum_{k=1}^{n} g_0(x_k)\, l_k(x) \equiv T_n(x)\, L_n[g_0, x], \end{aligned}$$

where

$$g_0(x) = \frac{(1 - x^2)\, U_{n-1}(x)}{n} P_n'(x).$$

Now, on using Theorem A of Erdös and Feldheim [6] and $|g_0(x)| \leqslant c_{15} w_f(1/n)$ we have

$$\begin{aligned} &\left(\int_{-1}^{1} |P_n(x) - H_n(P_n, x)|^{2p} (1 - x^2)^{-1/2}\, dx \right)^{1/2p} \\ &\leqslant c_{16} \left(\int_{-1}^{1} |L_n[g_0, x]|^{2p} (1 - x^2)^{-1/2}\, dx \right)^{1/2p} \\ &\leqslant c_{17} \max_{-1 \leqslant x \leqslant 1} |g_0(x)| \\ &\leqslant c_{18} w_f(1/n). \end{aligned}$$

This proves (4.3) and at the same time inequality (1.8). Proof of the inequality of (1.9) is a direct consequence of (1.8), (1.6), and the known estimate [3] (see formula (1.4)) $|H_n(f, \pm 1) - f(\pm 1)| \leqslant c_{19} w_f(1/n)$.

This proves Theorem 1. Next we turn to prove Theorem 2.

Proof of Theorem 2. One can easily see that for $-1 \leqslant x \leqslant 1$

$$H_n^*(f, x) - f(x) = H_n^*(f, x) - H_n^*(S_{2n-1}, x) + S_{2n-1}(x) - f(x), \quad (4.4)$$

where $S_{2n-1}(x)$ is the polynomial of best approximation of $f(x)$ and $H_n^*(f, x)$ is given by (1.11). From (4.4) it follows that

$$\int_{-1}^{1} [H_n^*(f, x) - f(x)]^2 (1 - x^2)^{-1/2}\, dx$$

$$\leqslant 2 \int_{-1}^{1} [H_n^*(f - S_{2n-1}, x)]^2 (1 - x^2)^{-1/2}\, dx$$

$$+ 2 \int_{-1}^{1} [S_{2n-1}(x) - f(x)]^2 (1 - x^2)^{-1/2}\, dx \equiv \Delta_1 + \Delta_2. \quad (4.5)$$

From the definition of $S_{2n-1}(x)$ we have for $-1 \leqslant x \leqslant 1$,

$$|S_{2n-1}(x) - f(x)| \leqslant E_{2n-1}(f), \quad (4.6)$$

where $E_{2n-1}(f)$ is the best approximation of $f(x)$. Consequently due to (4.6) we get

$$\Delta_2 \leqslant 2(E_{2n-1}(f))^2 \int_{-1}^{1} (1 - x^2)^{-1/2}\, dx = 2\pi E_{2n-1}^2(f). \quad (4.7)$$

Next, we turn to estimate Δ_1. We have

$$\Delta_1 \leqslant 4 \int_{-1}^{1} \left[\sum_{k=1}^{n} |f(x_k) - S_{2n-1}(x_k)|\, h_k(x) \right]^2 (1 - x^2)^{-1/2}\, dx$$

$$+ 4 \int_{-1}^{1} \left[\sum_{k=1}^{n} (f'(x_k) - S'_{2n-1}(x_k))\, \sigma_k(x) \right]^2 (1 - x^2)^{-1/2}\, dx$$

$$= \Delta_3 + \Delta_4. \quad (4.8)$$

Now, from (4.6) and (2.2) it follows that

$$\Delta_3 \leqslant 4E_{2n-1}^2(f) \int_{-1}^{1} \left[\sum_{k=1}^{n} h_k(x) \right]^2 (1 - x^2)^{-1/2}\, dx$$

$$= 4\pi E_{2n-1}^2(f). \quad (4.9)$$

Further, on using Lemma 3.2 we have

$$\Delta_4 = 4 \int_{-1}^{1} \sum_{k=1}^{n} (f'(x_k) - S'_{2n-1}(x_k))^2 \sigma_k^2(x)(1-x^2)^{-1/2}\, dx$$
$$= \frac{2\pi}{n^3} \sum_{k=1}^{n} (f'(x_k) - S'_{2n-1}(x_k))^2 (1-x_k^2). \tag{4.10}$$

Next, on using a theorem of J. Czipszer and G. Freud [4] and Corollary 1.44 of T. J. Rivlin [13, p. 23], we get

$$(1-x_k^2)^{1/2}\, |f'(x_k) - S'_{2n-1}(x_k)| \leqslant 40 E_{2n-2}(f'). \tag{4.11}$$

Consequently from (4.10), (4.11) it follows that

$$\Delta_4 \leqslant c_{20} n^{-2} E_{2n-2}^2(f'). \tag{4.12}$$

Thus, from (4.8), (4.9), and (4.12) we obtain

$$\Delta_1 \leqslant 4\pi E_{2n-1}^2(f) + c_{21} n^{-2} E_{2n-2}^2(f')$$
$$\leqslant c_{22} n^{-2} E_{2n-2}^2(f'). \tag{4.13}$$

Due to Rivlin [13, p. 23], we have

$$E_{2n-1}(f) \leqslant \frac{6}{2n-1} E_{2n-2}(f'). \tag{4.14}$$

On using (4.13), (4.7), and (4.14) we have (1.13). This completes the proof of Theorem 2 as well.

Proof of Theorem 3. First we need to show that for $f \in C[-1, 1]$,

$$\int_{-1}^{1} (Q_n^*(f, x) - f(x))^4 (1-x^2)^{-1/2}\, dx \leqslant c_{23}(w_f(1/n))^4. \tag{4.15}$$

Since

$$Q_n^*(f, x) - f(x) = Q_n^*(f - P_n, x) + Q_n^*(P_n, x) - P_n(x) + P_n(x) - f(x), \tag{4.16}$$

where $P_n(x)$ is the polynomial which satisfies (2.7) and (2.8), in view of (2.7) and since $Q_n^*(f, x)$ is such that $|f| \leqslant c_{24}$ implies $\|Q_n^*(f, x)\| \leqslant c_{24}$ we have

$$\|Q_n^*(f - P_n, x)\| \leqslant \|f - P_n\| \leqslant c_{25} w_f(1/n). \tag{4.17}$$

Next, we consider

$$\mu = \int_{-1}^{1} (P_n(x) - Q_n^*(P_n, x))^4 (1-x^2)^{-1/2}\, dx. \tag{4.18}$$

Further, one can easily see that

$$P_n(x) - Q_n^*(P_n, x) = (1-x^2)^{1/2} U_n(x) \sum_{k=1}^{n} g(t_k)\, \chi_k(x), \tag{4.19}$$

where

$$g(x) = -\frac{T_{n+1}(x)\, P_n'(x)(1-x^2)^{1/2}}{n+1} \tag{4.20}$$

and

$$\chi_k(x) = \frac{(1-x^2)^{1/2} \lambda_k(x)}{(1-t_k^2)^{1/2}}. \tag{4.21}$$

Also it is well known that for $-1 \leqslant x \leqslant 1$

$$(1-x^2)^{1/2}(U_n(x)) \leqslant 1. \tag{4.22}$$

Consequently from (4.19) and (4.22) it follows that

$$\mu \leqslant \int_{-1}^{1} \left(\sum_{k=1}^{n} g(t_k)\, \chi_k(x) \right)^4 (1-x^2)^{-1/2}\, dx \equiv \mu_1. \tag{4.23}$$

We also note (see (2.8) and (2.4)) that for $-1 \leqslant x \leqslant 1$

$$|\chi_k(x)| \leqslant \sqrt{2}, \qquad |g(x)| \leqslant c_{30} w_f(1/n). \tag{4.24}$$

Now, we may write

$$\mu \leqslant \mu_1 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{15}, \tag{4.25}$$

where on using (2.5) and (2.4)

$$\begin{aligned} \mu_{11} &= \int_{-1}^{1} \sum_{k=1}^{n} g^4(t_k)\, \chi_k^4(x)(1-x^2)^{-1/2}\, dx \\ &\leqslant c_{31}(w_f(1/n))^4 \int_{-1}^{1} \sum_{k=1}^{n} \chi_k^4(x)(1-x^2)^{-1/2}\, dx \leqslant c_{32}(w_f(1/n))^4. \end{aligned} \tag{4.26}$$

Next, we note that

$$\begin{aligned}\mu_{12} &\equiv \int_{-1}^{1} \sum_{k\neq j}\sum g^2(t_k)\, g^2(t_j)\, \chi_k^2(x)\, \chi_j^2(x)(1-x^2)^{-1/2}\, dx \\ &\leqslant \int_{-1}^{1} \sum_{k=1}^{n} \sum_{j=1}^{n} g^2(t_k)\, g^2(t_j)\, \chi_k^2(x)\, \chi_j^2(x)(1-x^2)^{-1/2}\, dx \\ &\leqslant c_{33}(w_f(1/n))^4 \int_{-1}^{1} \left(\sum_{k=1}^{n} \chi_k^2(x)\right)\left(\sum_{j=1}^{n} \chi_j^2(x)\right)(1-x^2)^{-1/2}\, dx \\ &\leqslant 4c_{33}(w_f(1/n))^4 \int_{-1}^{1} (1-x^2)^{-1/2}\, dx = 4c_{33}\pi(w_f(1/n))^4. \end{aligned} \tag{4.27}$$

Next, we observe that (Lemma 3.4)

$$\begin{aligned}\mu_{13} &\equiv \int_{-1}^{1} \sum_{k\neq j\neq i\neq m}\sum\sum\sum g(t_k)\, g(t_j)\, g(t_i)\, g(t_m) \cdot \\ &\qquad \times \chi_k(x)\, \chi_j(x)\, \chi_i(x)\, \chi_m(x)(1-x^2)^{-1/2}\, dx \\ &= 0. \end{aligned} \tag{4.28}$$

Next, we observe that

$$\begin{aligned}\mu_{14} &\equiv \int_{-1}^{1} \sum_{k\neq j}\sum g^3(t_k)\, g(t_j)\, \chi_k^3(x)\, \chi_j(x)(1-x^2)^{-1/2}\, dx \\ &= \int_{-1}^{1} \left[\left(\sum_{k=1}^{n} g^3(t_k)\, \chi_k^3(x)\right)\left(\sum_{k=1}^{n} g(t_k)\, \chi_k(x)\right)\right. \\ &\qquad \left. - \sum_{k=1}^{n} g^4(t_k)\, \chi_k^4(x)\right](1-x^2)^{-1/2}\, dx. \end{aligned} \tag{4.29}$$

Hence, on account of (2.4), (2.5), (4.24), (4.26), (3.10), and (3.13)

$$\begin{aligned}|\mu_{14}| &\leqslant c_{34}(w_f(1/n))^3 \int_{-1}^{1} \left|\sum_{k=1}^{n} g(t_k)\, \chi_k(x)\right| \frac{dx}{(1-x^2)^{1/2}} \\ &\qquad + c_{35}(w_f(1/n))^4 \\ &\leqslant c_{34}(w_f(1/n))^3 \pi \max_{-1\leqslant x\leqslant 1} |g(x)| + c_{35}(w_f(1/n))^4 \\ &\leqslant c_{34}(w_f(1/n))^3 \pi c_{30}\, w_f(1/n) + c_{35}(w_f(1/n))^4 \\ &\leqslant c_{36}(w_f(1/n))^4. \end{aligned} \tag{4.30}$$

Finally, we consider

$$\mu_{15}=\int_{-1}^{1} \sum\sum\sum_{k\neq j\neq i} g^2(t_k)\,\chi_k^2(x)\,g(t_j)\,\chi_j(x)\,g(t_i)\,\chi_i(x)(1-x^2)^{-1/2}\,dx$$
$$=\int_{-1}^{1} \sum_{k=1}^{n} g^2(t_k)\,\chi_k^2(x)$$
$$\times\left(\sum_{k=1}^{n} g(t_k)\,\chi_k(x)\right)^2 (1-x^2)^{-1/2}\,dx-\mu_{14}-\mu_{12}. \tag{4.31}$$

Therefore, on using (4.27), (4.30), and Lemma 3.4 we have

$$|\mu_{15}|\leqslant c_{37}(w_f(1/n))^2\int_{-1}^{1}\left(\sum_{k=1}^{n} g(t_k)\,\chi_k(x)\right)^2(1-x^2)^{-1/2}\,dx$$
$$+c_{38}(w_f(1/n))^4$$
$$\leqslant c_{37}(w_f(1/n))^2\int_{-1}^{1}\sum_{k=1}^{n} g^2(t_k)\,\chi_k^2(x)(1-x^2)^{-1/2}\,dx$$
$$+c_{38}(w_f(1/n))^4$$
$$\leqslant c_{39}(w_f(1/n))^4\int_{-1}^{1}\sum_{k=1}^{n}\chi_k^2(x)(1-x^2)^{-1/2}\,dx+c_{38}(w_f(1/n))^4$$
$$\leqslant c_{40}(w_f(1/n))^4. \tag{4.32}$$

On combining (4.32), (4.30), (4.27), (4.28), (4.26), and (4.25) we obtain

$$\mu\leqslant c_{41}(w_f(1/n))^4. \tag{4.33}$$

Now, on using (4.16), (4.17), (4.18), and (4.33) we obtain (1.20). Proof of (1.21) is a simple consequence of [3]

$$|R_n(f,\pm 1)-f(\pm 1)|\leqslant c_{42}\,w_f(1/n),$$

the representation given in the work of Bojanic, Prasad, and Saxena [3], and inequality (1.20).

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