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Integration over quantum permutation groups

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Abstract

We find a combinatorial formula for the Haar measure of quantum permutation groups. This leads to a dynamic formula for laws of diagonal coefficients, explaining the Poisson/free Poisson convergence result for characters.

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0. Introduction

A remarkable fact, discovered by Wang in [14], is that the set $X_n = \{1, \dots, n\}$ has a quantum permutation group. For $n = 1, 2, 3$ this is the usual symmetric group S_n . However, starting from $n = 4$ the situation is different: for instance the dual of $\mathbb{Z}_2 * \mathbb{Z}_2$ acts on X_4 . In other words, “quantum permutations” do exist. They form a compact quantum group Q_n , satisfying the axioms of Woronowicz in [15].

There are several motivations for study of Q_n :

- (1) This quantum group is supposed to be to noncommutative theories what the usual symmetric group is to classical theories. One can expect for instance that Q_n might be of help in connection with Connes’ approach [8].

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- (2) The subgroups of Q_n are in correspondence with subalgebras of the spin planar algebra [2], and several questions from Jones’ paper [11] should have formulations in terms of Q_n . Work here is in progress.
- (3) Some connections with Voiculescu’s free probability theory [13] are pointed out in [3,5]. The idea is that Q_n and its subgroups should provide new illustrations for the old principle “integrating = counting diagrams.”

In this paper we clarify some questions coming from free probability. The main tool is a Weingarten type formula, for Haar integration over Q_n :

$$\int u_{i_1 j_1} \dots u_{i_k j_k} = \sum_{pq} \delta_{pi} \delta_{qj} W_{kn}(p, q).$$

Here u_{ij} are the coefficients of the fundamental representation of Q_n . Their products are known to form a basis of the algebra of representative functions on Q_n , so the above formula gives indeed all integrals over Q_n .

The sum on the right is over non-crossing partitions, and W_{kn} is the Weingarten matrix, obtained as inverse of the Gram matrix of partitions G_{kn} .

This leads to a general formula for moments of diagonal coefficients of u :

$$\int (u_{11} + \dots + u_{ss})^k = \text{Tr}(G_{kn}^{-1} G_{ks}).$$

This is similar to the formula in our previous paper [5], for the free analogue of the orthogonal group $O(n)$. The change comes from the scalar product on the space of partitions, different from the one in [5]. This leads to a number of subtleties at level of applications, as well as at the conceptual level: for instance, the matrix G_{kn} in this paper is no longer equal to Di Francesco’s meander matrix [9].

As an application, we solve a problem arising from [3]. It is pointed out there that the character of u is Poisson for S_n with $n \rightarrow \infty$, and free Poisson for Q_n with $n \geq 4$. The fact that $S_n \rightarrow Q_n$ corresponds to a passage Poisson \rightarrow free Poisson is definitely positive, and not surprising. However, there is problem with correspondence of convergences, which is asymptotic \rightarrow exact.

We show here that a fully symmetric statement can be obtained by looking at laws of diagonal coefficients $u_{11} + \dots + u_{ss}$:

s	$u_{11} + \dots + u_{ss} \in \mathbb{C}(S_n)$	$u_{11} + \dots + u_{ss} \in \mathbb{C}(Q_n)$
1	projection (1/n)	projection (1/n)
$o(n)$	\simeq projection (s/n)	\simeq projection (s/n)
tn	\simeq Poisson (t)	\simeq free Poisson (t)
n	\simeq Poisson (1)	free Poisson (1)

As a conclusion, this paper, together with [5], gathers information obtained from direct application of Weingarten philosophy to free quantum groups. This is part of a general analytic approach to several classical and quantum enumeration problems, in the spirit of the Wick formula. See [3,4] and [6,7] for related work.

The paper is organized as follows. Section 1 is a preliminary, in Sections 2–3 we present the integration formulae, and in Sections 4–5 we discuss Poisson laws.

1. Quantum permutation groups

Let A be a \mathbb{C}^* -algebra. A projection is an element $p \in A$ satisfying $p^2 = p = p^*$. Two projections p, q are called orthogonal when $pq = 0$. A partition of unity is a set of mutually orthogonal projections, which sum up to 1.

Definition 1.1. A magic unitary is a square matrix over a \mathbb{C}^* -algebra, all whose rows and columns are partitions of unity with projections.

As a first example, consider the situation $G \curvearrowright X$ where a finite group acts on a finite set. The functions $f : G \rightarrow \mathbb{C}$ form a \mathbb{C}^* -algebra, with the sup norm and usual involution. Inside this algebra we have the characteristic functions

$$\chi_{ij} = \chi \{ \sigma \in G \mid \sigma(j) = i \}$$

which altogether form a magic unitary matrix. Indeed, when i is fixed and j varies, or vice versa, the corresponding sets form partitions of G .

Definition 1.2. χ is called magic unitary associated to $G \curvearrowright X$.

The interest in χ is that it encodes the structural maps of $G \curvearrowright X$. These are the multiplication, unit, inverse and action map:

$$\begin{aligned} m : G \times G &\rightarrow G, \\ u : \{ \cdot \} &\rightarrow G, \\ i : G &\rightarrow G, \\ a : X \times G &\rightarrow X. \end{aligned}$$

Consider the algebras $A = \mathbb{C}(G)$ and $V = \mathbb{C}(X)$. The dual structural maps of $G \curvearrowright X$, called comultiplication, counit, antipode and coaction, and denoted $\Delta, \varepsilon, S, \alpha$, are obtained as functional analytic duals of m, u, i, a :

$$\begin{aligned} \Delta : A &\rightarrow A \otimes A, \\ \varepsilon : A &\rightarrow \mathbb{C}, \\ S : A &\rightarrow A, \\ \alpha : V &\rightarrow V \otimes A. \end{aligned}$$

We denote by $e_i \in V$ the Dirac mass at $i \in X$. The following result is known since Wang’s paper [14]; for the magic unitary formulation, see [1,2].

Proposition 1.1. *The dual structural maps of $G \curvearrowright X$ are given by*

$$\begin{aligned} \Delta(\chi_{ij}) &= \sum \chi_{ik} \otimes \chi_{kj}, \\ \varepsilon(\chi_{ij}) &= \delta_{ij}, \\ S(\chi_{ij}) &= \chi_{ji}, \\ \alpha(e_i) &= \sum e_j \otimes \chi_{ji}, \end{aligned}$$

where χ is associated magic unitary matrix.

Proof. The structural maps are given by the following formulae:

$$\begin{aligned} m(\sigma, \tau) &= \sigma \tau, \\ u(\cdot) &= 1, \\ i(\sigma) &= \sigma^{-1}, \\ a(i, \sigma) &= \sigma(i). \end{aligned}$$

Thus the dual structural maps are given by the following formulae:

$$\begin{aligned} \Delta(f) &= (\sigma, \tau) \rightarrow f(\sigma \tau), \\ \varepsilon(f) &= f(1), \\ S(f) &= \sigma \rightarrow f(\sigma^{-1}), \\ \alpha(f) &= (i, \sigma) \rightarrow f(\sigma(i)). \end{aligned}$$

This gives all equalities in the statement, after a routine computation. \square

In the particular case of the symmetric group S_n acting on $X_n = \{1, \dots, n\}$, we have the following presentation result, which together with Proposition 1.1 gives a purely functional analytic description of $S_n \curvearrowright X_n$. See [1,2,14].

Theorem 1.1. $\mathbb{C}(S_n)$ is the universal commutative \mathbb{C}^* -algebra generated by n^2 elements χ_{ij} , with relations making χ a magic unitary matrix.

Proof. Let A be the universal algebra in the statement. We have an arrow $A \rightarrow \mathbb{C}(S_n)$, which is surjective by Stone–Weierstrass. As for injectivity, this follows from the fact that A with maps as in Proposition 1.1 is a commutative Hopf algebra coacting on X_n , hence corresponds to a group acting on X_n . \square

A free analogue of $\mathbb{C}(S_n)$ can be obtained by removing the commutativity relations in Theorem 1.1.

Definition 1.3. $A_s(n)$ is the universal \mathbb{C}^* -algebra generated by n^2 elements u_{ij} , with relations making u a magic unitary matrix.

This algebra fits into Woronowicz’s formalism in [15]. By using the universal property of $A_s(n)$ we can define maps as follows:

$$\begin{aligned} \Delta(u_{ij}) &= \sum u_{ik} \otimes u_{kj}, \\ \varepsilon(u_{ij}) &= \delta_{ij}, \\ S(u_{ij}) &= u_{ji}, \\ \alpha(e_i) &= \sum e_j \otimes u_{ji}. \end{aligned}$$

These satisfy the axioms for a comultiplication, counit, antipode and coaction. In other words, $A_s(n)$ is a Hopf algebra coacting on X_n .

The following fundamental result is due to Wang [14].

Theorem 1.2. $A_s(n)$ is the biggest Hopf algebra coacting on $X_n = \{1, \dots, n\}$.

Proof. The key remark here is that a linear map $\alpha : V \rightarrow V \otimes A$ is a morphism of \mathbb{C}^* -algebras if and only if its matrix of coefficients consists of projections, and has partitions of unity on all rows. Now if α is a coaction, one can use the antipode to get that the same is true for columns, and this leads to the result. \square

For $n = 1, 2, 3$ the canonical map $A_s(n) \rightarrow \mathbb{C}(S_n)$ is an isomorphism. This is because 1 or 4 projections which form a magic unitary mutually commute, and the same can be shown to happen for 9 projections. See [2,14].

For $n \geq 4$ the algebra $A_s(n)$ is non-commutative and infinite-dimensional. This is because 16 projections which form a magic unitary do not necessarily commute, and can generate an infinite-dimensional algebra. See [14].

2. Integration formula

For the rest of the paper, we assume $n \geq 4$. We use the notation $V = \mathbb{C}^n$.

In this section we find a formula for the Haar functional of $A_s(n)$. This is a linear form satisfying a certain bi-invariance condition, whose existence and uniqueness are shown by Woronowicz in [15]. We denote this form as an integral:

$$\int : A_s(n) \rightarrow \mathbb{C}.$$

The integrals of various combinations of generators u_{ij} can be computed by using Temperley–Lieb diagrams, by plugging results in [1] into the method in [5].

We present here an alternative approach, by using non-crossing partitions. The choice of partitions vs. diagrams is due to some simplifications in integration formulae, to become clear later on. Let us also mention that such partitions and diagrams are known to be in correspondence, via fatgraphs.

Definition 2.1. $NC(k)$ is the set of non-crossing partitions of $\{1, \dots, k\}$.

In this definition $\{1, \dots, k\}$ is regarded as an ordered set. The sets of the partition are unordered, and are called blocks. The fact that p is non-crossing means that we cannot have $a < b < c < d$ with both a, c and b, d in the same block of p .

Given an index set I , we can plug multi-indices $i = (i_1 \dots i_k)$ into partitions $p \in NC(k)$ in the following way: we take the partition p , and we replace each element $s \in \{1, \dots, k\}$ of the set which is partitioned by the corresponding index i_s . What we get is a collection of subsets with repetitions of the index set I .

The number δ_{pi} is defined to be 0 if some of these subsets contains two different indices of I , and to be 1 if not. Observe that we have $\delta_{pi} = 1$ if and only if each of the above subsets with repetitions contains a single element of I , repeated as many times as the cardinality of the subset with repetitions is.

We can summarize this definition in the following way.

Definition 2.2. Given a partition $p \in NC(k)$ and a multi-index $i = (i_1 \dots i_k)$, we can plug i into p , and we define the following number:

$$\delta_{pi} = \begin{cases} 0 & \text{if some block of } p \text{ contains two different indices of } i, \\ 1 & \text{if not.} \end{cases}$$

Consider now the fundamental corepresentation $u = (u_{ij})$ of the Hopf algebra $A_s(n)$. Its k th tensor power in the corepresentation sense is the following matrix, having as indices the multi-indices $i = (i_1 \dots i_k)$ and $j = (j_1 \dots j_k)$:

$$u^{\otimes k} = (u_{i_1 j_1} \dots u_{i_k j_k}).$$

Our first task is to find the fixed vectors of $u^{\otimes k}$. We recall that a vector ξ is fixed by a corepresentation r when we have $r(\xi \otimes 1) = \xi \otimes 1$.

The following result uses [16], and we refer to [1] for missing details.

Proposition 2.1. *The partitions in $NC(k)$ create tensors in $V^{\otimes k}$ via the formula*

$$p(\cdot) = \sum_i \delta_{pi} e_{i_1} \otimes \dots \otimes e_{i_k}$$

and we get in this way a basis for the space of fixed vectors of $u^{\otimes k}$.

Proof. The non-crossing partitions $p \in NC(l + k)$ transform tensors of $V^{\otimes l}$ into tensors of $V^{\otimes k}$, according to the following formula:

$$p(e_{j_1} \otimes \dots \otimes e_{j_l}) = \sum_i \delta_{p,ij} e_{i_1} \otimes \dots \otimes e_{i_k}.$$

Here the multi-index ij is obtained by concatenating the multi-indices i, j . Observe that this notation extends the one in the statement, where $l = 0$.

It is routine to check that linear maps corresponding to different partitions are linearly independent, so the abstract vector space $TL(l, k)$ spanned by $NC(l + k)$ can be viewed as space of linear maps between tensor powers of V :

$$TL(l, k) \subset Hom(V^{\otimes l}, V^{\otimes k}).$$

Consider now the following linear spaces:

$$\text{Hom}(u^{\otimes l}, u^{\otimes k}) \subset \text{Hom}(V^{\otimes l}, V^{\otimes k}).$$

These can be interpreted as follows:

- (1) The spaces on the right form a tensor subcategory $C(V)$ of the tensor category H of finite-dimensional Hilbert spaces.
- (2) The spaces on the left form a tensor subcategory $C(u)$ of the tensor category R of finite-dimensional corepresentations of $A_s(n)$.
- (3) The middle embeddings give an embedding of tensor categories $C(u) \subset C(V)$, which is the restriction of the canonical embedding $R \subset H$.

Now recall that Woronowicz’s Tannakian duality in [16] shows that $A_s(n)$ can be reconstructed from $R \subset H$. Moreover, the matrix version of duality, also from [16], shows that $A_s(n)$ can be reconstructed from $C(u) \subset C(V)$. In particular the presentation relations of $A_s(n)$ should correspond to some generation property of $C(u)$, and this is worked out in [1]: the conclusion is that $C(u)$ is the tensor subcategory of $C(V)$ generated by M and U , the multiplication and unit of V .

We have the following equality of subcategories of $C(V)$, where $\langle A \rangle$ is the category generated by a set of arrows A , and $1_k \in NC(k)$ is the k -block partition:

$$\begin{aligned} \{\text{Hom}(u^{\otimes l}, u^{\otimes k})\}_{lk} &= C(u) \\ &= \langle M, U \rangle \\ &= \langle 1_3, 1_1 \rangle \\ &= \{TL(l, k)\}_{lk}. \end{aligned}$$

Indeed, the first two equalities follows from the above discussion, the third equality follows from $M = 1_3$ and $U = 1_1$, and the fourth equality follows from a routine computation. With $l = 0$ this gives the result. \square

Definition 2.3. The Gram and Weingarten matrices are given by

$$\begin{aligned} G_{kn}(p, q) &= n^{|p \vee q|}, \\ W_{kn}(p, q) &= G_{kn}^{-1}(p, q), \end{aligned}$$

where indices p, q are partitions in $NC(k)$, and $|\cdot|$ is the number of blocks.

The matrix G_{kn} is indeed a Gram matrix, as shown by the following computation in $V^{\otimes k}$, with respect to the canonical scalar product:

$$\begin{aligned} \langle p(\cdot), q(\cdot) \rangle &= \left\langle \sum_i \delta_{pi} e_{i_1} \otimes \cdots \otimes e_{i_k}, \sum_i \delta_{qi} e_{i_1} \otimes \cdots \otimes e_{i_k} \right\rangle \\ &= \sum_i \delta_{pi} \delta_{qi} \end{aligned}$$

$$\begin{aligned}
 &= \sum_i \delta_{p \vee q, i} \\
 &= n^{|p \vee q|}.
 \end{aligned}$$

As for the matrix W_{kn} , this is indeed an analogue of the Weingarten matrix, as shown by the following result.

Theorem 2.1. *We have the integration formula*

$$\int u_{i_1 j_1} \dots u_{i_k j_k} = \sum_{pq} \delta_{pi} \delta_{qj} W_{kn}(p, q),$$

where the sum is over all partitions $p, q \in NC(k)$.

Proof. By standard results of Woronowicz in [16], the linear map

$$\pi(e_{i_1} \otimes \dots \otimes e_{i_k}) = \sum_j e_{j_1} \otimes \dots \otimes e_{j_k} \int u_{i_1 j_1} \dots u_{i_k j_k}$$

is the orthogonal projection onto the fixed point space $F = Hom(1, u^{\otimes k})$. For computing π , we use the factorisation

$$\begin{array}{ccc}
 V^{\otimes k} & \xrightarrow{\pi} & V^{\otimes k} \\
 \downarrow \gamma & & \cup \\
 F & \xrightarrow{\omega} & F,
 \end{array}$$

where the linear map γ is given by the formula

$$\gamma(x) = \sum_p \langle x, p \rangle p$$

and where ω is the inverse of the restriction of γ to F :

$$\omega = (\gamma|_F)^{-1}.$$

With the notation $e_i = e_{i_1} \otimes \dots \otimes e_{i_k}$, this gives:

$$\begin{aligned}
 \int u_{i_1 j_1} \dots u_{i_k j_k} &= \langle \pi(e_i), e_j \rangle \\
 &= \langle \omega \gamma(e_i), e_j \rangle \\
 &= \sum_p \langle e_i, p \rangle \langle \omega(p), e_j \rangle \\
 &= \sum_{pq} \langle e_i, p \rangle \langle q, e_j \rangle \langle \omega(p), q \rangle
 \end{aligned}$$

$$= \sum_{pq} \delta_{pi} \delta_{qj} \langle \omega(p), q \rangle.$$

Now the restriction of γ to F being the linear map corresponding to G_{kn} , its inverse ω is the linear map corresponding to W_{kn} . This gives the result. \square

Theorem 2.2. *We have the moment formula*

$$\int (u_{11} + \dots + u_{ss})^k = \text{Tr}(G_{kn}^{-1} G_{ks}),$$

where u is the fundamental corepresentation of $A_s(n)$.

Proof. We have the following computation:

$$\begin{aligned} \int (u_{11} + \dots + u_{ss})^k &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \int u_{i_1 i_1} \dots u_{i_k i_k} \\ &= \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \sum_{pq} \delta_{pi} \delta_{qi} G_{kn}^{-1}(p, q) \\ &= \sum_{pq} G_{kn}^{-1}(p, q) \sum_{i_1=1}^s \dots \sum_{i_k=1}^s \delta_{pi} \delta_{qi}. \end{aligned}$$

Now the last term on the right is an entry of the Gram matrix:

$$\begin{aligned} \int (u_{11} + \dots + u_{ss})^k &= \sum_{pq} G_{kn}^{-1}(p, q) G_{ks}(p, q) \\ &= \sum_{pq} G_{kn}^{-1}(p, q) G_{ks}(q, p) \\ &= \text{Tr}(G_{ks}^{-1} G_{ks}). \end{aligned}$$

This gives the result. \square

3. Numeric results

We know that the order k moments of diagonal coefficients of u can be explicitly computed, provided we know how to invert the Gram matrix G_{kn} . This matrix has integer entries, and its size is C_k , the k th Catalan number.

The sequence of Catalan numbers is as follows:

$$1, 2, 5, 14, 42, 132, 249, \dots$$

These numbers tell us that:

- (1) For $k = 1, 2, 3$ the moments can be computed directly.

- (2) For $k = 4$ we can use a computer.
 (3) For $k = 5$ we need a computer implementation of $NC(k)$.
 (4) For $k \geq 6$ we need a supercomputer (or a new idea).

In this section we compute the moments for $k = 1, 2, 3, 4$. The formulae below can be regarded as experimental data, illustrating the combinatorics of Gram and Weingarten matrices. They are useful for checking validity of various general statements, and this is how most results in next sections were obtained.

Theorem 3.1. *We have the moment formulae*

$$\begin{aligned} \int (u_{11} + \cdots + u_{ss}) &= \frac{s}{n}, \\ \int (u_{11} + \cdots + u_{ss})^2 &= \frac{s}{n} \cdot \frac{n + (s - 2)}{n - 1}, \\ \int (u_{11} + \cdots + u_{ss})^3 &= \frac{s}{n} \cdot \frac{n^2 + 3(s - 2)n + (s^2 - 9s + 10)}{(n - 1)(n - 2)}, \\ \int (u_{11} + \cdots + u_{ss})^4 &= \frac{s}{n} \cdot \frac{P_s(n)}{(n - 1)(n - 2)(n^2 - 3n + 1)}, \end{aligned}$$

where $P_s(n)$ is the following polynomial:

$$\begin{aligned} P_s(n) &= n^4 + (6s - 12)n^3 \\ &\quad + (6s^2 - 46s + 52)n^2 \\ &\quad + (s^3 - 26s^2 + 104s - 88)n \\ &\quad + (12s^2 - 38s + 28). \end{aligned}$$

Proof. Here are the matrices G_{2n} and G_{2n}^{-1} :

$$\begin{aligned} G_{2n} &= n \begin{pmatrix} 1 & 1 \\ 1 & n \end{pmatrix}, \\ G_{2n}^{-1} &= \frac{1}{n(n - 1)} \begin{pmatrix} n & -1 \\ -1 & 1 \end{pmatrix}. \end{aligned}$$

Here are the matrices G_{3n} and G_{3n}^{-1} :

$$G_{3n} = n \begin{pmatrix} n & 1 & 1 & 1 & n \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & n & 1 & n \\ 1 & 1 & 1 & n & n \\ n & 1 & n & n & n^2 \end{pmatrix},$$

$$G_{3n}^{-1} = \frac{1}{n(n-1)(n-2)} \begin{pmatrix} n-1 & -n & 1 & 1 & -1 \\ -n & n^2 & -n & -n & 2 \\ 1 & -n & n-1 & 1 & -1 \\ 1 & -n & 1 & n-1 & -1 \\ -1 & 2 & -1 & -1 & 1 \end{pmatrix}.$$

And here is G_{4n} , whose inverse will not be given here:

$$G_{4n} = n \begin{pmatrix} n & n & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n \\ n & n^2 & n & n & 1 & n & 1 & 1 & n & n & 1 & n & n & n^2 \\ 1 & n & n & 1 & 1 & n & 1 & 1 & 1 & 1 & 1 & n & 1 & n \\ 1 & n & 1 & n & 1 & 1 & 1 & 1 & n & n & 1 & 1 & 1 & n \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & n & n & 1 & 1 & n^2 & n & 1 & n & n & 1 & n & n & n^2 \\ 1 & 1 & 1 & 1 & 1 & n & n & 1 & n & 1 & 1 & 1 & n & n \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n & 1 & 1 & n & 1 & n \\ 1 & n & 1 & n & 1 & n & n & n & n^2 & n & 1 & n & n & n^2 \\ 1 & n & 1 & n & 1 & n & 1 & 1 & n & n^2 & n & n & n & n^2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & n & n & n & n & n \\ 1 & n & n & 1 & 1 & n & 1 & n & n & n & n & n^2 & n & n^2 \\ n & n & 1 & 1 & 1 & n & n & 1 & n & n & n & n & n^2 & n^2 \\ n & n^2 & n & n & 1 & n^2 & n & n & n^2 & n^2 & n & n^2 & n^2 & n^3 \end{pmatrix}.$$

By computing $\text{Tr}(G_{kn}^{-1}G_{ks})$ we get the formulae in the statement. \square

We would like now to point out the fact that some simplifications appear for $s = 2$. This is not surprising, because the operator $u_{11} + u_{22}$ is a sum of two projections, and such sums have in general reasonably simple combinatorics.

Proposition 3.1. *We have the following moment formulae:*

$$\begin{aligned} \int (u_{11} + u_{22}) &= \frac{2}{n}, \\ \int (u_{11} + u_{22})^2 &= \frac{2}{n-1}, \\ \int (u_{11} + u_{22})^3 &= \frac{2}{n-1} \cdot \frac{n+2}{n}, \\ \int (u_{11} + u_{22})^4 &= \frac{2}{n-1} \cdot \frac{n^2 + 2n - 12}{n^2 - 3n + 1}. \end{aligned}$$

Proof. This follows from Theorem 3.1. \square

4. Asymptotic laws

According to Voiculescu’s free probability theory, the free analogue of the Poisson law of parameter 1 is the following probability measure on $[0, 4]$:

$$\mu_1 = \frac{1}{2\pi} \sqrt{4x^{-1} - 1} dx.$$

This is also known as Marchenko–Pastur law of parameter 1. See [10,12]. For reasons that will become clear later on, we prefer the Poisson terminology.

The following result is pointed out in [3].

Proposition 4.1. *Let u be the fundamental corepresentation of $A_s(n)$.*

- (1) u_{11} is a projection of trace $1/n$.
- (2) $u_{11} + \dots + u_{nn}$ is free Poisson of parameter 1.

Proof. The first assertion is clear. The moments of the variable in the second assertion are the Catalan numbers

$$\begin{aligned} \int (u_{11} + \dots + u_{nn})^l &= \text{Tr}(G_{kn}^{-1} G_{kn}) \\ &= \text{Tr}(1) \\ &= \#NC(k) \\ &= C_k \end{aligned}$$

known to be equal to the moments of the free Poisson law. \square

The measure μ_1 is part of a one-parameter family of real measures. The free Poisson law of parameter $t > 0$ is the probability measure on the set

$$\{0\} \cup [(1 - \sqrt{t})^2, (1 + \sqrt{t})^2]$$

given by the following formula, with the notation $K = \max(0, 1 - t)$:

$$\mu_t = K\delta_0 + \frac{1}{2\pi x} \sqrt{4t - (x - 1 - t)^2} dx.$$

The free Poisson laws form a one-parameter semigroup with respect to free convolution, in the sense that we have $\mu_{s+t} = \mu_s \boxplus \mu_t$. See [10,12].

Lemma 4.1. *We have the estimates*

$$\begin{aligned} G_{kn} &= \Delta_{kn}^{1/2} (Id + O(n^{-1/2})) \Delta_{kn}^{1/2}, \\ G_{kn}^{-1} &= \Delta_{kn}^{-1/2} (Id + O(n^{-1/2})) \Delta_{kn}^{-1/2}, \end{aligned}$$

where $\Delta_{kn}(p, p) = n^{|p|}$ is the diagonal matrix formed by diagonal entries of G_{kn} .

Proof. We have the following formula:

$$(\Delta_{kn}^{-1/2} G_{kn} \Delta_{kn}^{-1/2})(p, q) = \Delta_{kn}^{-1/2}(p, p) G_{kn}(p, q) \Delta_{kn}^{-1/2}(q, q).$$

The (p, q) coefficient of this matrix is given by

$$n^{-\frac{|p|}{2}} n^{|p \vee q|} n^{-\frac{|q|}{2}} = n^{|p \vee q| - \frac{|p| + |q|}{2}}.$$

It is standard combinatorics to check that the last exponent is negative for $p \neq q$, and zero for $p = q$. This gives the result. \square

Theorem 4.1. Let u be the fundamental corepresentation of $A_s(n)$.

- (1) $u_{11} + \dots + u_{ss}$ with $s = o(n)$ is a projection of trace s/n .
- (2) $u_{11} + \dots + u_{ss}$ with $s = tn + o(n)$ is free Poisson of parameter t .

Proof. We use the following moment estimate:

$$\begin{aligned} \int (u_{11} + \dots + u_{ss})^k &= \text{Tr}(G_{kn}^{-1} G_{ks}) \\ &\simeq \text{Tr}(\Delta_{kn}^{-1} \Delta_{ks}) \\ &= \sum_p n^{-|p|} s^{|p|} \\ &= \sum_p (s/n)^{|p|}. \end{aligned}$$

(1) With $s = o(n)$ the terms that count are those corresponding to minimal values of $|p|$. But the minimal value is $|p| = 1$, and this value appears only once. Thus the above sum is asymptotically equal to s/n , and this gives the assertion.

(2) With $s = tn + o(n)$ we have the following estimate:

$$\int (u_{11} + \dots + u_{ss})^k \simeq \sum_p t^{|p|}.$$

On the other hand, the term on the right is the order k moment of the free Poisson law of parameter t , and we are done (see [10]). \square

Corollary 4.1. Let $u_{ij}(n)$ be the fundamental corepresentation of $A_s(n)$. Then for any $t \in (0, 1]$ the following limit converges:

$$\rho_t = \lim_{n \rightarrow \infty} \text{law} \left(\sum_{i=1}^{[tn]} u_{ii}(n) \right)$$

and we get a one-parameter (truncated) semigroup with respect to free convolution.

Proof. This is clear from Theorem 4.1 and from the fact that free Poisson laws form a one-parameter semigroup with respect to free convolution. \square

5. Symmetric groups

We present here classical analogues of results in previous section. These justify the table and comments in the introduction.

The following result is pointed out in [3] in the case $s = n$:

Lemma 5.1. *We have the law formula*

$$\text{law}(u_{11} + \dots + u_{ss}) = \frac{s!}{n!} \sum_{p=0}^s \frac{(n-p)!}{(s-p)!} \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!},$$

where u is the fundamental corepresentation of $\mathbb{C}(S_n)$.

Proof. We have the moment formula

$$\int (u_{11} + \dots + u_{ss})^k = \frac{1}{n!} \sum_{f=0}^s m_f f^k,$$

where m_f is the number of permutations of $\{1, \dots, n\}$ having exactly f fixed points in the set $\{1, \dots, s\}$. Thus the law in the statement, say ν_{sn} , is the following average of Dirac masses:

$$\nu_{sn} = \frac{1}{n!} \sum_{f=0}^s m_f \delta_f.$$

Permutations contributing to m_f are obtained by choosing f points in the set $\{1, \dots, s\}$, then by permuting the remaining $n - f$ points in $\{1, \dots, n\}$ in such a way that there is no fixed point in $\{1, \dots, s\}$. These latter permutations are counted as follows: we start with all permutations, we subtract those having one fixed point, we add those having two fixed points, and so on. We get:

$$\begin{aligned} \nu_{sn} &= \frac{1}{n!} \sum_{f=0}^s \binom{s}{f} \left(\sum_{k=0}^{s-f} (-1)^k \binom{s-f}{k} (n-f-k)! \right) \delta_f \\ &= \sum_{f=0}^s \sum_{k=0}^{s-f} (-1)^k \frac{1}{n!} \cdot \frac{s!}{f!(s-f)!} \cdot \frac{(s-f)!(n-f-k)!}{k!(s-f-k)!} \delta_f \\ &= \frac{s!}{n!} \sum_{f=0}^s \sum_{k=0}^{s-f} \frac{(-1)^k (n-f-k)!}{f!k!(s-f-k)!} \delta_f. \end{aligned}$$

We continue the computation by using the index $p = f + k$:

$$\begin{aligned} v_{sn} &= \frac{s!}{n!} \sum_{p=0}^s \sum_{k=0}^p \frac{(-1)^k (n-p)!}{(p-k)! k! (s-p)!} \delta_{p-k} \\ &= \frac{s!}{n!} \sum_{p=0}^s \frac{(n-p)!}{(s-p)! p!} \sum_{k=0}^p (-1)^k \binom{p}{k} \delta_{p-k} \\ &= \frac{s!}{n!} \sum_{p=0}^s \frac{(n-p)!}{(s-p)!} \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!}. \end{aligned}$$

Here $*$ is convolution of real measures, and the assertion follows. \square

The Poisson law of parameter 1 is the following real probability measure:

$$v_1 = \frac{1}{e} \sum_{p=0}^{\infty} \frac{1}{p!} \delta_p.$$

The following result is a classical analogue of Proposition 4.1.

Proposition 5.1. *Let u be the fundamental corepresentation of $A_s(n)$.*

- (1) u_{11} is a projection of trace $1/n$.
- (2) $u_{11} + \dots + u_{nn}$ with $n \rightarrow \infty$ is Poisson.

Proof. The first assertion is clear. For the second one, we have

$$\text{law}(u_{11} + \dots + u_{nn}) = \sum_{p=0}^n \frac{(\delta_1 - \delta_0)^{*p}}{p!}$$

and the measure on the right converges with $n \rightarrow \infty$ to the Poisson law. \square

The Poisson law of parameter $t \in (0, 1]$ is the following real probability measure:

$$v_t = e^{-t} \sum_{p=0}^{\infty} \frac{t^p}{p!} \delta_p.$$

The following results are classical analogues of Theorem 5.1 and Corollary 5.1.

Theorem 5.1. *Let u be the fundamental corepresentation of $\mathbb{C}(S_n)$.*

- (1) $u_{11} + \dots + u_{ss}$ with $s = o(n)$ is a projection of trace s/n .
- (2) $u_{11} + \dots + u_{ss}$ with $s = tn + o(n)$ is Poisson of parameter t .

Proof. (1) With s fixed and $n \rightarrow \infty$ we have the estimate

$$\begin{aligned} \text{law}(u_{11} + \dots + u_{ss}) &= \sum_{p=0}^s \frac{(n-p)!}{n!} \cdot \frac{s!}{(s-p)!} \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!} \\ &= \delta_0 + \frac{s}{n}(\delta_1 - \delta_0) + O(n^{-2}) \end{aligned}$$

and the law on the right is that of a projection of trace s/n .

(2) We have a law formula of the following type:

$$\text{law}(u_{11} + \dots + u_{ss}) = \sum_{p=0}^s c_p \cdot \frac{(\delta_1 - \delta_0)^{*p}}{p!}.$$

The coefficients c_p can be estimated by using the Stirling formula:

$$\begin{aligned} c_p &= \frac{(tn)!}{n!} \cdot \frac{(n-p)!}{(tn-p)!} \\ &\simeq \frac{(tn)^{tn}}{n^n} \cdot \frac{(n-p)^{n-p}}{(tn-p)^{tn-p}} \\ &= \left(\frac{tn}{tn-p}\right)^{tn-p} \left(\frac{n-p}{n}\right)^{n-p} \left(\frac{tn}{n}\right)^p. \end{aligned}$$

The last expression is estimated by using the definition of exponentials:

$$c_p \simeq e^p e^{-p} t^p = t^p.$$

We compute now the Fourier transform with respect to a variable y :

$$\mathcal{F}(\text{law}(u_{11} + \dots + u_{ss})) \simeq \sum_{p=0}^s t^p \cdot \frac{(e^y - 1)^p}{p!}.$$

The sum of the series on the right is $e^{t(e^y-1)}$, and this is known to be the Fourier transform of the Poisson law ν_t . This gives the second assertion. \square

Corollary 5.1. *Let $u_{ij}(n)$ be the fundamental corepresentation of $\mathbb{C}(S_n)$. Then for any $t \in (0, 1]$ the following limit converges:*

$$\rho_t = \lim_{n \rightarrow \infty} \text{law} \left(\sum_{i=1}^{\lfloor tn \rfloor} u_{ii}(n) \right)$$

and we get a one-parameter (truncated) semigroup with respect to usual convolution.

Proof. This is clear from Theorem 5.1 and from the fact that Poisson laws form a one-parameter semigroup with respect to usual convolution. \square

This result is to be compared to Corollary 4.1, which asserts that for $A_s(n)$ we get a one-parameter semigroup with respect to free convolution.

It follows from [5,7] that similar statements hold for $\mathbb{C}(O(n))$, $\mathbb{C}(U(n))$ and for $A_o(n)$, $A_u(n)$. We believe that further work in this direction can lead to an abstract notion of free Hopf algebra, but we do not have any other example so far.

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