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Analytic and Rational Automorphisms of Complex Algebraic Groups

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Let G be a connected affine algebraic group over the field of complex numbers. Viewing G also as a complex analytic group, we consider the group $\mathscr{A}(G)$ of all complex analytic automorphisms of G. As is well known, $\mathscr{A}(G)$ has a natural structure of a complex Lie group, being identifiable with a closed complex Lie subgroup of the group of all automorphisms of the Lie algebra $\mathscr{L}(G)$ of G. The rational automorphisms of G evidently constitute a subgroup $\mathscr{W}(G)$ of $\mathscr{A}(G)$. It is our main purpose here to determine the situation of $\mathscr{W}(G)$ within $\mathscr{A}(G)$.

In fact, the results of [1] enable us to establish the precise result stated in Theorem 1 below without much additional work. This opens up the possibility of obtaining information about the structure of $\mathscr{A}(G)$ via an examination of $\mathscr{W}(G)$. In particular, by using Theorem 1 in conjunction with the results of [2] concerning $\mathscr{W}(G)$, we obtain a structural characterization of those groups G for which $\mathscr{A}(G)$ is algebraic (when viewed as a subgroup of the automorphism group of $\mathscr{L}(G)$), and we show that the identity components of $\mathscr{A}(G)$ and $\mathscr{W}(G)$ are always algebraic.

The statement and the proof of Theorem 1 make reference to certain subgroups of G, as follows. Let G_u denote the unipotent radical of G, i.e., the unique maximum unipotent normal algebraic subgroup of G. Let M denote the maximum nilpotent normal analytic subgroup of G, and let N be the radical of the commutator subgroup $[G, G]$ of G (actually, N coincides with $[G, G]_u$). Each of these is clearly a connected algebraic subgroup of G, and we have $N \subset G_u \subset M$. It is clear from the definitions that N and M are stable under the action of $\mathscr{A}(G)$, and that G_u is stable under the action of $\mathscr{W}(G)$. Moreover, since M is a normal algebraic subgroup of G containing G_u, we have $M_u = G_u$. Since M is nilpotent, it follows from standard structure theory of affine algebraic groups that M is the direct product $G_u \times M_s$, where M_s is a (complex) toroid and consists

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precisely of the semisimple elements of M. Therefore, M_s is the unique maximum reductive subgroup of M, whence it is stable under the action of $\mathscr{A}(G)$. Finally, consider the factor commutator group $G/[G, G]$. This is a connected abelian affine algebraic group, and therefore (as for M just above) has a unique maximum reductive subgroup. Let Q denote the full inverse image of this reductive group in G. It is clear from the definition that Q is stable under the action of $\mathscr{A}(G)$.

Now we define a subgroup $\mathscr{V}(G)$ of $\mathscr{A}(G)$ as follows. The elements of $\mathscr{V}(G)$ are those automorphisms α of G which leave the elements of Q fixed and satisfy $\alpha(x)x^{-1} \in M_s$ for every element x of M. It is readily seen that $\mathscr{V}(G)$ is a closed complex Lie subgroup of $\mathscr{A}(G)$. From the fact that Q, M and M_s are stable under the action of $\mathscr{A}(G)$, it follows that $\mathscr{V}(G)$ is normal in $\mathscr{A}(G)$.

Noting that G_u is a *nilpotent nucleus* of G in the sense of [1], we have from [1, Theorem 8.2] that an element of $\mathscr{A}(G)$ belongs to $\mathscr{W}(G)$ if and only if it sends G_u onto itself. In particular, this shows that $\mathscr{W}(G)$ is a closed complex Lie subgroup of $\mathscr{A}(G)$.

THEOREM 1. *Let G be a connected complex affine algebraic group. As a complex Lie group, the group $\mathscr{A}(G)$ of all analytic automorphisms of G is the semidirect product of the normal subgroup $\mathscr{V}(G)$ defined above and the group $\mathscr{W}(G)$ of all rational automorphisms of G. Moreover, the canonical image of $\mathscr{V}(G)$ in the full linear group on the Lie algebra of G is an algebraic vector subgroup of that full linear group.*

Proof. Let us fix a maximal reductive subgroup H of G, so that G is the semidirect product $G_u \cdot H$. We claim that the group Q used in defining $\mathscr{V}(G)$ coincides with NH. In order to see this, note that $[G, G]$ is the semidirect product $N \cdot [H, H]$, whence $G/[G, G]$ is the direct product of the vector group G_u/N and the toroid $H/[H, H]$. Thus Q is the full inverse image of $H/[H, H]$ in G, which is evidently NH.

Next, we show that M_s lies in the center of G and coincides with $M \cap H$. Since $M \cap H$ is normal in H, it is reductive, so that $M \cap H \subset M_s$. Now consider the conjugation action of G on M_s. Since M_s is a toroid, the group $\mathscr{A}(M_s)$ of analytic automorphisms of M_s is discrete. The natural map $G \to \mathscr{A}(M_s)$ is continuous, and G is connected. Hence the conjugation action of G on M_s must be trivial, which means that M_s lies in the center of G. Since M_s is reductive, a conjugate of M_s lies in H, so that $M_s \subset H$. With the inclusion established above, this gives $M_s = M \cap H$.

Let τ be any element of $\mathscr{V}(G)$, and let η be any element of the Lie algebra $\mathscr{L}(M)$. If we identify τ with the corresponding automorphism of $\mathscr{L}(G)$, the definition of $\mathscr{V}(G)$ gives $\tau(\eta) - \eta \in \mathscr{L}(M_s)$. We define the linear map

$\tau'\colon \mathscr{L}(M) \to \mathscr{L}(M_s)$ by setting $\tau'(\eta) = \tau(\eta) - \eta$. A simple direct verification shows that the map sending each element τ of $\mathscr{V}(G)$ onto τ' is an injective group homomorphism of $\mathscr{V}(G)$ into the vector group of all linear maps $\mathscr{L}(M) \to \mathscr{L}(M_s)$ that annihilate $\mathscr{L}(N) + \mathscr{L}(M_s)$. Moreover, we claim that this homomorphism is surjective. The proof of this assertion is contained in [1, Section 8], but we shall reproduce the necessary argument here, for greater intelligibility.

Since $\mathscr{L}(H)$ is a reductive Lie algebra whose center contains $\mathscr{L}(M_s)$, there is an ideal J of $\mathscr{L}(H)$ such that $\mathscr{L}(H)$ is the direct Lie algebra sum of J and $\mathscr{L}(M_s)$. Now we have $\mathscr{L}(G) = \mathscr{L}(G_u) + \mathscr{L}(H) = \mathscr{L}(G_u) + \mathscr{L}(M_s) + J = \mathscr{L}(M) + J$, and $\mathscr{L}(M) \cap J = (0)$. Let δ be any linear map $\mathscr{L}(M) \to \mathscr{L}(M_s)$ that annihilates $\mathscr{L}(N) + \mathscr{L}(M_s)$. Extend δ to a linear map $\mathscr{L}(G) \to \mathscr{L}(M_s)$ by making $\delta(J) = (0)$. Now define the linear map $\delta_1\colon \mathscr{L}(G) \to \mathscr{L}(G)$ by $\delta_1(\mu) = \mu + \delta(\mu)$. It is easy to verify directly that δ_1 is a Lie algebra automorphism of $\mathscr{L}(G)$. The universal covering group G^0 of G may be written as a semidirect product $G_u \cdot H^0$, where H^0 is the universal covering group of H. Our Lie algebra automorphism δ_1 defines an analytic automorphism τ^0 of G^0 in the natural fashion. Since δ_1 coincides with the identity map on $\mathscr{L}(H)$, this automorphism τ^0 leaves the elements of H^0 fixed. Hence, via the covering $G^0 \to G$, the automorphism τ^0 induces an analytic automorphism τ of G. It is seen directly from this construction that τ belongs to $\mathscr{V}(G)$ and that $\tau' = \delta$.

We have just seen that when $\mathscr{A}(G)$ is identified with its natural image in the full linear group on $\mathscr{L}(G)$ then the subgroup $\mathscr{V}(G)$ becomes the algebraic vector subgroup consisting of all linear maps $\sigma\colon \mathscr{L}(G) \to \mathscr{L}(G)$ such that, if i denotes the identity map on $\mathscr{L}(G)$, the map $\sigma - i$ sends $\mathscr{L}(M)$ into $\mathscr{L}(M_s)$ and annihilates $\mathscr{L}(N) + \mathscr{L}(H)$. This proves the second part of the statement of Theorem 1.

Next, let us show that $\mathscr{V}(G) \cap \mathscr{W}(G) = (1)$. Let α be an element of $\mathscr{V}(G) \cap \mathscr{W}(G)$, and let x be an element of G_u. Since α is a rational automorphism of G, we have $\alpha(G_u) = G_u$. Hence $\alpha(x)x^{-1} \in G_u$. On the other hand, since α belongs to $\mathscr{V}(G)$, we have $\alpha(x)x^{-1} \in M_s = M \cap H$. Thus $\alpha(x)x^{-1}$ belongs to $H \cap G_u = (1)$, showing that α leaves the elements of G_u fixed. Since α leaves fixed also the elements of $H \subset Q$, it follows that α is the identity automorphism, as we wished to show.

Now let α be any element of $\mathscr{A}(G)$. Then $\alpha(G_u)$ and G_u are nilpotent nuclei of G, in the sense of [1]. Hence we can apply [1, Theorem 8.4] (keeping in view also the construction made in the proof of that theorem), which shows that there is an element τ in $\mathscr{V}(G)$ such that $\tau(G_u) = \alpha(G_u)$. Now $\tau^{-1}\alpha$ is an analytic automorphism of G that sends G_u onto itself. As we have already noted just before stating Theorem 1, this implies that $\tau^{-1}\alpha$ belongs to $\mathscr{W}(G)$. Hence we conclude that $\mathscr{A}(G) = \mathscr{V}(G)\mathscr{W}(G)$.

In order to obtain the proper semidirect product decomposition of $\mathscr{A}(G)$, let us construct the semidirect product of the complex analytic group $\mathscr{V}(G)$ and the complex Lie group $\mathscr{W}(G)$, using the conjugation action of $\mathscr{W}(G)$ on $\mathscr{V}(G)$. The multiplication map of this semidirect product into $\mathscr{A}(G)$ is clearly a morphism of complex Lie groups. By the above, this morphism is bijective. From the fact that $\mathscr{W}(G)$ is isomorphic with a closed complex Lie subgroup of the full linear group on $\mathscr{L}(G)$, we find that our semidirect product is a complex Lie group with only a countable set of connected components. Hence Pontrjagin's well-known theorem on homomorphisms of topological groups shows that our morphism is also an open map. Therefore, $\mathscr{A}(G)$ is the semidirect product $\mathscr{V}(G) \cdot \mathscr{W}(G)$ in the strict sense of complex Lie groups, so that Theorem 1 is proved.

COROLLARY 2. *In the notation of Theorem* 1, *we have* $\mathscr{A}(G) = \mathscr{W}(G)$ *if and only if* G_u *coincides with* N *or with* M.

Proof. We have $M = G_u \times M_s$. Hence, if $G_u = M$ then M_s is trivial, and the definition of $\mathscr{V}(G)$ gives that $\mathscr{V}(G)$ is trivial. If $G_u = N$ then the definition of $\mathscr{V}(G)$, together with the fact that $Q = NH$, shows that $\mathscr{V}(G)$ is trivial. Conversely, if $\mathscr{V}(G)$ is trivial, then our description of the image of $\mathscr{V}(G)$ in the full linear group on $\mathscr{L}(G)$ shows that either $\mathscr{L}(M_s) = (0)$, in which case $M_s = (1)$ and $M = G_u$, or else $\mathscr{L}(M) = \mathscr{L}(N) + \mathscr{L}(M_s)$, in which case $M = N \times M_s$, whence $G_u = N$.

If P is any subgroup of $\mathscr{A}(G)$ we shall say that P is *algebraic* if the canonical image of P in the full linear group on $\mathscr{L}(G)$ is an algebraic subgroup of that full linear group.

THEOREM 3. *Let* G *be a connected complex affine algebraic group. The connected component of the neutral element in the group* $\mathscr{W}(G)$ *of rational automorphisms of* G *is algebraic.*

Proof. Let H be a maximal reductive subgroup of G, and let T be the connected component of the neutral element in the center of H. Let X denote the stabilizer of T in $\mathscr{W}(G)$. Denote the canonical image of G in $\mathscr{W}(G)$ by G'. It follows at once from the conjugacy of the maximal reductive subgroups of G that $\mathscr{W}(G) = G'X$. Indicating connected components of the neutral element by a subscript 1, we wish to show first that $\mathscr{W}(G)_1 = G'X_1$. Let $G' \cdot X$ denote the semidirect product constructed from the complex analytic group G' (a complex analytic subgroup of $\mathscr{W}(G)$) and the complex Lie group X (a closed complex Lie subgroup of $\mathscr{W}(G)$), using the natural action of X on G'. Then the multiplication map $G' \cdot X \to \mathscr{W}(G)$ is a surjective morphism of complex Lie groups. The same argument we used at the end of the proof of Theorem 1 shows that this morphism is also an open map. Hence, if K denotes the kernel of this morphism, $\mathscr{W}(G)$ is isomorphic (as a topological group) with $(G' \cdot X)/K$. Now K consists of the elements (u, u^{-1}) with u in $G' \cap X$. Let G_u' and H' denote the natural images of G_u and H in G'. Then we have $G' = G_u'H'$, and clearly $H' \subset X$. Hence $G' \cap X = (G_u' \cap X)H'$. The group $G_u' \cap X$ is the canonical image of the normalizer of T in G_u. This normalizer actually coincides with the centralizer of T in G_u, for if x is an element of the normalizer we have $xtx^{-1}t^{-1} \in T \cap G_u = (1)$ for every element t of T. Via the exponential map $\mathscr{L}(G_u) \to G_u$, which is bijective, because G_u is nilpotent and simply connected, we see that the centralizer of T in G_u is connected. Hence $G_u' \cap X$ is connected. Since H' is connected, it follows that $G' \cap X$ is connected. Hence the kernel K of our morphism $G' \cdot X \to \mathscr{W}(G)$ is connected. Since $\mathscr{W}(G)$ is isomorphic with $(G' \cdot X)/K$, it follows that $\mathscr{W}(G)_1$ coincides with the image of $(G' \cdot X)_1 = G' \cdot X_1$, i.e., that $\mathscr{W}(G)_1 = G'X_1$.

Since T is a toroid, we know that $\mathscr{A}(T)$ is discrete, whence X_1 must leave the elements of T fixed. Thus we have $\mathscr{W}(G)_1 \subset G'Z$, where Z denotes the element-wise fixer of T in $\mathscr{W}(G)$.

Now let us consider the algebra A of all polynomial functions on G. Via the compositions $f \to f \circ \alpha$, where $f \in A$ and $\alpha \in \mathscr{W}(G)$, we have a right $\mathscr{W}(G)$-module structure on A in which $\mathscr{W}(G)$ acts by algebra automorphisms on A. In proving Lemma 3.1 of [2], we have shown that A is locally finite with respect to the action of the subgroup $G'Z$ of $\mathscr{W}(G)$. *A fortiori*, A is locally finite as a $\mathscr{W}(G)_1$-module.

With every pair (x, f), where x is an element of G and f is an element of A, we associate the complex-valued function x/f on $\mathscr{W}(G)_1$, where $(x/f)(\alpha) = f(\alpha(x))$. Let B denote the smallest algebra of functions on $\mathscr{W}(G)_1$ that contains all the functions x/f and is stable under the translation actions, as well as under the involution corresponding to the inversion map. From the fact that A is locally finite as a $\mathscr{W}(G)_1$-module, it follows that B is a Hopf subalgebra of the Hopf algebra of all analytic representative functions on $\mathscr{W}(G)_1$. Moreover, from the fact that A is finitely generated as an algebra, it follows that the same is true for B. Hence the algebra homomorphisms of B into the field of complex numbers constitute an affine algebraic group $\mathscr{G}(B)$ whose algebra of polynomial functions may be identified with B. Our original group $\mathscr{W}(G)_1$ may be identified in the natural fashion with a complex analytic subgroup of $\mathscr{G}(B)$.

Exactly as in the proof of [2, Theorem 2.1], one sees that $\mathscr{G}(B)$ is naturally isomorphic with a subgroup of $\mathscr{W}(G)$, and one sees at the same timc that the natural map $\mathscr{G}(B) \to \mathscr{W}(G)$, as defined in [2], is a morphism of complex Lie groups. The image of $\mathscr{G}(B)$ in $\mathscr{W}(G)$ contains $\mathscr{W}(G)_1$, so that it is actually an open complex Lie subgruop of $\mathscr{W}(G)$. Therefore, we may identify $\mathscr{G}(B)$, as a complex Lie group, with its image in $\mathscr{W}(G)$, so that $\mathscr{W}(G)_1 \subset \mathscr{G}(B) \subset \mathscr{W}(G)$. Now let us identify these groups with their canonical images in the full linear group on $\mathscr{L}(G)$. An evident slight adjustment of [2, Proposition 2.2] shows that $\mathscr{G}(B)$ thus becomes an *algebraic* subgroup of the full linear group on $\mathscr{L}(G)$. (The canonical map is a morphism of affine algebraic groups from $\mathscr{G}(B)$ to the full linear group on $\mathscr{L}(G)$.) The above inclusions show that $\mathscr{W}(G)_1$ is the identity component of $\mathscr{G}(B)$ in the topological sense. But this is also the identity component of $\mathscr{G}(B)$ in the sense of algebraic groups. Hence $\mathscr{W}(G)_1$ is algebraic, and Theorem 3 is established.

COROLLARY 4. *Let G be a connected complex affine algebraic group. Then $\mathscr{A}(G)_1$ is algebraic, and the following three conditions are mutually equivalent.*

(1) *$\mathscr{A}(G)$ is algebraic.*

(2) *$\mathscr{W}(G)$ is algebraic.*

(3) *Either the connected component of the neutral element in the center of G is unipotent, or the dimension of the center of G/G_u is at most equal to* 1.

Proof. From Theorem 1, we have $\mathscr{A}(G)_1 = \mathscr{V}(G) \cdot \mathscr{W}(G)_1$, and $\mathscr{V}(G)$ is a connected algebraic group. By Theorem 3, the same is true for $\mathscr{W}(G)_1$. Being the group generated by two connected algebraic subgroups of the full linear group on $\mathscr{L}(G)$, the group $\mathscr{A}(G)_1$ is also algebraic, by a standard result of algebraic group theory. An evident slight extension of this argument shows that (2) implies (1). Now suppose that (1) is satisfied. Then $\mathscr{A}(G)/\mathscr{A}(G)_1$ is finite. Hence, using Theorem 1, we have that $\mathscr{W}(G)/\mathscr{W}(G)_1$ is finite. Since $\mathscr{W}(G)_1$ is algebraic, it follows that $\mathscr{W}(G)$ is algebraic. Thus (1) and (2) are equivalent. Finally, the equivalence of (2) and (3) is known from [2, Theorems 3.2 and 3.3].

An interesting special case is the case where the identity component of the center of G is unipotent. In this case, we have from Corollary 4 that $\mathscr{A}(G)$ is algebraic. Moreover, since the group M_s figuring in the definition of $\mathscr{V}(G)$ is a central toroid, it must be trivial in the present case. Hence $\mathscr{V}(G)$ is trivial. Thus, *if the identity component of the center of G is unipotent, then $\mathscr{A}(G)$ is algebraic and coincides with $\mathscr{W}(G)$.*

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