Physical limits on information processing

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Abstract

We derive a fundamental upper bound on the rate at which a device can process information (i.e., the number of logical operations per unit time), arising from quantum mechanics and general relativity. In Planck units a device of volume \( V \) can execute no more than the cube root of \( V \) operations per unit time. We compare this to the rate of information processing performed by nature in the evolution of physical systems, and find a connection to black hole entropy and the holographic principle.

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In this note we derive an upper bound on the rate at which a device can process information. We define this rate as the number of logical operations per unit time, denoted as the ops rate \( R \). The operations in question can be those of either classical or quantum computers. The basis of our result can be stated very simply: information processing requires energy, and general relativity limits the energy density of any object that does not collapse to a black hole. Replacing information processing by information in the previous sentence leads to holography or black hole entropy bounds, a connection we will explore further below. For related work on fundamental physical limits to computation, see [1] and [2]. We use Planck units throughout, in which the speed of light, Planck’s constant and the Planck mass (equivalently, Newton’s constant) are unity.

Our result is easily deduced using the Margolus–Levitin (ML) theorem [3] from quantum mechanics, and the hoop conjecture from general relativity, originally formulated by Thorne [4].

The Margolus–Levitin theorem states that a quantum system with average energy \( E \) requires at least \( \Delta t > \frac{\pi}{2} E^{-1} \) to evolve into an orthogonal (distinguishable) state. It is easy to provide a heuristic justification of this result. For an energy eigenstate of energy \( E, E^{-1} \) is the time required for its phase to change by order one. In a two state system the energy level splitting \( E \) is at most of order the average energy of the two levels. Then, the usual energy-time uncertainty principle suggests that the system cannot be made to undergo a controlled quantum jump on timescales much less than \( E^{-1} \), as this would introduce energy larger than the splitting into the system.

The hoop conjecture gives a criteria for gravitational collapse. It states that a system of total energy \( M \), if confined to a sphere of radius \( L < \eta M \) (\( \eta \) is a coefficient of order one, which we neglect below), must inevitably evolve into a black hole. The condition \( L < M \) is readily motivated by the Schwarzschild radius \( R_s = 2M \). This conjecture has been confirmed in astrophysically-motivated numerical simulations, and has been placed on even stronger footing by recent results on black hole formation from relativistic particle collisions [5]. These results show that, even in the case when all of the energy \( M \) is provided by the kinetic energy of two highly boosted particles, a black hole forms whenever the particles pass within a distance of order \( M \) of each other. Two particle collisions had seemed the most likely to provide a counterexample to the conjecture, since the considerable energy of each particle might have allowed escape from gravitational collapse. One can think of the hoop conjecture as requiring that the average energy density of an object of size \( L \) be bounded above by \( L^{-2} \) in order not to collapse to a black hole. Thus, large objects which are not black holes must be less and less dense.
Our main result follows directly. Consider a device of size $L$ and volume $V \sim L^3$, comprised of $n$ individual components of average energy $E$. Then, the ML theorem gives an upper bound on the total number of operations per unit time

$$\mathcal{R} < nE,$$  

while the hoop conjecture requires $M \sim nE < L$. Combined, we obtain

$$\mathcal{R} < L \sim V^{1/3}. \tag{2}$$

It is interesting to compare this bound to the rate of information processing performed by nature in the evolution of physical systems. At first glance, there appears to be a problem since one typically assumes the number of degrees of freedom in a region is proportional to $V$ (is extensive). Then, the amount of information processing necessary to evolve such a system in time grows much faster than our bound (2) as $V$ increases. Recall that for $n$ degrees of freedom (for simplicity, qubits), the dimension of Hilbert space $H$ is $N = \dim H = 2^n$ and the entropy is $S = \ln N \sim n$. In the extensive case, $n \sim S \sim V$.

However, gravity also constrains the maximum information content (entropy $S$) of a region of space. ‘t Hooft [6] showed that if one excludes states from the Hilbert space whose energies are so large that they would have already caused gravitational collapse, one obtains $S = \ln N < A^{3/4}$, where $N$ is the number of degrees of freedom and $A$ the surface area. To deduce this result, ‘t Hooft replaces the system under study with a thermal one. The number of states of a system with constant total energy $M$ is given to high accuracy by the thermal result in the large volume limit (recall the relation between the microcanonical and canonical ensembles in statistical mechanics). Given a thermal region of radius $L$ and temperature $T$, we have $S \sim T^3 L^3$ and $M \sim T^4 L^3$. Requiring $M < L$ then implies $T \sim L^{-1/2}$ and $S < L^{3/2} \sim A^{3/4}$. We stress that the thermal replacement is just a calculational trick: temperature plays no role in the results, which can also be obtained by direct counting. In [7], it was shown that imposing the condition $\text{Tr} [\rho H] < L$ on a density matrix $\rho$ implies a similar bound $S_N < A^{3/4}$ on the von Neumann entropy $S_N = - \text{Tr} \rho \ln \rho$. For $\rho$ a pure state the result reduces to the previous Hilbert space counting. We note that these bounds are more restrictive than the bound obtained from black hole entropy: $S < A$ [8]. One can interpret this discrepancy as a consequence of higher entropy density of gravitational degrees of freedom relative to ordinary matter [9].

Using these results we can calculate the maximum rate of information processing necessary to simulate any physical system of volume $V$ which is not a black hole. The rate $\mathcal{R}$ is given by the number of degrees of freedom $S \sim L^{3/2}$ times the maximal ML rate $T \sim L^{-1/2}$. This yields $\mathcal{R} \sim L$ as in our bound (2).

Finally, we note that black holes themselves appear to saturate our bound. If we take the black hole entropy to be $S \sim A \sim L^2$, and the typical energy of its modes to be the Hawking temperature $T_H \sim L^{-1}$, we again obtain $\mathcal{R} \sim L$.

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References


1 We identify $n$ individual components for later comparison with entropy. The ML theorem could, of course, be applied to the entire device as well.