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Sets of generators blocking all generators in finite classical polar spaces

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ABSTRACT

We introduce generator blocking sets of finite classical polar spaces. These sets are a generalisation of maximal partial spreads. We prove a characterization of these minimal sets of the polar spaces $Q(2n,q)$, $Q^-(2n+1,q)$ and $H(2n,q^2)$, in terms of cones with vertex a subspace contained in the polar space and with base a generator blocking set in a polar space of rank 2.

1. Introduction and definitions

A (finite) *generalized quadrangle* (GQ) is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{G}, \mathrm{I})$ in which $\mathcal{P}$ and $\mathcal{G}$ are disjoint non-empty sets of objects called *points* and *lines* (respectively), and for which $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{G}) \cup (\mathcal{G} \times \mathcal{P})$ is a symmetric point-line incidence relation satisfying the following axioms:

(i) Each point is incident with $1+t$ lines $(t \geqslant 1)$ and two distinct points are incident with at most one line.
(ii) Each line is incident with $1+s$ points $(s \geqslant 1)$ and two distinct lines are incident with at most one point.
(iii) If X is a point and l is a line not incident with X, then there is a unique pair $(Y, m) \in \mathcal{P} \times \mathcal{G}$ for which $X \mathrm{I} m \mathrm{I} Y \mathrm{I} l$.

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The integers s and t are the *parameters* of the GQ and $\mathcal{S}$ is said to have *order* (s,t). If $\mathcal{S}=(\mathcal{P},\mathcal{G},\mathrm{I})$ is a GQ of order (s,t), we say that $\mathcal{S}'=(\mathcal{P}',\mathcal{G}',\mathrm{I}')$ is a *subquadrangle* of order (s',t') if and only if $\mathcal{P}'\subseteq\mathcal{P}$, $\mathcal{G}'\subseteq\mathcal{G}$, and $\mathcal{S}'=(\mathcal{P}',\mathcal{G}',\mathrm{I}')$ is a generalized quadrangle with I' the restriction of I to $\mathcal{P}'\times\mathcal{G}'$.

The *finite classical polar spaces* are the geometries consisting of the totally isotropic, respectively, totally singular, subspaces of non-degenerate sesquilinear, respectively, non-degenerate quadratic forms on a projective space $\mathrm{PG}(n,q)$. So these geometries are the non-singular symplectic polar spaces $\mathrm{W}(2n+1,q)$, the non-singular parabolic quadrics $\mathrm{Q}(2n,q)$, $n\geqslant 2$, the non-singular elliptic and hyperbolic quadrics $\mathrm{Q}^-(2n+1,q)$, $n\geqslant 2$, and $\mathrm{Q}^+(2n+1,q)$, $n\geqslant 1$, respectively, and the non-singular hermitian varieties $\mathrm{H}(d,q^2)$, $d\geqslant 3$. For q even, the parabolic polar space $\mathrm{Q}(2n,q)$ is isomorphic to the symplectic polar space $\mathrm{W}(2n-1,q)$. For our purposes, it is sufficient to recall that every non-singular parabolic quadric in $\mathrm{PG}(2n,q)$ can, up to a coordinate transformation, be described as the set of projective points satisfying the equation $X_0^2+X_1X_2+\cdots+X_{2n-1}X_{2n}=0$. Every non-singular elliptic quadric of $\mathrm{PG}(2n+1,q)$ can, up to a coordinate transformation, be described as the set of projective points satisfying the equation $g(X_0,X_1)+X_2X_3+\cdots+X_{2n}X_{2n+1}=0$, $g(X_0,X_1)$ an irreducible homogeneous quadratic polynomial over $\mathrm{GF}(q)$. Finally, the hermitian variety $\mathrm{H}(n,q^2)$ can, up to a coordinate transformation, be described as the set of projective points satisfying the equation $X_0^{q+1}+X_1^{q+1}+\cdots+X_n^{q+1}=0$.

The *generators* of a classical polar space are the totally isotropic or totally singular subspaces of maximal dimension. If the generators are of dimension $r-1$, then the polar space is said to be of *rank* r.

Finite classical polar spaces of rank 2 are examples of generalized quadrangles, and are called *finite classical generalized quadrangles*. These are the non-singular parabolic quadrics $\mathrm{Q}(4,q)$, the non-singular elliptic quadrics $\mathrm{Q}^-(5,q)$, the non-singular hyperbolic quadrics $\mathrm{Q}^+(3,q)$, the non-singular hermitian varieties $\mathrm{H}(3,q^2)$ and $\mathrm{H}(4,q^2)$, and the symplectic generalized quadrangles $\mathrm{W}(3,q)$. The GQs $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$ are dual to each other, and have both order (q,q). The GQs $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$ are self-dual if and only if q is even. Finally, the GQs $\mathrm{H}(3,q^2)$ and $\mathrm{Q}^-(5,q)$ are dual to each other, and have respective order (q^2,q) and (q,q^2). The GQ $\mathrm{H}(4,q^2)$ has order (q^2,q^3), and the GQ $\mathrm{Q}^+(3,q)$ has order $(q,1)$. By taking hyperplane sections in the ambient projective space, it is clear that $\mathrm{Q}^+(3,q)$ is a subquadrangle of $\mathrm{Q}(4,q)$, that $\mathrm{Q}(4,q)$ is a subquadrangle of $\mathrm{Q}^-(5,q)$, and that $\mathrm{H}(3,q^2)$ is a subquadrangle of $\mathrm{H}(4,q^2)$. These well-known facts can be found in e.g. [9].

Next consider the projective space $\mathrm{PG}(n,q)$. A set $\mathcal{B}$ of points of $\mathrm{PG}(n,q)$ is a *blocking set* if each hyperplane of $\mathrm{PG}(n,q)$ contains at least one point of $\mathcal{B}$. It is well known that a line of $\mathrm{PG}(n,q)$ is the smallest blocking set of $\mathrm{PG}(n,q)$. For $n=2$, we call a blocking set containing a line *trivial*. For a given q, let ϵ_q denote the integer number such that $q+\epsilon_q$ is the size of the smallest non-trivial blocking set of $\mathrm{PG}(2,q)$. It is also well known that any blocking set B of $\mathrm{PG}(n,q)$, $n>2$, such that $|B|<q+\epsilon_q$, contains a line [2]. The following proposition will serve as a motivation to introduce generator blocking sets of finite classical polar spaces.

Proposition 1.1. *Suppose that $\mathcal{L}$ is a set of lines of* $\mathrm{Q}(4,q)$ *with the property that every line of* $\mathrm{Q}(4,q)$ *meets at least one line of $\mathcal{L}$. If $|\mathcal{L}|$ is smaller than the size of a non-trivial blocking set of* $\mathrm{PG}(2,q)$*, then $\mathcal{L}$ contains the pencil of $q+1$ lines through a point of* $\mathrm{Q}(4,q)$ *or $\mathcal{L}$ contains a regulus contained in* $\mathrm{Q}(4,q)$.

Proof. Using the duality between $\mathrm{Q}(4,q)$ and $\mathrm{W}(3,q)$, the set $\mathcal{L}$ translates to a set $\mathcal{B}$ of points of $\mathrm{W}(3,q)$, such that each point of $\mathrm{W}(3,q)$ is collinear in $\mathrm{W}(3,q)$ to at least one point of $\mathcal{B}$. If φ is the symplectic polarity defining $\mathrm{W}(3,q)$, this means that for each point $P\in\mathrm{W}(3,q)$, the plane P^φ meets $\mathcal{B}$. If π is any plane of $\mathrm{PG}(3,q)$, then π^φ is a point of $\mathrm{W}(3,q)$, and so every plane of $\mathrm{PG}(3,q)$ is the pole of a point of $\mathrm{W}(3,q)$ with relation to the defining polarity φ. Hence, $\mathcal{B}$ is a blocking set with respect to planes of $\mathrm{PG}(3,q)$ and hence, by the assumption on the size of $|\mathcal{L}|$, and by [2], it contains a line l. If l is a line of $\mathrm{W}(3,q)$, it corresponds with the pencil of $q+1$ lines through a point of $\mathrm{Q}(4,q)$. If l is not a line of $\mathrm{W}(3,q)$, it corresponds with a regulus contained in $\mathrm{Q}(4,q)$. The proposition follows. $\square$

Consider a finite classical polar space $\mathcal{S}$ of rank $r \geqslant 2$. A set $\mathcal{L}$ of generators of $\mathcal{S}$ is called a *generator blocking set* if it has the property that every generator of $\mathcal{S}$ meets at least one element of $\mathcal{L}$ non-trivially. We generalize this definition to non-classical GQs, and we say that $\mathcal{L}$ is a generator blocking set of a GQ $\mathcal{S}$ if $\mathcal{L}$ has the property that every line of $\mathcal{S}$ meets at least one element of $\mathcal{L}$. Clearly, for finite classical generalized quadrangles, both definitions coincide. Suppose that $\mathcal{L}$ is a generator blocking set of a finite classical polar space, respectively a GQ. We call an element π of $\mathcal{L}$ *essential* if and only if there exists a generator, respectively line, of $\mathcal{S}$, meeting no element of $\mathcal{L} \setminus \{\pi\}$. We call $\mathcal{L}$ *minimal* if and only if all of its elements are essential.

A *spread* of a finite classical polar space is a set $\mathcal{C}$ of generators such that every point is contained in exactly one element of $\mathcal{C}$. Hence, the generators in the set $\mathcal{C}$ are pairwise disjoint. A *cover* is a set $\mathcal{C}$ of generators such that every point is contained in at least one element of $\mathcal{C}$. Hence, a spread is a cover consisting of pairwise disjoint generators. From the definitions, it follows that spreads and covers are particular examples of generator blocking sets.

In this paper, we will study small generator blocking sets of the polar spaces $\mathrm{Q}(2n, q)$, $\mathrm{Q}^-(2n+1, q)$ and $\mathrm{H}(2n, q^2)$, $n \geqslant 2$, all of rank n. The following theorems, inspired by Proposition 1.1, will be proved in Section 2.

Theorem 1.2. *Let $\mathcal{L}$ be a generator blocking set of a finite generalized quadrangle of order (s, t), with $|\mathcal{L}| = t + 1$. Then $\mathcal{L}$ is the pencil of $t + 1$ lines through a point, or $t \geqslant s$ and $\mathcal{L}$ is a spread of a subquadrangle of order $(s, t/s)$.*

Theorem 1.3.

(a) *Let $\mathcal{L}$ be a generator blocking set of $\mathrm{Q}^-(5, q)$, with $|\mathcal{L}| = q^2 + \delta + 1$. If $\delta \leqslant \frac{1}{2}(3q - \sqrt{5q^2 + 2q + 1})$, then $\mathcal{L}$ contains the pencil of $q^2 + 1$ generators through a point or $\mathcal{L}$ contains a cover of $\mathrm{Q}(4, q)$ embedded as a hyperplane section in $\mathrm{Q}^-(5, q)$.*

(b) *Let $\mathcal{L}$ be a generator blocking set of $\mathrm{H}(4, q^2)$, with $|\mathcal{L}| = q^3 + \delta + 1$. If $\delta < q - 3$, then $\mathcal{L}$ contains the pencil of $q^3 + 1$ generators through a point.*

Section 3 is devoted to a generalization of Proposition 1.1 and Theorem 1.3 to finite classical polar spaces of any rank.

2. Generalized quadrangles

In this section, we study minimal generator blocking sets $\mathcal{L}$ of GQs of order (s, t). After general observations and the proof of Theorem 1.2, we devote two subsections to the particular cases $\mathcal{S} = \mathrm{Q}^-(5, q)$ and $\mathcal{S} = \mathrm{H}(4, q^2)$. We remind that for a GQ $\mathcal{S} = (\mathcal{P}, \mathcal{G}, \mathrm{I})$ of order (s, t), $|\mathcal{P}| = (st + 1)(s + 1)$ and $|\mathcal{G}| = (st + 1)(t + 1)$, see e.g. [9]. Suppose that P is a point of $\mathcal{S}$, then we denote by $P^\perp$ the set of all points of $\mathcal{S}$ collinear with P. By definition, $P \in P^\perp$. For a classical GQ $\mathcal{S}$ with point set $\mathcal{P}$, the set $P^\perp = \pi \cap \mathcal{P}$, with π the tangent hyperplane to $\mathcal{S}$ in the ambient projective space at the point P [5,9]. Therefore, when P is a point of a classical GQ $\mathcal{S}$, we also use the notation $P^\perp$ for the tangent hyperplane π. From the context, it will always be clear whether $P^\perp$ refers to the point set or to the tangent hyperplane.

We denote by $\mathcal{M}$ the set of points of $\mathcal{P}$ covered by the lines of $\mathcal{L}$, and we call any point of $\mathcal{P}$ a *covered point* if it belongs to $\mathcal{M}$. Suppose that $\mathcal{P} \neq \mathcal{M}$, and consider a point $P \in \mathcal{P} \setminus \mathcal{M}$. Since a GQ does not contain triangles, different lines on P meet different lines of $\mathcal{L}$. As every point lies on $t + 1$ lines, this implies that $|\mathcal{L}| = t + 1 + \delta$ with $\delta \geqslant 0$. For each point $P \in \mathcal{M}$, we define $w(P)$ as the number of lines of $\mathcal{L}$ on P. Also, we define

$$W := \sum_{P \in \mathcal{M}} \bigl(w(P) - 1\bigr),$$

then clearly $|\mathcal{M}| = |\mathcal{L}|(s + 1) - W$.

We denote by b_i the number of lines of $\mathcal{G} \setminus \mathcal{L}$ that meet exactly i lines of $\mathcal{L}$, $0 \leqslant i$. Derived from this notation, for $P \notin \mathcal{M}$ and $1 \leqslant i$, we denote by $b_i(P)$ the number of lines on P that meet exactly i lines of $\mathcal{L}$. Remark that there is no a priori upper bound on the number of lines of $\mathcal{L}$ that meet a line of $\mathcal{G} \setminus \mathcal{L}$. In the next lemmas however, we will search for completely covered lines not in $\mathcal{L}$, and therefore we denote by $\tilde{b}_i$ the number of lines of $\mathcal{G} \setminus \mathcal{L}$ that contain exactly i covered points, $0 \leqslant i \leqslant s+1$, and we denote by $\tilde{b}_i(P)$ the number of lines on $P \notin \mathcal{M}$ containing exactly i covered points, $0 \leqslant i \leqslant s+1$.

Lemma 2.1. *Suppose that $\delta < s - 1$.*

(a) *Let X be a point of $\mathcal{P} \setminus \mathcal{M}$. Then $\sum_i b_i(X)(i-1) = \delta$ and*

$$\sum_{P \in X^{\perp} \cap \mathcal{M}} \bigl(w(P) - 1\bigr) \leqslant \delta.$$

(b) *A line not contained in $\mathcal{M}$ can meet at most $\delta + 1$ lines of $\mathcal{L}$. In particular, $\tilde{b}_i = b_i = 0$ for $i = 0$ and for $\delta + 1 < i < s + 1$.*

(c) $$\sum_{i=2}^{\delta+1} \tilde{b}_i(i-1) \leqslant \sum_{i=2}^{\delta+1} b_i(i-1).$$

(d) *If P_0 is a point of $\mathcal{M}$ that lies on a line l meeting $\mathcal{M}$ only in P_0, then*

$$\sum_{P \in \mathcal{M} \setminus P_0^{\perp}} \bigl(w(P) - 1\bigr) \leqslant \delta s.$$

(e) $$(s-\delta) \sum_{i=1}^{\delta+1} b_i(i-1) \leqslant (st - t - \delta)(s+1)\delta + W\delta.$$

(f) *If not all lines on a point P belong to $\mathcal{L}$, then at most $\delta + 1$ lines on P belong to $\mathcal{L}$, and less than $t/s + 1$ lines on P not in $\mathcal{L}$ are completely contained in $\mathcal{M}$.*

Proof. From the assumption that $|\mathcal{L}| = t + 1 + \delta$, $\delta < s - 1$, it follows that not all points of $\mathcal{P}$ can be covered. So we have that $\mathcal{P} \neq \mathcal{M}$.

(a) Consider a point $X \in \mathcal{P} \setminus \mathcal{M}$. Each of the $t+1$ lines on X meets a line of $\mathcal{L}$, and every line of $\mathcal{L}$ meets exactly one of these $t+1$ lines. Hence

$$\bigl|X^{\perp} \cap \mathcal{M}\bigr| \geqslant t + 1 = \sum_i b_i(X).$$

Furthermore,

$$\sum_{P \in X^{\perp} \cap \mathcal{M}} w(P) = \sum_i b_i(X) i = |\mathcal{L}| = t + 1 + \delta.$$

Both assertions follow immediately.

(b) Since every line of $\mathcal{S}$ meets a line of $\mathcal{L}$, it follows that $\tilde{b}_0 = b_0 = 0$. Consider any line $l \notin \mathcal{L}$ containing a point $P \notin \mathcal{M}$. The t lines different from l on P are blocked by at least t lines of $\mathcal{L}$ not meeting l. So at most $|\mathcal{L}| - t = \delta + 1$ lines of $\mathcal{L}$ can meet l.

(c) Consider a line l containing i covered points with $0 < i \leqslant \delta + 1$. Then l must meet at least i lines of $\mathcal{L}$, and, by (b), at most $\delta + 1$ lines of $\mathcal{L}$. On the left hand side, this line is counted exactly $i - 1$ times, on the right hand side this line is counted at least $i - 1$ times. This gives the inequality.

(d) Each point P, with $P \notin P_0^\perp$, is collinear to exactly one point $X \neq P_0$ of l. For $X \in l$, $X \neq P_0$, the inequality of (a) gives $\sum_{P \in X^\perp \cap \mathcal{M}} (w(P) - 1) \leqslant \delta$. Summing over the s points on l different from P_0 gives the expression.

(e) It follows from (b) that every line with a point not in $\mathcal{M}$ has at least $s - \delta$ points not in $\mathcal{M}$. Taking the sum over all points P not in $\mathcal{M}$ and using the equality of (a), one finds

$$\sum_{i=1}^{\delta+1} b_i (s-\delta)(i-1) \leqslant \sum_{P \notin \mathcal{M}} \sum_{i=1}^{\delta+1} b_i(P)(i-1) = \big(|\mathcal{P}| - |\mathcal{M}|\big)\delta.$$

As $|\mathcal{M}| = |\mathcal{L}|(s+1) - W$, the assertion follows.

(f) Suppose that the point P lies on exactly $x \geqslant 1$ lines that are not elements of $\mathcal{L}$. It is not possible that all these x lines are contained in $\mathcal{M}$, since this would require xs lines of $\mathcal{L}$ that are not on P, and then $|\mathcal{L}| \geqslant t+1-x+xs \geqslant t+s$, a contradiction with $\delta < s-1$. Thus we find a point $P_0 \in P^\perp \setminus \mathcal{M}$. Then the t lines on P_0, different from $\langle P, P_0 \rangle$, must be blocked by a line of $\mathcal{L}$ not on P, hence at most $\delta + 1$ lines of $\mathcal{L}$ can contain P.

If y lines on P do not belong to $\mathcal{L}$, but are completely contained in $\mathcal{M}$, then at least $1 + ys$ lines contained in $\mathcal{L}$ meet the union of these y lines, so $1 + ys \leqslant |\mathcal{L}| = t+1+\delta$, so $y < t/s+1$ as $\delta < s$. $\square$

Lemma 2.2. *Suppose that $\delta = 0$. If two lines of $\mathcal{L}$ meet, then $\mathcal{L}$ is a pencil of $t+1$ lines through a point P.*

Proof. The lemma follows immediately from Lemma 2.1 (f). $\square$

Lemma 2.3. *Suppose that $\delta = 0$. If $\mathcal{L}$ is not a pencil, then $t \geqslant s$ and $\mathcal{L}$ is a spread of a subquadrangle of order $(s, t/s)$.*

Proof. We may suppose that $\mathcal{L}$ is not a pencil, so that the lines of $\mathcal{L}$ are pairwise skew by Lemma 2.2. Consider the set $\mathcal{G}'$ of all lines completely contained in $\mathcal{M}$. The set $\mathcal{G}'$ contains at least all the elements of $\mathcal{L}$, so $\mathcal{G}'$ is not empty. If $l \in \mathcal{G}'$ and $P \in \mathcal{M}$ not on l, then there is a unique line $g \in \mathcal{G}$ on P meeting l. As this line contains already two points of $\mathcal{M}$, it is contained in $\mathcal{M}$ by Lemma 2.1 (b), that is $g \in \mathcal{G}'$. This shows that $(\mathcal{M}, \mathcal{G}')$ is a GQ of some order (s, t') and hence it has $(t's+1)(s+1)$ points. As $|\mathcal{M}| = (t+1)(s+1)$, then $t's = t$, that is $t' = t/s$ and hence $t \geqslant s$. $\square$

This lemma proves Theorem 1.2.

2.1. The case $\mathcal{S} = \mathrm{Q}^-(5, q)$

In this subsection, $\mathcal{S} = \mathrm{Q}^-(5,q)$, so $(s,t) = (q, q^2)$, and $|\mathcal{L}| = q^2+1+\delta$. We suppose that $\mathcal{L}$ contains no pencil and we will show for small δ that $\mathcal{L}$ contains a cover of a parabolic quadric $\mathrm{Q}(4,q) \subseteq \mathcal{S}$.

The set $\mathcal{M}$ of covered points blocks all the lines of $\mathrm{Q}^-(5,q)$. An easy counting argument shows that $|\mathcal{M}| \geqslant q^3+1$ (in fact, it follows from [8] that $|\mathcal{M}| \geqslant q^3+q$, but we will not use this stronger lower bound). Thus $W = |\mathcal{L}|(q+1) - |\mathcal{M}| \leqslant (q+1)(q+\delta)$.

Lemma 2.4. *If $\delta \leqslant \frac{q-1}{2}$, then $W \leqslant \delta(q+2)$.*

Proof. Denote by $\mathcal{B}$ the set of all lines not in $\mathcal{L}$, meeting exactly i lines of $\mathcal{L}$ for some i, with $2 \leqslant i \leqslant \delta+1$. We count the number of pairs (l, m), $l \in \mathcal{L}$, $m \in \mathcal{B}$, l meets m. The number of these pairs is $\sum_{i=2}^{\delta+1} b_i i$.

It follows from Lemma 2.1 (e), $W \leqslant (q+1)(q+\delta)$, and $\delta \leqslant \frac{q-1}{2}$, that

$$\begin{aligned}\sum_{i=2}^{\delta+1} b_i i &\leqslant 2\sum_{i=1}^{\delta+1} b_i(i-1) \leqslant 2 \cdot \frac{(q^3-q^2-\delta)(q+1)\delta + W\delta}{q-\delta} \\ &\leqslant 2\frac{(q+1)\delta(q^3-q^2+q)}{q-\delta} \leqslant 2(q-1)\big(q^3-q^2+q\big) =: c.\end{aligned}$$

Hence, some line l of $\mathcal{L}$ meets at most $\lfloor c/|\mathcal{L}|\rfloor$ lines of $\mathcal{B}$, where $\lfloor r\rfloor$, r a real number, denotes the largest integer n such that $n \leqslant r$. Denote by $\mathcal{B}_1$ the set of lines not in $\mathcal{L}$ that meet exactly one line of $\mathcal{L}$. If a point P does not lie on a line of $\mathcal{B}_1$, then it lies on at least $q^2-q-\delta$ lines of $\mathcal{B}$ (by Lemma 2.1 (f) and since $\mathcal{L}$ contains no pencil). As $\delta \leqslant \frac{q-1}{2}$, then $c/|\mathcal{L}| < 2(q^2-q-\delta)$, so at most one point of l can have this property. Thus l has $x \geqslant q$ points that lie on a line of $\mathcal{B}_1$, so l is the only line of $\mathcal{L}$ meeting such a line. Apply Lemma 2.1 (d) on these x points. As every point not on l is collinear with at most one of these x points, it follows that

$$\sum_{P\in\mathcal{M}\setminus l}\bigl(w(P)-1\bigr) \leqslant \frac{x\delta q}{x-1} \leqslant \frac{\delta q^2}{q-1} < \delta(q+1)+1.$$

All but at most one point of l lie on a line of $\mathcal{B}_1$, so l is the only line of $\mathcal{L}$ on these points. One point of l can be contained in more than one line of $\mathcal{L}$, but then it is contained in at most $\delta+1$ lines of $\mathcal{L}$ by Lemma 2.1 (f). Hence $\sum_{P\in l}(w(P)-1) \leqslant \delta$, and therefore $W \leqslant \delta(q+2)$. $\square$

Lemma 2.5. *If $\delta \leqslant \frac{q-1}{2}$, then*

$$\tilde{b}_{q+1} \geqslant q^3+q-\delta-\frac{(q^3+q^2-q\delta-q+1)\delta}{q-\delta}.$$

Proof. We count the number of incident pairs (P,l), $P \in \mathcal{M}$ and l a line of $\mathrm{Q}^-(5,q)$, to see

$$|\mathcal{M}|\bigl(q^2+1\bigr) = |\mathcal{L}|(q+1)+\sum_{i=1}^{q+1}\tilde{b}_i i.$$

As $\mathrm{Q}^-(5,q)$ has $(q^2+1)(q^3+1) = |\mathcal{L}|+\sum_{i=1}^{q+1}\tilde{b}_i$ lines, then

$$\begin{aligned}|\mathcal{L}|q+\sum_{i=1}^{q+1}\tilde{b}_i(i-1) &= |\mathcal{L}|(q+1)+\sum_{i=1}^{q+1}\tilde{b}_i i-\bigl(q^2+1\bigr)\bigl(q^3+1\bigr)\\ &= |\mathcal{M}|\bigl(q^2+1\bigr)-\bigl(q^2+1\bigr)\bigl(q^3+1\bigr)\\ &= \bigl(q^2+1\bigr)(q+1)(q+\delta)-W\bigl(q^2+1\bigr).\\ &\geqslant \bigl(q^2+1\bigr)(q+1)q-\delta\bigl(q^2+1\bigr),\end{aligned}$$

where we used $W \leqslant \delta(q+2)$ from Lemma 2.4. From Lemma 2.1 (c) and (e) and $W \leqslant \delta(q+2)$, we have

$$(q-\delta)\sum_{i=2}^{\delta+1}\tilde{b}_i(i-1) \leqslant (q-\delta)\sum_{i=2}^{\delta+1}b_i(i-1) \leqslant \bigl(q^3-q^2\bigr)(q+1)\delta+\delta^2.$$

Together this gives

$$\bigl(|\mathcal{L}|+\tilde{b}_{q+1}\bigr)q \geqslant \bigl(q^2+1\bigr)(q+1)q-\delta\bigl(q^2+1\bigr)-\frac{(q^3-q^2)(q+1)\delta+\delta^2}{q-\delta}.$$

Using $|\mathcal{L}| = q^2+1+\delta$, the assertion follows. $\square$

Lemma 2.6. *If $\delta \leqslant \frac{1}{2}(3q-\sqrt{5q^2+2q+1})$, then $|\mathcal{L}|(|\mathcal{L}|-1)\delta < \tilde{b}_{q+1}(q+1)q$.*

Proof. First note that the upper bound on δ implies that $\delta \leqslant \frac{1}{2}(q-1)$. Using the lower bound on $\tilde{b}_{q+1}$ from the previous lemma, we find

$$\begin{aligned}&2(q-\delta)\big(\tilde{b}_{q+1}(q+1)q-|\mathcal{L}|\big(|\mathcal{L}|-1\big)\delta\big)\\&\quad\geqslant 2q^4\cdot g(\delta)+(q-1-2\delta)\big(-2\delta^2q^2+\delta^2q+3q^4+3q^3+2q^2+q\big)\\&\qquad+2\delta^4+2\delta^3+q\delta^2+3q^2\delta^2+q+q^2+3q^3+\frac{5}{2}q^4,\end{aligned}$$

with

$$g(\delta):=q^2-\frac{1}{2}q-\frac{1}{4}-3q\delta+\delta^2.$$

The smaller zero of g is $\delta_1=\frac{1}{2}(3q-\sqrt{5q^2+2q+1})$. Hence, if $\delta\leqslant\delta_1$, then $\delta\leqslant\frac{1}{2}(q-1)$ and $g(\delta)\geqslant 0$, and therefore $|\mathcal{L}|(|\mathcal{L}|-1)\delta<\tilde{b}_{q+1}(q+1)q$. $\square$

Lemma 2.7. *If $\delta\leqslant\frac{1}{2}(3q-\sqrt{5q^2+2q+1})$, then there exists a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathcal{M}$.*

Proof. Count triples (l_1,l_2,g), where l_1,l_2 are skew lines of $\mathcal{L}$ and $g\notin\mathcal{L}$ is a line meeting l_1 and l_2 and being completely contained in $\mathcal{M}$. Then

$$|\mathcal{L}|\big(|\mathcal{L}|-1\big)z\geqslant\tilde{b}_{q+1}(q+1)q,$$

where z is the average number of transversals contained in $\mathcal{M}$ and not contained in $\mathcal{L}$, of two skew lines of $\mathcal{L}$. By Lemma 2.6, we find that $z>\delta$. Hence, we find two skew lines $l_1,l_2\in\mathcal{L}$ such that $\delta+1$ of their transversals are contained in $\mathcal{M}$. The lines l_1 and l_2 generate a hyperbolic quadric $\mathrm{Q}^+(3,q)$ contained in $\mathrm{Q}^-(5,q)$, denoted by $\mathcal{Q}^+$. If some point P of $\mathcal{Q}^+$ is not contained in $\mathcal{M}$, then the line on it meeting l_1,l_2 has at least two points in $\mathcal{M}$ and the second line of $\mathcal{Q}^+$ on it has at least $\delta+1$ points in $\mathcal{M}$. This is not possible (cf. Lemma 2.1 (a)). Hence, $\mathcal{Q}^+$ is contained in $\mathcal{M}$. $\square$

Lemma 2.8. *If $\delta\leqslant\frac{1}{2}(3q-\sqrt{5q^2+2q+1})$, then $\mathcal{M}$ contains a parabolic quadric $\mathrm{Q}(4,q)$.*

Proof. We may suppose that $\delta>0$, since the case $\delta=0$ is handled by Lemma 2.3. Lemma 2.7 shows that $\mathcal{M}$ contains a hyperbolic quadric $\mathrm{Q}^+(3,q)$, which will be denoted by $\mathcal{Q}^+$. We also know that $|\mathcal{M}|=|\mathcal{L}|(q+1)-W\geqslant q^3+q^2+q+1-\delta$ by Lemma 2.4. There are $q+1$ hyperplanes through $\mathcal{Q}^+$, necessarily intersecting $\mathrm{Q}^-(5,q)$ in parabolic quadrics $\mathrm{Q}(4,q)$.

Hence, there exists a parabolic quadric $Q(4,q)$, denoted by $\mathcal{Q}$, containing $\mathcal{Q}^+$ such that

$$c:=\big|\big(\mathcal{Q}\setminus\mathcal{Q}^+\big)\cap\mathcal{M}\big|\geqslant\frac{|\mathcal{M}|-(q+1)^2}{q+1}>q^2-q-1.$$

Hence, $c\geqslant q^2-q$. From now on we mean in this proof by a *hole* of $\mathcal{Q}$ a point of $\mathcal{Q}$ that is not in $\mathcal{M}$. Each of the q^3-q-c holes of $\mathcal{Q}$ can be perpendicular to at most δ points of $(\mathcal{Q}\setminus\mathcal{Q}^+)\cap\mathcal{M}$ (cf. Lemma 2.1 (a)). Thus we find a point $P\in(\mathcal{Q}\setminus\mathcal{Q}^+)\cap\mathcal{M}$ that is perpendicular to at most

$$\frac{(q^3-q-c)\delta}{c}\leqslant q\delta$$

holes of $\mathcal{Q}$. The point P lies on $q+1$ lines of $\mathcal{Q}$ and if such a line is not contained in $\mathcal{M}$, then it contains at least $q-\delta$ holes of $\mathcal{Q}$ (cf. Lemma 2.1 (b)). Thus the number of lines of $\mathcal{Q}$ on P that are not contained in $\mathcal{M}$ is at most $q\delta/(q-\delta)$. The hypothesis on δ guarantees that this number is less than $q+1-\delta$. Thus, P lies on at least $r\geqslant\delta+1$ lines of the set $\mathcal{Q}$ that are contained in $\mathcal{M}$. These r lines meet $\mathcal{Q}^+$ in r points of the conic $C:=P^\perp\cap\mathcal{Q}^+$. Denote this set of r points by C'.

Assume that $\mathcal{Q}\setminus P^\perp$ contains a hole R. For $X\in C'$, the hole R has a unique neighbor Y on the line PX; if this is not the point X, then the line RY has at least two points in $\mathcal{M}$, namely Y and the point $RY\cap\mathcal{Q}^+$. So if $|R^\perp\cap C'|=\emptyset$, then there are at least $r\geqslant\delta+1$ lines on the hole R with at least two points in $\mathcal{M}$. This contradicts Lemma 2.1 (a). Therefore $|R^\perp\cap C'|\geqslant r-\delta\geqslant 1$. As every point of C' lies on $q+1$ lines of $\mathcal{Q}$, two of which are in $\mathcal{Q}^+$ and one other is contained in $\mathcal{M}$, then every point of

C' has at most $(q-2)q$ neighbors in $\mathcal{Q}$ that are holes. Counting pairs (X,Y) of perpendicular points $X \in C'$ and holes $R \in \mathcal{Q} \setminus P^\perp$, it follows that $\mathcal{Q} \setminus P^\perp$ contains at most $r(q-2)q/(r-\delta) \leqslant (\delta+1)q(q-2)$ holes. Since $P^\perp \cap \mathcal{Q}$ contains at most $q\delta$ holes, we see that $\mathcal{Q}$ has at most $q\delta + (\delta+1)q(q-2)$ holes. As $\delta \leqslant (q-1)/2$, this number is less than $\frac{1}{2}q(q^2-1)$. Hence, $c > |\mathcal{Q}| - |\mathcal{Q}^+| - \frac{1}{2}q(q^2-1) = \frac{1}{2}q(q^2-1)$. It follows that P is perpendicular to at most

$$\frac{(q^3-q-c)\delta}{c} < \delta$$

holes of $\mathcal{Q}$. This implies that all $q+1$ lines of $\mathcal{Q}$ on P are contained in $\mathcal{M}$. Then every hole of $\mathcal{Q}$ must be connected to at least $q+1-\delta$ and thus all points of the conic C. Apart from P, there is only one such point in $\mathcal{Q}$, so $\mathcal{Q}$ has at most one hole. Then Lemma 2.1 (a) shows that $\mathcal{Q}$ has no hole. □

Lemma 2.9. *If $\mathcal{M}$ contains a parabolic quadric $\mathrm{Q}(4,q)$, denoted by $\mathcal{Q}$, and $|\mathcal{L}| \leqslant q^2+q$, then $\mathcal{L}$ contains a cover of $\mathcal{Q}$.*

Proof. Consider a point $P \in \mathcal{Q}$. As $|P^\perp \cap \mathcal{Q}| = q^2+q+1$, some line of $\mathcal{L}$ must contain two points of $P^\perp \cap \mathcal{Q}$. Then this line is contained in $\mathcal{Q}$ and contains P. □

In this subsection we assumed that $\mathcal{L}$ contains no pencil. The assumption that $\delta \leqslant \frac{1}{2}(3q - \sqrt{5q^2+2q+1})$ then implies that $\mathcal{L}$ contains a cover of a $\mathrm{Q}(4,q) \subseteq \mathrm{Q}^-(5,q)$. Hence, we may conclude the following theorem.

Theorem 2.10. *If $\mathcal{L}$ is a generator blocking set of $\mathrm{Q}^-(5,q)$, $|\mathcal{L}| = q^2+1+\delta$, $\delta \leqslant \frac{1}{2}(3q - \sqrt{5q^2+2q+1})$, then $\mathcal{L}$ contains the pencil of q^2+1 lines through a point of $\mathrm{Q}^-(5,q)$ or $\mathcal{L}$ contains a cover of an embedded parabolic quadric $\mathrm{Q}(4,q) \subset \mathrm{Q}^-(5,q)$.*

2.2. *The case $\mathcal{S} = \mathrm{H}(4,q^2)$*

In this subsection, $\mathcal{S} = \mathrm{H}(4,q^2)$, so $(s,t) = (q^2,q^3)$. We suppose that $\mathcal{L}$ contains no pencil and that $|\mathcal{L}| = q^3+1+\delta$, and we show that this implies that $\delta \geqslant q-3$. The set $\mathcal{M}$ of covered points must block all the lines of $\mathrm{H}(4,q^2)$. It follows from [3] that $|\mathcal{M}| \geqslant q^5+q^2$, and hence $W = |\mathcal{L}|(q^2+1) - |\mathcal{M}| \leqslant (q^2+1)(q+\delta)$.

Lemma 2.11. *If $\delta < q-1$, then $W \leqslant \delta(q^2+3)$.*

Proof. Denote by $\mathcal{B}$ the set of all lines not in $\mathcal{L}$, meeting exactly i lines of $\mathcal{L}$ for some i, with $2 \leqslant i \leqslant \delta+1$. We count the number of pairs (l,m), $l \in \mathcal{L}$, $m \in \mathcal{B}$, l meets m. The number of these pairs is $\sum_{i=2}^{\delta+1} b_i i$.

It follows from Lemma 2.1 (e), $W \leqslant (q^2+1)(q+\delta)$, and $\delta < q-1$, that

$$\begin{aligned}\sum_{i=2}^{\delta+1} b_i i \leqslant 2\sum_{i=1}^{\delta+1} b_i(i-1) &\leqslant \frac{2(q^5-q^3-\delta)(q^2+1)\delta + 2W\delta}{q^2-\delta}\\ &\leqslant \frac{2(q^2+1)\delta(q^5-q^3+q)}{q^2-\delta} \leqslant 2\big(q^6+1\big) =: c.\end{aligned}$$

Hence, some line l of $\mathcal{L}$ meets at most $\lfloor c/|\mathcal{L}| \rfloor$ lines of $\mathcal{B}$. Denote by $\mathcal{B}_1$ the set of lines not in $\mathcal{L}$ that meet exactly one line of $\mathcal{L}$. If a point P does not lie on a line of $\mathcal{B}_1$, then it lies on at least $q^3-q-\delta$ lines of $\mathcal{B}$ (by Lemma 2.1 (f) and since $\mathcal{L}$ contains no pencil). As $\delta < q-1$, then $c/|\mathcal{L}| < 3(q^3-q-\delta)$, so at most two points of l can have this property. Thus l has $x \geqslant q^2-1$ points that lie on a line of $\mathcal{B}_1$, so l is the only line of $\mathcal{L}$ meeting such a line. Apply Lemma 2.1 (d) on these x points. As every point not on l is collinear with at most one of these x points, it follows that

$$\sum_{P\notin l}\bigl(w(P)-1\bigr)\leqslant\frac{x\delta q^2}{x-1}\leqslant\delta\bigl(q^2+1\bigr)+\frac{2\delta}{q^2-2}<\delta\bigl(q^2+1\bigr)+1.$$

Hence, $\sum_{P\notin l}(w(P)-1)\leqslant\delta(q^2+1)$.

All but at most two points of l lie on a line of $\mathcal{B}_1$, so l is the only line of $\mathcal{L}$ on these at least q^2-1 points. At most two points of l can be contained in more than one line of $\mathcal{L}$, but each such point is contained in at most $\delta+1$ lines of $\mathcal{L}$ by Lemma 2.1 (f). Hence $\sum_{P\in l}(w(P)-1)\leqslant 2\delta$, and therefore $W\leqslant\delta(q^2+3)$. □

Lemma 2.12. *If $\delta\leqslant q-2$, then*

$$\tilde{b}_{q^2+1}\geqslant q^4+q-\delta-\frac{(q^5+2q^3-2q\delta-q+2)\delta}{q^2-\delta}.$$

Proof. We count the number of incident pairs (P,l), $P\in\mathcal{M}$ and l a line of $\mathrm{H}(4,q^2)$, to see

$$|\mathcal{M}|\bigl(q^3+1\bigr)=|\mathcal{L}|\bigl(q^2+1\bigr)+\sum_{i=1}^{q^2+1}\tilde{b}_i i.$$

Since $\mathrm{H}(4,q^2)$ has $(q^3+1)(q^5+1)=|\mathcal{L}|+\sum_{i=1}^{q^2+1}\tilde{b}_i$ lines,

$$\begin{aligned}|\mathcal{L}|q^2+\sum_{i=1}^{q^2+1}\tilde{b}_i(i-1)&=|\mathcal{L}|\bigl(q^2+1\bigr)+\sum_{i=1}^{q^2+1}\tilde{b}_i i-\bigl(q^3+1\bigr)\bigl(q^5+1\bigr)\\&=|\mathcal{M}|\bigl(q^3+1\bigr)-\bigl(q^5+1\bigr)\bigl(q^3+1\bigr)\\&=\bigl(q^3+1\bigr)\bigl(q^3+q^2+\delta\bigl(q^2+1\bigr)\bigr)-W\bigl(q^3+1\bigr)\\&\geqslant\bigl(q^3+1\bigr)(q+1)q^2-2\delta\bigl(q^3+1\bigr).\end{aligned}$$

From Lemma 2.1 (c) and (e) and Lemma 2.11, we have

$$\bigl(q^2-\delta\bigr)\sum_{i=2}^{\delta+1}\tilde{b}_i(i-1)\leqslant\bigl(q^2-\delta\bigr)\sum_{i=2}^{\delta+1}b_i(i-1)\leqslant\bigl(q^5-q^3\bigr)\bigl(q^2+1\bigr)\delta+2\delta^2.$$

Together this gives

$$\bigl(|\mathcal{L}|+\tilde{b}_{q^2+1}\bigr)q^2\geqslant\bigl(q^3+1\bigr)(q+1)q^2-2\delta\bigl(q^3+1\bigr)-\frac{(q^5-q^3)(q^2+1)\delta+2\delta^2}{q^2-\delta}.$$

Using $|\mathcal{L}|=q^3+1+\delta$, the assertion follows. □

Lemma 2.13. *If $\delta\leqslant q-4$, then $|\mathcal{L}|(|\mathcal{L}|-1)3q<\tilde{b}_{q^2+1}(q^2+1)q^2$.*

Proof. First note that by the assumption on δ, we may use the lower bound on $\tilde{b}_{q^2+1}$ from the previous lemma, and so we find

$$\begin{aligned}&\bigl(q^2-\delta\bigr)\bigl(\tilde{b}_{q^2+1}\bigl(q^2+1\bigr)q^2-|\mathcal{L}|\bigl(|\mathcal{L}|-1\bigr)3q\bigr)\\&\quad\geqslant(q-4-\delta)\bigl(q^6-\delta\bigr)\bigl(q^3+q^2+5q+5\delta+21\bigr)+r(q,\delta),\end{aligned}$$

with

$$r(q,\delta) = \left(81+33\delta+5\delta^2\right)q^6 + \left(1-2\delta+2\delta^2\right)q^5 + \left(\delta+7\delta^2\right)q^4 - \left(2\delta^2+6\delta\right)q^3 - \delta q^2 + \left(\delta+3\delta^2+3\delta^3\right)q - 84\delta - 41\delta^2 - 5\delta^3.$$

Since $r(q,\delta) > 0$ if $\delta \leqslant q-4$, the lemma follows. □

Lemma 2.14. *If $\mathcal{L}$ contains no pencil, then $\delta \geqslant q-3$.*

Proof. Assume that $\delta < q-3$. Consider a hermitian variety $\mathrm{H}(3,q^2)$, denoted by $\mathcal{H}$, contained in $\mathrm{H}(4,q^2)$. A cover of $\mathcal{H}$ contains at least q^3+q lines by [8], so $\mathcal{H}$ contains at least one hole P. Of all lines through P in $\mathrm{H}(4,q^2)$, q^3-q are not contained in $\mathcal{H}$. They must all meet a line of $\mathcal{L}$, so at most $q+1+\delta$ lines of $\mathcal{L}$ can be contained in $\mathcal{H}$. Hence, at most $|\mathcal{L}|+(q+1+\delta)q^2 = 2q^3+q^2+1+\delta(q^2+1) < (q^2+1)(2q+\delta+1)$ points of $\mathcal{H}$ are covered.

Counting the number of triples (l_1,l_2,g), where l_1,l_2 are skew lines of $\mathcal{L}$ and $g \notin \mathcal{L}$ is a line meeting l_1 and l_2 and being completely contained in $\mathcal{M}$, it follows that

$$|\mathcal{L}|\left(|\mathcal{L}|-1\right)z \geqslant \tilde{b}_{q^2+1}\left(q^2+1\right)q^2,$$

where z is the average number of transversals contained in $\mathcal{M}$ but not belonging to $\mathcal{L}$, of two skew lines of $\mathcal{L}$. By Lemma 2.13, we find that $z > 3q$. So there exist skew lines l_1 and l_2 in $\mathcal{L}$ such that at least $3q+1$ transversals to both lines are contained in $\mathcal{M}$. These transversals are pairwise skew, so the hermitian variety $\mathrm{H}(3,q^2)$ induced in the 3-space generated by l_1 and l_2 contains at least $z(q^2+1) \geqslant (3q+1)(q^2+1) > (q^2+1)(2q+\delta+1)$ points of $\mathcal{M}$. This is a contradiction. □

We have shown that $\delta \geqslant q-3$ if $\mathcal{L}$ contains no pencil. Note that we have no result for $q \in \{2,3\}$. Hence, we have proved the following result.

Theorem 2.15. *If $\mathcal{L}$ is a generator blocking set of $\mathrm{H}(4,q^2)$, $q>3$, $|\mathcal{L}| = q^3+1+\delta$, $\delta < q-3$, then $\mathcal{L}$ contains the pencil of q^3+1 lines through a point.*

3. Polar spaces of higher rank

Consider a subspace V and a point set $\mathcal{B}$ in a projective space, such that the subspace $W := \langle\mathcal{B}\rangle$ has no point in common with V. The *cone with vertex V and base $\mathcal{B}$*, denoted by $V\mathcal{B}$, is the union of the point sets of the subspace V and all the subspaces in the set $\{\langle V,P\rangle \parallel P \in \mathcal{B}\}$. Note that $V\mathcal{B} = \mathcal{B}$ when V is the empty subspace and that $V\mathcal{B} = V$ when $\mathcal{B}$ is the empty set.

In this section, we denote a polar space of rank r by $\mathcal{S}_r$. The parameters (s,t) refer in this section always to (q,q), (q,q^2), (q^2,q^3) respectively, for the polar spaces $\mathrm{Q}(2n,q)$, $\mathrm{Q}^-(2n+1,q)$, $\mathrm{H}(2n,q^2)$. The term *polar space* refers from now on always to a finite classical polar space. Consider a point P in a polar space $\mathcal{S}$. If $\mathcal{S}$ is determined by a polarity ϕ of the ambient projective space, which is true for all polar spaces except for $\mathrm{Q}(2n,q)$ when q is even, then $P^\perp$ denotes the hyperplane P^ϕ. The set $P^\perp \cap \mathcal{S}$ is exactly the set of points of $\mathcal{S}$ collinear with P, including P. For any point set A of the ambient projective space, we define $A^\perp := \langle A\rangle^\phi$.

When $\mathcal{S} = \mathrm{Q}(2n,q)$ and q is even, for P a point of $\mathcal{S}$, let $P^\perp$ denote the tangent hyperplane to $\mathcal{S}$ at P. For any point set A containing at least one point of $\mathcal{S}$, we define the notation $A^\perp$ as

$$A^\perp := \bigcap_{X \in A\cap\mathcal{S}} X^\perp.$$

Using this notation, we can formulate the following property. Consider any polar space $\mathcal{S}_n$ of rank n, and any subspace π of dimension $l \leqslant n-1$, completely contained in $\mathcal{S}_n$. Then $\pi^\perp \cap \mathcal{S}_n = \pi\mathcal{S}_{n-l-1}$ is the cone with vertex π and base $\mathcal{S}_{n-l-1}$ which is a polar space of the same type of rank $n-l-1$ [5,6].

A minimal generator blocking set of $\mathcal{S}_n$, $n \geqslant 3$, can be constructed in a cone as follows. Consider an $(n-3)$-dimensional subspace π_{n-3} completely contained in $\mathcal{S}_n$, hence $\pi_{n-3}^\perp \cap \mathcal{S}_n = \pi_{n-3}\mathcal{S}_2$. Suppose

Table 1
Small examples in rank n.

Polar space	(s,t)	Cone	Base set	Dimension
$\mathrm{Q}(2n,q)$	(q,q)	$\pi_{n-2}\mathrm{Q}(2,q)$	$\mathrm{Q}(2,q)$	$n+1$
		$\pi_{n-3}\mathrm{Q}^+(3,q)$	a spread of $\mathrm{Q}^+(3,q)$	$n+1$
$\mathrm{Q}^-(2n+1,q)$	(q,q^2)	$\pi_{n-2}\mathrm{Q}^-(3,q)$	$\mathrm{Q}^-(3,q)$	$n+2$
		$\pi_{n-3}\mathrm{Q}(4,q)$	a cover of $\mathrm{Q}(4,q)$	$n+2$
$\mathrm{H}(2n,q^2)$	(q^2,q^3)	$\pi_{n-2}\mathrm{H}(2,q^2)$	$\mathrm{H}(2,q^2)$	$n+1$

that $\mathcal{L}$ is a minimal generator blocking set of $\mathcal{S}_2$, then $\mathcal{L}$ consists of lines. Each element of $\mathcal{L}$ spans together with π_{n-3} a generator of $\mathcal{S}_n$, and these $|\mathcal{L}|$ generators of $\mathcal{S}_n$ constitute a minimal generator blocking set of $\mathcal{S}_n$ of size $|\mathcal{L}|$.

Using the smallest generator blocking sets of the mentioned polar spaces of rank 2, we obtain examples of the same size in general rank, listed in Table 1. The notation π_i refers to an i-dimensional subspace. When the cone is $\pi_i B$, the example consists of the generators through the vertex π_i, contained in the cone $\pi_i B$, meeting the base of the cone in the elements of the base set, and the size of the example equals the size of the base set. We will call π_i the *vertex* of the generator blocking set.

The natural question is whether these examples are the smallest ones. The answer is yes, and the following theorem, proved by induction on n, gives our new result.

Theorem 3.1.

(a) *Let $\mathcal{L}$ be a generator blocking set of* $\mathrm{Q}(2n,q)$*, with $|\mathcal{L}| = q+1+\delta$. Let ϵ be the natural number such that $q+1+\epsilon$ is the size of the smallest non-trivial blocking set in* $\mathrm{PG}(2,q)$*. If $q>3$ and $\delta < \min\{\frac{q-2}{2}, \epsilon\}$, then $\mathcal{L}$ contains one of the two examples listed in Table 1 for* $\mathrm{Q}(2n,q)$*.*
(b) *Let $\mathcal{L}$ be a generator blocking set of* $\mathrm{Q}^-(2n+1,q)$*, with $|\mathcal{L}| = q^2+1+\delta$. If $\delta \leqslant \frac{1}{2}(3q-\sqrt{5q^2+2q+1})$, then $\mathcal{L}$ contains one of the two examples listed in Table 1 for* $\mathrm{Q}^-(2n+1,q)$*.*
(c) *Let $\mathcal{L}$ be a generator blocking set of* $\mathrm{H}(2n,q^2)$*, $q>3$, with $|\mathcal{L}| = q^3+1+\delta$. If $\delta < q-3$, then $\mathcal{L}$ contains the example listed in Table 1 for* $\mathrm{H}(2n,q^2)$*.*

3.1. Preliminaries

The following lemma will be useful.

Lemma 3.2.

(a) *If a quadric $\pi_{n-4}\mathrm{Q}^+(3,q)$ or $\pi_{n-3}\mathrm{Q}(2,q)$ in $\mathrm{PG}(n,q)$ is covered by generators, then for any hyperplane T of $\mathrm{PG}(n,q)$, at least $q-1$ of the generators in the cover are not contained in T.*
(b) *If a quadric $\pi_{n-4}\mathrm{Q}(4,q)$ or $\pi_{n-3}\mathrm{Q}^-(3,q)$ in $\mathrm{PG}(n+1,q)$ is covered by generators, then for any hyperplane T, at least q^2-q of the generators in the cover are not contained in T.*
(c) *If a hermitian variety $\pi_{n-3}\mathrm{H}(2,q^2)$ in $\mathrm{PG}(n,q^2)$ is covered by generators, then for any hyperplane T of $\mathrm{PG}(n,q^2)$, at least q^3-q of the generators in the cover are not contained in T.*

Proof. (a) This is clear if T does not contain the vertex of the quadric (i.e. the subspace π_{n-4}, π_{n-3} respectively). If T contains the vertex, then going to the quotient space of the vertex, it is sufficient to handle the cases $\mathrm{Q}(2,q)$ and $\mathrm{Q}^+(3,q)$. The case $\mathrm{Q}(2,q)$ is degenerate but obvious, since any line contains at most two points of $\mathrm{Q}(2,q)$. So suppose that C is a cover of $\mathrm{Q}^+(3,q) \subset \mathrm{PG}(3,q)$, then T is a plane. If $T \cap \mathrm{Q}^+(3,q)$ contains lines, then it contains exactly two lines of $\mathrm{Q}^+(3,q)$. Since at least $q+1$ lines are required to cover $\mathrm{Q}^+(3,q)$, at least $q-1$ lines in C do not lie in T.

(b) Again, we only have to consider the case that T contains the vertex, and so it is sufficient to consider the two cases $\mathrm{Q}^-(3,q)$ and $\mathrm{Q}(4,q)$ in the quotient geometry of the vertex T. For $\mathrm{Q}^-(3,q)$, the assertion is obvious. Suppose finally that C is a cover of $\mathrm{Q}(4,q) \subset \mathrm{PG}(4,q)$. Then T has dimension three. If $T \cap \mathrm{Q}(4,q)$ contains lines at all, then $T \cap \mathrm{Q}(4,q)$ is a hyperbolic quadric $\mathrm{Q}^+(3,q)$ or a cone over a conic $\mathrm{Q}(2,q)$. As these can be covered by $q+1$ lines and since a cover of $\mathrm{Q}(4,q)$ needs at least q^2+1 lines, the assertion is obvious also in this case.

(c) Now we only have to handle the case $\mathrm{H}(2,q^2)$. Since all lines of $\mathrm{PG}(2,q^2)$ contain at most $q+1$ points of $\mathrm{H}(2,q^2)$, the assertion is obvious. □

From now on, we always assume that $\mathcal{S}_n \in \{\mathrm{Q}(2n,q), \mathrm{Q}^-(2n+1,q), \mathrm{H}(2n,q^2)\}$. In this section, $\mathcal{L}$ denotes a generator blocking set of size $|\mathcal{L}| = t+1+\delta$ of a polar space $\mathcal{S}_n$.

Section 2 was devoted to the case $n=2$ of Theorem 3.1 (b) and (c), the case $n=2$ of Theorem 3.1 (a) is Proposition 1.1. The case $n=2$ serves as the induction basis. From now on assume that $n \geqslant 3$. The induction hypothesis is that if $\mathcal{L}$ is a generator blocking set of $\mathcal{S}_{n-1}$ of size $t+1+\delta$, with $\delta < \delta_0$, then $\mathcal{L}$ contains one of the examples listed in Table 1. The number δ_0 can be derived from the case $n=2$ in Theorem 3.1.

The polar space $\mathcal{S}_n$ has $\mathrm{PG}(2n+e,s)$ as the ambient projective space. Here $e=1$ if and only if $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$, and $e=0$ otherwise. Call a point P of $\mathcal{S}_n$ a *hole* if it is not covered by a generator of $\mathcal{L}$. If P is a hole, then $P^\perp$ meets every generator of $\mathcal{L}$ in an $(n-2)$-dimensional subspace. In the polar space $\mathcal{S}_{n-1}$, which is induced in the quotient space of P by projecting from P, these $(n-2)$-dimensional subspaces induce a generator blocking set $\mathcal{L}'$, $|\mathcal{L}'| \leqslant |\mathcal{L}|$. Applying the induction hypothesis, $\mathcal{L}'$ contains one of the examples of $\mathcal{S}_{n-1}$ described in Table 1, living in dimension $n+e$; we will denote this example by $\mathcal{L}^P$. Hence, the $(n+1+e)$-space on P containing the $(n-2)$-dimensional subspaces that are projected from P on the elements of $\mathcal{L}^P$, is a cone with vertex P and base the $(n+e)$-dimensional subspace containing a minimal generator blocking set of $\mathcal{S}_{n-1}$ described in Table 1. We denote this $(n+1+e)$-space on P by S_P.

Lemma 3.3. *Consider a polar space $\mathcal{S}_n \in \{\mathrm{Q}(2n,q), \mathrm{Q}^-(2n+1,q), \mathrm{H}(2n,q^2)\}$, and a generator blocking set of size $t+1+\delta$. If P is a hole and T is an $(n+e)$-dimensional space π on P and in S_P, then at least $t - t/s$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace not contained in T.*

Proof. This assertion follows by going to the quotient space of P, and using Lemma 3.2 and the induction hypothesis of this section. □

We recall the following facts from [6]. Consider a quadric $\mathcal{Q}$ in a projective space $\mathrm{PG}(n,q)$. An i-dimensional subspace π_i of $\mathrm{PG}(n,q)$ will intersect $\mathcal{Q}$ again in a possibly degenerate quadric $\mathcal{Q}'$. If $\mathcal{Q}'$ is degenerate, then $\pi_i \cap \mathcal{Q} = \mathcal{Q}' = R\mathcal{Q}''$, where R is a subspace completely contained in $\mathcal{Q}$, and where $\mathcal{Q}''$ is a non-singular quadric. We call R the *radical* of $\mathcal{Q}'$. Clearly, all generators of $\mathcal{Q}'$ contain R. We recall that $\mathcal{Q}''$ does not have necessarily the same type as $\mathcal{Q}$.

Consider a hermitian variety $\mathcal{H}$ in a projective space $\mathrm{PG}(n,q^2)$. An i-dimensional subspace π_i of $\mathrm{PG}(n,q^2)$ will intersect $\mathcal{H}$ again in a possibly degenerate hermitian variety $\mathcal{H}'$. If $\mathcal{H}'$ is degenerate, then $\pi_i \cap \mathcal{H} = \mathcal{H}' = R\mathcal{H}''$, where R is a subspace completely contained in $\mathcal{H}$, and $\mathcal{H}''$ is a non-singular hermitian variety. We call R the *radical* of $\mathcal{H}'$. Clearly, all generators of $\mathcal{H}'$ contain R.

Lemma 3.4. *Let $\mathcal{L}$ be a minimal generator blocking set of size $t+1+\delta$ of $\mathcal{S}_n$. If an $(n+1+e)$-dimensional subspace Π of $\mathrm{PG}(2n+e,s)$ contains more than $t/s+1+\delta$ generators of $\mathcal{L}$, then $\mathcal{L}$ is one of the examples listed in Table 1.*

Proof. First we show that Π is covered by the generators of $\mathcal{L}$. Assume not and let P be a hole of Π. If $\Pi \cap \mathcal{S}_n$ is degenerate, then its radical is contained in all generators of $\Pi \cap \mathcal{S}_n$, so P is not in the radical. Hence, $P^\perp \cap \Pi$ has dimension $n+e$ and thus $S_P \cap \Pi$ has dimension at most $n+e$. Lemma 3.3 shows that at least $t - t/s$ generators of $\mathcal{L}$ meet S_P in an $(n-2)$-dimensional subspace that is not contained in Π. Hence, Π contains at most $t/s+1+\delta$ generators of $\mathcal{L}$. This contradiction shows that Π is covered by the generators of $\mathcal{L}$.

The subspace Π is an $(n+1+e)$-dimensional subspace containing generators of $\mathcal{S}_n$. This leaves a restricted number of possibilities for $\Pi \cap \mathcal{S}_n$:

(1) $\Pi \cap \mathcal{S}_n \in \{\pi_{n-3}\mathrm{Q}^+(3,q), \pi_{n-2}\mathrm{Q}(2,q)\}$ when $\mathcal{S}_n = \mathrm{Q}(2n,q)$,
(2) $\Pi \cap \mathcal{S}_n \in \{\pi_{n-4}\mathrm{Q}^+(5,q), \pi_{n-3}\mathrm{Q}(4,q), \pi_{n-2}\mathrm{Q}^-(3,q)\}$ when $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$, and
(3) $\Pi \cap \mathcal{S}_n \in \{\pi_{n-3}\mathrm{H}(3,q^2), \pi_{n-2}\mathrm{H}(2,q^2)\}$ when $\mathcal{S}_n = \mathrm{H}(2n,q^2)$.

Case 1. $\Pi \cap \mathcal{S}_n = \pi_{n-2}\mathcal{S}_1$ ($\mathcal{S}_1 = \mathrm{Q}(2,q), \mathrm{Q}^-(3,q)$, or $\mathrm{H}(2,q^2)$).

A generator of $\mathcal{L}$ contained in Π contains the vertex π_{n-2}. If one of the $t+1$ generators on π_{n-2} is not contained in $\mathcal{L}$, then at least s generators of $\mathcal{L}$ are required to cover its points outside of π_{n-2}. Hence, if x of the $t+1$ generators on π_{n-2} are not contained in $\mathcal{L}$, then $|\mathcal{L}| \geqslant t+1-x+xs$. Since $|\mathcal{L}| = t+1+\delta$, with $\delta < s-1$, this implies $x=0$. So $\mathcal{L}$ contains the pencil of generators of $\pi_{n-2}\mathcal{S}_1$, and by the minimality of $\mathcal{L}$, it is equal to this pencil.

Case 2. $\Pi \cap \mathcal{S}_n \in \{\pi_{n-3}\mathrm{Q}^+(3,q), \pi_{n-3}\mathrm{Q}(4,q)\}$.

Recall that $\Pi \cap \mathcal{S}_n = \pi_{n-3}\mathrm{Q}^+(3,q)$ when $\mathcal{S}_n = \mathrm{Q}(2n,q)$ and then $(s,t) = (q,q)$, and that $\Pi \cap \mathcal{S}_n = \pi_{n-3}\mathrm{Q}(4,q)$ when $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$ and then $(s,t) = (q,q^2)$.

All generators of $\mathcal{L}$ contained in Π must contain the vertex π_{n-3}. We will show that the generators of $\mathcal{L}$ contained in Π already cover $\Pi \cap \mathcal{S}_n$; then $\mathcal{L}$ contains (by minimality) no further generator and thus $\mathcal{L}$ is one of the two examples.

Assume that some point P of $\Pi \cap \mathcal{S}_n$ does not lie on any generator of $\mathcal{L}$ contained in Π. As all generators of $\mathcal{L}$ contained in Π contain the vertex π_{n-3}, then P is not in this vertex. Hence, $P^\perp \cap \Pi \cap \mathcal{S}_n$ is a pencil of $t/s+1$ generators $g_0, \ldots, g_{t/s}$ on the subspace $\pi_{n-2} = \langle P, \pi_{n-3}\rangle$. None of the generators g_i is contained in $\mathcal{L}$. Therefore, at least $s+1$ generators of $\mathcal{L}$ are required to cover g_i. One such generator of $\mathcal{L}$ may contain the vertex π_{n-2} and counts for each generator g_i, but this still leaves at least $(t/s+1)s+1$ generators in $\mathcal{L}$ necessary to cover all the generators g_i. But $|\mathcal{L}| < t+s$, a contradiction.

Case 3. $\Pi \cap \mathcal{S}_n \in \{\pi_{n-4}\mathrm{Q}^+(5,q), \pi_{n-3}\mathrm{H}(3,q^2)\}$, and we will show that this case is impossible.

Recall that $\Pi \cap \mathcal{S}_n = \pi_{n-4}\mathrm{Q}^+(5,q)$ when $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$ and then $(s,t) = (q,q^2)$, and that $\Pi \cap \mathcal{S}_n = \pi_{n-3}\mathrm{H}(3,q^2)$ when $\mathcal{S}_n = \mathrm{H}(2n,q^2)$ and then $(s,t) = (q^2,q^3)$. In both cases, $t/s = q$. Denote by V the vertex of $\Pi \cap \mathcal{S}_n$.

All generators of $\mathcal{L}$ contained in Π must contain the vertex V. We will show that the generators of $\mathcal{L}$ contained in Π already cover $\Pi \cap \mathcal{S}_n$.

Assume that some point P of $\Pi \cap \mathcal{S}_n$ does not lie on any generator of $\mathcal{L}$ contained in Π. As all generators of $\mathcal{L}$ contained in Π contain the vertex V, then P is not in V. When $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$, then $P^\perp \cap \Pi \cap \mathcal{S}_n$ contains $2(q+1)$ generators on the subspace $\pi = \langle P, V\rangle$. None of these generators is contained in $\mathcal{L}$. These $2(q+1)$ generators split into two classes, corresponding with the two classes of generators of the hyperbolic quadric $\mathrm{Q}^+(3,q)$, the base of the cone $\pi\mathrm{Q}^+(3,q) = P^\perp \cap \Pi \cap \mathcal{S}_n$. Consider one such class of generators, denoted by $g_0, \ldots, g_q$. When $\mathcal{S}_n = \mathrm{H}(2n,q^2)$, then $P^\perp \cap \Pi \cap \mathcal{S}_n$ contains $q+1$ generators on the subspace $\pi = \langle P, V\rangle$, and none of these generators is contained in $\mathcal{L}$. Also denote these generators by $g_0, \ldots, g_q$. So in both cases we consider $t/s+1 = q+1$ generators $g_0, \ldots, g_q$ on the subspace $\pi = \langle P, V\rangle$, not contained in $\mathcal{L}$. Consider now any generator g_i, then at least $s+1$ generators of $\mathcal{L}$ are required to cover g_i. One such generator of $\mathcal{L}$ may contain the vertex π and counts for each generator g_i, but this still leaves at least $(t/s+1)s+1$ generators in $\mathcal{L}$ necessary to cover all the generators g_i. But $|\mathcal{L}| < t+s$, a contradiction.

Hence in the quotient geometry of the vertex V, the generators of $\mathcal{L}$ contained in Π induce either a cover of $\mathrm{Q}^+(5,q)$, which has size at least q^2+q (see [4]) or a cover of $\mathrm{H}(3,q^2)$, which has size at least q^3+q^2 (see [8]). In both cases, this is in contradiction with the assumed upper bound on $|\mathcal{L}|$. $\square$

3.2. The polar spaces $\mathrm{Q}^-(2n+1,q)$ *and* $\mathrm{H}(2n,q^2)$

This subsection is devoted to the proof of Theorem 3.1 (b) and (c).

Lemma 3.5. *Suppose that $\mathcal{C}$ is a line cover of* $\mathrm{Q}(4,q)$ *with $q^2+1+\delta$ lines. Then each conic and each line of* $\mathrm{Q}(4,q)$ *meet at most $(\delta+1)(q+1)$ lines of $\mathcal{C}$.*

Proof. If $w(P)+1$ is defined as the number of lines of $\mathcal{C}$ on a point P, then the sum of the weights $w(P)$ over all points of $\mathrm{Q}(4,q)$ is $\delta(q+1)$. Hence, a conic can meet at most $(\delta+1)(q+1)$ lines of $\mathcal{C}$, and the same holds for lines. □

Lemma 3.6. *Suppose that $\mathcal{S}_n \in \{\mathrm{Q}^-(2n+1,q), \mathrm{H}(2n,q^2)\}$, $n \geqslant 3$. Suppose that $\mathcal{L}$ is a minimal generator blocking set of size $t+1+\delta$ of $\mathcal{S}_n$, $\delta < \delta_0$. If there exists a hole P that projects $\mathcal{L}$ on a generator blocking set containing a minimal generator blocking set of $\mathcal{S}_{n-1}$ that has a non-trivial vertex, then $\mathcal{L}$ is one of the examples in Table 1.*

Proof. Part 1. We show that there exists a line l containing a hole and meeting $t+1$ generators $g_0,\dots,g_t$ of $\mathcal{L}$. Consider a hole as described in the assertion, so that the mentioned minimal example has a non-trivial vertex. According to Table 1, the minimal example has at least $t+1$ generators that all contain the vertex. So for l we can take any line on P projecting to a point of the vertex.

Part 2. We show that there exists a plane π of $\mathcal{S}_n$ on l that meets at most one generator of $\mathcal{L}$ in a line, and such that $\pi \setminus l$ contains a hole Q.

To see this, first note that $l^\perp \cap \mathcal{S}_n = l\mathcal{S}_{n-2}$, hence the number of planes on the line l that are contained in $\mathcal{S}_n$ equals $|\mathcal{P}_{n-2}|$ ($\mathcal{P}_{n-2}$ is the point set of $\mathcal{S}_{n-2}$). If a generator g of $\mathcal{L}$ meets some plane on l in a line, then g meets l. In this case, as P is a hole and thus not contained in g, the subspace $l^\perp \cap g$ has dimension $n-2$, so the number of planes on l meeting g in a line equals θ_{n-3}.

Consequently, we find such a plane π on l meeting at most $m := |\mathcal{L}| \cdot \theta_{n-3}/|\mathcal{P}_{n-2}|$ generators g_i in a line. As $n \geqslant 3$, a calculation shows that $m < 2$, so π meets at most one of the generators of $\mathcal{L}$ in a line. Then the generators of $\mathcal{L}$ cover at most $|\mathcal{L}|+s$ points of π (recall that s is the order of the underlying field), so we find a hole Q in π that is not on l.

Part 3. We show that there exists a point contained in at least $t-\delta$ generators of $\mathcal{L}$.

Choose π and Q according to Part 2, and consider the minimal generator blocking set $\mathcal{L}_0^Q$ contained in $\mathcal{L}^Q$. Its structure is described in Table 1, which implies that it consists of at least $t+1$ generators. These come from generators in $\mathcal{L}$ and at least $2(t+1)-|\mathcal{L}| = t+1-\delta$ of these are among the generators $g_0,\dots,g_t$. By Part 2, at most one of these meets π in a line. Thus we may assume for $i \leqslant t-\delta$ that g_i meets S_Q in an $(n-2)$-subspace projected from Q on one of the elements of $\mathcal{L}_0^Q$.

The point $P_i := l \cap g_i$ belongs to the $(n-2)$-subspace $Q^\perp \cap g_i$ and hence to S_Q, $1 \leqslant i \leqslant t-\delta$. If l is not contained in S_Q, then it follows that $l \cap S_Q$ is a point equal to all points $P_1,\dots,P_{t-\delta}$ and we are done. So suppose for the rest of Part 3 that l is contained in S_Q, and that not all $t-\delta$ generators $g_1,\dots,g_{t-\delta}$ pass through the same point of l. Denote the projection of l from Q by l_Q. This line is covered by the elements of $\mathcal{L}_0^Q$.

Table 1 gives the possible structures of $\mathcal{L}_0^Q$, which live in a $\pi_{n-4}Q(4,q)$, $\pi_{n-3}Q^-(3,q)$ or $\pi_{n-3}H(2,q^2)$ and cover this cone. As l is contained in S_Q, then l_Q is contained in the cone. We claim that the vertex of $\mathcal{L}_0^Q$ is non-empty and that l_Q meets this vertex non-trivially. This is clear for the latter two structures, since their bases do not contain lines. In the first case, it follows from Lemma 3.5 and the fact that l_Q meets $t+1-\delta$ of the generators of $\mathcal{L}_0^Q$. In fact, if the base of $\mathcal{L}_0^Q$ is a parabolic quadric $\mathrm{Q}(4,q)$, then $\mathcal{S}_n = Q^-(2n+1,q)$ (see Table 1) and thus $t=q^2$ and hence $t+1-\delta > (\delta+1)(q+1)$, so the lemma can be applied.

The fact that l_Q meets the vertex of the described cones means that Q lies on a line l' meeting l and such that l' projects from Q to the vertex. Then $g_0,\dots,g_{t-\delta}$ meet l', since their projections belong to $\mathcal{L}_0^Q$ and thus contain the vertex. Hence, these generators meet the lines l and l' of the plane π. Since at most one of these generators meets π in a line, then $t-\delta$ of these generators must contain the point $l \cap l'$.

Part 4. We show that $\mathcal{L}$ is one of the examples in Table 1. Let X be a point contained in $t-\delta$ generators of $\mathcal{L}$, and consider a hole R not in the perp of X. Then S_R meets at least $(t-\delta+t+1)-(t+1+\delta)=t-2\delta$ of the generators on X in an $(n-2)$-subspace. These generators are therefore contained in $T := \langle S_R, X\rangle$. Finally, consider a hole R' not in T and not in the perp of X. Then at least $t-3\delta > t/s+1+\delta$ of the generators that contain X and are contained in T meet $S_{R'}$ in an $(n-2)$-subspace. These generators lie therefore in $\langle S_{R'} \cap T, X\rangle$, which has dimension $n+1+e$. Now Lemma 3.4 completes the proof. $\square$

Corollary 3.7. *Theorem* 3.1 (c) *is true for* $\mathrm{H}(2n, q^2)$, $n \geqslant 3$.

Proof. Theorem 2.15 guarantees that the assumption of Lemma 3.6 is true for $\mathcal{S}_n = \mathrm{H}(2n, q^2)$ and $n=3$. Theorem 3.1 (c) then follows from the induction hypothesis. $\square$

We may now assume that $\mathcal{S}_n = \mathrm{Q}^-(2n+1, q)$, $n=3$, and that the projection of $\mathcal{L}$ from every hole contains a generator blocking set with a trivial vertex, i.e. a cover of $Q(4,q)$. As $n=3$, then $\mathcal{L}$ is a set of planes.

Lemma 3.8. *If a hyperplane T contains more than $q+1+3\delta$ elements of $\mathcal{L}$, then $\mathcal{L}$ is one of the two examples in* $\mathrm{Q}^-(7,q)$ *from Table* 1.

Proof. Denote by $\mathcal{L}'$ the set of the generators of $\mathcal{L}$ that are contained in T. If P is a hole not contained in T, then S_P meets all except at most δ planes of $\mathcal{L}$ in a line, and hence more than $q+1+2\delta$ of these planes are contained in T. Recall that S_P is a cone with vertex P over $S_P \cap T$, and $S_P \cap T$ has dimension four.

Note that $P^\perp \cap \mathrm{Q}^-(7,q) = P\mathcal{Q}_5$ with $\mathcal{Q}_5$ an elliptic quadric $\mathrm{Q}^-(5,q)$, and we may suppose that $\mathcal{Q}_5 \subseteq T$. Denote by $\mathcal{Q}_4$ the parabolic quadric $\mathrm{Q}(4,q)$ contained in $\mathcal{Q}_5$ such that $S_P = P\mathcal{Q}_4$, then $T \cap S_P \cap \mathrm{Q}^-(7,q) = \mathcal{Q}_4$. Consider any point $Q \in (\mathrm{Q}^-(7,q) \cap P^\perp) \setminus (S_P \cup \mathcal{Q}_5)$. Clearly $W := Q^\perp \cap T \cap S_P$ meets $\mathrm{Q}^-(7,q)$ in an elliptic quadric $\mathrm{Q}^-(3,q)$. There are $(q^4-q^2)(q-1)$ points like Q, and at most $(q^2-q)(q+1)$ of them are covered by elements of $\mathcal{L}$, since we assumed that more than $q+1+3\delta$ elements of $\mathcal{L}$ are contained in T. So at least $q^5-q^4-2q^3+q^2+q>0$ points Q of $(\mathrm{Q}^-(7,q) \cap P^\perp) \setminus (S_P \cup \mathcal{Q}_5)$ are holes and have the property that $W := Q^\perp \cap T \cap S_P$ meets $\mathrm{Q}^-(7,q)$ in an elliptic quadric $\mathrm{Q}^-(3,q)$. As before, $S_Q \cap T$ has dimension four and meets at least $|\mathcal{L}'|-\delta$ planes of $\mathcal{L}'$ in a line. Then at least $|\mathcal{L}'|-2\delta$ planes of $\mathcal{L}'$ meet $S_P \cap T$ and $S_Q \cap T$ in a line. As $S_P \cap S_Q \cap T \subseteq W$ does not contain singular lines, it follows that these $|\mathcal{L}'|-2\delta$ planes of $\mathcal{L}'$ are contained in the subspace $H := \langle S_P \cap T, S_Q \cap T\rangle$.

We have $W \cap \mathrm{Q}^-(7,q) = \mathrm{Q}^-(3,q)$, so in the quotient geometry of P, the $|\mathcal{L}'|-2\delta$ planes induce $|\mathcal{L}'|-2\delta$ lines all meeting this $\mathrm{Q}^-(3,q)$. Now $\mathcal{L}$ is projected from P on a cover of a parabolic quadric $\mathrm{Q}(4,q)$ with at most $q^2+1+\delta$ lines. Then $|\mathcal{L}'|-2\delta$ lines of the cover must meet more than $q+1$ points of this elliptic quadric $\mathrm{Q}^-(3,q)$. It follows that $S_Q \cap T$ contains more than $q+1$ points of the elliptic quadric $\mathrm{Q}^-(3,q)$ in W and hence $W \subseteq S_Q$. Then $S_P \cap T$ and $S_Q \cap T$ meet in W, so the subspace H they generate has dimension five. As $|\mathcal{L}'|-2\delta > q+1+\delta$ planes of $\mathcal{L}$ lie in H, Lemma 3.4 completes the proof. $\square$

Lemma 3.9. *Suppose that $\mathcal{L}$ is a minimal generator blocking set of size $t+1+\delta$ of* $\mathrm{Q}^-(7,q)$, $\delta < \delta_0$. *If there exists a hole P that projects $\mathcal{L}$ on a generator blocking set containing a cover of* $\mathrm{Q}(4,q)$, *then $\mathcal{L}$ is one of the examples in Table* 1.

Proof. Consider a hole P. Then $S_P \cap \mathrm{Q}^-(7,q) = P\mathrm{Q}(4,q)$. Denote the base of this cone by $\mathcal{Q}_4$. The assumption of the lemma is that $\mathcal{L}^P$ is a minimal cover $\mathcal{C}$ of $\mathcal{Q}_4$. Consider a point $X \in \mathcal{Q}_4$ contained in exactly one line of $\mathcal{C}$. Then $X^\perp \cap \mathcal{Q}_4 = X\mathrm{Q}(2,q)$, and each line on X is covered completely, so $X^\perp \cap \mathcal{Q}_4$ meets at least q^2+1 lines of $\mathcal{C}$.

The lines of $\mathcal{C}$ are projections from P of the intersections of elements of $\mathcal{L}$ with the subspace S_P, call $\mathcal{C}'$ this set of intersections that is projected on $\mathcal{C}$. Thus the line $h = PX$ of S_P on P meets exactly one line of $\mathcal{C}'$ and $h^\perp \cap S_P \cap \mathrm{Q}^-(7,q) = h\mathrm{Q}(2,q)$ meets at least q^2+1 lines of $\mathcal{C}'$. At most δ elements of $\mathcal{L}$ are possibly not intersecting S_P in an element of $\mathcal{C}'$, so we find a hole Q on h with $Q \neq P$. There are at least q^2+1 elements in $\mathcal{C}'$, so at least $q^2+1-\delta$ elements come from planes $\pi \in \mathcal{L}$ with $\pi \cap \mathrm{Q}^\perp \subset S_Q$. For each such element, its intersection with $h\mathrm{Q}(2,q)$ lies in S_Q. Thus either $S_P \cap S_Q = h^\perp \cap S_P$ or $S_P \cap S_Q$ is a 3-dimensional subspace of $h^\perp \cap S_P$ that contains a cone $Y\mathrm{Q}(2,q)$.

In the second case, the vertex Y must be the point Q (as $Q \in S_Q$); but then projecting from Q we see a cover of $\mathrm{Q}(4,q)$ containing a conic meeting at least $q^2+1-\delta$ of the lines of the cover. In this situation, Lemma 3.5 gives $q^2+1-\delta \leqslant (\delta+1)(q+1)$, that is $\delta > q-3$, a contradiction.

Hence, $S_P \cap S_Q$ has dimension four, so $T = \langle S_P, S_Q \rangle$ is a hyperplane. At least q^2 planes of $\mathcal{L}$ meet S_P in a line that is not contained in $S_P \cap S_Q$. At least $q^2-\delta$ of these also meet S_Q in a line and hence are contained in T. It follows from $\delta < q/2$ that $q^2-\delta > q+1+3\delta$, and then Lemma 3.8 completes the proof. $\square$

Corollary 3.10. *Theorem* 3.1 (b) *is true for* $\mathrm{Q}^-(2n+1,q)$, $n \geqslant 3$.

Proof. Theorem 2.10 guarantees that for $\mathcal{S}_n = \mathrm{Q}^-(7,q)$ and $n=3$, the assumption of either Lemma 3.6 or Lemma 3.9 is true. Hence, Theorem 3.1 (b) follows for $n=3$. But then the assumption of Lemma 3.6 is true for $\mathcal{S}_n = \mathrm{Q}^-(2n+1,q)$ and $n=4$, and then Theorem 3.1 (b) follows from the induction hypothesis. $\square$

3.3. The polar space $\mathrm{Q}(2n,q)$

This subsection is devoted to the proof of Theorem 3.1 (a). Lemma 3.6 can also be translated to this case, but only for a bad upper bound on δ. Therefore we treat the polar space $\mathrm{Q}(2n,q)$ separately. Recall that for $\mathrm{Q}(2n,q)$, $\delta_0 = \min\{\frac{q-2}{2}, \epsilon\}$, with ϵ such that $q+1+\epsilon$ is the size of the smallest non-trivial blocking set of $\mathrm{PG}(2,q)$.

We suppose that $\mathcal{L}$ is a generator blocking set of $\mathrm{Q}(2n,q)$, $n \geqslant 3$, of size $q+1+\delta$, $\delta < \delta_0$. Recall that $\mathcal{L}^R$ is the minimal generator blocking set of $\mathrm{Q}(2n-2,q)$ contained in the projection of $\mathcal{L}$ from a hole R. So when $n=3$, it is possible that $\mathcal{L}^R$ is a generator blocking set of $\mathrm{Q}(4,q)$ with a trivial vertex.

For Lemmas 3.11, 3.12, and 3.13, the assumption is that $n=3$, and that for any hole R, $\mathcal{L}^R$ has a trivial vertex, i.e. $\mathcal{L}^R$ is a regulus.

So let R be a hole such that $\mathcal{L}^R$ is a regulus. Let g_i, $i = 1,\ldots, q+1+\delta$, be the elements of $\mathcal{L}$ and denote by l_i the intersection of $R^\perp \cap g_i$. At least $q+1$ of the lines l_i are projected on the lines of the regulus $\mathcal{L}^R$. We denote the $q+1$ lines of the regulus $\mathcal{L}^R$ by $\tilde{l}_i$, $i = 1,\ldots,q+1$. The opposite lines of the regulus $\mathcal{L}^R$ are denoted by $\tilde{m}_i$, $i = 1,\ldots,q+1$.

Lemma 3.11. *Suppose that $\tilde{m}_j$ is a line of the opposite regulus and that B_j is the set of points that are the intersection of the lines l_i with $\langle R, \tilde{m}_j\rangle$. Then B_j contains a line.*

Proof. As $|\mathcal{L}|$ is smaller than the smallest non-trivial blocking set in a plane, it suffices to show that every line k of $\pi := \langle R, \tilde{m}_j\rangle$ meets B_j. So let k be a line of π. We may assume that k contains a hole R'. By the assumption made before this lemma, $\mathcal{L}^{R'}$ is also a generator blocking set with a trivial vertex, i.e. a regulus $\mathcal{R}'$. Hence, at least $2(q+1) - |\mathcal{L}| > 2$ planes g_i of $\mathcal{L}$ are projected from R to different lines of $\mathcal{R}$ and from R' to different lines of $\mathcal{R}'$. Thus, these planes g_i meet π in points, and moreover different such g_i give points $P_i = g_i \cap \pi$ that span different lines P_iR' with R'. Thus π is contained in $S_{R'}$ and hence all lines of π' on R meet a plane of $\mathcal{L}$ (even one of the planes that projects to a line of $\mathcal{R}'$). $\square$

We denote the line contained in the set B_j by m_j, and so m_j is projected from R on $\tilde{m}_j$. Now we consider again the hole R and the regulus $\mathcal{L}^R$.

Lemma 3.12. *The generator blocking set $\mathcal{L}^R$ arises as the projection from R of a regulus, of which the lines are contained in the elements of $\mathcal{L}$.*

Proof. An element $g_i \in \mathcal{L}$ that is projected from R on the line $\tilde{m}_j$ must meet the plane $\langle R, \tilde{m}_j\rangle$ in a line. But an element $g_i \in \mathcal{L}$ cannot meet a plane $\langle R, \tilde{l}_i\rangle$ and a plane $\langle R, \tilde{m}_j\rangle$ in a line, since then g_i would be a generator of $Q(6,q)$ contained in $R^\perp$ not containing R, a contradiction. So at most δ elements of $\mathcal{L}$ meet S_R in a line that is projected on a line $\tilde{m}_j$. Hence, at least $q+1-\delta$ planes $\langle R, \tilde{m}_j\rangle$ do not contain a line l_i, so, by Lemma 3.11, there are at least $q+1-\delta$ lines $m_j \subseteq B_j$ not coming from the intersection of an element of $\mathcal{L}$ and S_R, that are projected on a line of the opposite regulus of $\mathcal{L}^R$. Number these $n \geqslant q+1-\delta$ lines from 1 to n.

Suppose that $l_1, l_2, \dots, l_{q+1}$ are transversal to m_1. Since $\delta \leqslant \frac{q-1}{2}$, a second transversal m_2 has at least $\frac{q+3}{2}$ common transversals with m_1. So we find lines $l_1, \dots, l_{\frac{q+3}{2}}$ lying in the same 3-space $\langle m_1, m_2\rangle$. A third line m_j, $j \neq 1,2$, has at least 2 common transversals with m_1 and m_2, so all transversals m_j lie in $\langle m_1, m_2\rangle$. Suppose that we find at most q lines $l_1, \dots, l_q$ which are transversal to $m_1, \dots, m_{q+1-\delta}$. Then $q+1-\delta$ remaining points on the lines m_j must be covered by the $\delta+1$ remaining lines l_i, so $\delta+1 \geqslant q+1-\delta$, a contradiction with the assumption on δ. So we find a regulus of lines $l_1, \dots, l_{q+1}$ that is projected on $\mathcal{L}^R$ from R. $\square$

Lemma 3.13. *Let $q>3$. The set $\mathcal{L}$ contains $q+1$ generators through a point P, which are projected from P on a regulus.*

Proof. Consider the hole R. By Lemma 3.12, $R^\perp$ contains a regulus $\mathcal{R}_1$ of $q+1$ lines l_i contained in planes of $\mathcal{L}$. Denote the 3-dimensional space containing $\mathcal{R}_1$ by π_3. Consider any hole $R' \in Q(6,q) \setminus \pi_3^\perp$. By the assumption made before Lemma 3.11 and Lemma 3.12, R' gives rise to a regulus $\mathcal{R}_2$ of $q+1$ lines contained in planes of $\mathcal{L}$. Since $R' \in Q(6,q) \setminus \pi_3^\perp$, $\mathcal{R}_1 \neq \mathcal{R}_2$. Hence, at least $\frac{q+3}{2}$ planes of $\mathcal{L}$ contain a line of both $\mathcal{R}_1$ and $\mathcal{R}_2$ and in at most one plane, the reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ can share the same line. The reguli $\mathcal{R}_1$ and $\mathcal{R}_2$ define a 4- or 5-dimensional space Π.

If Π is 4-dimensional, then $\Pi \cap Q(6,q) = \langle P, \mathcal{Q}\rangle$, for some point P and some hyperbolic quadric $Q^+(3,q)$, denoted by $\mathcal{Q}$. For $\mathcal{Q}$ we may choose the hyperbolic quadric containing $\mathcal{R}_1$. There are at least $\frac{q+1}{2}$ planes of $Q(6,q)$, completely contained in Π, containing a line of $\mathcal{R}_1$ and a different line of $\mathcal{R}_2$. These planes are necessarily planes of $\mathcal{L}$. Consider now a plane π_2 of $Q(6,q)$, completely contained in Π, only containing a line of $\mathcal{R}_1$ and not containing a different line of $\mathcal{R}_2$. If π_2 is not a plane of $\mathcal{L}$, it contains a hole Q. Then $Q^\perp$ intersects the at least $\frac{q+1}{2}$ planes of $\mathcal{L}$ on P in a line, and the projection of these at least $\frac{q+1}{2}$ lines from Q is one line l. If this line l belongs to $\mathcal{L}^Q$, then at least q more elements of $\mathcal{L}$ are projected from Q on the q other elements of $\mathcal{L}^Q$, hence, $q + \frac{q+1}{2} \leqslant q+1+\delta$, a contradiction with $\delta < \frac{q-1}{2}$. Hence, π_2 is a plane of $\mathcal{L}$, and $\mathcal{L}$ contains $q+1$ generators of $Q(6,q)$ through P, which are projected from P on a regulus.

If Π is 5-dimensional, then its intersection with $Q(6,q)$ is a cone $P\mathcal{Q}$, $\mathcal{Q}$ a parabolic quadric $Q(4,q)$, or a hyperbolic quadric $Q^+(5,q)$. If $\Pi \cap Q(6,q) = PQ(4,q)$, then the base $\mathcal{Q}$ can be chosen in such a way that $\mathcal{R}_1 \subset \mathcal{Q}$. But then the same arguments as in the case that Π is 4-dimensional apply, and the lemma follows.

So assume that $\Pi \cap Q(6,q) = Q^+(5,q)$. Consider again the $n \geqslant \frac{q+1}{2}$ planes $\pi^1, \dots, \pi^n$ of $\mathcal{L}$ containing a line of $\mathcal{R}_1$ and a different line of $\mathcal{R}_2$. Then half of these planes lie in the same equivalence class of planes of $Q^+(5,q)$ and so intersect mutually in a point. If $q>3$, then we can assume that the two planes π^1 and π^2 intersect in a point P, hence, $\langle \pi^1, \pi^2\rangle$ is a 4-dimensional space necessarily intersecting $Q(6,q)$ in a cone $P\mathcal{Q}$, $\mathcal{Q}$ a hyperbolic quadric $Q^+(3,q)$. Clearly, since two distinct lines of $\mathcal{R}_1$ span $\langle \mathcal{R}_1\rangle$, and two distinct lines of $\mathcal{R}_2$ span $\langle \mathcal{R}_2\rangle$, the reguli $\mathcal{R}_1, \mathcal{R}_2 \subseteq \langle \pi^1, \pi^2\rangle$. But since the planes $\pi^3, \dots, \pi^n$ contain a different line from $\mathcal{R}_1$ and $\mathcal{R}_2$, these $n \geqslant \frac{q+1}{2}$ planes of $\mathcal{L}$ are completely contained in $\langle \pi^1, \pi^2 \rangle$. But then again the same arguments as in the case that Π is 4-dimensional apply, and the lemma follows. □

From now on we assume that $n \geqslant 3$, and that there exists a hole R such that $\mathcal{L}^R$ has a non-trivial vertex α. This means that also for $n = 3$, this vertex is non-trivial. This assumption will be in use for Lemmas 3.14, 3.16, 3.17, 3.18, and Corollary 3.15. Remark that also the induction hypothesis is used. We will call the subspace $\langle R, \alpha \rangle$ the *vertex of* S_R.

A *nice point* is a point that lies in at least $q - \delta - 1$ elements of $\mathcal{L}$. In the next lemma, for a hole X, we denote by $\bar{\mathcal{L}}^X$ the set of generators of $\mathcal{L}$ that are projected from X on the elements of $\mathcal{L}^X$. Hence, the generators of $\bar{\mathcal{L}}^X$ intersect S_X in $(n-2)$-dimensional subspaces.

Lemma 3.14. *Call α the vertex of $\mathcal{L}^R$. Then there exists a nice point N on every line through R meeting α.*

Proof. Part 1. We show the existence of a particular plane π on l.

Let l be a line on R projecting to a point of α. Consider $g \in \mathcal{L}$. As R is a hole, either g meets l in a point S or $g \cap l = \emptyset$. In the first case, $l^\perp \cap g = R^\perp \cap S^\perp \cap g$ is an $(n-2)$-dimensional space, hence we find θ_{n-3} planes on l meeting g in a line. In the second case, $l^\perp \cap g$ is an $(n-3)$-dimensional space, so we find θ_{n-3} planes on l meeting g in one point. As there are θ_{2n-5} planes of $Q(2n, q)$ on l, a double counting argument shows that there exists such a plane π such that $\pi \setminus l$ meets at most $|\mathcal{L}|\theta_{n-3}/\theta_{2n-5} < 2$ generators of $\mathcal{L}$. Then we find a point $Q \in \pi \setminus l$ not being covered by $\mathcal{L}$. In case there exists a generator $g \in \mathcal{L}$ meeting $\pi \setminus l$ in a single point T, which implies that g is skew to l, then we can choose Q in such a way that Q, R and T are not collinear.

Part 2. We show the existence of a nice point.

Since $\bar{\mathcal{L}}^Q$ and $\bar{\mathcal{L}}^R$ share at least $q + 1 - \delta$ generators, then $q + 1 - \delta$ generators of $\bar{\mathcal{L}}^Q$ meet l, and at most one of these contains a point of $\pi \setminus l$. Hence, we find $q - \delta$ generators $g_i \in \bar{\mathcal{L}}^Q \cap \bar{\mathcal{L}}^R$, each of them meeting π in one point, which is on l. Assume now that l has no nice point, then at least two of the $q - \delta$ generators, say g_1 and g_2, do not meet l in a common point.

Case 1. First assume that the generators of $\bar{\mathcal{L}}^Q$ are projected from Q on a generator blocking set with an $(n-3)$-dimensional vertex and base a conic $Q(2, q)$. The points $P_i := g_i \cap l$, $i = 1, 2$, are collinear, so are projected from Q onto two collinear points in distinct elements of $\mathcal{L}^Q$. So they must be contained in their intersection, which is the vertex of $\mathcal{L}^Q$. Then π is a plane in the vertex of S_Q, so all the generators of $\bar{\mathcal{L}}^Q$ meet π in a line different from l, which is a contradiction with the choice of π. Hence, the $q - \delta$ generators meeting l in a point must all meet l in the same point X, and we are done.

Case 2. Now assume that the generators of $\bar{\mathcal{L}}^Q$ are projected from Q on a generator blocking set with an $(n-4)$-dimensional vertex, and base a regulus $\mathcal{R}$. Consider again the two distinct generators g_1 and g_2 meeting l in two distinct points P_1 and P_2. If $n = 3$, then the vertex of S_Q is just the point Q, and hence the line l is skew to the vertex of S_Q. So let $n \geqslant 4$ and suppose that l meets the vertex of S_Q in one point U. The generators g_1 and g_2 contain the vertex of $\mathcal{L}^Q$ after projection from Q, so must contain a point on the line $\langle Q, U \rangle$ before projection. At most one of the generators g_1 and g_2, say g_1, can contain the point U itself. Then g_2 meets π in a line containing a point of l not on $\langle Q, U \rangle$ and a point on the line $\langle Q, U \rangle$. Now consider any other generator $g \in (\bar{\mathcal{L}}^Q \cap \bar{\mathcal{L}}^R) \setminus \{g_1, g_2\}$. As g contains the vertex of $\mathcal{L}^Q$ after projection from Q, g meets the line $\langle Q, U \rangle$ before projection. Assume that g does not contain U, then g contains a point of l different from U, and a point of $\langle Q, U \rangle \setminus \{U\}$, so g_2 meets π in a line different from l. But now two generators, g and g_2, meet π in a line, a contradiction. So necessarily, g must contain the point U. So assuming that l meets S_Q in one point U, we found that at least $q - \delta - 2$ generators in $(\bar{\mathcal{L}}^Q \cap \bar{\mathcal{L}}^R) \setminus \{g_1, g_2\}$ meet l in one point U. Together with the generator g_1, we find $q - \delta - 1$ generators meeting l in one point U, and we are done.

So we may now assume that l is skew to the vertex of S_Q, and this handles also the case $n = 3$. A line of the regulus $\mathcal{R}$ is contained in exactly one element of $\mathcal{L}^Q$ and meets no other elements of $\mathcal{L}^Q$, so l must be projected from the vertex of S_Q on a line of the opposite regulus $\mathcal{R}'$. Hence, each line on Q must meet a generator of $\bar{\mathcal{L}}^Q$. By the choice of the plane π, at most one generator meets $\pi \setminus l$ in a point T not on the line $\langle R, Q\rangle$ or at most one generator meets π in a line $m \neq l$. This leads immediately to a contradiction in the first case. In the second case, the line $\langle Q, R\rangle$ meets the line m, which is the intersection line of π and of some generator $g' \in \mathcal{L}$. Necessarily, g' is projected on a line of $\mathcal{R}$. But then any other line on Q must meet another generator different from g'. Again, this is a contradiction with the choice of π. This finalizes Case 2 completely, and we are done. □

Corollary 3.15. *If R is a hole such that $\mathcal{L}^R$ has a non-trivial vertex, and $N \in R^\perp$ is a nice point, then N lies in the vertex of S_R.*

Proof. A nice point lies in at least $q - \delta - 1$ generators of $\mathcal{L}$ and at least $q - 2\delta - 1 \geqslant 2$ of these must belong to $\bar{\mathcal{L}}^R$. As two elements of $\bar{\mathcal{L}}^R$ necessarily meet in a point of the vertex of S_R, the assertion follows. □

Lemma 3.16. *Let $n \geqslant 4$. If β denotes the subspace generated by all nice points, then* $\dim(\beta) \geqslant n - 3$.

Proof. Suppose that R is a hole. If $n \geqslant 4$, then by the induction hypothesis, the vertex of $\mathcal{L}^R$ has dimension at least $n - 4$. Hence, using Lemma 3.14, the nice points generate a subspace γ of dimension at least $n - 4$. Suppose that $\dim(\gamma) = n - 4$, then $\gamma \subseteq \mathrm{Q}(2n, q)$ by Corollary 3.15 and so $\dim(\gamma^\perp) = n + 3 < 2n$, and so we find a hole $P \notin \gamma^\perp$. Consider this hole P, then the same argument gives us a subspace γ' spanned by nice points in $P^\perp$ of dimension at least $n-4$, different from γ. So $\dim(\beta) \geqslant n - 3$. □

Lemma 3.17. *There exists a hole R such that the vertex of S_R is an $(n-2)$-dimensional subspace and there exists a generator g on the vertex of S_R such that g meets exactly one element of $\mathcal{L}$ in an $(n-2)$-dimensional subspace and such that all other elements of $\mathcal{L}$ do not meet g or meet g only in points of the vertex of S_R.*

Proof. First let $n = 3$. By the assumption, there exists a hole R such that $\mathcal{L}^R$ has a non-trivial vertex, which is a point X. So the vertex of S_R is the line RX and has dimension $n - 2$.

Now let $n \geqslant 4$. By Lemma 3.16, we find a subspace γ of dimension $n - 3$ spanned by nice points. Consider a hole $R \in \gamma^\perp \setminus \gamma$. Clearly, the vertex of S_R will be spanned by R and the projection of γ from R, so has dimension $n - 2$.

So for $n \geqslant 3$, we always find a hole R such that the vertex V of S_R has dimension $n - 2$, and $V = \langle R, \pi_{n-3}\rangle$, with π_{n-3} the vertex of $\mathcal{L}^R$. There are $q+1$ generators π^i on V, which in the quotient of V form a conic $\mathrm{Q}(2, q)$. Hence, any generator of $\mathrm{Q}(2n, q)$ can meet at most one of these in a point not in V. There are also $q + 1$ elements τ^i of $\mathcal{L}$ which provide the example $\mathcal{L}^R$, and $\tau^i \mapsto \pi^i := \langle R, \tau^i \cap R^\perp\rangle$ is a one-to-one correspondence between the $q + 1$ subspaces τ^i and π^i. As the remaining δ elements of $\mathcal{L}$ can each meet at most one of the generators π^i not in V, this proves the statement. □

Lemma 3.18. *Let $n \geqslant 3$. There exists an $(n-3)$-dimensional subspace contained in at least q elements of $\mathcal{L}$.*

Proof. By Lemma 3.17, we may consider a hole R, such that the vertex of S_R is an $(n-2)$-dimensional subspace and there exists a generator g on the vertex of S_R such that g meets exactly one element of $\mathcal{L}$ in an $(n-2)$-dimensional subspace and such that all other elements of $\mathcal{L}$ do not meet g or meet g only in points of the vertex of S_R. Call again $V = \langle R, \pi_{n-3}\rangle$ the vertex of S_R, with π_{n-3} the vertex of $\mathcal{L}^R$. Denote the elements of $\mathcal{L}$ intersecting S_R in an $(n-2)$-dimensional subspace by g_i. Note that g is projected from R onto an element of $\mathcal{L}^R$, and we may call g_1 the unique element of $\mathcal{L}$ meeting g in an $(n-2)$-dimensional space. We find also a hole $Q \neq R$, $Q \in g \setminus V$.

Part 1. We show that $\mathcal{L}^Q$ is a cone $\pi'_{n-3}\mathrm{Q}(2,q)$. Clearly, at least $q-\delta$ elements of $\mathcal{L}$ that meet S_R in an $(n-2)$-dimensional subspace, also meet S_Q in an $(n-2)$-dimensional subspace and are projected on elements of $\mathcal{L}^Q$. Consider now the hole Q, and suppose that $\mathcal{L}^Q$ is a cone $\pi_{n-4}\mathcal{R}$, $\mathcal{R}$ a regulus. The generator g_1 is projected from Q on a subspace $\tilde{g}_1$ not in $\mathcal{L}^Q$, since $\tilde{g}_1$ meets at least $q-\delta$ of the projected spaces g_i, $i \neq 1$, in an $(n-3)$-dimensional space, which has larger dimension than the vertex of $\mathcal{L}^Q$. But $\tilde{g}_1$ lies in $\pi_{n-4}\mathcal{R}$, since it intersects at least $q-\delta$ spaces g_i in an $(n-3)$-dimensional subspace. Hence, $\tilde{g}_1$ meets the $q+1$ elements of $\mathcal{L}^Q$ in different $(n-3)$-spaces and is completely covered. So the projection of R from Q is covered by elements of $\mathcal{L}^Q$, and hence, the line $l=\langle R,Q\rangle$ must meet an element of $\mathcal{L}\setminus\{g_1\}$, a contradiction. So $\mathcal{L}^Q$ is a cone $\pi'_{n-3}\mathrm{Q}(2,q)$.

Part 2. We show that g is contained in S_Q. Consider two of the at least $q-\delta-1$ generators, both different from g_1 and meeting V in an $(n-3)$-dimensional subspace, call them g_2 and g_3, denote $\pi^i_{n-3} := g_i \cap V$, $i=2,3$. Both generators are projected from Q onto two elements of $\mathcal{L}^Q$. If $\pi^2_{n-3}=\pi^3_{n-3}$, then the projection of g from Q also contains the projection of π^2_{n-3} from Q, and so g is contained in S_Q. If $\pi^2_{n-3}\neq\pi^3_{n-3}$, then choose a point $P\in\pi^2_{n-3}\setminus\pi^3_{n-3}$ and a point $P'\in\pi^3_{n-3}\setminus\pi^2_{n-3}$. The collinear points P and P' are projected from Q on collinear points, contained in the projection of π^2_{n-3} and π^3_{n-3} respectively, and hence the projection of g from Q contains two collinear points, which is a contradiction unless g is projected on g_2 or g_3. In both cases, g is contained in S_Q.

Part 3. We prove the final statement. It follows that $\tilde{g}_1$, the projection of g_1 from Q, is contained in $\mathcal{L}^Q$, so $\pi'_{n-3}\subset\tilde{g}_1$, and g_1 and V are projected from Q on $\tilde{g}_1$. Before projection from R, the elements g_i meet V in $(n-3)$-dimensional subspaces contained in V.

The subspace π'_{n-3} lies in the projection from Q of elements of $\mathcal{L}$ meeting $\langle\pi'_{n-3},Q\rangle$ in an $(n-3)$-dimensional subspace. But the choice of g implies that there is only a unique element of $\mathcal{L}$ meeting $\langle\pi'_{n-3},Q\rangle$ in an $(n-3)$-dimensional subspace and in points outside of V (the element meeting g in g_1), so at least q other elements of $\mathcal{L}$ intersect V in the same $(n-3)$-dimensional subspace. □

The following lemma summarizes in fact Lemmas 3.14, 3.16 and 3.17, 3.18, and Corollary 3.15. The condition on δ enables the use of the induction hypothesis.

Lemma 3.19. *Let $n\geqslant 3$. Suppose that $\mathcal{L}$ is a minimal generator blocking set of size $q+1+\delta$ of* $\mathrm{Q}(2n,q)$, $\delta\leqslant\delta_0$. *If there exists a hole R that projects $\mathcal{L}$ on a generator blocking set containing a minimal generator blocking set of* $\mathrm{Q}(2n-2,q)$ *that has a non-trivial vertex, then $\mathcal{L}$ is a generator blocking set of* $\mathrm{Q}(2n,q)$ *listed in Table* 1.

Proof. By Lemma 3.18, we can find an $(n-3)$-dimensional subspace α of $\mathrm{Q}(2n,q)$ that is contained in at least q elements of $\mathcal{L}$. Consider now a hole $H\notin\alpha^\perp$. Then $H^\perp\cap\alpha^\perp$ is an $(n+1)$-dimensional space, which meets the elements of $\mathcal{L}$ inducing the example $\mathcal{L}^H$ and passing through α, in $(n-2)$-dimensional subspaces. Hence, at least $q-\delta$ elements of $\mathcal{L}$ on α meet $H^\perp$ in an $(n-2)$-dimensional subspace that is projected from H onto elements of $\mathcal{L}^H$. These $(n-2)$-dimensional subspaces are all containing the $(n-4)$-dimensional subspace $H^\perp\cap\alpha$. Since S_H is $(n+1)$-dimensional, these $q-\delta$ $(n-2)$-dimensional subspaces lie in the n-dimensional space $S_H\cap\alpha^\perp$. Hence, we find in the $(n+1)$-dimensional space $\langle\alpha,S_H\cap\alpha^\perp\rangle$ at least $q-\delta>\delta+2$ elements of $\mathcal{L}$. Lemma 3.4 assures that $\mathcal{L}$ is one of the generator blocking sets of $\mathrm{Q}(2n,q)$ listed in Table 1. □

Finally, we can prove Theorem 3.1 (a). Note that the technical condition $q>3$ is due to Lemma 3.13.

Lemma 3.20. *Theorem* 3.1 (a) *is true for* $\mathrm{Q}(2n,q)$, $n\geqslant 3$, $q>3$.

Proof. Proposition 1.1 assures that the assumptions of either Lemma 3.13 or Lemma 3.19, $n=3$, are true. Hence, Theorem 3.1 (a) follows for $n=3$. But then the assumption of Lemma 3.19 is true for $\mathrm{Q}(2n,q)$ and $n=4$, and then Theorem 3.1 (a) follows by induction. □

Table 2
Bounds on the size of small maximal partial spreads.

Polar space	Lower bound
$Q^-(2n+1,q)$	$n \geqslant 3$: $q^2 + \frac{1}{2}(3q - \sqrt{5q^2+2q+1})$
$Q^+(4n+3,q)$	$n \geqslant 1$, $q \geqslant 7$: $2q+1$
$Q(2n,q)$, q odd	$n \geqslant 3$, $q > 3$: $q+1+\delta_0$, with $\delta_0 = \min\{\frac{q-2}{2}, \epsilon\}$, ϵ such that $q+1+\epsilon$ is the size of the smallest non-trivial blocking set in $PG(2,q)$.
$Q(2n+2,q)$, q even; $W(2n+1,q)$, q odd and even	$n \geqslant 2$, $q \geqslant 5$: $2q+1$
$H(2n,q^2)$	$n \geqslant 3$: q^3+q-2
$H(2n+1,q^2)$	$q \geqslant 13$ and $n \geqslant 2$: $2q+3$

4. Remarks

We mentioned already that a maximal partial spread is in fact a special generator blocking set. The results of Theorem 3.1 imply an improvement of the lower bound on the size of maximal partial spreads in the polar spaces $Q^-(2n+1,q)$, $Q(2n,q)$, and $H(2n,q^2)$ when the rank is at least 3. In Table 2, we summarize the known lower bounds on the size of small maximal partial spreads of polar spaces. The results for $Q^+(2n+1,q)$, $W(2n+1,q)$ and $H(2n+1,q^2)$ are proved in [7]. Note that the polar space $Q(2n+2,q)$, q even, is isomorphic with the polar space $W(2n+1,q)$, q even. The isomorphism is induced by projecting $Q(2n+2,q)$, q even, from its nucleus. As the lower bounds mentioned in Table 2 for the polar spaces $Q(2n,q)$ and $W(2n+1,q)$ are in both cases true for even and odd q, the lower bound for $Q(2n+2,q)$, q even, is not as good as the previous known lower bound found through the polar space $W(2n+1,q)$, q even.

One can wonder what happens with generator blocking sets of the polar spaces $Q^+(2n+1,q)$, $W(2n+1,q)$, q odd, and $H(2n+1,q^2)$. Unfortunately, the approach presented in Section 2 for these polar spaces fails, which makes the complete approach of this paper not usable for these polar spaces in higher rank.

In [1], an overview of the size of the smallest non-trivial blocking sets of $PG(2,q)$ is given. If q is an odd prime, then $\epsilon = \frac{q+1}{2}$. So if q is an odd prime, the condition on δ in the case of generator blocking sets of $Q(2n,q)$, $n \geqslant 3$, drops to $\delta < \frac{q-1}{2}$.

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