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ASYMPTOTIC PROPERTIES OF AUTOREGRESSIVE INTEGRATED MOVING AVERAGE PROCESSES

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In this paper we study the asymptotic behavior of so-called autoregressive integrated moving average processes. These processes constitute a large class of stochastic difference equations which includes among many other well-known processes the simple one-dimensional random walk. They were dubbed by G.E.P. Box and G.M. Jenkins who found them to provide useful models for studying and controlling the behavior of certain economic variables and various chemical processes. We show that autoregressive integrated moving average processes are asymptotically normally distributed, and that the sample paths of such processes satisfy a law of the iterated logarithm. We also establish a law which determines the time spent by a sample path on one or the other side of the "trend line" of the process.

probability theory econometrics
time series

1. Introduction and statement of results

In this paper we study the asymptotic behavior of so-called autoregressive integrated moving average processes. These processes constitute a large class of stochastic difference equations which includes among many other well-known processes the simple one-dimensional random walk. They were dubbed by G.E.P. Box and G.M. Jenkins who found them to provide useful models for studying and controlling the behavior of certain economic variables and various chemical processes [2, pp. 85–125].

An autoregressive integrated moving average process (hereafter an ARIMA process) is defined as follows:

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Definition 1. Let $x \equiv \{x(t): t = -n+1, -n+2, ...\}$ be a family of real-valued random variables. Then x is an *ARIMA process* if and only if it satisfies the following conditions:

(1) There exist constants $\bar{x}_t$, $-n+1 \leqslant t \leqslant 0$, such that

$$x(t) = \bar{x}_t \quad \text{w.p. 1 } (\equiv \text{ with probability 1}), \quad t = -n+1, ..., 0. \tag{1.1}$$

(2) There exists a family of non-degenerate, identically and independently distributed real-valued random variables $\eta \equiv \{\eta(t): t = ..., -1, 0, 1, ...\}$ with mean zero and finite variance σ_η^2, and two sequences of constants $\{a_k: k = 0, ..., n\}$, $\{\alpha_s: s = ..., -1, 0, 1, ...\}$ such that $a_0 = \alpha_0 = 1$, $a_n \neq 0$, and

$$\sum_{s=-\infty}^{\infty} \alpha_s^2 < \infty, \tag{1.2}$$

$$\sum_{k=0}^{n} a_k x(t-k) = \sum_{s=-\infty}^{\infty} \alpha_s \eta(t+s), \quad t = 1, 2, ... \tag{1.3}$$

(3) There exists a positive integer l_0, non-negative integers l_j, and complex constants z_j, $j = 1, ..., l$, such that

$$\sum_{k=0}^{n} a_k z^{n-k} = (z-1)^{l_0} \prod_{j=1}^{l} (z - z_j)^{l_j}, \tag{1.4}$$

$$|z_j| < 1, \quad j = 1, ..., l. \tag{1.5}$$

In interpreting this definition note that, when $n = 1$ and $\alpha_s = 0$ for $s \neq 0$, then x is a simple one-dimensional random walk. Note also that Box and Jenkins always assumed that $n = l_0$, that $\alpha_s = 0$ for $s > 0$, and that $|\alpha_s| \leqslant K\beta^{|s|}$ for some $\beta \in (0, 1)$ and some suitably large constant K. Since the latter assumptions are not needed to establish our results, we have not insisted on them being valid here.

The behavior of x can be characterized in the following way:[1]

Theorem 1. *Suppose that x is an ARIMA process and let*

$$y(t) \equiv \sum_{s=-\infty}^{\infty} \alpha_s \eta(t+s), \quad t = 1, 2, ... \tag{1.6}$$

[1] A similar result is stated and proved in [6, pp. 177–178].

Then there exists a function $\varphi(\cdot)$ and a sequence of real constants γ_s, $s = 0, 1, \ldots$, which satisfy the following conditions:

$$\gamma_0 = 1, \tag{1.7}$$

$$\sum_{k=0}^{v} a_k \gamma_{v-k} = 0, \quad v = 1, \ldots, n-1, \tag{1.8}$$

$$\sum_{k=0}^{n} a_k \gamma_{v-k} = 0, \quad v = n, n+1, \ldots, \tag{1.9}$$

$$\varphi(t) = \bar{x}_t, \quad t = -n+1, \ldots, 0, \tag{1.10}$$

$$\sum_{k=0}^{n} a_k \varphi(t-k) = 0, \quad t = 1, 2, \ldots, \tag{1.11}$$

$$x(t) = \varphi(t) + \sum_{s=0}^{t-1} \gamma_s y(t-s), \quad t = 1, 2, \ldots . \tag{1.12}$$

In interpreting this theorem note that (1.10), (1.11) and (1.4) imply that there exist constants A_{jk} ($j = 1, \ldots, l$; $k = 0, \ldots, l_j - 1$) and B_k ($k = 0, \ldots, l_0 - 1$) such that

$$\varphi(t) = \sum_{j=1}^{l} \sum_{k=0}^{l_j-1} A_{jk}(t^k z_j^t) + \sum_{k=0}^{l_0-1} B_k t^k, \quad t = -n+1, \ldots . \tag{1.13}$$

Since $|z_j| < 1$ for all j we can conclude from (1.13) that, for large t, the "trend line" $\varphi(\cdot)$ satisfies the approximate relation

$$\varphi(t)\, t^{-(l_0-1)} \sim B_{l_0-1}. \tag{1.14}$$

Next note that (1.7)–(1.9), (1.3) and (1.4) imply that there exist constants C_{jk} ($j = 1, \ldots, l$; $k = 0, \ldots, l_j - 1$) and D_k ($k = 0, \ldots, l_0 - 1$) such that

$$D_{l_0-1} \neq 0, \tag{1.15}$$

$$\gamma_s = \sum_{j=1}^{l} \sum_{k=0}^{l_j-1} C_{jk}(s^k z_j^s) + \sum_{k=0}^{l_0-1} D_k s^k. \tag{1.16}$$

Thus γ_s satisfies for large enough s the approximate relation

$$\gamma_s\, s^{-(l_0-1)} \sim D_{l_0-1}. \tag{1.17}$$

Finally, note that, if we assume that

$$0 < \left| \sum_{s=-\infty}^{\infty} \alpha_s \right| \tag{1.18}$$

and that the function

$$f_y(\lambda) \equiv \left| \sum_{s=-\infty}^{\infty} \alpha_s \, \mathrm{e}^{-\mathrm{i}s\lambda} \right|^2 \sigma_\eta^2, \quad \lambda \in [-\pi, \pi), \tag{1.19}$$

is piecewise continuous on $[-\pi, \pi)$ and continuous in a neighborhood of $\lambda = 0$, then [10, Lemma 1] implies that, for all non-negative integers q,

$$\lim_{T \to \infty} \mathrm{P}\left[\sum_{t=1}^{T} t^q \, y(t) \{T^{2q+1} f_y(0)/(2q+1)\}^{-1/2} < z \right] =$$

$$= (2\pi)^{-1/2} \int_{-\infty}^{z} \exp[-\tfrac{1}{2}u^2] \, \mathrm{d}u. \tag{1.20}$$

But if that is so, and

$$\mathfrak{S}_t \equiv D_{l_0 - 1} \sum_{s=0}^{t-1} s^{l_0 - 1} \, y(t-s)$$

$$= D_{l_0 - 1} \sum_{k=0}^{l_0 - 1} \binom{l_0 - 1}{k} t^{l_0 - 1 - k} \sum_{v=1}^{t} (-v)^k \, y(v),$$

then there exists a normally distributed random variable $\mathcal{G}$ with mean zero and variance

$$\tilde{\sigma}^2 \equiv f_y(0) \, D_{l_0 - 1}^2 \sum_{k=0}^{l_0 - 1} \sum_{m=0}^{l_0 - 1} \binom{l_0 - 1}{k} \binom{l_0 - 1}{m} (-1)^{k+m} (k+m+1)^{-1}$$

$$= f_y(0) \, D_{l_0 - 1}^2 / (2l_0 - 1) \tag{1.21}$$

such that, for large enough t and for all $a \in (-\infty, \infty)$,

$$\mathrm{P}[\mathfrak{S}_t \, t^{-(l_0 - 1/2)} < a] \sim \mathrm{P}[\mathcal{G} < a]. \tag{1.22}$$

Evidently, if (1.18) is satisfied, then (1.14), (1.15), (1.17), (1.20) and (1.22) imply that for large t the behavior of $x(t)$ is completely dominated by the behavior of $\mathfrak{S}_t$. From this fact and (1.22) we can infer the validity of Theorem 2 below. A formal proof of it is given in Section 2.

Theorem 2. *Suppose that x is an ARIMA process that satisfies* (1.18), *and let $\tilde{\sigma}^2$ be as defined in* (1.21). *Moreover, assume that $f_y(\cdot)$ is piecewise continuous on $[-\pi, \pi)$ and continuous in a neighborhood of $\lambda = 0$. Then, for all $z \in (-\infty, \infty)$,*

$$\lim_{t\to\infty} \mathbf{P}[x(t)\tilde{\sigma}^{-1}t^{-(l_0-1/2)} < z] = \lim_{t\to\infty} \mathbf{P}[(x(t)-\varphi(t))\tilde{\sigma}^{-1}t^{-(l_0-1/2)} < z]$$

$$= (2\pi)^{-1/2}\int_{-\infty}^{z} \exp[-\tfrac{1}{2}u^2]\,du. \quad (1.23)$$

The latter theorem characterizes the asymptotic distribution of $x(t)$. We can also characterize the asymptotic behavior of x by giving upper bounds on the growth of $|x(t)|$ and by estimating the fraction of time $x(t)$ spends above the trend-line $\varphi(\cdot)$. This is done in Theorems 3 and 4.

Before we state Theorems 3 and 4 we must introduce a certain notational convention and state an invariance principle: Let C denote the set of continuous functions on $[0, 1]$, let $\mathcal{S}$ denote the class of Borel sets in C, and let P be a probability measure on $(C, \mathcal{S})$. If $\pi_\tau(\cdot)$ is a function on $(C, \mathcal{S})$ with value $\pi_\tau(X) = X(\tau)$ at $X \in C$, then $\pi_\tau(\cdot)$ is a random variable on $(C, \mathcal{S}, P)$. Moreover, $\{\pi_\tau(\cdot): \tau \in [0, 1]\}$ is a well-defined random process on $(C, \mathcal{S}, P)$. Next, let D be the set of functions on $[0, 1]$ that are right continuous and have left-hand limits, and let $\mathcal{D}$ be the class of Borel sets for the Skorohod topology in D (cf. [1, pp. 111–123]). Both P and the $\pi_\tau(\cdot)$ can be extended to $(D, \mathcal{D})$. We denote by X_τ both the function $\pi_\tau(\cdot)$ on $(D, \mathcal{D})$ and the value of $\pi_\tau(\cdot)$ at $X \in D$.

The invariance principle we seek to establish concerns all processes consisting of independently and identically distributed random variables with finite fourth moments.

An Invariance Principle. Let η be as above, let $q \geqslant 0$ be an integer, let $A_0 \equiv 0$, and let

$$A_i \equiv \sum_{j=1}^{i} (i-j)^q\, \eta(j), \quad i = 1, 2, \ldots . \quad (1.24)$$

Moreover, for each $\tau \in [0, 1]$ and each $n = 1, 2, \ldots$, let $[n\tau]$ denote the largest integer k such that $k \leqslant n\tau$, let

$$\hat{X}_n(\tau) \equiv (2q+1)^{1/2}\, \sigma_\eta^{-1}\, n^{-(q+1/2)}\, A_{[n\tau]}, \quad (1.25)$$

and note that $\hat{X}_n(\cdot) \in D$ w.p. 1 for all $n = 1, 2, \ldots$. Finally, let $\hat{P}_n$ denote the distribution of $\hat{X}_n(\cdot)$ on $(D, \mathcal{D})$. Then there exists a probability measure $\hat{W}_q$ on $(D,\mathcal{D})$ with the following three properties:

(i) For each $\tau \in (0, 1]$ and $\alpha \in (-\infty, \infty)$,

$$\hat{W}_q\{X: X_\tau < \alpha\} = (2\pi\tau^{2q+1})^{-1/2} \int_{-\infty}^{\alpha} \exp[-\tfrac{1}{2}u^2 \tau^{-(2q+1)}]\, du, \quad (1.26)$$

and for $\tau = 0$,

$$\hat{W}_q\{X: X_0 = 0\} = 1. \quad (1.27)$$

(ii) For each finite m-tuple $\{\tau_1, ..., \tau_m\}$ such that $0 \leqslant \tau_1 < ... < \tau_m \leqslant 1$, the vector $(X_{\tau_1}, ..., X_{\tau_m})$ is normally distributed with mean zero and covariance matrix

$$\Gamma(\tau_1, ..., \tau_m) \equiv \{\mu_{\tau_i,\tau_j}\}_{1 \leqslant i,j \leqslant m}, \quad (1.28)$$

where, for $k \equiv \min\{i, j\}$ and $p \equiv \max\{i, j\}$,

$$\mu_{\tau_i,\tau_j} = (2q+1)\, \tau_k^{q+1} \sum_{u=0}^{q} \binom{q}{u} (\tau_p - \tau_k)^u\, \tau_k^{q-u} (2q+1-u)^{-1}. \quad (1.29)$$

(iii) The stochastic process $\{X_\tau: \tau \in [0, 1]\}$ is continuous w.p. 1 under $\hat{W}_q$.

Moreover, if $E\{\eta(t)^4\} < \infty$, then the $\hat{P}_n$ converge weakly to $\hat{W}_q$.

For $q = 0$ this invariance principle (without the fourth moment assumption on η) is due to Donsker. To the best of our knowledge it is new for $q > 0$. A similar invariance principle for weighted sums of the form

$$\sum_{j=1}^{i} j^q \eta(j)$$

is proved in [10] (cf. [10, Lemma 4, p. 17]) without the fourth moment assumption on η.

Now the theorems.

Theorem 3. *Suppose that x is an ARIMA process that satisfies the conditions of Theorem 2. Moreover, let*

$$N_T \equiv \sum_{t=1}^{T} \psi(x(t) - \varphi(t)), \quad T = 1, 2, ..., \quad (1.30)$$

where

$$\psi(s) \equiv \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{otherwise.} \end{cases} \quad (1.31)$$

Then

$$\lim_{T\to\infty} P[N_T/T < \beta] = \hat{W}_{l_0-1}\{X: h(X) < \beta\}, \tag{1.32}$$

where

$$h(X) \equiv \int_{\{\tau\in[0,1]:\, X(\tau)>0\}} d\tau, \tag{1.33}$$

and β ranges over the set of continuity points of the distribution of $h(\cdot)$ under $\hat{W}_{l_0-1}$. Finally, when $l_0 = 1$,

$$\hat{W}_0\{X: h(X) < \beta\} = (2/\pi)\arcsin \beta^{1/2}, \beta \in [0,1]. \tag{1.34}$$

Theorem 4. *Suppose that x is an ARIMA process that satisfies the conditions of Theorem 2, and let $\tilde{\sigma}^2$ be as defined in (1.21). Then*

$$\limsup_{t\to\infty} \{|(x(t) - \varphi(t))\,[2\tilde{\sigma}^2\, t^{2l_0-1} \log\log t]^{-1/2}|\} = 1 \quad w.p.\ 1. \tag{1.35}$$

Of the two theorems the last is the easiest to interpret. It simply says that, whatever be the positive value of ϵ, with probability one

$$\varphi(t) - [2\tilde{\sigma}^2\, t^{2l_0-1} \log\log t]^{1/2}\,(1+\epsilon) \leqslant$$

$$\leqslant x(t) \leqslant \varphi(t) + [2\tilde{\sigma}^2\, t^{2l_0-1} \log\log t]^{1/2}\,(1+\epsilon)$$

for all but a finite number of values of t, and for infinitely many t either

$$x(t) \leqslant \varphi(t) - [2\tilde{\sigma}^2\, t^{2l_0-1} \log\log t]^{1/2}(1-\epsilon),$$

or

$$x(t) \geqslant \varphi(t) + [2\tilde{\sigma}^2\, t^{2l_0-1} \log\log t]^{1/2}\,(1-\epsilon).$$

While harder to interpret, Theorem 3 is from an econometric point of view by far the most interesting of the two theorems. When $l_0 = 1$, (1.32) and (1.34) imply that the chances are one in ten that $x(t)$ will be larger than $\varphi(t)$ for more than 97.6% of the time. The chances are one in five that $x(t)$ will be larger than $\varphi(t)$ for at least 90.5% of the time. Similar estimates hold for the likelihood that $x(t)$ will be less than $\varphi(t)$. The fact that $x(t)$ with such a large probability will either be greater than $\varphi(t)$ most of the time or smaller than $\varphi(t)$ most of the time, and the fact that $\varphi(t)$ for large t is dominated by $\tilde{S}_t$ (cf. (1.20)) makes it nearly impossible, when $l_0 = 1$, to use observations on $x(t)$ to estimate $\varphi(t)$.

We have not been able to derive the distribution of $h(\cdot)$ under $\hat{W}_{l_0-1}$ for $l_0 > 1$. However, the results of the simulation experiment on an

ARIMA process with $l_0 = 2$ presented in Example 1 below suggest that the chances of estimating $\varphi(t)$ from time-series observations on $x(\cdot)$ are, if anything, poorer when $l_0 > 1$ than when $l_0 = 1$. To see why, compare the result of Feller's simulation experiment on the standard random walk [4, Fig. 5, p. 84] with the result of our simulation experiment (cf. Figs. 1 and 2). One striking difference is that Feller's process seems to change sign much more frequently than our process. In fact the number of changes of sign of an ordinary random walk grows (very roughly spoken) as some constant multiple of $\sqrt{t}$, while the number of changes of sign of our process for $l_0 > 1$ grows as some constant multiple of log t.

Example 1. Let $x = \{x(t)\colon t = -2, -1, \ldots\}$ be an ARIMA process which satisfies the equations

$$x(t) - 2.5\,x(t-1) + 2\,x(t-2) - 0.5\,x(t-3) = \eta(t), \qquad (1.36)$$

$$x(-2) = 1, \qquad x(-1) = 0.7, \qquad x(0) = 0.5. \qquad (1.37)$$

In this case $n = 3$ and

$$\sum_{k=0}^{3} a_k z^{3-k} = (z-1)^2\,(z-0.5). \qquad (1.38)$$

Moreover,

$$\varphi(t) = 0.4 - 0.1\,t + 0.1\,(0.5)^t, \quad t = -2, -1, \ldots, \qquad (1.39)$$

$$\gamma_s = 0.2 + 1.9\,s + 0.8\,(0.5)^s, \quad s = 0, 1, \ldots, \qquad (1.40)$$

$$\tilde{\sigma}^2 = \tfrac{1}{3}\,(1.9)^2\,\sigma_\eta^2. \qquad (1.41)$$

Consequently, for all $z \in (-\infty, \infty)$,

$$\lim_{t\to\infty} \mathrm{P}[x(t)[\tfrac{1}{3}\,(1.9)^2\,\sigma_\eta^2\,t^3]^{-1/2} < z] = (2\pi)^{-1/2} \int_{-\infty}^{z} \exp[-\tfrac{1}{2}u^2]\,\mathrm{d}u, \qquad (1.42)$$

$$\lim_{t\to\infty} \sup\{|(x(t) - 0.4 + 0.1\,t - 0.1\,(0.5)^t) \times [\tfrac{2}{3}\,(1.9)^2\,\sigma_\eta^2\,t^3 \log\log t]^{-1/2}|\} = 1 \quad \text{w.p. 1.} \qquad (1.43)$$

Moreover, for all $\beta \in (0,1]$ which are continuity points of the distribution of $h(\cdot)$ under $\hat{W}_1$,

$$\lim_{T\to\infty} \mathrm{P}[T^{-1} \sum_{t=1}^{T} \psi(x(t) - 0.4 + 0.1\,t - 0.1\,(0.5)^t) < \beta] = \hat{W}_1\{X\colon h(X) < \beta\}. \qquad (1.44)$$

To bring home the implications of (1.42) and (1.44) for time-series analysis of X we have simulated the behavior of $x(t)$, for $t = 1, 2, \ldots, 10000$. The results of the simulations are presented in the two graphs of Figs. 1 and 2 which picture one (!) realization of X under the assumption that η is a (pseudo-) Bernoulli process with

$$P\{\eta(t) = 1\} = P\{\eta(t) = -1\} = \tfrac{1}{2}.$$

The generating method we used is described in the Reference Manual for the 1108 computer at the University of Wisconsin Computing Center. In studying the figures note that the maximum value of $x(\cdot)$ in Fig. 2 is 985236 0. In contrast $\varphi(10000) = -999.6$.

In concluding this section we should point out that the proofs of Theorems 3 and 4 are based on the validity of two general theorems concerning the asymptotic behavior of sums of the form

$$\sum_{s=1}^{T} (T-s)^q\, y(s), \tag{1.45}$$

where the $y(s)$ are as defined in (1.6) and q is a non-negative integer. Since these theorems are of interest in themselves, we state them below, and prove them in Section 2.

Theorem 5. *Suppose that* $y \equiv \{y(t)\colon t = 1, 2, \ldots\}$ *satisfies* (1.6), (1.2) *and* (1.18). *Moreover, suppose that the function* $f_y(\cdot)$ *defined in* (1.19) *is piecewise continuous on* $[-\pi, \pi)$ *and continuous in a neighborhood of* $\lambda = 0$. *Finally, let*

$$S_T \equiv \sum_{s=1}^{T} (T-s)^q\, y(s), \quad T = 1, 2, \ldots,$$

where q *is a non-negative integer, let* $\psi(\cdot)$ *be as defined in* (1.31), *and let* $h(\cdot)$ *be as defined in* (1.33). *Then*

$$\lim_{T\to\infty} P\left[T^{-1} \sum_{t=1}^{T} \psi(S_t) < \beta\right] = \hat{W}_q\{X\colon h(X) < \beta\}, \tag{1.46}$$

where $\beta \in [0,1]$ *ranges over the set of continuity points of the distribution of* $h(\cdot)$ *under* $\hat{W}_q$.

This theorem for the case $q = 0$ was established in [10] (cf. [10, Theorem 1, p. 3]).

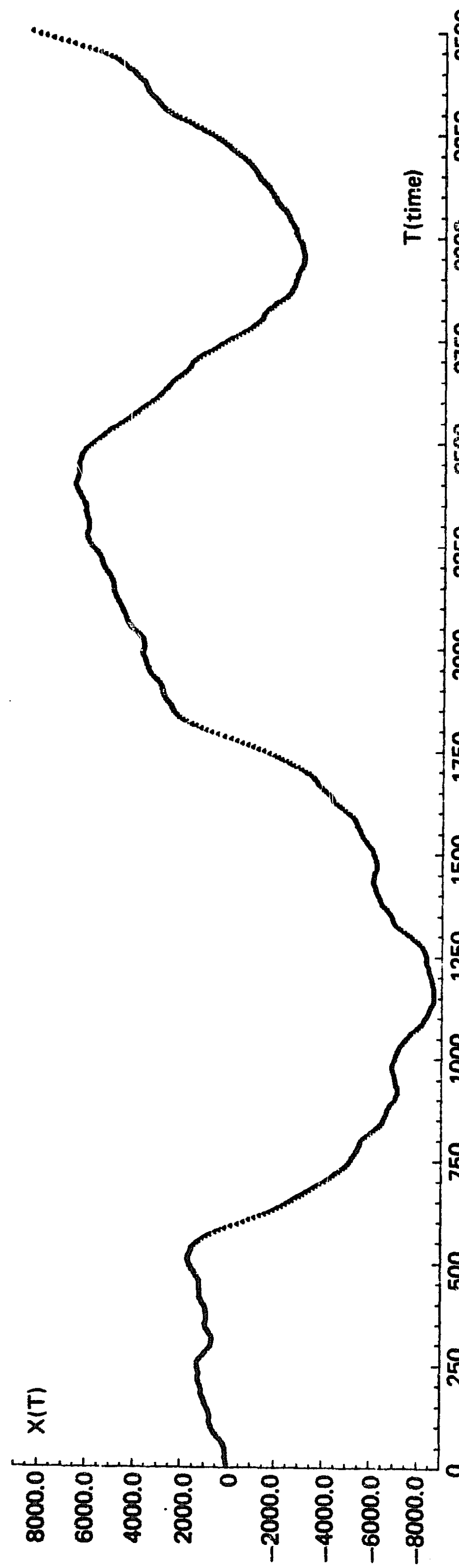
X(T)
8000.0
6000.0
4000.0
2000.0
0
−2000.0
−4000.0
−6000.0
−8000.0
0
250
500
750
1000
1250
1500
1750
2000
2250
2500
2750
3000
3250
3500
T(time)

Fig. 1.

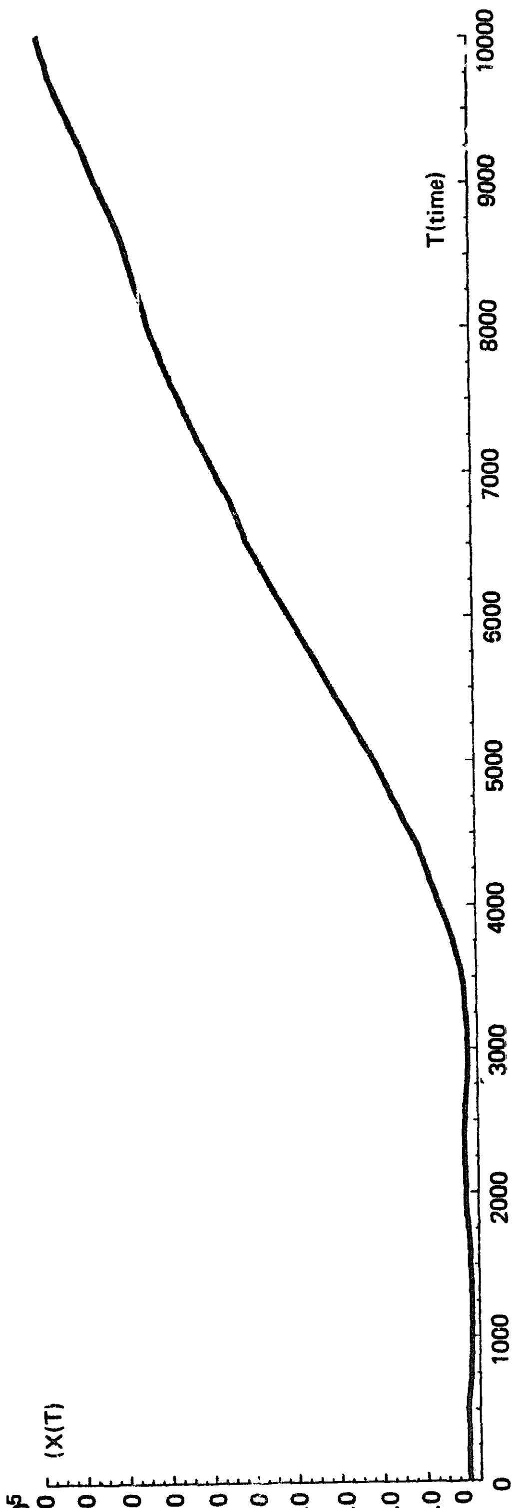
x10⁵
10.0
9.0
8.0
7.0
6.0
5.0
4.0
3.0
2.0
1.0
0
(X(T)
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
T(time)

Fig. 2.

Theorem 6. *Suppose that* $y \equiv \{y(t)\colon t = 1, 2, \ldots\}$ *satisfies the conditions of Theorem 5. Then, for any integer* $q \geqslant 0$,

$$\limsup_{T\to\infty} \left\{ \left| \{2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T\}^{-1/2} \sum_{t=0}^{T-1} t^q\, y(T-t) \right| \right\} = 1 \quad w.p.\ 1. \tag{1.47}$$

This theorem is an extension of theorems previously obtained by Gapoškin [5, Theorem 1, p. 412] and Oodaira [8, Corollary to Theorem 3, p. 3].

2. Proofs of Theorems 1–6

In this section we will give brief proofs of Theorems 1–6. We begin with Theorem 1, which is basically a well-known theorem.

2.1. Proof of Theorem 1. The existence of a function $\varphi(\cdot)$ and a set of constants γ_s that satisfy (1.10) and (1.12) is easy to verify. So we will not prove it here.

To establish (1.7)–(1.9) and (1.11) we use (1.3), (1.6) and (1.12) to note that, for all $t = 1, 2, \ldots$ and $n' \equiv \min\{n, t-1\}$,

$$\begin{aligned} y(t) &= \sum_{k=0}^{n} a_k x(t-k) \\ &= \sum_{k=0}^{n} a_k \varphi(t-k) + \sum_{k=0}^{n'} a_k \sum_{s=0}^{t-k-1} \gamma_s y(t-k-s) \\ &= \sum_{k=0}^{n} a_k \varphi(t-k) + \sum_{k=0}^{n'} a_k \sum_{v=k}^{t-1} \gamma_{v-k} y(t-v) \\ &= \sum_{k=0}^{n} a_k \varphi(t-k) + \sum_{v=0}^{n'-1} \left(\sum_{k=0}^{v} a_k \gamma_{v-k} \right) y(t-v) \\ &\quad + \sum_{v=n'}^{t-1} \left(\sum_{k=0}^{n'} a_k \gamma_{v-k} \right) y(t-v). \end{aligned} \tag{2.1}$$

Since (2.1) is an identity in $y(t)$, (2.1) implies the validity of (1.6)–(1.8) and (1.10). □

2.2. Proof of Theorem 2. It follows from (1.14) that

$$\lim_{t\to\infty} \{\varphi(t)\, t^{-(l_0 - 1/2)}\} = 0. \tag{2.2}$$

Moreover, it follows from [9, Lemma 6] and some algebra that, for all $j = 1, \ldots, l$ and $k = 0, \ldots, l_j - 1$,

$$\lim_{t\to\infty} \left\{ \sum_{s=0}^{t-1} s^k\, z_j^s\, y(t-s)\, t^{-(l_0 - 1/2)} \right\} = 0 \quad \text{w.p. 1.} \tag{2.3}$$

Finally, if we let $R_y(t-s) \equiv \mathrm{E}\{y(t)y(s)\}$, then (cf. [10, equation (2.3)]), for $k = 0, \ldots, l_0 - 1$,

$$\mathrm{E}\left\{ \sum_{s=0}^{t-1} s^k\, y(t-s)\, t^{-(l_0 - 1/2)} \right\}^2 = t^{-(2l_0 - 1)} \sum_{s,r=0}^{t-1} s^k\, r^k\, R_y(r-s), \tag{2.4}$$

$$\lim_{t\to\infty} \left\{ t^{-(2k+1)} \sum_{s,r=0}^{t-1} s^k\, r^k\, R_y(r-s) \right\} = (2k+1)^{-1} f_y(0)\,, \tag{2.5}$$

$$2l_0 - 1 - 2k - 1 = 2(l_0 - k) - 2. \tag{2.6}$$

From (2.4)–(2.6) and the Borel–Cantelli Lemma it follows that, if $l_0 > 1$, then

$$\lim_{t\to\infty} \sum_{s=0}^{t-1} s^k\, y(t-s)\, t^{-(l_0 - 1/2)} = 0 \quad \text{w.p. 1} \quad \text{for } k = 0, \ldots, l_0 - 2. \tag{2.7}$$

But if that is so, then (2.2), (2.3), (2.7), (1.12), (1.15)–(1.16), (1.21), [10, Lemma 1], and the easily verifiable fact that

$$\lim_{t\to\infty} \mathrm{E}\{\tilde{S}_t^2\, t^{-2(l_0 - 1/2)}\} = \tilde{\sigma}^2 \tag{2.8}$$

imply that, for all $z \in (-\infty, \infty)$,

$$\begin{aligned}\lim_{t\to\infty} \mathrm{P}[x(t)\, \tilde{\sigma}^{-1}\, t^{-(l_0 - 1/2)} < z] &= \lim_{t\to\infty} \mathrm{P}[(x(t) - \varphi(t))\, \tilde{\sigma}^{-1}\, t^{-(l_0 - 1/2)} < z] \\ &= \lim_{t\to\infty} \mathrm{P}[\tilde{S}_t\, \tilde{\sigma}^{-1}\, t^{-(l_0 - 1/2)} < z] \\ &= (2\pi)^{-1/2} \int_{-\infty}^{z} \exp[-\tfrac{1}{2}u^2]\, \mathrm{d}u, \end{aligned} \tag{2.9}$$

which proves the theorem. □

2.3. Proof of the Invariance Principle. We begin by establishing the existence of $\hat{W}_q$: Let $\{G_i: i = 1, 2, ...\}$ be a sequence of independently and identically distributed Gaussian random variables with mean zero and variance one, let $R_0 \equiv 0$, and let

$$R_i \equiv \sum_{j=1}^{i} (i-j)^q G_j, \qquad i = 1, 2, ..., n, \quad n = 1, 2, \ldots .$$

Moreover, for each $\tau \in [0, 1]$ and each $n = 1, 2, ...$, let

$$X_n(\tau) \equiv (2q+1)^{1/2} n^{-(q+1/2)} R_{[n\tau]}.$$

Finally, let $(\Omega, \mathfrak{F}, P)$ denote the probability space of the G_i, and let P_n denote the distribution of $X_n(\cdot)$ on $(D, \mathcal{D})$. We will show that the P_n converge weakly to $\hat{W}_q$. To do that we first fix $\tau \in (0,1]$, and observe that $X_n(\tau)$ is normally distributed with mean zero and variance

$$\sigma_\tau^2(n) \equiv (2q+1)\, n^{-(2q+1)} \sum_{j=1}^{[n\tau]} ([n\tau]-j)^{2q} \tag{2.10}$$

and that $\sigma_\tau^2(n)$ converges to τ^{2q+1}. Since $X_n(0) \equiv 0$ w.p. 1 for all n, it follows that for all $\tau \in [0,1]$ the distribution of $X_n(\tau)$ converges weakly to the distribution specified for X_τ under $\hat{W}_q$ in (1.26) and (1.27).

Next observe that, if $\{\tau_i: i = 1, ..., m\}$ is an m-tuple such that $0 < \tau_1 < ... < \tau_m \leq 1$, then $(X_n(\tau_1), ..., X_n(\tau_m))$ is normally distributed with mean zero and covariance matrix

$$\Gamma_n(\tau_1, ..., \tau_m) \equiv \begin{bmatrix} \sigma_{\tau_1}^2(n) & \mu_{\tau_1,\tau_2}(n) & \cdots & \mu_{\tau_1,\tau_m}(n) \\ \mu_{\tau_2,\tau_1}(n) & \sigma_{\tau_2}^2(n) & \cdots & \mu_{\tau_2,\tau_m}(n) \\ \vdots & & \ddots & \vdots \\ \mu_{\tau_m,\tau_1}(n) & \mu_{\tau_m,\tau_2}(n) & \cdots & \sigma_{\tau_m}^2(n) \end{bmatrix}, \tag{2.11}$$

where for $k = \min\{i,j\}$ and $p = \max\{i,j\}$, $i \neq j$,

$$\mu_{\tau_i,\tau_j}(n) \equiv (2q+1)\, n^{-(2q+1)} \sum_{j=1}^{[n\tau_k]} ([n\tau_k]-j)^q ([n\tau_p]-j)^q. \tag{2.12}$$

If $\Gamma(\tau_1, ..., \tau_m)$ is as defined in (1.28) and (1.29), it is easy to see that

$$\Gamma(\tau_1, ..., \tau_m) = \lim_{n\to\infty} \Gamma_n(\tau_1, ..., \tau_m). \tag{2.13}$$

Consequently, the distribution of $(X_n(\tau_1), ..., X_n(\tau_m))$ converges weakly to the distribution specified for $(X(\tau_1), ..., X(\tau_m))$ under $\hat{W}_q$.

To conclude the proof of the existence of $\hat{W}_q$ on $(D, \mathcal{D})$ it now suffices (cf. [1, Theorem 6.1]) to show that $\{P_n\}$ is a tight sequence, and that the $\hat{W}_q$ with the finite-dimensional distributions specified in (1.26)–(1.29) assigns measure 1 to C. We first show that $\{P_n\}$ is a tight sequence. This we can do (cf. [1, Theorem 15.6]) by showing that for any $0 \leqslant \tau_1 \leqslant \tau \leqslant \tau_2$ there exists a finite positive constant K, independent of τ_1 and τ_2, such that

$$\mathrm{E}\{(X_n(\tau) - X_n(\tau_1))^2\,(X_n(\tau_2) - X_n(\tau))^2\} \leqslant K(\tau_2 - \tau_1)^2. \quad (2.14)$$

The proof of the validity of (2.14) involves very lengthy calculations. Since they are all of an elementary nature we will only sketch the barest outline of a proof here. Note first that the existence of K is hard to ascertain only because the difference $\tau_2 - \tau_1$ may become arbitrarily small. Next note that (2.14) is trivially satisfied if $n \leqslant (\tau_2 - \tau_1)^{-1}$ since then $[n\tau]$ must equal either $[n\tau_1]$ or $[n\tau_2]$. Consequently, we need only worry about (2.14) for large n. Finally note that, for all $t \in [0, 1]$,

$$t - [nt]/n < n^{-1},$$

and that, for all integers k,

$$\lim_{n\to\infty} \left\{ n^{-(k+1)} \sum_{s=1}^{n} s^k \right\} = (k+1)^{-1}.$$

By repeated use of the last two observations we can show that there exist finite positive constants K_1, K_2, K_3, independent of n, such that, for all $n > (\tau_2 - \tau_1)^{-1}$,

$$\begin{aligned}
&\mathrm{E}\{(X_n(\tau) - X_n(\tau_1))^2\,(X_n(\tau_2) - X_n(\tau_1))^2\} \leqslant \\
&\leqslant K_1 n^{-1}\{n^{-1}([n\tau_2] - [n\tau_1])\} + K_2\{n^{-1}([n\tau_2] - [n\tau_1])\}^2 \\
&\quad + K_3 \sum_{k,j=0}^{q-1} \binom{q}{k} \{n^{-1}([n\tau_2] - [n\tau_1])\}^{2q-k-j} \\
&\leqslant K_4 n^{-1}(\tau_2 - \tau_1) + K_5(\tau_2 - \tau_1)^2 \leqslant (K_4 + K_5)(\tau_2 - \tau_1)^2
\end{aligned} \quad (2.15)$$

for suitably large finite constants K_4 and K_5. Hence (2.14) is valid with $K = K_4 + K_5$.

The preceding observations imply that there exists a measure $\hat{W}_q$ on $(D, \mathcal{D})$ which satisfies (1.26)–(1.29) in the subset of $[0, 1]$ in which $\pi_\tau(\cdot)$ is continuous except at points forming a set of $\hat{W}_q$-measure 0. We will show that this subset is all of [0,1] by showing that $\{X_\tau : \tau \in [0,1]\}$ is continuous w.p. 1 under $\hat{W}_q$. To do the latter we need only observe (cf. [7, Theorem 4, pp. 969–970]) that, with respect to $\hat{W}_q$,

$$\mathrm{E}\{(X(\tau_2) - X(\tau_1))^2\} = \tau_2^{2q+1} - 2\mu_{\tau_1,\tau_2} + \tau_1^{2q+1}$$

$$\leqslant \tau_2^{2q+1} - \tau_1^{2q+1} \leqslant (2q+1)(\tau_2 - \tau_1)$$

$$\text{for } 0 \leqslant \tau_1 < \tau_2 \leqslant 1, \qquad (2.16)$$

and that

$$\int_0^\epsilon u^{-1/2} (\log u^{-1})^{-1/2}\, \mathrm{d}u \leqslant 2\, \epsilon^{1/2} (\log \epsilon^{-1})^{-1/2} < \infty. \qquad (2.17)$$

We have shown that a $\hat{W}_q$ with the required properties exists on $(D, \mathcal{D})$. We have also shown that the P_n converge weakly to $\hat{W}_q$. With the only exception that we have to appeal to the Central Limit Theorem in [10, pp. 6–9] to show that the finite-dimensional distributions of $\hat{X}_n(\tau)$ converge to the finite-dimensional distributions of the X_τ under $\hat{W}_q$, the proof of the weak convergence of the $\hat{P}_n$ to $\hat{W}_q$ is identical with the proof that P_n converges weakly to $\hat{W}_q$. Since there is no need to repeat the proof, we can consider the invariance principle established. □

2.4. Proof of Theorem 5. To begin with, let $k > 1$ be fixed; let

$$T_i = [i(T/k)], \qquad i = 0, 1, ..., k, \quad T \geqslant 1, \qquad (2.18)$$

and let

$$S_i(T) \equiv \sum_{t=1}^{T_i} (T_i - t)^q\, y(t) = \sum_{l=0}^{q} \binom{q}{l} T_i^{q-l} (-1)^l \sum_{t=1}^{T_i} t^l y(t),$$

$$i = 0, 1, ..., k. \qquad (2.19)$$

Then it follows from [10, eq. (2.18)] that, for all $a \in (-\infty, \infty)$,

$$\lim_{T\to\infty} \mathrm{P}\{S_i(T)\, T^{-(q+1/2)} < a\} = (2\pi)^{-1/2}\, \sigma_i^{-1} \int_{-\infty}^{a} \exp[-\tfrac{1}{2} u^2 \sigma_i^{-2}]\, \mathrm{d}u, \qquad (2.20)$$

where

$$\sigma_i^2 = f_y(0)\, (2q+1)^{-1}\, k^{-(2q+1)} i^{2q+1}. \qquad (2.21)$$

It is also easy to show that, for large T and $i < m$,

$$(S_i(T)\, T^{-(q+1/2)}, S_m(T)\, T^{-(q+1/2)})$$

is approximately normally distributed with mean zero and covariance matrix

$$\Gamma_{i,m} \equiv \begin{pmatrix} \sigma_i^2 & \sigma_{i,m} \\ \sigma_{i,m} & \sigma_m^2 \end{pmatrix}, \tag{2.22}$$

where

$$\sigma_{i,m} = f_y(0)\, k^{-(2q+1)}\, i^{q+1}\, m^q \sum_{l,p=0}^{q} \binom{q}{l}\binom{q}{p} (i/m)^p\, (-1)^{l+p}\, (l+p+1)^{-1}. \tag{2.23}$$

Since most of the detailed calculations needed to establish this fact are given in [10, pp. 21–25], we omit them here for brevity's sake. Similar calculations for an n-tuple $\{i_1, ..., i_n\}$, $1 \leqslant i_1 < ... < i_n \leqslant k$, show that, for large T,

$$(S_{i_1}(T)\, T^{-(q+1/2)}, ..., S_{i_n}(T)\, T^{-(q+1/2)})$$

is approximately normally distributed with mean zero and covariance matrix

$$\Gamma_{i_1, ..., i_n} = \begin{bmatrix} \sigma_{i_1}^2 & \rho_{i_1,i_2} & \cdots & \rho_{i_1,i_n} \\ \rho_{i_1,i_2} & \sigma_{i_2}^2 & \cdots & \rho_{i_2,i_n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{i_1,i_n} & \rho_{i_2,i_n} & \cdots & \sigma_{i_n}^2 \end{bmatrix}. \tag{2.24}$$

Next note that for $0 \leqslant l \leqslant 2q$ and $i < m$ there exist constants K_6 and K_7 that are independent of l, i and m, and satisfy

$$\left| 1 - (l+1)\, i^{-(l+1)} \sum_{j=1}^{i} (i-j)^l \right| \leqslant K_6 i^{-1}, \tag{2.25}$$

and

$$\Bigg| i^{-(q+1/2)}\, m^{-(q+1/2)} \sum_{j=1}^{i} (i-j)^q\, (m-j)^q - (i/m)^{1/2} \sum_{l,p=0}^{q} \binom{q}{l}\binom{q}{p} (i/m)^{l+p}\, (-1)^{l+p}\, (l+p+1)^{-1} \Bigg| \leqslant$$

$$\leqslant K_7 (im)^{-1/2}. \tag{2.26}$$

Consequently, there exists a sequence of normally distributed random variables $\{G_i\colon i = 1, 2, ...\}$ with mean zero and variance one such that, for large i and T, and for all $a \in (-\infty, \infty)$,

$$\mathbf{P}[\{(2q+1)k^{2q+1}/f_y(0)\}^{1/2}\, i^{-(q+1/2)}\, S_i(T)\, T^{-(q+1/2)} < a]$$

$$\sim \mathbf{P}\left[(2q+1)^{1/2}\, i^{-(q+1/2)} \sum_{j=1}^{i} (i-j)^q G_j < a\right]. \tag{2.27}$$

Moreover, for large enough i_1 and T, and any n-tuple $\{i_1, ..., i_n\}$, $i_1 < i_2 < ... < i_n$,

$$\mathbf{P}[\{(2q+1)\, k^{2q+1}/f_y(0)\}^{1/2}\, T^{-(q+1/2)}\, (S_{i_1}(T)\, i_1^{-(q+1/2)}, ...,$$

$$S_{i_n}(T)\, i_n^{-(q+1/2)}) < (a_1, ..., a_n)]$$

$$\sim \mathbf{P}\left[(2q+1)^{1/2}\left(i_1^{-(q+1/2)} \sum_{j=1}^{i_1} (i_1-j)^q G_j, ..., i_n^{-(q+1/2)} \sum_{j=1}^{i_n} (i_n-j)^q G_j\right)\right.$$

$$\left. < (a_1, ..., a_n)\right], \tag{2.28}$$

where $(a_1, ..., a_n)$ is an n-dimensional real vector; i.e., if

$$R_i \equiv \sum_{j=1}^{i} (i-j)^q G_j, \quad i = 1, 2, ...,$$

then the distributions of vectors of the form

$$((2q+1)\, k^{2q+1}/f_y(0))^{1/2}\, T^{-(q+1/2)}(S_{i_1}(T)\, i_1^{-(q+1/2)}, ..., S_{i_n}(T)\, i_n^{-(q+1/2)})$$

are, for large enough i_1 and T, approximately equal to the distributions of the corresponding vectors

$$(2q+1)^{1/2}(R_{i_1}\, i_1^{-(q+1/2)}, ..., R_{i_n}\, i_n^{-(q+1/2)}).$$

Finally, lengthy calculations based on (2.21), (2.23)–(2.26) will also show that, if $\beta \in [0, 1] - \{\text{a countable set}\}$, then for each $\gamma > 0$ there is a k_γ such that, for $k > k_\gamma$ and sufficiently large T,

$$\mathbf{P}\left[k^{-1} \sum_{i=1}^{k} \psi(R_i) < \beta\right] - \gamma \leqslant \mathbf{P}\left[k^{-1} \sum_{i=1}^{k} \psi(S_i(T)) < \beta\right]$$

$$\leqslant \mathbf{P}\left[k^{-1} \sum_{i=1}^{k} \psi(R_i) < \beta\right] + \gamma. \tag{2.29}$$

The basic idea underlying the proof of (2.29) can be described briefly as follows: For each k, let B_i, $i = 1, ..., k$, be normally distributed random variables with joint distributions equal to the limiting joint distributions of the $\{(2q+1)\,k^{2q+1}/f_y(0)\}^{1/2}\,S_i(T)\,T^{-(q+1/2)}$, $i = 1, \ldots, k$. Moreover, let $B_0 \equiv 0$, and let

$$U_k(\tau) \equiv k^{-(q+1/2)}\,B_{[k\tau]}, \quad \tau \in [0,1].$$

Then $\{U_k(\tau)\colon \tau \in [0,1]\}$ belongs to $(D, \mathcal{D})$, and it is easy to verify that, if $\widetilde{P}_k$ denotes the distribution of $U_k(\cdot)$ on $(D, \mathcal{D})$, then the finite-dimensional distributions of the $\widetilde{P}_k$ converge weakly to the corresponding finite-dimensional distributions of $\hat{W}_q$. After lengthy calculations one can also show that there exists a finite constant K_8 such that, for any triple $0 < t_1 \leqslant t \leqslant t_2 \leqslant 1$ and all $k \geqslant 1$,

$$\mathrm{E}\{[U_k(\tau) - U_k(\tau_1)]^2\,[U_k(\tau_2) - U_k(\tau)]^2\} \leqslant K_8(\tau_2 - \tau_1)^2. \quad (2.30)$$

Hence the $\widetilde{P}_k$ constitute a tight sequence which converges weakly to $\hat{W}_q$.

If we now let

$$g_k(\beta) \equiv \mathbf{P}\left[k^{-1}\sum_{i=1}^{k}\psi(R_i) < \beta\right], \quad \beta \in (0,1],$$

$$g_k^*(\beta) \equiv \mathbf{P}\left[k^{-1}\sum_{i=1}^{k}\psi(B_i) < \beta\right], \quad \beta \in (0,1],$$

$$g(\beta) \equiv \hat{W}_q\{X\colon h(X) < \beta\}, \quad \beta \in (0,1],$$

then it follows from the proof of the Invariance Principle and the easily established fact that $h(\cdot)$ is continuous except at points forming a set of $\hat{W}_q$-measure zero (cf. [1, p. 231] for the necessary arguments) that $g(\cdot)$ is well-defined. Finally it follows from the proof of the Invariance Principle, [1, Corollary 1 to Theorem 5.1, pp. 30–31], arguments similar to those used to prove [10, Lemma 6], and the tightness of the $\widetilde{P}_k$ that at each continuity point β of $g(\cdot)$

$$\lim_{k\to\infty} g_k(\beta) = \lim_{k\to\infty} g_k^*(\beta) = g(\beta). \quad (2.31)$$

For brevity's sake we have left out most of the necessary details underlying the proofs of (2.30)–(2.31). Evidently, (2.31) and (2.25)–(2.28) establish the validity of (2.29) for all $\beta \in [0,1]$ that are continuity points of $g(\cdot)$.

Before concluding the proof of the theorem we must define one more sum and determine its asymptotic behavior.[2] So let

[2] While the details of the calculations in (2.32)–(2.39) are our own, we have borrowed the idea of them from Erdös and Kac's paper [3, pp. 1012–1014].

$$S_0 = 0,$$

$$S_r \equiv \sum_{t=1}^{r} (r-t)^q\, y(t), \quad r = 1, 2, \dots .$$

Moreover, let

$$H_T \equiv T^{-1} \sum_{i=1}^{k} \sum_{r=T_{i-1}+1}^{T_i} \{\psi(S_i(T)) - \psi(S_r)\}.$$

Then

$$\mathrm{E}\{|H_T|\} \leqslant T^{-1} \sum_{i=1}^{k} \sum_{r=T_{i-1}+1}^{T_i} \mathrm{E}\{|\psi(S_i(T)) - \psi(S_r)|\}. \tag{2.32}$$

Now, for $r \in (T_{i-1}, T_i]$,

$$\mathrm{E}\{|\psi(S_i(T)) - \psi(S_r)|\} = \mathrm{P}\left[\sum_{t=1}^{T_i} (1-t/T_i)^q\, y(t) > 0, \sum_{t=1}^{r} (1-t/r)^q\, y(t) \leqslant 0\right] + \mathrm{P}\left[\sum_{t=1}^{T_i} (1-t/T_i)^q\, y(t) \leqslant 0, \sum_{t=1}^{r} (1-t/r)^q\, y(t) > 0\right] \tag{2.33}$$

Moreover, for large T,

$$\mathrm{P}\left[\sum_{t=1}^{T_i} (1-t/T_i)^q\, y(t) > 0, \sum_{t=1}^{r} (1-t/r)^q\, y(t) \leqslant 0\right] \leqslant$$

$$\leqslant \mathrm{P}\left[\sum_{t=1}^{T_i} (1-t/T_i)^q\, y(t) > \epsilon\, T_i^{1/2}, \sum_{t=1}^{r} (1-t/r)^q\, y(t) \leqslant 0\right] + \mathrm{P}\left[0 < \sum_{t=1}^{T_i} (1-t/T_i)^q\, y(t) \leqslant \epsilon\, T_i^{1/2}\right]$$

$$\leqslant \mathrm{P}\left[\sum_{t=1}^{r} [(1-t/T_i)^q - (1-t/r)^q]\, y(t) > \tfrac{1}{2}\epsilon\, T_i^{1/2}\right] + \mathrm{P}\left[\sum_{t=r+1}^{T_i} (1-t/T_i)^q\, y(t) > \tfrac{1}{2}\epsilon\, T_i^{1/2}\right] + K_9\epsilon$$

$$\leqslant [(\tfrac{1}{2}\epsilon)^2 (T_i/r)]^{-1} \sum_{m,n=0}^{q} \Big\{ \tbinom{q}{m}\tbinom{q}{n}(-1)^{m+n} \times [1-(r/T_i)^m]\,[1-(r/T_i)^n]\, r^{-(n+m+1)} \sum_{t,s=1}^{r} t^n s^m R_y(t-s) \Big\}$$

$$+ (\tfrac{1}{2}\epsilon)^{-2}[(T_i - T_{i-1})/T_i](T_i - T_{i-1})^{-1}$$

$$\times \sum_{s,t=T_{i-1}+1}^{T_i} (1-t/T_i)^q (1-s/T_i)^q R_y(t-s) + K_9\epsilon$$

$$\leqslant K_{10}\epsilon^{-2}(1-(T_{i-1}/T_i)^q)^2 + K_{11}\epsilon^{-2} i^{-1} + K_9\epsilon \tag{2.34}$$

for suitably large constants K_9, K_{10}, K_{11}, all of which are independent of ϵ. Similar calculations show that, for large T,

$$\mathrm{P}\left[\sum_{t=1}^{T_i}(1-t/T_i)^q y(t) \leqslant 0, \sum_{t=1}^{r}(1-t/r)^q y(t) > 0\right] \leqslant$$

$$\leqslant \tilde{K}_{10}\epsilon^{-2}(1-(T_{i-1}/T_i)^q)^2 + \tilde{K}_{11}\epsilon^{-2} i^{-1} + \tilde{K}_9\epsilon \tag{2.35}$$

for suitably large constants $\tilde{K}_9$, $\tilde{K}_{10}$ and $\tilde{K}_{11}$. From (2.32)–(2.35) it follows that with $K_i^* = 2\max\{\tilde{K}_i, K_i\}$, $i = 9, 10, 11$,

$$\mathrm{E}\{|H_T|\} \leqslant k^{-1}\sum_{i=1}^{k}\{K_{10}^*\epsilon^{-2}(1-(T_{i-1}/T_i)^q)^2 + K_{11}^*\epsilon^{-2}i^{-1}\} + K_9^*\epsilon$$

$$\equiv M(T, \epsilon; k). \tag{2.36}$$

Since, for large T,

$$(1-(T_{i-1}/T_i)^q)^2 = \mathrm{O}(i^{-2}),$$

it follows from (2.36) that, for some large K_{12},

$$\limsup_{T\to\infty} \mathrm{E}\{|H_T|\} \leqslant K_{12}\{\epsilon^{-2}k^{-1}(1+\log k) + \epsilon\}. \tag{2.37}$$

Now, for any $\delta > 0$,

$$\mathrm{P}\left[\left|\sum_{t=1}^{T} T^{-1}\psi(S_t) - T^{-1}\sum_{i=1}^{k}(T_i - T_{i-1})\,\psi(S_i(T))\right| \geqslant \delta\right] =$$

$$= \mathrm{P}[|H_T| \geqslant \delta] \leqslant \delta^{-1}M(T, \epsilon; k). \tag{2.38}$$

Hence, for any $\beta \in (0,1)$,

$$\mathrm{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t) < \beta\right] \leqslant$$

$$\leqslant \mathrm{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t) < \beta, |H_T| < \delta\right] + \mathrm{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t) < \beta, |H_T| \geqslant \delta\right]$$

$$\leqslant \mathrm{P}\left[T^{-1}\sum_{i=1}^{k}(T_i - T_{i-1})\,\psi(S_i(T)) < \beta + \delta\right] + \delta^{-1}M(T, \epsilon; k), \tag{2.39}$$

$$\mathbf{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t)<\beta\right]$$

$$\geqslant \mathbf{P}\left[T^{-1}\sum_{i=1}^{k}(T_i-T_{i-1})\psi(S_i(T))<\beta-\delta\right] - \delta^{-1}M(T,\epsilon;k). \tag{2.40}$$

From (2.39) and (2.40), from the fact that

$$\lim_{T\to\infty}\{T^{-1}(T_i-T_{i-1})\}=k^{-1},$$

from (2.37), and from (2.29) and (2.31) it follows that, if $\beta\in(0,1]$ is a continuity point of $g(\cdot)$, then, for any given $\epsilon>0$ and $\delta<\frac{1}{2}\beta$, and for $\gamma\equiv 2\delta^{-1}\epsilon K_{12}$ and $k>k_\gamma$,

$$g_k(\beta)-\gamma\leqslant \lim_{T\to\infty}\mathbf{P}\left[T^{-1}\sum_{i=1}^{k}(T_i-T_{i-1})\,\psi(S_i(T))<\beta\right]$$

$$\leqslant g_k(\beta)+\gamma, \tag{2.41}$$

$$g(\beta-2\delta)-2\,\delta^{-1}K_{12}\{\epsilon^{-2}k^{-1}(1+\log k)+2\epsilon\}\leqslant$$

$$\leqslant \liminf_{T\to\infty}\mathbf{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t)<\beta\right]\leqslant \limsup_{T\to\infty}\mathbf{P}\left[T^{-1}\sum_{t=1}^{T}\psi(S_t)<\beta\right]$$

$$\leqslant g(\beta+2\delta)+2\delta^{-1}K_{12}\{\epsilon^{-2}k^{-1}(1+\log k)+2\epsilon\}. \tag{2.42}$$

By letting $k\to\infty$ and $(\epsilon,\delta)\to 0$ appropriately, the validity of the relation (1.49) of Theorem 5 becomes an immediate consequence of (2.42). □

2.5 Proof of Theorem 3. In light of (1.12), (1.15), (1.16), (2.2), (2.3) and (2.7) the validity of equation (1.32) of Theorem 3 becomes a simple corollary of Theorem 5 which needs no further proof. The validity of equation (1.34) was established in [10]. □

2.6. Proof of Theorem 6. To establish (1.47) we begin by showing that, for any given integer $q\geqslant 1$,

$$\limsup_{t\to\infty}\left\{\left|\{2(2q+1)^{-1}\sigma_\eta^2\,t^{2q+1}\log\log t\}^{-1/2}\sum_{s=0}^{t-1}s^q\,\eta(t-s)\right|\right\}$$

$$=1\quad \text{w.p. 1}. \tag{2.43}$$

To do so we fix the value of q and pick an arbitrary $\epsilon > 0$. Moreover we let $\theta > 1$ be a constant whose value we will determine later, and we let $N_s = [\theta^{2s}]$, $s = 1, 2, \ldots$. Finally, we let

$$t_N^{-1} \equiv \{2(2q+1)^{-1}\,\sigma_\eta^2\, N \log\log N\}^{1/2}, \quad N = 1, 2, \ldots .$$

It follows easily from [10, Lemma 1] and Feller's [4, Theorem VIII.5] that, for large s,

$$\mathbf{P}\left[t_{N_s} \sum_{t=1}^{N_s} (1-t/N_s)^q \eta(t) \geq 1+\epsilon\right] \leq \exp[-(1+\epsilon)^2 \log\log N_s]$$
$$= (\log N_s)^{-(1+\epsilon)^2} \sim (2s \log\theta)^{-(1+\epsilon)^2} . \tag{2.44}$$

Consequently, by the Borel–Cantelli lemma,

$$\limsup_{s\to\infty} \left\{ t_{N_s} \sum_{t=1}^{N_s} (1-t/N_s)^q\, \eta(t) \right\} \leq 1+\epsilon \quad \text{w.p. 1.} \tag{2.45}$$

Next, for $s = 1, 2, \ldots$ let

$$R_s \equiv \sup_{N_{s-1}<N<N_s} \left\{ t_N \left| \sum_{t=1}^{N} (1-t/N)^q \eta(t) - \sum_{t=1}^{N_{s-1}} (1-t/N_{s-1})^q \eta(t) \right| \right\}, \tag{2.46}$$

$$R_s^1 \equiv \sup_{N_{s-1}<N<N_s} \left\{ t_N \left| \sum_{t=1}^{N_{s-1}} [(1-t/N)^q - (1-t/N_{s-1})^q]\, \eta(t) \right| \right\}, \tag{2.47}$$

$$R_s^2 \equiv \sup_{N_{s-1}<N<N_s} \left\{ t_N \left| \sum_{t=N_{s-1}+1}^{N} (1-t/N)^q \eta(t) \right| \right\} . \tag{2.48}$$

Then[3]

$$R_s \leq R_s^1 + R_s^2, \quad s = 1, 2, \ldots . \tag{2.49}$$

Moreover, if we let

$$S_i^0 \equiv \sum_{t=N_{s-1}+1}^{i} \eta(t), \quad i > N_{s-1} ,$$

and observe that

$$\sum_{t=N_{s-1}+1}^{N} (1-t/N)^q \eta(t) = \sum_{t=N_{s-1}+1}^{N-1} [(1-t/N)^q - (1-(t+1)/N)^q]\, S_t^0 ,$$

[3] The details of the calculations in (2.49)–(2.55) are our own. However, we have borrowed the idea of them from Gapoškin (cf. [5, pp. 414–415]).

it is easy to see that

$$R_s^2 \leqslant t_{N_{s-1}} (1-(N_{s-1}+1)/N_s)^q \sup_{N_{s-1}<i<N} |S_i^0| . \tag{2.50}$$

Since, for large s, $t_{N_{s-1}}/t_{N_s} \sim \theta$, and by [10, Theorem 2]

$$\limsup_{N\to\infty} \{t_N |S_N^0|\} \leqslant 1 \quad \text{w.p. } 1,$$

(2.50) implies that there is a $\theta_1(\epsilon) > 1$ such that, for all $1 < \theta \leqslant \theta_1(\epsilon)$,

$$\limsup_{s\to\infty} R_s^2 \leqslant \tfrac{1}{2}\epsilon \quad \text{w.p. } 1. \tag{2.51}$$

Finally, if we observe that

$$R_s^1 = \sup_{N_{s-1}<N<N_s} \left\{ t_N \left| \sum_{m=0}^{q} \binom{q}{m} \{N^{-m} - N_{s-1}^{-m}\} \sum_{t=1}^{N_{s-1}} (-t)^m \eta(t) \right| \right\}$$

$$\leqslant \sup_{N_{s-1}<N<N_s} \left\{ \sum_{m=0}^{q} \binom{q}{m} \left| \frac{N_{s-1}^m - N^m}{(NN_{s-1})^m} \right| |t_N/t_{N_{s-1}}| \left| t_{N_{s-1}} \sum_{t=1}^{N_{s-1}} (-t)^m \eta(t) \right| \right\}$$

$$\leqslant \sup_{N_{s-1}<N<N_s} \left\{ \sum_{m=0}^{q} \binom{q}{m} (1-\theta^{-2m}) |t_N/t_{N_{s-1}}| \left| t_{N_{s-1}} \sum_{t=1}^{N_{s-1}} (-t/N_{s-1})^m \eta(t) \right| \right\},$$

we can use [10, Theorem 2] to conclude that

$$\limsup_{s\to\infty} R_s^1 \leqslant \sum_{m=0}^{q} \binom{q}{m} (1-\theta^{-2m}) (m+1)^{-1} (2q+1). \tag{2.52}$$

From (2.52) it follows that there is a number $1 < \theta_2(\epsilon) \leqslant \theta_1(\epsilon)$ such that, if $1 < \theta < \theta_2(\epsilon)$ and $\theta - 1 < \tfrac{1}{4}\epsilon$, then

$$\limsup_{s\to\infty} R_s^1 \leqslant \tfrac{1}{2}\epsilon \quad \text{w.p. } 1. \tag{2.53}$$

Since

$$\sup_{N_{s-1}<N<N_s} \left\{ \left| t_N \sum_{t=1}^{N} (1-t/N)^q \eta(t) - t_{N_{s-1}} \sum_{t=1}^{N_{s-1}} (1-t/N_{s-1})^q \eta(t) \right| \right\} \leqslant$$

$$\leqslant R_s + t_{N_s} |1-(t_{N_{s-1}}/t_{N_s})| \left| \sum_{s=1}^{N_{s-1}} (1-t/N_{s-1})^q \eta(t) \right|, \tag{2.54}$$

we can conclude from (2.45), (2.49), (2.51), (2.52) and (2.53) that

$$\limsup_{N\to\infty} t_N \sum_{t=1}^{N}(1-t/N)^q \eta(t) \leqslant 1+4\epsilon \quad \text{w.p. 1.} \tag{2.55}$$

But if that is so, then the fact that $\epsilon > 0$ was chosen arbitrarily implies

$$\limsup_{T\to\infty} \left\{[2(2q+1)^{-1}\sigma_\eta^2 T^{2q+1}\log\log T]^{-1/2} \sum_{t=0}^{T-1} t^q \eta(T-t)\right\} \leqslant 1 \quad \text{w.p. 1.} \tag{2.56}$$

Since (2.35) is obviously true for $-\eta(\cdot)$ as well as for $\eta(\cdot)$, we have shown that

$$\limsup_{T\to\infty} \left\{[2(2q+1)^{-1}\sigma_\eta^2 T^{2q+1}\log\log T]^{-1/2} \left|\sum_{t=0}^{T-1} t^q \eta(T-t)\right|\right\} \leqslant 1 \quad \text{w.p. 1.} \tag{2.57}$$

To establish the "converse" of (2.57) we proceed as follows. Let $\epsilon > 0$ be chosen arbitrarily, and let d be an integer such that

$$[(d-1)/d]^{2q+1} > \delta > 1-\epsilon, \tag{2.58}$$

where δ is a constant whose value will be determined later. Also, let $T_r = d^r$, $r = 1, 2, \ldots$, and let

$$B_r = \left\{\sum_{t=T_{r-1}}^{T_r-1} t^q \eta(T_r-t) > \delta\{2(2q+1)^{-1}\sigma_\eta^2 T_r^{2q+1}\log\log T_r\}^{1/2}\right\}. \tag{2.59}$$

Then with $\tilde{K}^2 \equiv d^{2q+1}(d-1)^{-(2q+1)} - (d-1)^{-(2q+1)} \geqslant 1$

$$\begin{aligned}\mathrm{P}[B_r] = \mathrm{P}\Bigg[&(2q+1)^{1/2}\sigma_\eta^{-1}(T_r-T_{r-1})^{-(q+1/2)} \sum_{t=T_{r-1}}^{T_r-1} t^q \eta(T_r-t) \\ &> \delta\{2T_r^{2q+1}(T_r-T_{r-1})^{-(2q+1)}\log\log T_r\}^{1/2}\Bigg] \\ \geqslant \mathrm{P}\Bigg[&(2q+1)^{1/2}\sigma_\eta^{-1}(T_r-T_{r-1})^{-(q+1/2)} \sum_{t=0}^{T_r-T_{r-1}-1} (t+T_{r-1})^q \eta(T_r-t) \\ &> (2\delta\log\log T_r)^{1/2}\Bigg]\end{aligned}$$

$$\geqslant \mathbf{P}\Big[(2q+1)^{1/2}\,\sigma_\eta^{-1}\,\widetilde{K}^{-1}(T_r-T_{r-1})^{-(q+1/2)} \sum_{t=1}^{T_r-T_{r-1}} (T_r-t)^q\,\eta(T_{r-1}+t)$$

$$> (2\delta \log\log T_r)^{1/2}\Big]$$

$$= \mathbf{P}\Big[(2q+1)^{1/2}\,\sigma_\eta^{-1}\,\widetilde{K}^{-1}(T_r-T_{r-1})^{-(q+1/2)} \sum_{t=1}^{T_r-T_{r-1}} (T_r-t)^q\,\eta(t)$$

$$> (2\delta \log\log T_r)^{1/2}\Big]. \tag{2.60}$$

It follows from [10, Lemma 1], some algebra, Feller's [4, Theorem VIII.5] and (2.60) that, for large r,

$$\mathbf{P}[B_r] > (2\delta \log\log T_r)^{-1} \exp\,[-\delta \log\log T_r]$$

$$= \{(2\delta \log\log T_r)\,(\log T_r)^{\delta}\}^{-1} > r^{-1}. \tag{2.61}$$

Consequently,

$$\sum_{r=1}^{\infty} \mathbf{P}[B_r] = \infty. \tag{2.62}$$

Next note that (by (2.44)), for any $\gamma > 0$ and for large r,

$$\mathbf{P}\Big[\sum_{t=0}^{T_{r-1}-1} t^q\,\eta(T_r-t) \geqslant (1+\gamma)\,[2(2q+1)^{-1}\,\sigma_\eta^2\,T_{r-1}^{2q+1}\log\log T_{r-1}]^{1/2}\Big] =$$

$$= \mathbf{P}\Big[\sum_{t=0}^{T_{r-1}-1} t^q\,\eta(T_{r-1}-t) \geqslant (1+\gamma)\,[2(2q+1)^{-1}\,\sigma_\eta^2\,T_{r-1}^{2q+1}\log\log T_{r-1}]^{1/2}\Big]$$

$$\leqslant \exp[-(1+\gamma)^2 \log\log T_{r-1}] = (\log T_{r-1})^{-(1+\gamma)^2} = ((r-1)\log d)^{-(1+\gamma)^2}. \tag{2.63}$$

Since (2.63) holds for $-\eta(\cdot)$ as well, we can use (2.63) to find for each $\varphi > 0$ an N so large that for all $r > N$ the probability is greater than or equal to $1-\varphi$ that

$$\Big|\sum_{t=0}^{T_{r-1}-1} t^q\,\eta(T_r-t)\Big| \leqslant 2[2(2q+1)^{-1}\,\sigma_\eta^2\,T_{r-1}^{2q+1}\log\log T_{r-1}]^{1/2}. \tag{2.64}$$

So, if we choose δ so close to 1 that

$$1-\delta<\tfrac{1}{4}(\delta+\epsilon-1)^2\,, \tag{2.65}$$

we find that

$$4T_{r-1}^{2q+1}=4(T_r/d)^{2q+1}<T_r^{2q+1}(\delta+\epsilon-1)^2\,, \tag{2.66}$$

and hence that (by (2.64)), with probability greater than or equal to $1-\varphi$,

$$\sum_{t=0}^{T_{r-1}-1} t^q\eta(T_r-t)>-(\delta+\epsilon-1)\,[2(2q+1)^{-1}\sigma_\eta^2\,T_r^{2q+1}\log\log T_r]^{1/2}\,. \tag{2.67}$$

When we add (2.67) to both sides of the inequality defining B_r, we see immediately from (2.62) and the Borel–Cantelli lemma that the event

$$[2(2q+1)^{-1}\sigma_\eta^2\,T^{2q+1}\log\log T]^{-1/2}\sum_{t=0}^{T-1} t^q\eta(T-t)\geqslant 1-\epsilon$$

happens infinitely often w.p. 1. Since ϵ was arbitrary, we have shown that

$$\limsup_{T\to\infty}\left\{[2(2q+1)^{-1}\sigma_\eta^2\,T^{2q+1}\log\log T]^{-1/2}\sum_{t=0}^{T-1} t^q\eta(T-t)\right\}\geqslant 1 \text{ w.p. } 1. \tag{2.68}$$

The validity of (2.43) now follows from (2.68) and (2.57), and the fact that (2.68) remains valid if we replace $\eta(\cdot)$ by $-\eta(\cdot)$. So much for (2.43).

To conclude the proof of (1.47) we next let N be a large positive integer, and we let

$$\hat{y}(t)\equiv\sum_{s=-N}^{N}\alpha_s\,\eta(t+s),\quad t=1,2,\ldots,$$

$$\begin{aligned}\hat{S}_T&\equiv\sum_{t=0}^{T-1} t^q\,\hat{y}(T-t)\\&=\sum_{t=0}^{T-1}\sum_{s=-N}^{N} t^q\alpha_s\,\eta(T-t-s)\\&=\sum_{s=-N}^{N}\alpha_s\sum_{v=s}^{T+s-1}(v-s)^q\,\eta(T-v)\\&=\sum_{m=0}^{q}\binom{q}{m}\sum_{s=-N}^{N}\alpha_s(-s)^{q-m}\sum_{v=s}^{T+s-1}v^m\,\eta(T-v)\,.\end{aligned} \tag{2.69}$$

Then (2.43) implies that

$$\limsup_{T\to\infty} \hat{S}_T[2(2q+1)^{-1} f_{\hat{y}}(0)\, T^{2q+1} \log\log T]^{-1/2} = 1 \quad \text{w.p. 1.} \tag{2.70}$$

Since we can replace $\hat{y}(\cdot)$ by $-\hat{y}(\cdot)$ in the definition of $\hat{S}_T$ without changing the conclusion (2.70), we also find that

$$\limsup_{T\to\infty} \{|\hat{S}_T\, [2(2q+1)^{-1} f_{\hat{y}}(0)\, T^{2q+1} \log\log T]^{-1/2}|\} = 1 \quad \text{w.p. 1.} \tag{2.71}$$

The arguments used to establish (2.71) suffice to show that, for any N and M,

$$\limsup_{T\to\infty} \left\{\left|\sum_{t=0}^{T-1} \sum_{s=-N}^{N} t^q\, \alpha_s\, \eta(T-t-s)\, [2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T]^{-1/2}\right|\right\}$$

$$= \left|\sum_{s=-N}^{N} \alpha_s\right| \left|\sum_{s=-\infty}^{\infty} \alpha_s\right|^{-1} \quad \text{w.p. 1,} \tag{2.72}$$

$$\limsup_{T\to\infty} \left\{\left|\sum_{t=0}^{T-1} \sum_{|s|=N+1}^{N+M} t^q \{\alpha_s\, \eta(T-t-s)\, [2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T]^{-1/2}\right|\right\}$$

$$= \left|\sum_{|s|=N+1}^{N+M} \alpha_s\right| \left|\sum_{s=-\infty}^{\infty} \alpha_s\right|^{-1}$$

$$\leqslant \left|\sum_{|s|=N+1}^{k_N} \alpha_s\right| \left|\sum_{s=-\infty}^{\infty} \alpha_s\right|^{-1} \quad \text{w.p. 1,} \tag{2.73}$$

where k_N is the integer which maximizes the value of $|\Sigma_{|s|=N+1}^{n} \alpha_s|$.

Next we note that

$$y(t) = \lim_{N\to\infty} \left\{\sum_{s=-N}^{N} \alpha_s\, \eta(t-s)\right\} \quad \text{w.p. 1,} \qquad t = 1, 2, \ldots .$$

Thus, if for each N we let

$$\sum_{|s|>N} \alpha_s\, \eta(t-s) \equiv y(t) - \sum_{|s|\leqslant N} \alpha_s\, \eta(t-s), \qquad t = 1, 2, \ldots ,$$

then (2,72) and (2.73) can easily be seen to imply that

$$\limsup_{T\to\infty} \left\{\left|\sum_{t=0}^{T-1} t^q\, y(T-t)\, [2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T]^{-1/2}\right|\right\}$$

$$\leqslant \left\{\left|\sum_{|s|\leqslant N} \alpha_s\right| + \left|\sum_{|s|=N+1}^{k_N} \alpha_s\right|\right\} \left|\sum_{s=-\infty}^{\infty} \alpha_s\right|^{-1} \quad \text{w.p. 1.} \tag{2.74}$$

From (2.74) it follows that

$$\limsup_{T\to\infty}\left\{\left|\sum_{t=0}^{T-1} t^q y(T-t)\,[2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T]^{-1/2}\right|\right\} \leqslant 1 \quad \text{w.p. 1.} \tag{2.75}$$

Finally, it is easy to see that (2.72), (2.73) and

$$\begin{aligned}\left|\sum_{t=0}^{T-1} t^q \sum_{|s|\leqslant N} \alpha_s \eta(T-t-s)\right| &= \left|\sum_{t=0}^{T-1} t^q y(T-t) - \sum_{t=0}^{T-1} t^q \sum_{|s|>N} \alpha_s \eta(T-t-s)\right| \\ &\leqslant \left|\sum_{t=0}^{T-1} t^q y(T-t)\right| + \left|\sum_{t=0}^{T-1} t^q \sum_{|s|>N} \alpha_s \eta(T-t-s)\right|\end{aligned} \tag{2.76}$$

imply that, for any $\epsilon > 0$,

$$\limsup_{T\to\infty}\left\{\left|\sum_{t=0}^{T-1} t^q y(T-t)\,[2(2q+1)^{-1} f_y(0)\, T^{2q+1} \log\log T]^{-1/2}\right|\right\} \geqslant 1-\epsilon \quad \text{w.p. 1.} \tag{2.77}$$

This concludes the proof of Theorem 6. □

2.7. Proof of Theorem 4. The validity of (1.35) is an immediate consequence of (1.12), (1.15), (1.16), (2.2), (2.3), (2.7), [10, Theorem 2], and (1.47) for $q = l_0 - 1$ and, therefore, needs no further proof. □

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