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Large deviations and phase transition for random walks in random nonnegative potentials

Markus Flury*

Universität Zürich, Institut für Mathematik, Winterthurerstr. 190, CH-8057 Zürich, Switzerland

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Abstract

We establish large deviation principles and phase transition results for both quenched and annealed settings of nearest-neighbor random walks with constant drift in random nonnegative potentials on $\mathbb{Z}^d$. We complement the analysis of M.P.W. Zerner [Directional decay of the Green's function for a random nonnegative potential on $\mathbb{Z}^d$, Ann. Appl. Probab. 8 (1996) 246–280], where a shape theorem on the Lyapunov functions and a large deviation principle in absence of the drift are achieved for the quenched setting.

Keywords: Random walk; Random potential; Path measure; Lyapunov function; Shape theorem; Large deviation principle; Phase transition

1. Introduction

Let $\mathcal{S} = (S(n))_{n\in\mathbb{N}_0}$ be a symmetric nearest-neighbor random walk on $\mathbb{Z}^d$ starting at the origin, and denote by P, respectively E, the associated probability measure, respectively expectation. The aim of this article is a probabilistic description of the long-time behavior of the random walk, endowed with a drift and evolving in a random environment given by a random potential on the lattice. This description will be done for concrete realizations of the environment, the *quenched* setting, as well as for the averaged environment, the so-called *annealed* setting. For details, we make the following assumptions:

(Qu) $\mathbb{V} = (V_x)_{x\in\mathbb{Z}^d}$ is a family of independent, identically and not trivially distributed random variables in $L^d(\Omega, \mathcal{F}, \mathbb{P})$, which is independent of the random walk itself and satisfies ess inf $V_x = 0$.

* Tel.: +41 4463 55843; fax: +41 4463 55705.
E-mail address: mflury@amath.unizh.ch.

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(An) $\varphi : [0, \infty) \to [0, \infty)$ is a non constant, non decreasing and concave function with $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t)/t = 0$.

For $\omega \in \Omega$, $n \in \mathbb{N}$ and $h \in \mathbb{R}^d$, the *quenched path measure* $Q^h_{n,\omega}$ for the random walk $\mathcal{S}$ with constant drift h under the path potential

$$\Psi(n, \omega) \stackrel{\text{def}}{=} \sum_{1\le m\le n} V_{S(m)}(\omega)$$

is defined by means of the density function

$$\frac{\mathrm{d}Q^h_{n,\omega}}{\mathrm{d}P} \stackrel{\text{def}}{=} \frac{1}{Z^h_{n,\omega}} \exp(h \cdot S(n) - \Psi(n, \omega)) ,$$

where $Z^h_{n,\omega}$ denotes the corresponding (quenched) normalization. Notice that $Q^h_{n,\cdot}$ is a random probability measure, the randomness coming from the random potential $\Psi(n, \cdot)$.

For $x \in \mathbb{Z}^d$, let now

$$l_x(n) \stackrel{\text{def}}{=} \sharp\{1 \le m \le n : \ S(m) = x\}$$

denote the number of the random walk's visits to the site x up to time n. The *annealed path measure* Q^h_n for the random walk $\mathcal{S}$ with constant drift h under the path potential

$$\Phi(n) \stackrel{\text{def}}{=} \sum_{x\in\mathbb{Z}^d} \varphi(l_x(n))$$

is defined by means of the density function

$$\frac{\mathrm{d}Q^h_n}{\mathrm{d}P} \stackrel{\text{def}}{=} \frac{1}{Z^h_n} \exp(h \cdot S(n) - \Phi(n)) ,$$

where Z^h_n is the corresponding (annealed) normalization constant.

The model we come to introduce is a discrete-setting model for a particle moving in a random media. In the quenched setting, the walker jumps from site to site, thereby trying to avoid those regions where the potential takes on high values. The drift however implies a restriction in the search of such an "optimal strategy" by imposing a particular direction to the walk.

We shall point out that in the definition of the annealed path measures we are making a slight abuse of standard terminology. To clarify this aspect, consider

$$\varphi_{\mathbb{V}}(t) \stackrel{\text{def}}{=} -\log \mathbb{E} \exp(-t V_x), \quad t \in [0, \infty),$$

for a given potential $\mathbb{V}$. By Hölder inequality, dominated convergence and the assumption $\operatorname{ess\,inf} V_x = 0$, it is easy to see that $\varphi_{\mathbb{V}}$ fulfills the requirements (An). Let $Q^h_{\mathbb{V},n}$ denote the annealed path measure corresponding to $\varphi_{\mathbb{V}}$. The quenched potential can be rewritten as

$$\Psi(n, \omega) = \sum_{x\in\mathbb{Z}^d} l_x(n) V_x(\omega).$$

By the independence assumption on the potential, it now is easily seen that

$$\frac{\mathrm{d}Q^h_{\mathbb{V},n}}{\mathrm{d}P} = \frac{1}{\mathbb{E}Z^h_{n,\cdot}} \mathbb{E}\left[\exp(h \cdot S(n) - \Psi(n, \cdot))\right]$$

for any drift h and all $n \in \mathbb{N}$, which is the "classical" annealed path measure. Our results cover this standard case, but do not rely on the particular form of $\varphi_{\mathbb{V}}$ in the above definition.

An interesting example of such a potential $\varphi_{\mathbb{V}}$ is considered at the so-called *hard obstacle* or *trap model*. There, one assumes $V_x : \Omega \to \{0, \infty\}$ with positive probability for both values. The name of the model comes from the fact that

$$Z^h_{n,\omega} = E\left[\exp(h \cdot S(n))\,;\, V_{S(m)}(\omega) = 0 \text{ for } 1 < m \le n\right],$$

which describes the probability for the drifted random walk not to step into one of the "traps" $\{x \in \mathbb{Z}^d : V_x(\omega) = \infty\}$ up to time $n \in \mathbb{N}$. Such a potential is not in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ and consequently does not satisfy assumption (Qu). Yet, the function $\varphi_{\mathbb{V}}(t)$, associated to the classical annealed terms, does fulfill the required properties (An), and satisfies

$$\varphi_{\mathbb{V}}(t) = \begin{cases} -\log \mathbb{P}\left[V_x = 0\right] & \text{if } t > 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Again, the expected probability $\mathbb{E}Z^h_{n,\cdot}$ of not stepping into a trap equals the annealed normalization constant $Z^h_{\mathbb{V},n}$ corresponding to $\varphi_{\mathbb{V}}$. We thus have

$$\mathbb{E}Z^h_{n,\cdot} = E\left[\exp(h \cdot S(n) - \gamma \sharp\{S(m) : 1 \le m \le n\})\right]$$

with $\gamma = -\log \mathbb{P}[V_x = 0]$ and $n \in \mathbb{N}$.

We come back to the general setup of a random walk in a random potential. A similar model in a continuous setting, namely Brownian motion in a Poissonian potential, was first studied by A.S. Sznitman. By means of the powerful method of *enlargement of obstacles*, Sznitman established a precise picture in both quenched and annealed settings. He achieved results such as a shape theorem, large deviation principles (LDP's) and an accurate description of the transition between small and large drift. We refer the reader to Chapter 5 of [6] for a complete review of these results. In the discrete setting, an ample study of the random walk under the influence of the quenched potential was made by Zerner [8]. His results, however, are limited to the case where no drift is present.

The aim of the present work is to add the missing pieces to Zerner's analysis, recovering the larger picture for the random walk with drift in both quenched and annealed settings. The organization of the article is as follows: In Section 2, we state the main results. In Section 3, we follow Zerner's analysis and prove a shape theorem for the directed random walk. Section 4 is devoted to the proof of the LDP's. In Section 5, we closely follow Sznitman's path to analyze phase transitions in the long-time behavior of the random walk, related to the size of the drift.

2. Main results

The essential quantities in our study of the large time asymptotics of the random walk are the so-called *Lyapunov functions* on $\mathbb{R}^d$. Let

$$H(x) \stackrel{\text{def}}{=} \inf\{n \in \mathbb{N}_0 : S(n) = x\}$$

denote the time of the random walk's first visit to the lattice site $x \in \mathbb{Z}^d$. For $\lambda \ge 0$, $\omega \in \Omega$ and $x \in \mathbb{Z}^d$, we define the two-point functions

$$a_\lambda(x, \omega) \stackrel{\text{def}}{=} -\log E\left[\exp(-\lambda H(x) - \Psi(H(x), \omega))\,;\, H(x) < \infty\right],$$

$$b_\lambda(x) \stackrel{\text{def}}{=} -\log E\left[\exp(-\lambda H(x) - \Phi(H(x)))\,;\, H(x) < \infty\right].$$

Our first result introduces the Lyapunov functions α_λ and β_λ, and sets them in relation to the asymptotic behavior of a_λ and b_λ.

Theorem A (*Shape Theorem*).

(a) [8] *There is a family* $(\alpha_\lambda)_{\lambda\geq 0}$ *of norms on* $\mathbb{R}^d$*, such that for any* $\lambda \geq 0$ *and all sequences* $(x_k)_{k\in\mathbb{N}}$ *on* $\mathbb{Z}^d$ *with* $\|x_k\|_1 \to \infty$ *as* $k \to \infty$*, we have*

$$\lim_{k\to\infty} \frac{a_\lambda(x_k, \omega)}{\alpha_\lambda(x_k)} = 1 \tag{1}$$

on a set Ω_λ *of full* $\mathbb{P}$*-measure and in* $L^1(\Omega, \mathcal{F}, \mathbb{P})$*. Moreover,* $\alpha_\lambda(x)$ *is continuous in* $(\lambda, x) \in [0,\infty) \times \mathbb{R}^d$*, concave increasing in* $\lambda \in [0,\infty)$*, and satisfies*

$$\|x\|_1 (\lambda - \log \mathbb{E}\exp(-V_x)) \leq \alpha_\lambda(x) \leq \|x\|_1 (\lambda + \log(2d) + \mathbb{E}V_x)\,. \tag{2}$$

(b) *There is a family* $(\beta_\lambda)_{\lambda\geq 0}$ *of norms on* $\mathbb{R}^d$*, such that for any* $\lambda \geq 0$ *and all sequences* $(x_k)_{k\in\mathbb{N}}$ *on* $\mathbb{Z}^d$ *with* $\|x_k\|_1 \to \infty$ *as* $k \to \infty$*, we have*

$$\lim_{k\to\infty} \frac{b_\lambda(x_k)}{\beta_\lambda(x_k)} = 1. \tag{3}$$

Moreover, $\beta_\lambda(x)$ *is continuous in* $(\lambda, x) \in [0,\infty) \times \mathbb{R}^d$*, concave increasing in* $\lambda \in [0,\infty)$*, and satisfies*

$$\|x\|_1 (\lambda + \varphi(1)) \leq \beta_\lambda(x) \leq \|x\|_1 (\lambda + \log(2d) + \varphi(1))\,.$$

The first part of Theorem A, accounting for the *quenched Lyapunov functions* α_λ, is taken from [8]. We will not repeat the proof, which relies on the subadditive ergodic theorem, but refer the reader to the original paper. The second part of the theorem on the *annealed Lyapunov functions* β_λ is proven in Section 3 with the help of the subadditive limit theorem.

The Lyapunov functions play an important role in the large deviation principles. For $x \in \mathbb{R}^d$, we set

$$I(x) \overset{\text{def}}{=} \sup_{\lambda\geq 0}(\alpha_\lambda(x) - \lambda) \quad \text{and} \quad J(x) \overset{\text{def}}{=} \sup_{\lambda\geq 0}(\beta_\lambda(x) - \lambda)\,.$$

Both the functions I and J are continuous and convex increasing on their effective domains

$$D_I \overset{\text{def}}{=} \{x \in \mathbb{R}^d : I(x) < \infty\} \quad \text{and} \quad D_J \overset{\text{def}}{=} \{x \in \mathbb{R}^d : J(x) < \infty\},$$

of which both equal the closed unit ball of the 1-norm in $\mathbb{Z}^d$ (see p. 272 in [8] and Section 4 of the present article). In particular, I and J are lower semicontinuous functions with compact level-sets, which makes them *good rate functions* (see e.g. [1]).

Theorem B (*Large Deviation Principles*).

(a) *There is a set* Ω' *of full* $\mathbb{P}$*-measure, such that for all* $\omega \in \Omega'$ *and any drift* $h \in \mathbb{R}^d$*, we have*

$$\lim_{n\to\infty} \frac{\log Z^h_{n,\omega}}{n} = \sup_{x\in\mathbb{R}^d} (h \cdot x - I(x))\,, \tag{4}$$

and $S(n)/n$ *satisfies a large deviation principle under* $Q^h_{n,\omega}$ *with rate* n *and good rate function*

$$I_h(x) \overset{\text{def}}{=} I(x) - h \cdot x + \sup_{y\in\mathbb{R}^d} (h \cdot y - I(y))\,, \quad x \in \mathbb{R}^d,$$

as n tends to infinity. Namely, for any $\omega \in \Omega'$ and $h \in \mathbb{R}^d$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q^h_{n,\omega}[S(n) \in nA] \le - \inf_{x\in A} I_h(x),$$

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \log Q^h_{n,\omega}[S(n) \in nO] \ge - \inf_{x\in O} I_h(x)$$

for all closed subsets $A \subset \mathbb{R}^d$ and all open subsets $O \subset \mathbb{R}^d$.

(b) *For any drift $h \in \mathbb{R}^d$, we have*

$$\lim_{n\to\infty} \frac{\log Z^h_n}{n} = \sup_{x\in\mathbb{R}^d} (h \cdot x - J(x)), \tag{5}$$

and $S(n)/n$ satisfies a large deviation principle under Q^h_n with rate n and good rate function

$$J_h(x) \stackrel{\text{def}}{=} J(x) - h \cdot x + \sup_{y\in\mathbb{R}^d} (h \cdot y - J(y)), \quad x \in \mathbb{R}^d,$$

as n tends to infinity. Namely, for any $h \in \mathbb{R}^d$,

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q^h_n[S(n) \in nA] \le - \inf_{x\in A} J_h(x),$$

$$\underline{\lim_{n\to\infty}} \frac{1}{n} \log Q^h_n[S(n) \in nO] \ge - \inf_{x\in O} J_h(x)$$

for all closed subsets $A \subset \mathbb{R}^d$ and all open subsets $O \subset \mathbb{R}^d$.

The crucial case of Theorem B is the one of vanishing drift, which for the quenched setting already is proved in [8]. The extension to arbitrary drifts then follows by general principles (essentially Varadhan's lemma).

To describe the transition between small and large drift, we quantify the size of h in terms of the dual norms of the Lyapunov functions. For $\lambda \ge 0$, the dual norm of α_λ is defined by

$$\alpha^*_\lambda(\ell) \stackrel{\text{def}}{=} \sup_{x\neq 0}\left(\frac{\ell \cdot x}{\alpha_\lambda(x)}\right), \quad \ell \in \mathbb{R}^d,$$

while the dual norm of β_λ is defined by

$$\beta^*_\lambda(\ell) \stackrel{\text{def}}{=} \sup_{x\neq 0}\left(\frac{\ell \cdot x}{\beta_\lambda(x)}\right), \quad \ell \in \mathbb{R}^d.$$

It is plain to see that α^*_λ and β^*_λ indeed are norms again. Further elementary properties are established in Section 5.

As a corollary to Theorem A, we have the following "point to hyperplane" interpretation on the dual norms: For $\ell \neq 0$ and $u \ge 0$, let

$$H_\ell(u) \stackrel{\text{def}}{=} \inf\{n \ge 0 : \ell \cdot S(n) \ge u\}$$

be the time of the random walk's first entrance into the half-space $\{x \in \mathbb{R}^d : \ell \cdot x \ge u\}$.

Corollary C (*Point to Hyperplane Characterization of Dual Norms*).

(a) *There is a set of full $\mathbb{P}$-measure, on which for all $\lambda \in [0, \infty)$ and $\ell \in \mathbb{R}^d \setminus \{0\}$, we have*

$$\lim_{u\to\infty} \frac{1}{u} \log E\left[\exp(-\lambda H_\ell(u) - \Psi(H_\ell(u), \omega))\right] = -\frac{1}{\alpha^*_\lambda(\ell)}.$$

(b) *For all* $\lambda \in [0, \infty)$ *and* $\ell \in \mathbb{R}^d \setminus \{0\}$, *we have*

$$\lim_{u\to\infty} \frac{1}{u} \log E\left[\exp(-\lambda H_\ell(u) - \Phi(H_\ell(u)))\right] = -\frac{1}{\beta^*_\lambda(\ell)}.$$

Corollary C is the discrete counterpart to Sznitman's results for Brownian motion in a Poissonian potential [6, Corollary 2.11 and Corollary 3.6 of Chapter 5].

As the following theorem shows, the phase transition in the long-time behavior of the random walk is appropriately characterized by the size of the drift, measured in terms of the dual norms α^*_0 and β^*_0.

Theorem D (*Phase Transitions*).

(a) *On the set* Ω' *appearing in Theorem B, and for any* $h \in \mathbb{R}^d$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^h_{n,\omega} = \begin{cases} 0 & \text{if } \alpha^*_0(h) \le 1, \\ \lambda^{\mathrm{qu}}_h & \text{if } \alpha^*_0(h) > 1, \end{cases} \tag{6}$$

where $\lambda^{\mathrm{qu}}_h > 0$ *is the unique number with* $\alpha^*_{\lambda^{\mathrm{qu}}_h}(h) = 1$. *Again on* Ω', *we furthermore have the following limiting behavior: When* $\alpha^*_0(h) < 1$, *then*

$$\frac{S(n)}{n} \to 0 \quad \text{in } Q^h_{n,\omega} \text{ probability as } n \to \infty.$$

When $\alpha^*_0(h) > 1$, *then*

$$\operatorname{dist}\left(\frac{S(n)}{n}, M_h\right) \to 0 \quad \text{in } Q^h_{n,\omega} \text{ probability as } n \to \infty,$$

where $M_h \overset{\text{def}}{=} \{x \in \mathbb{R}^d : h \cdot x - I(x) = \lambda^{\mathrm{qu}}_h\}$ *is a compact set not containing the origin.*

(b) *For any* $h \in \mathbb{R}^d$, *we have*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^h_n = \begin{cases} 0 & \text{if } \beta^*_0(h) \le 1, \\ \lambda^{\mathrm{an}}_h & \text{if } \beta^*_0(h) > 1, \end{cases}$$

where $\lambda^{\mathrm{an}}_h > 0$ *is the unique number with* $\beta^*_{\lambda^{\mathrm{an}}_h}(h) = 1$. *We furthermore have the following limiting behavior: When* $\beta^*_0(h) < 1$,

$$\frac{S(n)}{n} \to 0 \quad \text{in } Q^h_n \text{ probability as } n \to \infty.$$

When $\beta^*_0(h) > 1$,

$$\operatorname{dist}\left(\frac{S(n)}{n}, N_h\right) \to 0 \quad \text{in } Q^h_n \text{ probability as } n \to \infty,$$

where $N_h \overset{\text{def}}{=} \{x \in \mathbb{R}^d : h \cdot x - J(x) = \lambda^{\mathrm{an}}_h\}$ *is a compact set containing the origin.*

Remark that for large drifts, since M_h and N_h are bounded away from the origin, Theorem D implies that the random walk $S(n)$ typically moves away from the origin, with distance of order $O(n)$ as $n \to \infty$. For small drifts, on the other hand, the dislocation rate $\|S(n)/n\|$ typically falls below any positive value in the limit $n \to \infty$. Theorem D thus displays two phase transitions, one for the quenched and one for the annealed setting, between *ballistic* behavior of the walk for large h and *sub-ballistic* behavior for small h. The unit spheres of α^*_0 and β^*_0 correspond to the sets of critical drifts.

The normalization for the asymptotics in Theorem D is appropriate in the ballistic regime. In the continuous model, more exact asymptotics for the sub-ballistic phase are established in [4, 5]. In the discrete setting, by analogy to the continuous model, we thus believe that convenient normalizations for $\log Z^h_{n,\omega}$, respectively $\log Z^h_n$, should be given by $n(\log n)^{-2/d}$, respectively $n^{d/d+2}$.

To conclude this section, we stress that Theorems A and B, Corollary C and Theorem D essentially are discrete counterparts to Sznitman's results for the Brownian motion in Poissonian potentials. We however would like to point out the introduction of λ_h^{qu} and λ_h^{an} in Theorem D, which we believe to be new: In order to obtain the ballistic behavior of the random walk, for either the continuous or the discrete setting, it actually suffices to show that the so-called *Lyapunov exponents*

$$\lim_{n\to\infty} \frac{1}{n} \log Z^h_{n,\omega} \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n} \log Z^h_n$$

are strictly positive (in fact, this is Sznitman's approach). By means of λ_h^{qu} and λ_h^{an}, on the other hand, we are able to express these limits in an implicit way, providing a useful relation to their counterparts in the simpler "point to hyperplane" setting of Corollary C. In fact, in the second, forthcoming paper [2], this relation is used in the context of a renewal formalism to transfer an exponential gap result from the "point to hyperplane" to the "fixed number of steps" setting, implying analyticity of the annealed Lyapunov exponent, and providing coincidence of the quenched and the annealed exponent for weak potentials in dimensions $d \geq 4$.

3. Lyapunov functions and shape theorem

The quenched part of Theorem A has been proved by M.P.W. Zerner in [8]: the existence of the norms α_λ and the bounds in (2) are part of Proposition 4, the asymptotic equivalence in (1) corresponds to his Theorem 8, and the further properties of α_λ are established on page 272. Observe that Zerner left out the condition ess inf $V_x = 0$ instead of introducing the parameter λ.

In the rest of this section, we follow Zerner's line to prove the remaining annealed part of Theorem A. Recall the two-point function

$$b_\lambda(x) = -\log E\left[\exp(-\lambda H(x) - \Phi(H(x)))\,;\, H(x) < \infty\right] \tag{7}$$

for $\lambda \geq 0$ and $x \in \mathbb{Z}^d$. The stopping time $H(x)$ denotes the time of the random walk's first visit to the lattice site x, and the path potential Φ is given by

$$\Phi(n) = \sum_{z\in\mathbb{Z}^d} \varphi(l_z(n)) \tag{8}$$

for $n \in \mathbb{N}$. Here, $\varphi : [0, \infty) \to [0, \infty)$ is a non-constant, concave increasing function, satisfying $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t)/t = 0$. By dominated convergence and Hölder inequality, it is plain that $b_\lambda(x)$, to any fixed $x \in \mathbb{Z}^d$, is continuous and concave increasing in the variable $\lambda \in [0, \infty)$. Moreover, we have $H(x) \geq \|x\|_1$ and thus $\Phi(H(x)) \geq \varphi(1)\|x\|_1$ for all $x \in \mathbb{Z}^d$ by the concavity of φ. For any $\lambda \geq 0$, this yields the lower bound

$$b_\lambda(x) \geq \|x\|_1 \left(\lambda + \varphi(1)\right), \tag{9}$$

while the upper bound

$$b_\lambda(x) \leq \|x\|_1 \left(\log(2d) + \lambda + \varphi(1)\right) \tag{10}$$

comes from restricting the expectation in (7) to a single $\|x\|_1$-step path from the origin to $x \in \mathbb{Z}^d$.

To a fixed λ, we want to establish the triangle inequality for b_λ as a function on $\mathbb{Z}^d$. To this end, let

$$H(x,z) \stackrel{\text{def}}{=} \inf\{m \geq H(x) : S(m) = z\}$$

be the time of the random walk's first visit to the site $z \in \mathbb{Z}^d$ after its first visit to the site $x \in \mathbb{Z}^d$, and set

$$\Phi(n,m) \stackrel{\text{def}}{=} \sum_{z \in \mathbb{Z}^d} \varphi\left(l_z(n) - l_z(m)\right)$$

for $n, m \in \mathbb{N}_0$ with $n \geq m$. Again by the concavity of φ, we have

$$\Phi(n) \leq \Phi(m) + \Phi(m,n) \tag{11}$$

for all $m \geq n$. The strong Markov property, applied to the stopping time $H(x)$, then implies

$$\begin{aligned} b_\lambda(x+y) &\leq -\log E\left[1_{\{H(x) \leq H(x,x+y) < \infty\}} \exp(-\lambda H(x) - \Phi(H(x))) \right.\\ &\qquad \left. \times \exp(\lambda H(x,x+y) - \Phi(H(x), H(x,x+y)))\right] \\ &= b_\lambda(x) + b_\lambda(y). \end{aligned} \tag{12}$$

Given the validity of the triangle inequality (12), we can now apply the subadditive limit theorem (see e.g. [3] Appendix II), which guarantees the existence of a function $\beta_\lambda : \mathbb{Z}^d \to [0, \infty)$, such that

$$\lim_{n \to \infty} \frac{1}{n} b_\lambda(nx) = \inf_{n \in \mathbb{N}} \frac{1}{n} b_\lambda(nx) = \beta_\lambda(x) \tag{13}$$

for every $x \in \mathbb{Z}^d$. It is easy to conclude that β_λ inherits from b_λ the same bounds as in (9) and (10), that is

$$\lambda + \varphi(1) \leq \frac{\beta_\lambda(x)}{\|x\|_1} \leq \log(2d) + \lambda + \varphi(1) \tag{14}$$

for all $x \in \mathbb{Z}^d \setminus \{0\}$, and that

$$\begin{aligned} &\beta_\lambda(nx) = n\beta_\lambda(x), \\ &\beta_\lambda(x+y) \leq \beta_\lambda(x) + \beta_\lambda(y) \end{aligned} \tag{15}$$

are satisfied for any $n \in \mathbb{N}$ and $x, y \in \mathbb{Z}$. Moreover, to fixed $x \in \mathbb{Z}^d$, $\beta_\lambda(x)$ is continuous and concave increasing in $\lambda \in [0, \infty)$: As a limit of concave functions, $\beta_\lambda(x)$ is concave again and thus lower semicontinuous (possibly being discontinuous in $\lambda = 0$). The upper semicontinuity, by the representation of $\beta_\lambda(x)$ as an infimum in (13), is derived from the continuity of $b_\lambda(nx)$ in λ for any $n \in \mathbb{N}$.

By setting $\beta_\lambda(qx) = q\beta_\lambda(x)$ for $q \in \mathbb{Q}$, we extend β_λ well-defined at first to a function on $\mathbb{Q}^d$, and then by continuity to a function on $\mathbb{R}^d$. Thereby, β_λ maintains its properties as a function of λ, and still satisfies (14) and (15). In particular, β_λ is a norm on $\mathbb{R}^d$. Moreover, since

$$\left|\beta_{\lambda_k}(x_k) - \beta_\lambda(x)\right| \leq \left|\beta_{\max_{j \in \mathbb{N}} \lambda_j}(x_k - x)\right| + \left|\beta_{\lambda_k}(x) - \beta_\lambda(x)\right|$$

for all sequences $(\lambda_k)_{k\in\mathbb{N}}$ and $(x_k)_{k\in\mathbb{N}}$ with $\lambda_k \to \lambda$ and $x_k \to x$, we obtain the joint continuity of $\beta_\lambda(x)$ in $(\lambda, x) \in [0,\infty) \times \mathbb{R}^d$ from the continuity in the single arguments.

It remains to prove the limiting behavior of b_λ/β_λ in (3). It suffices to show

$$\lim_{k\to\infty} \left| \frac{b_\lambda(x_k) - \beta_\lambda(x_k)}{\|x_k\|_1} \right| = 0 \tag{16}$$

for any sequence $(x_k)_{k\in\mathbb{N}}$ on $\mathbb{Z}^d$ with $\|x_k\|_1 \to \infty$. We yet can restrict to the case where $x_k/\|x_k\|_1 \to e$ for some point $e \in S^{d-1}$; if (16) was not true for an arbitrary sequence, it would not be true for a subsequence with this convergence property either.

To this end, for any $\varepsilon > 0$, choose $\tilde{e} \in \mathbb{Q}^d$ and $m \in \mathbb{N}$ such that $m\tilde{e} \in \mathbb{Z}^d$ and $\|e - \tilde{e}\|_1 < \varepsilon$, as well as $|\beta_\lambda(e) - \beta_\lambda(\tilde{e})| < \varepsilon$. We approximate $(x_k)_{k\in\mathbb{N}}$ by the sequence $(n_k x)_{k\in\mathbb{N}}$ on $\mathbb{Z}^d$, where

$$x = m\tilde{e} \in \mathbb{Z}^d \quad \text{and} \quad n_k = \left\lfloor \frac{\|x_k\|_1}{m} \right\rfloor.$$

Thereby $\lfloor\cdot\rfloor$ denotes the largest integer less than or equal to a real number. Notice first that $\lim_{k\to\infty} \|x_k\|_1/n_k = m$. We thus have

$$\begin{aligned} \|x_k - n_k x\|_1 &\le \left\| x_k - \frac{n_k m}{\|x_k\|_1} x_k \right\|_1 + \left\| \frac{n_k m}{\|x_k\|_1} x_k - n_k x \right\|_1 \\ &= \left(1 - \frac{n_k m}{\|x_k\|_1}\right) \|x_k\|_1 + n_k m \left\| \frac{x_k}{\|x_k\|_1} - \tilde{e} \right\|_1 \\ &< \varepsilon \|x_k\|_1 \end{aligned} \tag{17}$$

for k large enough. By the (inverted) triangle inequality (12) for b_λ, we get

$$\left| \frac{b_\lambda(x_k) - \beta_\lambda(x_k)}{\|x_k\|_1} \right| \le \frac{b_\lambda(x_k - n_k x)}{\|x_k\|_1} + \left| \frac{b_\lambda(n_k x)}{\|x_k\|_1} - \beta_\lambda(\tilde{e}) \right| + \left| \beta_\lambda(\tilde{e}) - \beta_\lambda\left(\frac{x_k}{\|x_k\|_1}\right) \right|.$$

The first summand on the right-hand side is bounded from above by εc_λ with $c_\lambda = \log(2d) + \lambda + \varphi(1)$ due to (10) and (17). The second summand tends to zero for k going to infinity since $\|x_k\|_1/n_k \to m$ and $b_\lambda(n_k x)/n_k \to \beta_\lambda(x)$. The last summand finally is smaller than ε for k large enough by the assumptions $\lim_{k\to\infty} x_k/\|x_k\|_1 = e$ and $|\beta_\lambda(\tilde{e}) - \beta_\lambda(e)| < \varepsilon$. Hence, letting ε tend to zero implies (16) and completes the proof of the shape theorem in the annealed setting.

4. Large deviation principles

The aim of this section is to prove Theorem B. The limit results (4) and (5) for arbitrary drifts as well as the LDP's for $Q_n^{h\neq 0}$ and $Q_{n,\omega}^{h\neq 0}$ thereby follow from the LDP's for $Q_n^{h=0}$ and $Q_{n,\omega}^{h=0}$ as an application of Varadhan's lemma (see e.g. [1] Theorem 2.1.10 and Exercise 2.1.24). To this purpose, we only need to establish the "exponential tightness estimates"

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \log E\left[\exp(h\cdot S(n) - \Psi(n,\omega))\,;\, h\cdot S(n) \ge nL\right] = -\infty,$$

for the quenched setting and

$$\lim_{L\to\infty} \overline{\lim_{n\to\infty}} \frac{1}{n} \log E\left[\exp(h\cdot S(n) - \Phi(n))\,;\, h\cdot S(n) \ge nL\right] = -\infty$$

for the annealed setting. But, since both the expectations in the above limits are bounded by

$$E\left[\exp(h \cdot S(n)); h \cdot S(n) \geq nL\right] \leq \exp(-nL) E\left[\exp(2h \cdot S(n))\right]$$
$$= \exp(-nL) E\left[\exp(2h \cdot S(1))\right]^n,$$

the exponential estimates follow immediately.

For vanishing drift, the limit in (4) and the large deviation property in the quenched setting have already been proved [8, Proposition 17 and Theorem 19].

We follow Zerner's line to prove the remaining annealed part of Theorem B for the case $h = 0$. That is, we investigate the large deviations of the symmetric random walk under the annealed path measures Q_n with density

$$\frac{\mathrm{d}Q_n}{\mathrm{d}P} \stackrel{\text{def}}{=} \frac{\exp(-\Phi(n))}{Z_n}$$

when $n \in \mathbb{N}$ tends to infinity, where the normalization constant Z_n is given by

$$Z_n \stackrel{\text{def}}{=} E\left[\exp(-\Phi(n))\right] = E\left[\exp\left(-\sum_{x \in \mathbb{Z}^d} \varphi(l_x(n))\right)\right].$$

Thereby, we have $Q_n = Q_n^{h=0}$ and $Z_n = Z_n^{h=0}$ according to the notations from Section 1.

We first take care of the normalization constant Z_n. Claim (5) in Theorem B reduces to

$$\lim_{n \to \infty} \frac{-\log Z_n}{n} = 0. \tag{18}$$

In fact, it turns out that the above limit equals $\lim_{n \to \infty} \varphi(n)/n$, which is assumed to be zero. To see this, observe that

$$\lim_{n \to \infty} \frac{\varphi(n)}{n} = \inf_{n \in \mathbb{N}} \frac{\varphi(n)}{n},$$

once again by the concavity of φ. By the definition of Φ, we therefore have

$$\lim_{n \to \infty} \frac{-\log Z_n}{n} \geq \lim_{n \to \infty} \frac{\varphi(n)}{n}.$$

It remains to prove the upper estimate. For any integer R and all $n \in \mathbb{N}$, we obviously have

$$Z_n \geq E\left[\exp(-\Phi(n)); \|S(m)\|_1 \leq R \text{ for } m \leq n\right].$$

In order to find a lower bound for the right-hand side of this inequality, observe that

$$\sum_{x \in \mathbb{Z}^d} \varphi(l_x(n)) 1_{\{\|S(m)\|_1 \leq R \text{ for } m \leq n\}} \leq (2R+1)^d \varphi(n)$$

is valid for all $n \in \mathbb{N}$, and that a $2R$-step path with start and end at the origin remains within the cube $\{x \in \mathbb{Z}^d : \|x\|_1 \leq R\}$. By the Markov property, we thus obtain

$$
\begin{aligned}
&-\varlimsup_{n\to\infty}\frac{1}{n}\log E\left[\exp(-\Phi(n));\ \|S(m)\|_1 \le R \text{ for } m\le n\right]\\
&\quad\le \lim_{n\to\infty}\frac{\varphi(n)}{n} - \varlimsup_{n\to\infty}\frac{1}{n}\log P\left[\|S(m)\|_1\le R \text{ for } m\le n\right]\\
&\quad\le \lim_{n\to\infty}\frac{\varphi(n)}{n} - \frac{1}{2R}\log P[S(2R)=0],
\end{aligned}
$$

of which the last summand vanishes when R tends to infinity by the local central limit theorem (see e.g. [7]). This completes the proof of (18).

We step forward to the large deviation principle. For the case $h = 0$, the rate function will be

$$J(x) = \sup_{\lambda\ge 0}(\beta_\lambda(x) - \lambda), \quad x \in \mathbb{R}^d.$$

As a supremum of continuous functions, J is lower semicontinuous. Furthermore, J inherits the convexity from the norms β_λ, and hence is upper semicontinuous on its effective domain

$$D_J = \{x \in \mathbb{R}^d : J(x) < \infty\}.$$

Moreover, the bounds for β_λ in (14) yield that D_J equals the closed unit ball of the 1-norm.

The rest of this section is devoted to the proof of the large deviations estimates: For any closed subset $A \subset \mathbb{R}^d$ and open subset $O \subset \mathbb{R}^d$,

$$\varlimsup_{n\to\infty}\frac{1}{n}\log Q_n[S(n)\in nA] \le -\inf_{x\in A} J(x), \tag{19}$$

$$\varliminf_{n\to\infty}\frac{1}{n}\log Q_n[S(n)\in nO] \ge -\inf_{x\in O} J(x). \tag{20}$$

We start with the upper estimate. Since $J(x) = \infty$ if $\|x\| > 1$ and $\|S(n)\|_1 \le n$ for $n \in \mathbb{N}$, we can restrict to the case where $A \subset D_J$ is compact. For $n \in \mathbb{N}$, we set $H(nA) = \inf\{H(x) : x \in nA\}$. Since $\{S(n) \in nA\} \subset \{H(nA) \le n\}$, we have

$$\log E\left[\exp(-\Phi(n));\ S(n)\in nA\right] \le -(b_\lambda(nA) - \lambda n)$$

for all $\lambda \ge 0$, where $b_\lambda(nA)$ is defined as in (7), but with $H(y)$ replaced by $H(nA)$. From the representation in (13) of $\beta_\lambda(x)$ as an infimum, and since A is bounded, we obtain

$$
\begin{aligned}
\frac{b_\lambda(nA)}{n} &\ge \frac{-\log|nA\cap\mathbb{Z}^d|}{n} + \sup_{x\in A\cap\frac{1}{n}\mathbb{Z}^d}\frac{b_\lambda(nx)}{n}\\
&\ge \frac{-\log(nK)}{n} + \sup_{x\in A}\beta_\lambda(a)
\end{aligned}
$$

for all $n \in \mathbb{N}$ and some constant $K = K(A, d)$. By (18), we then have

$$\varlimsup_{n\to\infty}\frac{1}{n}\log Q_n[S(n)\in nA] = -\sup_{\lambda\ge 0}\inf_{x\in A}\beta_\lambda(x) - \lambda. \tag{21}$$

However, in order to complete the proof of (19), we need to exchange infimum and supremum in (21). For any $\varepsilon > 0$, thanks to the compactness of A, there are $m \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_m > 0$, such that the compact sets

$$A_i = \left\{y \in A : \beta_{\lambda_i}(y) - \lambda_i \ge \inf_{x\in A} J(x) - \varepsilon\right\}, \quad i = 1, \dots, m,$$

cover A. From (21) applied to the sets A_i, we therefore obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n[S(n) \in nA] &\leq \max_{i=1,\ldots,m} \overline{\lim_{n\to\infty}} \frac{1}{n} \log Q_n[S(n) \in nA_i] \\ &\leq - \min_{i=1,\ldots,m} \inf_{x\in A_i} \beta_{\lambda_i}(x) - \lambda_i \\ &\leq - \inf_{y\in A} J(x) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, this proves (19).

We step forward to the proof of (20). That is, for an open set $O \subset \mathbb{R}^d$, we need to show

$$\underline{\lim}_{n\to\infty} \frac{1}{n} \log Q_n[S_n \in nO] \geq -J(x) \tag{22}$$

for all $x \in O \cap D_J \setminus \{0\}$, where the origin can be excluded since J is continuous on D_J. To do this, we first determine some number λ_0, at which $\beta_\lambda(x) - \lambda$ attains its maximum as a function of λ: Since $\beta_\lambda(x)$ is concave in λ, the right derivative

$$\dot{\beta}_\lambda^+(x) = \lim_{\varepsilon\downarrow 0} \frac{\beta_{\lambda+\varepsilon}(x) - \beta_\lambda(x)}{\varepsilon}$$

of $\beta_\lambda(x)$ is a well-defined and nondecreasing (but not necessarily continuous) function of $\lambda \in [0,\infty)$. Now, if $\dot{\beta}_\lambda^+(x) < 1$ for all $\lambda > 0$, the maximum is located at $\lambda_0 = 0$. Otherwise, we choose

$$\lambda_0 = \inf\{\lambda > 0 : \dot{\beta}_\lambda^+(x) < 1\},$$

which is finite since $x \in D_J \cap O$, and which is a transition point from nondecreasing to decreasing behavior for the map $\lambda \mapsto \beta_\lambda(x) - \lambda$. Hence, in both cases, we have

$$I(x) = \beta_{\lambda_0}(x) - \lambda_0. \tag{23}$$

We now take care of the fact that $\dot{\beta}_\lambda^+(x)$ might be discontinuous in λ_0. In that case, there is a non trivial "interval" in the half-line $\{sx : s \in [0,\infty)\}$, on which (23) remains true with the fixed constant λ_0. We express x by a linear combination of the end points of this interval. Let $\dot{\beta}_\lambda^-(x)$ denote the left derivative of $\beta_\lambda(x)$ with respect to $\lambda > 0$. Set

$$y^+ = \frac{x}{\dot{\beta}_{\lambda_0}^+(x)} \quad \text{and} \quad y^- = \begin{cases} 0 & \text{if } \lambda_0 = 0, \\ \dfrac{x}{\dot{\beta}_{\lambda_0}^-(x)} & \text{if } \lambda_0 > 0. \end{cases}$$

We then have

$$(1-t)y^- + ty^+ = x$$

with $t = \frac{|y-y^-|}{|y^+-y^-|} < 1$ if $\dot{\beta}_{\lambda_0}^+(x) < \dot{\beta}_{\lambda_0}^-(x)$, and with $t = 1$ in the continuous case. For a reason that will become clear later, we furthermore approximate y^- by slightly "smaller" sites $y_\rho^- \stackrel{\text{def}}{=} \rho y^-$. Since O is open, we can choose $\rho < 1$ large enough to fulfill

$$(1-t)y_\rho^- + ty^+ \in O. \tag{24}$$

Let finally $(y^-_{\rho,n})$ and (y^+_n) be two sequences in $\mathbb{Z}^d$, such that

$$\lim_{n\to\infty} \frac{y^-_{\rho,n}}{n} = (1-t)y^-_\rho \quad \text{and} \quad \lim_{n\to\infty} \frac{y^+_n}{n} = ty^+, \tag{25}$$

and set $x_n = y^-_{\rho,n} + y^+_n$ for $n \in \mathbb{N}$. Thereby, if $\lambda_0 = 0$ or $t = 1$, we may simply set $y^-_{\rho,n} = 0$.

We want to renew the Markov chain at the sites $y^-_{\rho,n}$ and x_n. To this end, let R be an arbitrary integer. Since O is open, we obtain from (24) and (25) that there exists some $n_0 \in \mathbb{N}$ such that $x_n + y \in nO$ is valid for all $\|y\|_1 \le R$ and all $n \ge n_0$. As a consequence,

$$\begin{aligned}\left\{H(y^-_{\rho,n}) \le (1-t)n\right\} \cap \left\{H(y^-_{\rho,n}, x_n) \le n\right\}& \\ \cap \left\{\|S(m) - x_n\|_1 \le R \text{ for } H(y^-_{\rho,n}, x_n) < m \le H(y^-_{\rho,n}, x_n) + n\right\}&\end{aligned}$$

is contained in $\{S(n) \in nO\}$ for n large enough. By the monotonicity of Φ and a double application of (11), we furthermore have

$$\Phi(n) \le \Phi\left(H(y^-_{\rho,n})\right) + \Phi\left(H(y^-_{\rho,n}), H(y^-_{\rho,n}, x_n)\right) + \Phi\left(H(y^-_{\rho,n}, x_n), H(y^-_{\rho,n}, x_n) + n\right).$$

From (18) and the strong Markov property, it thus follows that the left-hand side of (20) is not smaller than

$$\varliminf_{n\to\infty} \frac{1}{n} \log E\left[\exp\big(-\Phi(H(y^-_{\rho,n}))\big)\,;\, H(y^-_{\rho,n}) \le (1-t)n\right] \tag{26}$$

$$+ \varliminf_{n\to\infty} \frac{1}{n} \log E\left[\exp\big(-\Phi(H(y^+_n))\big)\,;\, H(y^+_n) \le tn\right] \tag{27}$$

$$+ \varliminf_{n\to\infty} \frac{1}{n} \log E\left[\exp(-\Phi(n))\,;\, \|S(m)\|_1 \le R \text{ for } m \le n\right],$$

of which the last summand vanishes when R tends to infinity, as we have seen in the proof of (18).

In order to bound the first and the second summand, we need the following result: For $\lambda \ge 0$ and $y \in \mathbb{Z}^d$, let the distribution P^y_λ be given by means of the density

$$\frac{dP^y_\lambda}{dP} = \frac{1}{Z^y_\lambda} \exp(-\lambda H(y) - \Phi(H(y)))\, 1_{\{H(y)<\infty\}},$$

where $Z^y_\lambda = b_\lambda(y)$ is the corresponding normalization constant.

Lemma 1. *Let (y_n) a be a sequence in $\mathbb{Z}^d$ with $\lim_{n\to\infty} y_n/n = y \in \mathbb{R}^d \setminus \{0\}$. For every $\lambda > 0$ and $\gamma < \dot\beta^+_\lambda(y) \le \dot\beta^-_\lambda(y) < \delta$, we have*

$$\lim_{n\to\infty} P^{y_n}_\lambda \left[H(y_n)/n \in [\gamma, \delta]\right] = 1. \tag{28}$$

Proof. For any $0 < \varepsilon < \lambda$, we have

$$\begin{aligned} &E\left[\exp(-\lambda H(y_n) - \Phi(H(y_n)))\,;\, H(y_n) \notin [\gamma n, \delta n]\right] \\ &\quad \le \exp(\varepsilon\gamma n) E\left[\exp(-(\lambda+\varepsilon)H(y_n) - \Phi(H(y_n)))\,;\, H(y_n) < \infty\right] \\ &\qquad + \exp(-\varepsilon\delta n) E\left[\exp(-(\lambda-\varepsilon)H(y_n) - \Phi(H(y_n)))\,;\, H(y_n) < \infty\right]. \end{aligned}$$

From the shape theorem therefore follows

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \log P_\lambda^{y_n} [H(y_n)/n \notin [\gamma, \delta]]$$
$$\leq -\varepsilon \min \left\{ \frac{\beta_{\lambda+\varepsilon}(y) - \beta_\lambda(y)}{\varepsilon} - \gamma, \delta - \frac{\beta_\varepsilon(y) - \beta_{\lambda-\varepsilon}(y)}{\varepsilon} \right\}.$$

By the assumptions on γ and δ, the right-hand side of this last expression is strictly negative for $\varepsilon > 0$ small enough, which then implies (28). $\square$

We are now able to complete the proof of (20). Suppose $\lambda > \lambda_0$ and $\gamma < \dot{\beta}_\lambda^+(y^+)$. The shape theorem provides that (27) is not smaller than

$$\gamma\lambda - \beta_\lambda(ty^+) + \lim_{n\to\infty} \frac{1}{n} \log P_\lambda^{y_n^+} \left[H(y_n^+)/n \in [\gamma, t]\right],$$

for which Lemma 1 applies because of $\dot{\beta}_\lambda^-(y^+) < \dot{\beta}_{\lambda_0}^+(y^+) = t$, the strict inequality coming from the choice of λ_0. Since $\dot{\beta}_\lambda^+(y^+)$ is upper semicontinuous in λ_0, we thus obtain that (27) is not smaller than

$$\sup_{\lambda>\lambda_0} \sup_{\gamma<\dot{\beta}_\lambda^+(y^+)} \gamma\lambda - \beta_\lambda(ty^+) \geq t\lambda_0 - \beta_{\lambda_0}(ty^+).$$

If $\lambda_0 = 0$ or $t = 1$ is the case, by setting $y_{\rho,n}^- = 0$ for $n \in \mathbb{N}$, this already proves (22). Suppose now $\lambda_0 > 0$ and $t < 1$, which implies $y_\rho^- \neq 0$. Since $\dot{\beta}_\lambda^-(y^-)$ is lower semicontinuous in λ_0, we have $\rho(1-t) = \dot{\beta}_{\lambda_0}^-(y_\rho^-) \leq \dot{\beta}_\lambda^+(y_\rho^-)$ and $\dot{\beta}_\lambda^-(y_\rho^-) < (1-t)$ whenever $\lambda < \lambda_0$ is large enough. The shape theorem and Lemma 1 with $\gamma = \rho^2(1-t)$ and $\delta = (1-t)$ then imply that (26) is not smaller than

$$\sup_{\lambda<\lambda_0} \rho^2(1-t)\lambda - \beta_\lambda((1-t)y_\rho^-) \geq \rho^2(1-t)\lambda_0 - \beta_{\lambda_0}(\rho(1-t)y^-).$$

Since $\rho < 1$ was arbitrary, we obtain (22). This completes the proof of (20).

5. Dual norms and phase transitions

The aim of this section is to prove Corollary C and Theorem D. Recall the definition of the dual norms

$$\alpha_\lambda^*(\ell) = \sup_{x\neq 0}\left(\frac{\ell \cdot x}{\alpha_\lambda(x)}\right) \quad \text{and} \quad \beta_\lambda^*(\ell) = \sup_{x\neq 0}\left(\frac{\ell \cdot x}{\beta_\lambda(x)}\right)$$

for $\lambda \geq 0$ and $\ell \in \mathbb{R}^d$. We first prove some elementary properties of α_λ^* and β_λ^*, similar to the ones of the Lyapunov functions α_λ and β_λ in Theorem A.

Lemma 2. (a) $1/\alpha_\lambda^*(\ell)$ *is continuous in* $(\lambda, \ell) \in [0, \infty) \times \mathbb{R}^d$ *and concave increasing in* $\lambda \in [0, \infty)$, *satisfying*

$$\frac{\|\ell\|_1}{\lambda + \log(2d) + \mathbb{E}V_x} \leq \alpha_\lambda^*(\ell) \leq \frac{\|\ell\|_1}{\lambda - \log \mathbb{E}\exp(-V_x)}.$$

(b) $1/\beta_\lambda^*(\ell)$ *is continuous in* $(\lambda, \ell) \in [0, \infty) \times \mathbb{R}^d$ *and concave increasing in* $\lambda \in [0, \infty)$, *satisfying*

$$\frac{\|\ell\|_1}{\lambda + \log(2d) + \varphi(1)} \leq \beta_\lambda^*(\ell) \leq \frac{\|\ell\|_1}{\lambda + \varphi(1)}.$$

Proof. Since the proof works the same way for either the quenched or the annealed case, we can restrict to the quenched setting. By the definition of $\alpha^*_\lambda(\ell)$, to fixed $\ell \in \mathbb{R}^d$, the concavity of $\alpha^*_\lambda(\ell)$ in λ is derived from the concavity of $\alpha_\lambda(x)$ in λ to every fixed $x \in \mathbb{R}^d$. The concavity then implies lower semicontinuity in λ, while the upper semicontinuity, again by the definition of $\alpha^*_\lambda(\ell)$, is derived from the continuity of $\alpha_\lambda(x)$ in λ. This proves continuity in the λ variable; continuity in the x variable is obvious. Let now $(\lambda_k)_{k\in\mathbb{N}}$ and $(\ell_k)_{k\in\mathbb{N}}$ be two sequences with $\lambda_k \to \lambda$ and $\ell_k \to \ell$. We then have

$$\left|\alpha^*_{\lambda_k}(\ell_k) - \alpha^*_\lambda(\ell)\right| \le \left|\alpha^*_{\max_{j\in\mathbb{N}} \lambda_j}(\ell_k - \ell)\right| + \left|\alpha^*_{\lambda_k}(\ell) - \alpha^*_\lambda(\ell)\right|$$

and thus $\lim_{k\to\infty} \alpha^*_{\lambda_k}(\ell_k) = \alpha^*_\lambda(\ell)$. This proves the joint continuity. The bounds for $\alpha^*_\lambda(\ell)$ finally follow from the bounds for $\alpha_\lambda(x)$ in (2) by standard calculations. □

The "point to hyperplane" interpretation on the dual norms in Corollary C is derived from the shape theorem (Theorem A). The proof is a modification of Sznitman's proof for the continuous setting [6]. Since it works in a similar way for either the quenched or annealed case, we restrict to the more complex quenched model. Here, we have to find a set of full $\mathbb{P}$-measure, on which

$$\lim_{u\to\infty} \frac{1}{u} \log E\left[\exp(-\lambda H_\ell(u) - \Psi(H_\ell(u), \omega))\right] = -\frac{1}{\alpha^*_\lambda(\ell)} \tag{29}$$

for all $\lambda \in [0, \infty)$ and $\ell \in \mathbb{R}^d \setminus \{0\}$, where $H_\ell(u)$ is the time of first entrance into the half-space $\{x \in \mathbb{R}^d : \ell \cdot x \ge u\}$. To this end, let $(u_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of numbers with $\lim_{n\to\infty} u_n = \infty$. For any fixed $\lambda \in [0, \infty)$, by the scalar linearity of α_λ on $\mathbb{R}^d$, we have

$$\alpha^*_\lambda(\ell) = \sup_{x\in\mathbb{R}^d : \ell\cdot x = 1} \frac{1}{\alpha_\lambda(x)}.$$

Consequently, since α_λ is continuous and $\lim_{\|x\|_1\to\infty} \alpha_\lambda(x) = \infty$, there exists $x^* \in \mathbb{R}^d$ such that $\ell \cdot x^* = 1$ and

$$\alpha^*_\lambda(\ell) = \frac{1}{\alpha_\lambda(x^*)}.$$

In order to find a lower bound for the left-hand side of (29), choose a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}^d$, such that $x_n \cdot \ell \ge u_n$ and $\lim_{n\to\infty} x_n/u_n = x^*$. We then have $H_\ell(u_n) \le H(x_n)$, and thus

$$\log E\left[\exp(-\lambda H_\ell(u) - \Phi(H_\ell(u), \omega))\right] \ge -a_\lambda(x_n, \omega)$$

for all $\omega \in \Omega$ and $n \in \mathbb{N}$. On the set Ω_λ of full $\mathbb{P}$-measure appearing in Theorem A, we consequently have

$$\lim_{n\to\infty} \frac{a_\lambda(x_n, \omega)}{u_n} = \alpha_\lambda(x^*).$$

Since the set Ω_λ does not depend on the sequence $(x_n)_{n\in\mathbb{N}}$, this proves the lower bound part of (29) on Ω_λ for a fixed λ and all $\ell \ne 0$.

For the upper estimate of the left-hand side of (29), choose a number R large enough, such that

$$\inf\{\alpha_\lambda(x) : \|x\|_1 \ge R\} \ge \alpha_\lambda(x^*), \tag{30}$$

which is possible since α_λ is a norm. For $\omega \in \Omega$ and any $K \subset \mathbb{Z}^d$, we set

$$a_\lambda(K,\omega) = -\log E\left[\exp(-\lambda H(K) - \Psi(H(K),\omega))\,;\, H(K) < \infty\right]$$

with $H(K) = \inf\{H(y) : y \in K\}$. For $n \in \mathbb{N}$, we furthermore set

$$D_n = \{x \in \mathbb{Z}^d : \|x\|_1 \geq Ru_n\} \cap \{x \in \mathbb{Z}^d : \ell \cdot x \geq u_n\},$$

whose interior boundary is

$$L_n = \{x \in D_n : \|x - y\|_1 = 1 \text{ for some } y \neq D_n\}.$$

Since we have $H(D_n) \leq H_\ell(u_n)$, the logarithmic expectation in (29) is bounded from above by

$$-a_\lambda(D_n,\omega) = -a_\lambda(L_n,\omega) \leq \log|L_n| - \min_{x\in L_n} a_\lambda(x,\omega). \tag{31}$$

Now, since $|L_n| \leq (2Ru_n+1)^d$, it only remains to take care of the minimum in (31), which we assume to be attained at a site $x_n \in L_n$. Again by the shape theorem, we have

$$\varliminf_{n\to\infty} \frac{a_\lambda(x_n,\omega)}{u_n} = \varliminf_{n\to\infty} \frac{\alpha(x_n)}{u_n} \geq \varliminf_{n\to\infty} \inf_{x\in D_n} \frac{\alpha_\lambda(x)}{u_n} \geq \alpha_\lambda(x^*)$$

on the same Ω_λ of full $\mathbb{P}$-measure as before, where the last estimate follows from (30). This completes the proof of (29) on Ω_λ for a fixed λ and all $\ell \neq 0$.

It remains to extend the result to all λ on a common set of full $\mathbb{P}$-measure. But, since the left-hand side in (29) is nondecreasing in λ, as well as the right-hand side is continuous in λ, such a set is given by $\bigcap_{\lambda'\in[0,\infty)\cap\mathbb{Q}} \Omega_{\lambda'}$.

We step forward to the proof of Theorem D. Again, we restrict to the quenched setting. The annealed part of the theorem then follows by a simple change of notations.

We first want to establish (6). By Theorem B, it suffices to show

$$\sup_{x\in\mathbb{R}^d} (h\cdot x - I(x)) = \begin{cases} 0 & \text{if } \alpha_0^*(h) \leq 1, \\ \lambda_h^{\mathrm{qu}} & \text{if } \alpha_0^*(h) > 1, \end{cases} \tag{32}$$

where $\lambda_h^{\mathrm{qu}} > 0$ is the unique number with $\alpha^*_{\lambda_h^{\mathrm{qu}}}(h) = 1$. Existence and uniqueness of λ_h^{qu}, as well as the property $\lambda_h^{\mathrm{qu}} > 0$, thereby follow from Lemma 2.

In the case $\alpha_0^*(h) \leq 1$, the lower estimate for the supremum is obvious. Assume now $\alpha_0^*(h) > 1$. From Theorem A, we know that $\alpha_\lambda(x)$, to fixed $x \in \mathbb{R}^d \setminus \{0\}$, is concave and strictly increasing in $\lambda \in [0,\infty)$. Therefore, the right derivative

$$\dot{\alpha}^+_{\lambda_h^{\mathrm{qu}}}(x) = \lim_{\varepsilon\downarrow 0} \frac{\alpha_{\lambda_h^{\mathrm{qu}}+\varepsilon}(x) - \alpha_{\lambda_h^{\mathrm{qu}}}(x)}{\varepsilon}$$

is well-defined and strictly positive. From $\alpha_{\lambda_h^{\mathrm{qu}}}(x)$, it furthermore inherits the scalar linearity in $x \in \mathbb{R}^d$. As a consequence, there exist $e \in \mathcal{S}^{d-1}$ and $y = e/\dot{\alpha}^+_{\lambda_h^{\mathrm{qu}}}(e)$, such that $\dot{\alpha}^+_{\lambda_h^{\mathrm{qu}}}(y) = 1$ and

$$1 = \alpha^*_{\lambda_h^{\mathrm{qu}}}(h) = \sup_{x\in\mathcal{S}^{d-1}} \left(\frac{h\cdot x}{\alpha_{\lambda_h^{\mathrm{qu}}}(x)}\right) = \frac{h\cdot e}{\alpha_{\lambda_h^{\mathrm{qu}}}(e)} = \frac{h\cdot y}{\alpha_{\lambda_h^{\mathrm{qu}}}(y)}.$$

By the first condition on y, the map $\lambda \mapsto \alpha_\lambda(y) - \lambda$ is nondecreasing for $\lambda \leq \lambda_h^{\mathrm{qu}}$ and nonincreasing for $\lambda > \lambda_h^{\mathrm{qu}}$. We therefore have $I(y) = \alpha_{\lambda_h^{\mathrm{qu}}}(y) - \lambda_h^{\mathrm{qu}}$. The second condition on y now implies

$$\sup_{x\in\mathbb{R}^d}(h\cdot x - I(x)) \geq h\cdot y - I(y) = \lambda_h^{\mathrm{qu}}.$$

For the reversed estimate, we additionally set $\lambda_h^{\mathrm{qu}} = 0$ when $\alpha_0^*(h) \leq 1$. We can assume $h \neq 0$. The definition of α_λ^* then yields

$$\sup_{y:h\cdot y\geq 0}(h\cdot y - \alpha_\lambda(y)) \leq \sup_{y\in\mathbb{R}^d}\left(h\cdot y - \frac{h\cdot y}{\alpha_\lambda^*(h)}\right) = \begin{cases} 0 & \text{if } \lambda = \lambda_h^{\mathrm{qu}}, \\ \infty & \text{if } \lambda \neq \lambda_h^{\mathrm{qu}}, \end{cases}$$

which leads to

$$\sup_{x\in\mathbb{R}^d}(h\cdot x - I(x)) \leq \inf_{\lambda\geq 0}\left(\sup_{y:h\cdot y\geq 0}(h\cdot y - \alpha_\lambda(y)) + \lambda\right) = \lambda_h^{\mathrm{qu}}.$$

This completes the proof of (32).

It remains to establish the limiting behavior of $S(n)/n$. To this end, observe that the rate function I_h satisfies

$$I_h(x) \geq \alpha(x) - h\cdot x$$

for all $x \in \mathbb{R}^d$. When $\alpha_0^*(h) < 1$, it only vanishes at the origin, and the sub-ballistic behavior follows by the large deviation estimates in the quenched part of Theorem B.

On the other hand, when $\alpha_0^*(h) > 1$, the rate function I_h only vanishes on the set M_h, which is compact by the continuity of I on its effective domain D_I (which itself is compact). Observe furthermore that M_h cannot contain the origin since $\lambda_h^{\mathrm{qu}} > 0$. The ballistic behavior now follows again by the large deviation estimates. This completes the proof of Theorem D.

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