Optimality and duality in multiobjective fractional programming involving $\rho$-semilocally type I-preinvex and related functions

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Abstract

Optimality conditions are obtained for a nonlinear fractional multiobjective programming problem involving $\eta$-semidifferentiable functions. Also, a general dual is formulated and a duality result is proved using concepts of generalized $\rho$-semilocally type I-preinvex functions.

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1. Introduction

Generalizations of convexity related to optimality conditions and duality for nonlinear singleobjective or multiobjective optimization problems have been of much interest in the recent past and many contributions have been made to this development, e.g., Antczak [1], Corley [2], Eguond [3], Geoffrion [4], Mishra [8], Mititelu [14], Mukherjee and Mishra [15], Rueda and Hanson [20], Yang and Li [23].

Under a convexity assumption and a regular hypothesis, there exists an equivalence between saddle-points of the Lagrangian and optima for an inequality constrained minimization problem. Jeyakumar discussed in [5] a class of nonsmooth nonconvex problems in which functions are lo-
cally Lipschitz and are satisfying some invex type conditions and he proved that duality theorems of Wolfe type hold for this class of problems.

Weir and Mond [22] considered preinvex functions. Mishra [6] and Mishra and Mukherjee [10] extended the class of \( \nu \)-invex functions to the case of continuous-time and established duality results for variational and control problems. Mishra and Mukherjee have also extended the concept of \( \nu \)-invex functions to nonsmooth case [11] and nonsmooth composite case [7] and [9].

Preda and Stancu-Minasian [18] gave optimality conditions for weak vector minima using \( \eta \)-semidifferentials and functions satisfying generalized semilocally preinvex properties and used these results to extend the Wolfe and Mond–Weir duals, generalizing results of Preda [16], Preda et al. [19].

Preda [17] considered necessary and sufficient optimality conditions for a nonlinear fractional multiple objective programming problem involving \( \eta \)-semidifferentiable functions. Also, a general dual was formulated and duality results were proved using concepts of generalized semilocally type I and related functions. Thus, results of Preda [16], Preda et al. [19], Preda and Stancu-Minasian [18] were generalized.


In this paper, using an idea of Mishra and Noor [12], we define \( \alpha \eta \)-locally starshaped sets and give an example. Then we consider optimality conditions for a nonlinear fractional multiple objective programming problem involving \( \eta \)-semidifferentiable functions. Also, a general dual is formulated and a duality result is proved using concepts of generalized \( \rho \)-semilocally type I and related functions. Thus, we extend the work of Mishra et al. [13] and generalize results obtained in the literature on this topic.

2. Definitions and preliminaries

For \( x, y \in \mathbb{R}^n \), by \( x \preceq y \) we mean \( x_i \leq y_i \) for all \( i \), \( x \preceq y \) means \( x_i \leq y_i \) for all \( i \) and \( x_j < y_j \) for at least one \( j \in \{1, \ldots, n\} \). By \( x < y \) we mean \( x_i < y_i \) for all \( i \) and by \( x \not\preceq y \) we mean the negation of \( x \preceq y \).

We denote \( M = \{1, 2, \ldots, m\} \) and \( P = \{1, 2, \ldots, p\} \).

Let \( X_0 \subseteq \mathbb{R}^n \) be a set, \( \alpha : X_0 \times X_0 \to \mathbb{R}_+ \) and \( \eta : X_0 \times X_0 \to \mathbb{R}_+ \) be a vectorial application.

We say that the set \( X_0 \) is \( \alpha \)-invex at \( \bar{x} \in X_0 \) if \( \bar{x} + \lambda \alpha(x, \bar{x})\eta(x, \bar{x}) \in X_0 \) for any \( x \in X_0 \) and \( \lambda \in [0, 1] \). We say that the set \( X_0 \) is \( \alpha \)-invex if \( X_0 \) is \( \alpha \)-invex at any \( \bar{x} \in X_0 \) (see [12]). If \( \alpha(x, \bar{x}) = 1 \), for any \( x, \bar{x} \in X_0 \), an \( \alpha \)-invex set becomes an invex set.

We remark that if \( \eta(x, \bar{x}) = x - \bar{x} \) for any \( x \in X_0 \) then \( X_0 \) is invex at \( \bar{x} \) if \( X_0 \) is a convex set at \( \bar{x} \).

**Definition 1.** We say that the set \( X_0 \subseteq \mathbb{R}^n \) is an \( \alpha \eta \)-locally starshaped set at \( \bar{x} \in X_0 \), if for any \( x \in X_0 \), there exists \( 0 < a_\eta(x, \bar{x}) \leq 1 \) such that \( \bar{x} + \lambda \alpha(x, \bar{x})\eta(x, \bar{x}) \in X_0 \) for any \( \lambda \in [0, a_\eta(x, \bar{x})] \). If \( \alpha(x, \bar{x}) = 1 \), for any \( x \in X_0 \), an \( \alpha \eta \)-locally starshaped set at \( \bar{x} \) becomes an \( \eta \)-locally starshaped set at \( \bar{x} \). We say that the set \( X_0 \) is \( \alpha \eta \)-locally starshaped if \( X_0 \) is \( \alpha \eta \)-locally starshaped at any \( \bar{x} \in X_0 \).

**Example 1.** Let \( a \in \mathbb{R}, a > 0 \). The set

\[
X_0 = \{ x \in \mathbb{R}^2 \mid x_1^{2/3} + x_2^{2/3} \leq a^{2/3} \}
\]

is an \( \alpha \eta \)-locally starshaped set, where
\[ \eta(x, \bar{x}) = -\bar{x}, \]
\[ \alpha(x, \bar{x}) = (\bar{x}^T \bar{x})^2 + 1, \]
\[ a_\eta(x, \bar{x}) = \min \left\{ 1, \frac{2}{(\bar{x}^T \bar{x})^2 + 1} \right\}. \]

(Here the symbol \(^T\) denotes the transpose of a matrix.)

Let \( \rho \in \mathbb{R}^n \) and \( \theta : X_0 \times X_0 \to \mathbb{R} \), such that \( \theta(x, y) \neq 0 \) if \( x \neq y \).

**Definition 2.** Let \( f : X_0 \to \mathbb{R}^n \) be a function, where \( X_0 \subseteq \mathbb{R}^n \) is an \( \eta \)-locally starshaped set at \( \bar{x} \in X_0 \). We say that \( f \) is:

(i1) \( \rho \)-semilocally preinvex (\( \rho \)-slpi) at \( \bar{x} \) if, corresponding to \( \bar{x} \) and each \( x \in X_0 \), there exists a positive number \( d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x}) \) such that \( f(\bar{x} + \lambda \eta(x, \bar{x})) \leq \lambda f(x) + (1 - \lambda) f(\bar{x}) - \lambda(1 - \lambda) \rho \theta^2(x, \bar{x}) \) for \( 0 < \lambda < d_\eta(x, \bar{x}) \); 

(i2) \( \rho \)-semilocally quasi-preinvex (\( \rho \)-slqpi) at \( \bar{x} \) if, corresponding to \( \bar{x} \) and each \( x \in X_0 \), there exists a positive number \( a_\eta(x, \bar{x}) \) such that \( f(x) \leq f(\bar{x}) \) and \( 0 < \lambda < d_\eta(x, \bar{x}) \) implies \( f(\bar{x} + \lambda \eta(x, \bar{x})) \leq f(\bar{x}) - \lambda(1 - \lambda) \rho \theta^2(x, \bar{x}) \).

If \( \rho = 0 \) in the above definition, \( f \) is semilocally preinvex (slpi) at \( \bar{x} \), respectively semilocally quasi-preinvex (slqpi) at \( \bar{x} \) [13].

**Definition 3.** Let \( f : X_0 \to \mathbb{R}^n \) be a function, where \( X_0 \subseteq \mathbb{R}^n \) is an \( \eta \)-locally starshaped set at \( \bar{x} \in X_0 \). We say that \( f \) is \( \eta \)-semidifferentiable at \( \bar{x} \) if \( (df)^+ (\bar{x}, \eta(x, \bar{x})) \) exists for each \( \bar{x} \in X_0 \), where
\[
(df)^+ (\bar{x}, \eta(x, \bar{x})) = \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left[ f(\bar{x} + \lambda \eta(x, \bar{x})) - f(\bar{x}) \right]
\]
(the right derivative at \( \bar{x} \) along the direction \( \eta(x, \bar{x}) \)).

If \( f \) is \( \eta \)-semidifferentiable at any \( \bar{x} \in X_0 \), then \( f \) is said to be \( \eta \)-semidifferentiable on \( X_0 \).

**Remark.** If \( \eta(x, \bar{x}) = x - \bar{x}, \) the \( \eta \)-semidifferentiability is the semidifferentiability notion. If a function is directionally differentiable, then it is semidifferentiable but the converse is not true.

**Lemma 1.** Let \( f : X_0 \to \mathbb{R}^n \) be an \( \eta \)-semidifferentiable function at \( \bar{x} \in X_0 \). If \( f \) is \( \rho \)-slipi at \( \bar{x} \) and \( f(x) \leq f(\bar{x}) \) then \( (df)^+ (\bar{x}, \eta(x, \bar{x})) \leq -\rho \theta^2(x, \bar{x}) \).

**Definition 4.** We say that \( f \) is \( \rho \)-semilocally pseudo-preinvex (\( \rho \)-splpi) at \( \bar{x} \) if for any \( x \in X_0 \),
\[
(df)^+ (\bar{x}, \eta(x, \bar{x})) \geq -\rho \theta^2(x, \bar{x}) \Rightarrow f(x) \geq f(\bar{x}).
\]

If \( f \) is \( \rho \)-splpi at any \( \bar{x} \in X_0 \), then \( f \) is said to be \( \rho \)-splpi on \( X_0 \).

If \( \rho = 0 \) in the above definition, \( f \) is semilocally pseudo-preinvex (splpi) [13].

**Definition 5.** Let \( X \) and \( Y \) be two subsets of \( X_0 \) and \( \bar{y} \in Y \). We say that \( Y \) is \( \alpha \eta \)-locally starshaped at \( \bar{y} \) with respect to \( X \) if for any \( x \in X \) there exists \( 0 < a_\eta(x, \bar{x}) \leq 1 \) such that \( \bar{y} + \lambda \alpha(x, \bar{x}) \eta(x, \bar{y}) \in Y \) for any \( 0 \leq \lambda \leq a_\eta(x, \bar{y}) \). If \( \alpha(x, \bar{x}) = 1 \), for any \( x \in X_0 \), an \( \alpha \eta \)-locally starshaped set at \( \bar{y} \) with respect to \( X \) becomes an \( \eta \)-locally starshaped set at \( \bar{y} \) with respect to \( X \).

**Definition 6.** Let \( Y \) be \( \eta \)-locally starshaped at \( \bar{y} \) with respect to \( X \) and \( f \) be an \( \eta \)-semidifferentiable function at \( \bar{y} \). We say that \( f \) is:
(a) $\rho$-slppi at $\bar{y} \in Y$ with respect to $X$, if for any $x \in X$, $(df)^+(\bar{y}, \eta(x, \bar{y})) \geq -\rho \theta^2(x, \bar{x}) \Rightarrow f(x) \geq f(\bar{y})$;
(b) $\rho$-strictly semilocally pseudo-preinvex ($\rho$-sslppi) at $\bar{y}$ with respect to $X$, if for each $x \in X$, $x \neq \bar{y}$, $(df)^+(\bar{y}, \eta(x, \bar{y})) \geq -\rho \theta^2(x, \bar{x}) \Rightarrow f(x) > f(\bar{y})$.

We say that $f$ is ($\rho$-slppi) $\rho$-sslppi on $Y$ with respect to $X$, if $f$ is ($\rho$-slppi) $\rho$-sslppi at any point of $Y$ with respect to $X$.

**Definition 7.** A function $f: X_0 \rightarrow \mathbb{R}^k$ is a convexlike function if for any $x, y \in X_0$ and $0 \leq \lambda \leq 1$, there is $z \in X_0$ such that

$$f(z) \leq \lambda f(x) + (1 - \lambda)f(y).$$

**Remark.** The convex and the preinvex functions are convexlike functions.

**Lemma 2.** (See [17].) Let $S$ be a nonempty set in $\mathbb{R}^n$ and $\psi: S \rightarrow \mathbb{R}^k$ be a convexlike function. Then either

$$\psi(x) < 0 \text{ has a solution } x \in S$$

or

$$\lambda^T \psi(x) \geq 0 \text{ for all } x \in S,$$

for some $\lambda \in \mathbb{R}^k$, $\lambda \geq 0$, but both alternatives are never true.

Using Lemma 2 from above instead of Lemma 2.9 from [18], we have that Theorems 3.4 and 3.5 stated there are still true. Thus, in the next section we will use the following version of Theorem 3.5 from [18].

**Lemma 3.** Let $\bar{x} \in X$ be a (local) weak minimum solution for the following problem:

$$\min(\varphi_1(x), \ldots, \varphi_p(x)),$$

subject to

$$\begin{cases}
h_j(x) \leq 0, & j \in M, \\
x \in X_0,
\end{cases}$$

where $\varphi = (\varphi_1, \ldots, \varphi_p): X_0 \rightarrow \mathbb{R}^p$ and $h_1, \ldots, h_m$ are $\eta$-semidifferentiable at $\bar{x}$. Also, we assume that $h_j$ ($j \in N(\bar{x})$) is a continuous function at $\bar{x}$ and $(d\varphi)^+ (\bar{x}, \eta(x, \bar{x}))$ and $(dh)^+ (\bar{x}, \eta(x, \bar{x}))$ are convexlike functions of $x$ on $X_0$. If $h$ satisfies a regularity condition at $\bar{x}$ (see [18]), then there exist $\lambda^0 \in \mathbb{R}^p$, $u^0 \in \mathbb{R}^m$ such that

$$\lambda^{0T} (d\varphi)^+ (\bar{x}, \eta(x, \bar{x})) + u^{0T} (dh)^+ (\bar{x}, \eta(x, \bar{x})) \geq 0 \text{ for all } x \in X_0,$$

$$u^{0T} h(\bar{x}) = 0,$$

$$h(\bar{x}) \leq 0,$$

$$\lambda^{0T} e = 1,$$

$$\lambda^0 \geq 0, \ u^0 \geq 0,$$

where $e = (1, \ldots, 1)^T \in \mathbb{R}^p$. 


In this paper we consider the following multiobjective nonlinear fractional programming problem:

\[
\begin{aligned}
\min & \left( \frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)} \right), \\
\text{subject to} & \quad h_j(x) \leq 0, \quad j = 1, 2, \ldots, m, \\
& \quad x \in X_0,
\end{aligned}
\]

where \( X_0 \subseteq \mathbb{R}^n \) is a nonempty set and \( g_i(x) > 0 \) for all \( x \in X_0 \) and each \( i = 1, \ldots, p \). Let \( f = (f_1, \ldots, f_p) \), \( g = (g_1, \ldots, g_p) \), \( h = (h_1, \ldots, h_m) \). We denote \( X = \{ x \in X_0 \mid h_j(x) \leq 0, \ j = 1, 2, \ldots, m \} \), the feasible set of problem (VFP).

Let \( \rho^1, \rho^2, \rho^3 \in \mathbb{R}^m \).

**Definition 8.** We say that the problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable type I-preinvex at \(\bar{x}\) if for any \( x \in X_0 \), we have

\[
\begin{aligned}
f_i(x) - f_i(\bar{x}) \geq (df_i)^+(\bar{x}, \eta(x, \bar{x})) - \rho_i^1 \theta^2(x, \bar{x}), & \quad \forall i \in P, \\
g_i(x) - g_i(\bar{x}) \leq (dg_i)^+(\bar{x}, \eta(x, \bar{x})) - \rho_i^2 \theta^2(x, \bar{x}), & \quad \forall i \in P, \\
-h_j(\bar{x}) \leq (dh_j)^+(\bar{x}, \eta(x, \bar{x})) - \rho_j^3 \theta^2(x, \bar{x}), & \quad \forall j \in M.
\end{aligned}
\]

The problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable pseudo-quasi-type I-preinvex on \(X_0\) if it is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable pseudo-quasi-type I-preinvex at any \(\bar{x} \in X_0\).

**Definition 9.** We say that the problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable pseudo-quasi-type I-preinvex at \(\bar{x}\) if for any \( x \in X_0 \), we have

\[
\begin{aligned}
(df_i)^+(\bar{x}, \eta(x, \bar{x})) \geq -\rho_i^1 \theta^2(x, \bar{x}) \quad \Rightarrow \quad f_i(x) \geq f_i(\bar{x}), & \quad \forall i \in P, \\
(dg_i)^+(\bar{x}, \eta(x, \bar{x})) \geq -\rho_i^2 \theta^2(x, \bar{x}) \quad \Rightarrow \quad g_i(x) \leq g_i(\bar{x}), & \quad \forall i \in P, \\
-h_j(\bar{x}) \leq 0 \quad \Rightarrow \quad (dh_j)^+(\bar{x}, \eta(x, \bar{x})) \leq -\rho_j^3 \theta^2(x, \bar{x}), & \quad \forall j \in M.
\end{aligned}
\]

The problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable pseudo-quasi-type I-preinvex on \(X_0\) if it is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable pseudo-quasi-type I-preinvex at any \(\bar{x} \in X_0\).

**Definition 10.** We say that the problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable quasi-pseudo-type I-preinvex at \(\bar{x}\) if for any \( x \in X_0 \), we have

\[
\begin{aligned}
f_i(x) \leq f_i(\bar{x}) \quad \Rightarrow \quad (df_i)^+(\bar{x}, \eta(x, \bar{x})) \leq -\rho_i^1 \theta^2(x, \bar{x}), & \quad \forall i \in P, \\
g_i(x) \leq g_i(\bar{x}) \quad \Rightarrow \quad (dg_i)^+(\bar{x}, \eta(x, \bar{x})) \geq -\rho_i^2 \theta^2(x, \bar{x}), & \quad \forall i \in P, \\
(h_j)(\bar{x}) \geq 0 \quad \Rightarrow \quad -h_j(\bar{x}) \geq 0, & \quad \forall j \in M.
\end{aligned}
\]

The problem (VFP) is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable quasi-pseudo-type I-preinvex on \(X_0\) if it is \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semidifferentiable quasi-pseudo-type I-preinvex at any \(\bar{x} \in X_0\).

**Definition 11.** For the problem (VFP), a point \(\bar{x} \in X\) is said to be a weak minimum if there exists no other feasible point \(x\) for which \(\frac{f(j)(\bar{x})}{g(j)(\bar{x})} > \frac{f(j)(x)}{g(j)(x)}\).

For \(\bar{x} \in X\) we put \(M(\bar{x}) = \{ j \in M \mid h(j)(\bar{x}) = 0 \}\), \(h^0 = (h(j))_{j \in M(\bar{x})}\) and \(N(\bar{x}) = M \setminus M(\bar{x})\).

**Definition 12.** We say that (VFP) satisfies the generalized Slater’s constraint qualification (GSCQ) at \(\bar{x} \in X\) if \(h^0\) is slppi at \(\bar{x}\) and there exists \(\hat{x} \in X\) such that \(h^0(\hat{x}) < 0\).
Lemma 4. (See [17, Lemma 13].) Let $\bar{x} \in X$ be a (local) weak minimum solution for (VFP). Further, we assume that $h_j$ is continuous at $\bar{x}$ for any $j \in N(\bar{x})$ and that $f, g, h^0$ are $\eta$-semidifferentiable at $\bar{x}$. Then, the system
\[
\begin{cases}
(df)^+(\bar{x}, \eta(x, \bar{x})) < 0, \\
(dg)^+(\bar{x}, \eta(x, \bar{x})) > 0, \\
(dh^0)^+(\bar{x}, \eta(x, \bar{x})) < 0,
\end{cases}
\]
has no solution $x \in X_0$.

Lemma 5 (Fritz–John Type Necessary Optimality Criteria [17, Theorem 14]). Let us suppose that $h_j$ is continuous at $\bar{x}$ for $j \in N(\bar{x})$, and $(df)^+(\bar{x}, \eta(x, \bar{x})), (dg)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dh^0)^+(\bar{x}, \eta(x, \bar{x}))$ are convexlike functions of $x$ on $X_0$. If $\bar{x}$ is a (local) weak minimum solution for (VFP), then there exist $\lambda^0 \in R^p, u^0 \in R^p, v^0 \in R^m$ such that
\[
\begin{align*}
\lambda^{0T} (df)^+(\bar{x}, \eta(x, \bar{x})) - u^{0T} (dg)^+(\bar{x}, \eta(x, \bar{x})) + v^{0T} (dh^0)^+(\bar{x}, \eta(x, \bar{x})) & \geq 0 \\
& \text{for all } x \in X_0, \\
v^{0T} h(\bar{x}) & = 0, \\
(\lambda^0, u^0, v^0) & \neq 0, \\
(\lambda^0, u^0, v^0) & \geq 0.
\end{align*}
\]

For each $u = (u_1, \ldots, u_p)^T \in R^p_+$, where $R^p_+$ denotes the positive orthant of $R^p$, we consider
\[
\begin{align*}
\text{minimize} & \quad (f_1(x) - u_1 g_1(x), \ldots, f_p(x) - u_p g_p(x)), \\
\text{subject to} & \quad \begin{cases} h_j(x) \leq 0, \\
x \in X_0. \end{cases}
\end{align*}
\]

(VFP$_u$)

The following lemma can be proved easy.

Lemma 6. If $\bar{x}$ is a (local) weak minimum for (VFP) then $\bar{x}$ is a (local) weak minimum for (VFP$_{u^0}$), where $u^0 = \frac{f(\bar{x})}{g(\bar{x})}$.

Using this lemma we can derive a Karush–Kuhn–Tucker type necessary optimality criterium for the problem (VFP).

Lemma 7 (Karush–Kuhn–Tucker Type Necessary Optimality Criterium [17, Theorem 16]). Let $\bar{x}$ be a (local) weak minimum solution for (VFP), let $h_j$ be continuous at $\bar{x}$ for $j \in N(\bar{x})$ and let $(df_i)^+(\bar{x}, \eta(x, \bar{x})), (dg_i)^+(\bar{x}, \eta(x, \bar{x})), i \in P$ and $(dh^0)^+(\bar{x}, \eta(x, \bar{x}))$ be convexlike functions of $x$ on $X_0$. If $g$ satisfies (GSCQ) at $\bar{x}$, then there exist $\lambda^0 \in R^p_+, u^0 \in R^p_+, v^0 \in R^m$ such that
\[
\begin{align*}
\sum_{i=0}^p \lambda_0^i ((df_i)^+(\bar{x}, \eta(x, \bar{x})) - u_0^i (dg_i)^+(\bar{x}, \eta(x, \bar{x}))) + v^{0T} (dh)^+(\bar{x}, \eta(x, \bar{x})) & \geq 0 \\
& \text{for all } x \in X_0, \\
v^{0T} h(\bar{x}) & = 0, \\
h(\bar{x}) & \leq 0, \\
\lambda^{0T} e & = 1, \\
\lambda^0 & \geq 0, \\
u^0 & \geq 0, \\
v^0 & \geq 0,
\end{align*}
\]
where $e = (1, \ldots, 1)^T \in R^p$. 

**Remark.** In the above theorem we can suppose, for any \( i \in P \), that \((df_i)^+ (\bar{x}, \eta(x, \bar{x})) - u_i^0 (dg_i)^+ (\bar{x}, \eta(x, \bar{x}))\) is convexlike on \( X_0 \), where \( u_i^0 = \frac{f_i (\bar{x})}{g_i (\bar{x})} \) instead of considering that \((df_i)^+ (\bar{x}, \eta(x, \bar{x}))\) and \((dg_i)^+ (\bar{x}, \eta(x, \bar{x}))\) are convexlike on \( X_0 \), for any \( i \in P \).

### 3. Sufficient optimality criteria

In this section, using the concept of (local) weak optimality, we give some sufficient optimality conditions for the (VFP) problem.

**Theorem 1.** Let \( \bar{x} \in X \) and (VFP) be \((\rho^1, \rho^2, \rho^3)\) \( \eta \)-semilocally type I-preinvex at \( \bar{x} \). Also, we assume that there exists \( \lambda^0 \in R^p \), \( u^0 \in R^p \) and \( v^0 \in R^m \) such that \( \lambda^0^T \rho^1 + v^0^T \rho^3 \leq 0 \), \( \rho^2 \geq 0 \) and

\[
\begin{align*}
\sum_{i=0}^{p} \lambda^0_i (df_i)^+ (\bar{x}, \eta(x, \bar{x})) + v^0^T (dh)^+ (\bar{x}, \eta(x, \bar{x})) &\geq 0 \quad \text{for all } x \in X, \quad (3.1) \\
(dg_i)^+ (\bar{x}, \eta(x, \bar{x})) &\leq 0, \quad \forall x \in X, \quad \forall i \in P, \quad (3.2) \\
v^0 h(\bar{x}) &= 0, \quad (3.3) \\
h(\bar{x}) &\leq 0, \quad (3.4) \\
\lambda^0 e &= 1, \quad (3.5) \\
\lambda^0 &\geq 0, \quad u^0 \geq 0, \quad v^0 \geq 0, \quad (3.6)
\end{align*}
\]

where \( e = (1, \ldots, 1)^T \in R^p \). Then \( \bar{x} \) is a weak minimum solution for (VFP).

**Proof.** We proceed by contradicting. Hence there exists \( \tilde{x} \in X \) such that

\[
\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \quad \text{for any } i \in P. \quad (3.7)
\]

Since (VFP) is \((\rho^1, \rho^2, \rho^3)\) \( \eta \)-semilocally type I-preinvex at \( \bar{x} \), we get

\[
\begin{align*}
f_i(\tilde{x}) - f_i(\bar{x}) &\geq (df_i)^+ (\bar{x}, \eta(x, \bar{x})) - \rho^1_i \theta^2(\bar{x}, \bar{x}), \quad i \in P, \quad (3.8) \\
g_i(\tilde{x}) - g_i(\bar{x}) &\geq (dg_i)^+ (\bar{x}, \eta(x, \bar{x})) - \rho^2_i \theta^2(\bar{x}, \bar{x}), \quad i \in P, \quad (3.9) \\
-h_j(\bar{x}) &\geq (dh_j)^+ (\bar{x}, \eta(x, \bar{x})) - \rho^3_j \theta^2(\bar{x}, \bar{x}), \quad j \in M. \quad (3.10)
\end{align*}
\]

Multiplying (3.8) by \( \lambda^0_i \geq 0, \quad i \in P, \quad \lambda^0 \in R^p_+ \), (3.10) by \( v^0_j \geq 0, \quad j \in M \), and then summing the obtained relations, we get

\[
\begin{align*}
\sum_{i=0}^{p} \lambda^0_i (f_i(\tilde{x}) - f_i(\bar{x})) - \sum_{j=1}^{m} v^0_j h_j(\bar{x}) &\geq \sum_{i=0}^{p} \lambda^0_i (df_i)^+ (\bar{x}, \eta(x, \bar{x})) + \sum_{j=1}^{m} v^0_j (dh_j)^+ (\bar{x}, \eta(x, \bar{x})) - (\lambda^0^T \rho^1 + v^0^T \rho^3) \theta^2(\bar{x}, \bar{x}) \\
&\geq 0,
\end{align*}
\]
where the last inequality is according to (3.1) and $\lambda^0 \rho^1 + v^0 \rho^3 \leq 0$. Hence,

$$
\sum_{i=0}^{p} \lambda^0_i (f_i(\tilde{x}) - f_i(\bar{x})) - \sum_{j=1}^{m} v^0_j h_j(\tilde{x}) \geq 0.
$$

(3.11)

By (3.3) we get

$$
\sum_{i=0}^{p} \lambda^0_i (f_i(\tilde{x}) - f_i(\bar{x})) \geq 0.
$$

(3.12)

Using (3.6) and (3.12), we obtain that there exists $i_0 \in P$ such that

$$
f_{i_0}(\tilde{x}) \geq f_{i_0}(\bar{x}).
$$

(3.13)

By (3.2), (3.9) and $\rho^2 \geq 0$ it follows

$$
g_i(\tilde{x}) \leq g_i(\bar{x}), \quad i \in P.
$$

(3.14)

Now, using (3.13), (3.14), $f \geq 0$ and $g > 0$, we obtain

$$
\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},
$$

which is in contradiction to (3.7). Thus, the theorem is proved and $\tilde{x}$ is a weak minimum solution for (VFP).

Theorem 2. Let $\tilde{x} \in X$ and (VFP) be $(\rho^1, \rho^2, \rho^3) \eta$-semilocally type I-preinvex at $\tilde{x}$. Also, we assume that there exist $\lambda^0 \in R^p$, $u^0_i = \frac{f_i(\tilde{x})}{g_i(\tilde{x})}$, $i \in P$, and $v^0 \in R^m$ such that $\lambda^0^T \rho^1 - \sum_{i=1}^{p} \lambda^0_i u^0_i \rho^2_i + v^0 \rho^3 \leq 0$,

$$
\sum_{i=0}^{p} \lambda^0_i ((d f_i)^+(\tilde{x}, \eta(x, \tilde{x})) - u^0_i (d g_i)^+(\tilde{x}, \eta(x, \tilde{x}))) + v^0 (d h)^+(\tilde{x}, \eta(x, \tilde{x})) \geq 0,
$$

for all $x \in X$.

(3.15)

$$
v^0 \rho^1 h(\tilde{x}) = 0.
$$

(3.16)

$$
h(\tilde{x}) \leq 0.
$$

(3.17)

$$
\lambda^0 e = 1.
$$

(3.18)

$$
\lambda^0 \geq 0, \quad u^0 \geq 0, \quad v^0 \geq 0.
$$

(3.19)

Then $\tilde{x}$ is a weak minimum solution for (VFP).

Proof. We proceed by contradicting. Then if $\tilde{x}$ is not a weak minimum solution for (VFP), we have that there exists $\bar{x} \in X$ such that

$$
\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}
$$

for any $i \in P$,

i.e.,

$$
f_i(\tilde{x}) < u^0_i g_i(\tilde{x})
$$

for any $i \in P$.

(3.20)

Since (VFP) is $(\rho^1, \rho^2, \rho^3) \eta$-semilocally type I-preinvex at $\tilde{x}$, we get
Using these inequalities and (3.19), we get

\[
\sum_{i=0}^{p} \lambda_i^0 (f_i(\bar{x}) - f_i(\tilde{x})) - \sum_{i=0}^{p} \lambda_i^0 u_i^0 (g_i(\bar{x}) - g_i(\tilde{x})) - \sum_{j=1}^{m} v_j^0 h_j(\tilde{x}) \\
\geq \sum_{i=0}^{p} \lambda_i^0 ((df_i)^+(\bar{x}, \eta, \tilde{x})) - u_i^0 (dg_i)^+(\bar{x}, \eta, \tilde{x})) \\
+ \sum_{j=1}^{m} v_j^0 (dh_j)^+(\bar{x}, \eta, \tilde{x})) - \left(\lambda^0 \rho^1 - \sum_{i=1}^{p} \lambda_i^0 u_i^0 \rho_i^2 + v^0 \rho^3 \right) \theta^2(\tilde{x}, \bar{x}) \\
\geq 0,
\]

where the last inequality is according to (3.15) and \(\lambda^0 \rho^1 - \sum_{i=1}^{p} \lambda_i^0 u_i^0 \rho_i^2 + v^0 \rho^3 \leq 0\). Therefore,

\[
\sum_{i=0}^{p} \lambda_i^0 [(f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})) - (f_i(\bar{x}) - u_i^0 g_i(\tilde{x}))] - \sum_{j=1}^{m} v_j^0 h_j(\tilde{x}) \geq 0.
\]

Since \(u_i^0 = \frac{f_i(\tilde{x})}{g_i(\tilde{x})}, i \in P\), we obtain

\[
\sum_{i=0}^{p} \lambda_i^0 (f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})) - v^0 h(\tilde{x}) \geq 0.
\]

Now, (3.16) give

\[
\sum_{i=0}^{p} \lambda_i^0 (f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})) \geq 0. \tag{3.21}
\]

Since \(\lambda_i^0 \geq 0, \lambda^0 e = 1\), we get that there exists \(i_0 \in P\) such that:

\[
(f_{i_0}(\tilde{x}) - u_{i_0}^0 g_{i_0}(\tilde{x}) \geq 0,
\]

i.e.,

\[
\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},
\]

which is in contradiction with (3.20). Hence \(\bar{x}\) is a weak minimum solution for (VFP) and the proof is complete. \(\square\)

**Theorem 3.** Let \(\tilde{x} \in X, u_i^0 = \frac{f_i(\tilde{x})}{g_i(\tilde{x})}, i \in P\) and \(\lambda^0 \in \mathbb{R}^p, v^0 \in \mathbb{R}^m\) such that the conditions (3.15)--(3.19) of Theorem 2 hold. Moreover, we assume that (VFP\(_{u^0}\)) is \((\rho^1, \rho^2, \rho^3, \eta)-\text{semilocally pseudo-quasi-type I-preinvex}\) at \(\tilde{x}\) and \(\sum_{i=0}^{p} \lambda_i^0 \rho_i^1 + \sum_{j=1}^{m} v_j^0 \rho_j^3 \geq 0\). Then \(\tilde{x}\) is a weak minimum solution for (VFP).
Proof. We suppose that \( \bar{x} \) is not a weak minimum solution for (VFP). Then there exists \( \tilde{x} \in X \) such that
\[
\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})}
\]
for any \( i \in P \), i.e.,
\[
f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < 0 \quad \text{for any } i \in P,
\]
which is equivalent to
\[
f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < f_i(\bar{x}) - u_i^0 g_i(\bar{x}) \quad \text{for any } i \in P.
\]
Now, by the \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semilocally pseudo-quasi-type I-preinvexity of (VFP) at \( \bar{x} \) we get
\[
(d f_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - (d g_i)^+(\bar{x}, \eta(\bar{x}, \bar{x})) < -\rho_i \theta(\tilde{x}, \bar{x}) \quad \text{for any } i \in P.
\]
Using \( \lambda^0_i \in R^p_+ \), \( e^T \lambda^0_0 = 1 \) we obtain
\[
\sum_{i=0}^p \lambda^0_i ((d f_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - (d g_i)^+(\bar{x}, \eta(\bar{x}, \bar{x}))) < -\sum_{i=1}^p \lambda^0_i \rho_i \theta(\tilde{x}, \bar{x}).
\]
(3.22)
For \( j \in M(\bar{x}) \), \( h_j(\bar{x}) = 0 \). Hence \( -h_j(\bar{x}) \leq 0 \) for any \( j \in M(\bar{x}) \). Now, by the \((\rho^1, \rho^2, \rho^3)\) \(\eta\)-semilocally pseudo-quasi-type I-preinvexity of (VFP) at \( \bar{x} \), we obtain
\[
(d h_j)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) \leq -\rho_j \theta(\tilde{x}, \bar{x}) \quad \text{for any } j \in M(\bar{x}).
\]
But \( v^0 \in R^m_+ \) and \( v^0_j = 0 \) for \( j \in N(\bar{x}) \) and then we get
\[
\sum_{j=1}^m v^0_j (d h_j)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) \leq -\sum_{j=1}^m v^0_j \rho_j \theta(\tilde{x}, \bar{x}).
\]
(3.23)
Now, by (3.22), (3.23) and \( \sum_{i=0}^p \lambda^0_i \rho_i^1 + \sum_{j=1}^m v^0_j \rho_j^3 \geq 0 \) we obtain
\[
\sum_{i=0}^p \lambda^0_i ((d f_i)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) - (d g_i)^+(\bar{x}, \eta(\bar{x}, \bar{x})))
\]
\[
+ \sum_{j=1}^m v^0_j (d h_j)^+(\bar{x}, \eta(\tilde{x}, \bar{x})) < -\left( \sum_{i=0}^p \lambda^0_i \rho_i^1 + \sum_{j=1}^m v^0_j \rho_j^3 \right) \theta(\tilde{x}, \bar{x}) \leq 0
\]
which is a contradiction to (3.15). Hence \( \bar{x} \) is a weak minimum for (VFP) and the theorem is proved. \( \square \)

4. Duality

We consider, for (VFP), a general Mond–Weir dual (FMWD) as
\[
\max \psi(y, \lambda, u, v) = u - v^T_I h_I(y)e,
\]
subject to:
\[
\sum_{i=0}^{p} \lambda_i \left( (df_i)^+(y, \eta(x, y)) - u_i (dg_i)^+(y, \eta(x, y)) \right) + v^T (dh)^+(y, \eta(x, y)) \geq 0
\]
for all \( x \in X \),
\( f_i(y) - u_ig_i(y) \geq 0 \) for any \( i \in P \),
\( v^T h_{I_s}(y) \geq 0 \) \((1 \leq s \leq \gamma)\),
\( \lambda^T e = 1, \lambda \geq 0, \lambda \in \mathbb{R}^p \),
\( u \geq 0, u \in \mathbb{R}^p, v \geq 0, y \in X_0 \),
where \( \gamma \geq 1, I_s \cap I_t = \emptyset \) for \( s \neq t \) and \( \bigcup_{s=0}^{v} I_s = M \). (Here \( v_{I_s} = (v_j)_{j \in I_s}, h_{I_s} = (h_j)_{j \in I_s} \).)

Let \( W \) denote the set of all feasible solutions of (FMWD). Also, we define the following sets
\[
A = \{ (\lambda, u, v) \in \mathbb{R}^p \times \mathbb{R}^p \times \mathbb{R}^m \mid (y, \lambda, u, v) \in W \text{ for some } y \in X_0 \}
\]
and, for \((\lambda, u, v) \in A\),
\[
B(\lambda, u, v) = \{ y \in X_0 \mid (y, \lambda, u, v) \in W \}.
\]

We put \( B = \bigcup_{(\lambda, u, v) \in A} B(\lambda, u, v) \) and note that \( B \subset X_0 \). Also, we note that if \((y, \lambda, u, v) \in W \) then \((\lambda, u, v) \in A \) and \( y \in B(\lambda, u, v) \).

Now we establish a certain duality result between (VFP) and (FMWD). Assume that \( f, g \) and \( h \) are \( \eta \)-semidifferentiable on \( X \).

**Theorem 4 (Weak Duality).** Assume that for all feasible solutions \( x \in X \) and \((y, \lambda, u, v) \in W \) for (VFP) and (FMWD) respectively, we have
\[
(df_i)^+(y, \eta(x, y)) - u_ig_i(y) \geq 0 \Rightarrow f_i(x) - u_ig_i(x) + v^T h_{I_0}(x) \geq f_i(y) - u_ig_i(y) + v^T h_{I_0}(y)
\]
for all \( i \in P \),
\( \forall i \in P \),
and
\[
-v^T h_{I_s}(y) \leq 0 \Rightarrow \sum_{j \in I_s} v_j (dh_j)^+(y, \eta(x, y)) \leq -\rho^3 \theta^2(x, y)
\]
for \( 1 \leq s \leq \gamma \),
hold on \( B(\lambda, u, v) \) and \( \sum_{i=1}^{p} \lambda_i \rho^1_i + \sum_{s=1}^{v} \rho^3_s \geq 0 \). Then the following cannot hold:
\[
f_i(x) - u_ig_i(x) \leq v^T h_{I_0}(y) \quad \text{for any } i \in P,
\]
and
\[
f_{i_0}(x) - u_{i_0}g_{i_0}(x) < v^T h_{I_0}(y) \quad \text{for some } i_0 \in P.
\]

**Proof.** Using (4.3) and (4.7), we obtain
\[
\sum_{j \in I_s} v_j (dh_j)^+(y, \eta(x, y)) \leq -\rho^3 \theta^2(x, y), \quad 1 \leq s \leq \gamma.
\]
Now we suppose to the contrary of the result of the theorem that (4.8) and (4.9) hold. Hence if (4.8) and (4.9) hold for some feasible \( x \) for (VFP) and \( (y, \lambda, u, v) \) feasible for (FMWD), we obtain
\[
f_i(x) - u_i g_i(x) \leq v_{i_0}^T h_{i_0}(y) \quad \text{for any } i \in P,
\]
and
\[
f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{i_0}^T h_{i_0}(y) \quad \text{for some } i_0 \in P.
\]
According to (4.2), (4.5) and the feasibility of \( x \) for (VFP), we have
\[
v_{i_0}^T h_{i_0}(x) \leq 0 \leq f_i(y) - u_i g_i(y) \quad \text{for all } i \in P.
\]
Combining (4.11)–(4.13) we get
\[
f_i(x) - u_i g_i(x) + v_{i_0}^T h_{i_0}(x) \leq f_i(y) - u_i g_i(y) + v_{i_0}^T h_{i_0}(y), \quad \forall i \in P,
\]
and
\[
f_{i_0}(x) - u_{i_0} g_{i_0}(x) + v_{i_0}^T h_{i_0}(x) < f_{i_0}(y) - u_{i_0} g_{i_0}(y) + v_{i_0}^T h_{i_0}(y) \quad \text{for some } i_0 \in P.
\]
By (4.6), (4.14) and (4.15) we obtain
\[
(d f_i)^+(y, \eta(x, y)) - u_i (d g_i)^+(y, \eta(x, y)) \leq \sum_{j \in I_0} v_j (d h_j)^+(y, \eta(x, y))
\]
\[
< - \rho^1 \theta^2(x, y) \quad \text{for any } i \in P.
\]
By (4.4) and (4.16) we get
\[
\sum_{i=0}^p \lambda_i ((d f_i)^+(y, \eta(x, y)) - u_i (d g_i)^+(y, \eta(x, y))) + \sum_{j \in I_0} v_j (d h_j)^+(y, \eta(x, y))
\]
\[
< - \sum_{i=0}^p \lambda_i \rho^1 \theta^2(x, y).
\]
Now, by (4.1) and \( \sum_{i=1}^p \lambda_i \rho^1_i + \sum_{s=1}^\gamma \rho^3_s \geq 0 \) we obtain
\[
\sum_{s=1}^\gamma \sum_{j \in I_s} v_j (d h_j)^+(y, \eta(x, y)) > \sum_{i=0}^p \lambda_i \rho^1_i \theta^2(x, y) \geq - \sum_{s=1}^\gamma \rho^3_s \theta^2(x, y)
\]
which is a contradiction to (4.10). Thus the theorem is proved. \( \square \)

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References