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Fréchet differentiability of the solutions of a semilinear abstract Cauchy problem

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Abstract

Sufficient conditions are found under which the solutions $z(t;q)$ of a semilinear abstract Cauchy problem of the form $\frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t))$ are Fréchet differentiable with respect to the parameter q. An explicit form is provided for the sensitivity equation satisfied by the Fréchet derivative $D_q z(t;q)$.

Keywords: Abstract Cauchy problem; Analytic semigroup; Infinitesimal generator; Fréchet differentiability; Fréchet derivative

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1. Introduction

In this article we consider the problem of dependence on an unknown parameter q of the solution $z(t;q)$ of the semilinear abstract Cauchy problem

$$(\mathcal{P})_q \quad \begin{cases} \frac{d}{dt}z(t) = A(q)z(t) + F(q,t,z(t)), & z(t) \in Z, \\ z(0) = z_0, & t \in [0,T], \end{cases}$$

where Z, a Banach space, $q \in Q_{\mathrm{ad}} \subset Q$, a normed linear space (Q_{ad} is an open subset of Q), and $A(q)$ is the infinitesimal generator of an analytic semigroup $T(t;q)$ on Z for all $q \in Q_{\mathrm{ad}}$. The spaces Z and Q are referred to as the state space and the parameter space, respectively, while Q_{ad} will be referred to as the admissible parameter set. The set Q_{ad} reflects the fact that sometimes not all elements of Q are "admissible" for the particular problem at hand, although quite often one has $Q_{\mathrm{ad}} = Q$.

Parameter identification problems for system $(\mathcal{P})_q$ and other similar type of equations [2,5,7] are usually solved by direct methods such as quasilinearization. For the application of these methods it is essential that solutions be differentiable with respect to the parameter q. For a concrete application of quasilinearization in a model similar to $(\mathcal{P})_q$ for the dynamics of Shape Memory Alloys see [10].

In 1977, Clark and Gibson [4] analyzed the differentiability of solutions in linear abstract Cauchy problems of the type

$$\frac{d}{dt}z(t) = A(q)z(t) + u(t),$$

where $A(q)$ generates a strongly continuous semigroup and $A(q) = A + B(q)$ where $B(q)$ is assumed to be bounded. That is, the dependence on q comes through a bounded component of $A(q)$.

Later on, in 1982 [1] this problem was studied under weaker assumptions, allowing for the parameter q to appear in unbounded terms of $A(q)$.

In 2000 Burns et al. [3] derived conditions under which the solutions of nonlinear Cauchy problems of the type

$$\frac{d}{dt}z(t) = Az(t) + F\big(q,t,z(t)\big),$$

are differentiable with respect to the parameter q. In this case, the parameter q was not allowed to appear in the linear part of the equation.

In this article we shall obtain conditions under which the solutions of the general abstract Cauchy problem $(\mathcal{P})_q$ are Fréchet differentiable with respect to q. To our knowledge, this problem has never been dealt with before. Moreover, we will prove that, under certain conditions, the corresponding Fréchet derivatives are solutions of particular nonhomogeneous evolution equations called the "sensitivity equations." We will provide an explicit form for these equations.

2. Preliminary results

Throughout this paper we shall consider the following standing hypotheses:

H1: There exist $\varepsilon_0 > 0$ such that the type of $T(t;q)$, call it w_q, is less than or equal to $-\varepsilon_0$ for all $q \in Q_{\text{ad}}$ and there exists $C_q > 0$ such that $\|T(t;q)\| \leqslant C_q e^{-\varepsilon_0 t}$ for all $t \geqslant 0$ and $q \in Q_{\text{ad}}$. The constant C_q depends on q but it can be chosen independent of q on compact subsets of Q_{ad}.

Note. Although we will make explicit use of the hypothesis $w_q \leqslant -\varepsilon_0$ for $q \in Q_{\text{ad}}$, this can be relaxed by requiring only that $\sup_{q\in Q_{\text{ad}}} w_q < \infty$.

H2: $\mathcal{D}(A(q)) = D$ is independent of q and D is a dense subspace of Z.

For $q \in Q_{\text{ad}}$, let $\sigma(A(q))$, $\rho(A(q))$ denote the spectrum and resolvent, respectively, of the operator $A(q)$. Since $w_q \leqslant -\varepsilon_0$, we have $\sup\{\text{Re}(\lambda), \lambda \in \sigma(A(q))\} \leqslant -\varepsilon_0$. For $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) > -\varepsilon_0$ the fractional powers $(\lambda I - A(q))^\delta$ of $\lambda I - A(q)$ are well defined, closed, linear, invertible operators in Z when $\delta \in [0,1]$ (see [13, Section 2.6]). We shall denote by $Z_{q,\delta}$ the space $\mathcal{D}((-A(q))^\delta)$ embedded with the norm of the graph of $(-A(q))^\delta$. For fixed δ, these spaces are all the same, independent of q, since $\mathcal{D}((-A(q))^\delta) = [D, Z]_{1-\delta}$ (the real interpolation space of order $1-\delta$ between D and Z), in the sense of an isomorphism (see [8, Corollary 2.2.3]). Hence, for all $q \in Q$, these spaces are all set theoretically equal and topologically isomorphic. In order to simplify the notation, we shall then denote $\mathcal{D}((-A(q))^\delta)$ with D_δ and $Z_{q,\delta}$ with Z_δ. Now, since $0 \in \rho(A(q))$, it follows that the graph norm is equivalent to $\|z\|_{q,\delta} = \|(-A(q))^\delta z\|$. Also, there exists a constant M_q such that $\|(-A(q))^\delta T(t;q)\| \leqslant M_q \frac{e^{-\epsilon_0 t}}{t^\delta}$, for all $t > 0$ (see [13, Theorem 2.6.13]).

H3: There exists a constant $\delta \in (0,1)$ such that the mapping $F : Q_{\text{ad}} \times [0,T] \times Z_{q,\delta} \to Z$ is locally Lipschitz continuous in t and z; i.e., for any $q \in Q_{\text{ad}}$ and any bounded subset U of $[0,T] \times Z_\delta$ there exists a constant $L = L(q,U)$ such that for $i = 1, 2$,

$$\left\| F(q,t_1,z_1) - F(q,t_2,z_2) \right\|_Z \leqslant L\big(|t_1 - t_2| + \|z_1 - z_2\|_{q,\delta}\big)$$

for $(t_i, z_i) \in U$, where the constant L can be chosen independent of q on any compact subset of Q_{ad}.

This regularity condition guarantees existence and uniqueness of solutions of problem $(\mathcal{P})_q$, provided that the initial condition z_0 is in Z_δ. See [12] and [11] for details.

We will now state and prove two results that will be needed later on.

Lemma 1. *Under hypotheses* H1 *and* H2, *for any* $q_1, q_2 \in Q_{\text{ad}}$ *and* $\delta \in (0,1)$ *we have*:

(i) $A(q_1)(-A(q_2))^{-\delta}$ *is bounded on* $Z_{1-\delta}$.

(ii) $A(q_1)T(\cdot;q_2) \in L^1(0,\infty;\mathcal{L}(Z))$ *and* $A(q_1)T(\cdot;q_2) \in L^\infty(\eta,\infty;\mathcal{L}(Z))$ *for each* $\eta > 0$.

(iii) $T(\cdot;q_2) \in L^1(0,\infty : \mathcal{L}(Z, Z_{q_1,\delta}))$ *and* $T(\cdot;q_2) \in L^\infty(\eta,\infty;\mathcal{L}(Z;Z_{q_1,\delta}))$ *for each* $\eta > 0$.

Proof. Since $A(q_2)$ is the infinitesimal generator of an analytic semigroup $T(t;q_2)$, this semigroup commutes with any fractional power of $-A(q_2)$. Hence, for any $z \in D((-A(q_2))^\delta)$ and any $t > 0$ we have

$$\begin{aligned} A(q_1)T(t;q_2)z &= A(q_1)T(t;q_2)\big(-A(q_2)\big)^{-\delta}\big(-A(q_2)\big)^{\delta} z \\ &= A(q_1)\big(-A(q_2)\big)^{-\delta}T(t;q_2)\big(-A(q_2)\big)^{\delta} z. \end{aligned}$$

Note that $(-A(q_2))^{-\delta} \in \mathcal{L}(Z)$ and $A(q_1)$ is closed. By the Closed Graph Theorem it follows that $A(q_1)(-A(q_2))^{-\delta}$ is bounded on $D((-A(q_2))^{1-\delta})$ (this proves (i)), which is dense in Z.

Hence, for $z \in D((-A(q_2))^\delta)$,

$$\begin{aligned} \big\|A(q_1)T(t;q_2)z\big\| &\leqslant \big\|A(q_1)\big(-A(q_2)\big)^{-\delta}\big\|_{\mathcal{L}(Z_{1-\delta},Z)}\big\|T(t;q_2)\big(-A(q_2)\big)^{\delta} z\big\| \\ &\leqslant C(q_1,q_2)M_{q_2}\frac{e^{-\varepsilon_0 t}}{t^\delta}\|z\|. \end{aligned} \tag{1}$$

Since $z \in D((-A(q_2))^\delta)$ is dense in Z and $A(q_1)T(t;q_2)$ is everywhere defined, it follows that (1) holds for all $z \in Z$.

Therefore

$$\big\|A(q_1)T(t;q_2)\big\|_{\mathcal{L}(Z)} \leqslant C(q_1,q_2)M_{q_2}t^{-\delta}e^{-\varepsilon_0 t}, \quad t > 0,$$

which clearly implies (ii) since $0 < \delta < 1$ and $\varepsilon_0 > 0$.

Finally,

$$\begin{aligned} \big\|\big(-A(q_1)\big)^{\delta}T(t;q_2)\big\|_{\mathcal{L}(Z)} &= \big\|\big(-A(q_1)\big)^{\delta-1}A(q_1)T(t;q_2)\big\|_{\mathcal{L}(Z)} \\ &\leqslant \big\|\big(-A(q_1)\big)^{\delta-1}\big\|_{\mathcal{L}(Z)}\big\|A(q_1)T(t;q_2)\big\|_{\mathcal{L}(Z)} \\ &\leqslant C(q_1)\big\|A(q_1)T(t;q_2)\big\|_{\mathcal{L}(Z)}. \end{aligned}$$

Thus, for $t > 0$,

$$\big\|T(t;q_2)\big\|_{\mathcal{L}(Z;Z_{q_1,\delta})} \leqslant C(q_1)\big\|A(q_1)T(t;q_2)\big\|_{\mathcal{L}(Z)} \leqslant \tilde{C}(q_1,q_2)t^{-\delta}e^{-\varepsilon_0 t}.$$

Hence (iii) follows and the desired result is established. □

Note. Although this result clearly implies that the operator $A(q_1)T(t;q_2)$ is bounded for $t > 0$, no uniform bound can be found for t near zero. For $q_1 = q_2 = q$, Lemma 1 implies, in particular, that the derivative $\frac{d}{dt}T(t;q)$ of the solution operator of the homogeneous equation associated with $(\mathcal{P})_q$ is integrable in a neighborhood of $t = 0$.

We will also require that $A(q)$ be "well-behaved" with respect to q in the following sense:

H4: For the δ in H3 and for any $q_1, q_2 \in Q_{\text{ad}}$ there are constants $M(q_1,q_2)$ and $C(q_1,q_2)$ both depending on q_1 and q_2, such that $\|(-A(q_1))^\delta(-A(q_2))^{-\delta}\|_{\mathcal{L}(Z)} \leqslant M(q_1,q_2)$, $\|A(q_1)[A(q_2)]^{-1} - I\| \leqslant C(q_1,q_2)$ and $C(q_1,q_2) \to 0$ as $q_1 \to q_2$.

For an example of a family of differential operators satisfying hypothesis H4 see [12, Lemma 2.4, Appendix].

Note. It is sufficient to request that H4 be true for $\delta = 1$, since in that case, it can be shown that the first inequality of H4 holds for any $0 < \delta \leqslant 1$ (see [9, Lemma 3.3]). We point out that we can establish Theorem 2 below replacing H4 with the following hypothesis:

H4′: For each $q_0 \in Q_{\text{ad}}$ there exists $C = C(q_0)$ such that

$$\left\| \big(A(q) - A(q_0)\big)z \right\| \leqslant C\|q - q_0\| \left\| A(q_0)z \right\|, \quad z \in D,\ q \in Q_{\text{ad}}.$$

Theorem 2. *Suppose* H1–H4 *hold. Then for any* $q_0 \in Q_{\text{ad}}$ *and* $\varepsilon > 0$, *there exists* $\tilde{\delta} > 0$ *such that*

$$\left\| A(q)T(\cdot, q_0)z - A(q_0)T(\cdot, q_0)z \right\|_{L^1(0,\infty;Z)} \leqslant \varepsilon \|z\|$$

for all $z \in Z$, *and for all* $q \in Q_{\text{ad}}$ *satisfying* $\|q - q_0\| < \tilde{\delta}$, *that is*

$$\left\| A(q)T(\cdot, q_0) - A(q_0)T(\cdot, q_0) \right\|_{L^1(0,\infty;\mathcal{L}(Z))} \leqslant \varepsilon,$$

or equivalently, for every fixed $q_0 \in Q_{\text{ad}}$ *the mapping from* Q *into* $L^1(0, \infty; \mathcal{L}(Z))$ *defined by*

$$q \to A(q)T(\cdot, q_0)$$

is continuous on Q_{ad}.

Proof. Let $\varepsilon > 0$, $q_0 \in Q_{\text{ad}}$. Then for $z \in Z$ we have

$$\begin{aligned}
&\left\| A(q)T(\cdot; q_0)z - A(q_0)T(\cdot; q_0)z \right\|_{L^1(0,\infty;Z)} \\
&\quad = \int_0^\infty \left\| A(q)T(t; q_0)z - A(q_0)T(t; q_0)z \right\|_Z dt \\
&\quad = \int_0^\infty \left\| \big(A(q)A(q_0)^{-1} - I\big)A(q_0)T(t; q_0)z \right\|_Z dt \\
&\quad \leqslant \left\| A(q)A(q_0)^{-1} - I \right\| \int_0^\infty \left\| A(q_0)T(t; q_0)z \right\|_Z dt \\
&\quad \leqslant C(q, q_0) \left\| A(q_0)T(\cdot, q_0) \right\|_{L^1(0,\infty;\mathcal{L}(Z))} \|z\| \\
&\quad \leqslant \varepsilon \|z\| \quad \text{for } \|q - q_0\| < \tilde{\delta},
\end{aligned}$$

where $C(q, q_0)$ is the constant in H4. We have used Lemma 1(ii) to obtain the next to last inequality while the last inequality follows by choosing $\tilde{\delta}$ small enough such that $C(q, q_0) \leqslant \varepsilon [\|A(q_0)T(\cdot; q_0)\|_{L^1(0,\infty;\mathcal{L}(Z))}]^{-1}$ and $q \in Q_{\text{ad}}$, for $\|q - q_0\| < \tilde{\delta}$. □

3. Main results

We now proceed to prove the main results. Recall that for $z_0 \in Z_\delta$, the solution $z(t;q)$ of $(\mathcal{P})_q$ satisfies the integral equation

$$\begin{aligned} z(t;q) &= T(t;q)z_0 + \int_0^t T(t-s;q)F\big(q,s,z(s;q)\big)\,ds \\ &\doteq T(t;q)z_0 + S(t;q), \quad t \in [0,T]. \end{aligned}$$

It is important to note here that while $S(t;q)$ is defined only for $t \in [0,T]$ (where solutions of $(\mathcal{P})_q$ are known to exist), $T(t;q)z_0$ is defined for all $t \geqslant 0$.

Consider now the following standing hypothesis concerning the q-regularity of $\frac{d}{dt}T(t;q)$:

H5: The mapping $q \to A(q)T(\cdot;q_0)$ from Q into $L^1(0,\infty;\mathcal{L}(Z))$ is Fréchet differentiable at q_0 for all $q_0 \in Q_{\text{ad}}$ (under H1–H4, we already know that this mapping is continuous, by virtue of Theorem 2).

Observation. No general conditions on the family of operators $A(q)$ are known to guarantee hypothesis H5. However, in some examples H5 does hold. For example, in the case of linear delay differential equations, with q denoting the vector of delays (see [1]).

Theorem 3. *Suppose* H1–H5 *hold. It follows that*

(i) *The mapping $q \to T(\cdot;q)$ from $Q \to L^\infty(0,\infty;\mathcal{L}(Z))$ is Fréchet differentiable at q_0, for each $q_0 \in Q_{\text{ad}}$. Moreover, for any $t > 0$ and $h \in Q_{\text{ad}}$ the q-Fréchet derivative of $T(t;q)$ evaluated at $q_0 \in Q_{\text{ad}}$ and applied to $h \in Q$, i.e., $[D_qT(t;q_0)]h$, is the solution $v_h(t)$ of the following linear IVP, the so-called "sensitivity equation" for $T(t;q)$, in $\mathcal{L}(Z)$,*

$$(S_1) \quad \begin{cases} \frac{d}{dt}v_h(t) = A(q_0)v_h(t) + [D_qA(q)T(t;q_0)|_{q=q_0}]h, \\ v_h(0) = 0, \end{cases}$$

and

(ii) *for every $q_0 \in Q_{\text{ad}}$, $D_qT(\cdot;q_0) = D_qT(\cdot;q)|_{q=q_0} \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))$.*

Proof. Let $q_0 \in Q_{\text{ad}}$. From Lemma 1(ii) and Theorem 2 it follows immediately that for $z_0 \in D$, $A(q)T(\cdot;q_0)z_0$, viewed as a mapping from Q into $L^1(0,\infty;Z)$ satisfies the hypotheses of Theorem 1 in [1] and therefore $T(t;q)z_0$ is Fréchet q-differentiable at q_0 as a mapping from Q into Z (in fact, our hypothesis H5 implies hypothesis H6 in [1], Lemma 1(ii) implies hypothesis H4 in [1] and Theorem 2 implies hypothesis H5 in [1]). Moreover, we have

$$\big[D_qT(t;q_0)z_0\big](\cdot) = \int_0^t T(t-s;q_0)\big[D_qA(q)T(s;q_0)z_0|_{q=q_0}\big](\cdot)\,ds. \tag{2}$$

It remains to show the Fréchet differentiability of the mapping $q \to T(\cdot;q)$ when viewed as a mapping form Q into $L^\infty(0,\infty;\mathcal{L}(Z))$, i.e., in the stronger $L^\infty(0,\infty;\mathcal{L}(Z))$ norm. Let $\varepsilon > 0$, $t > 0$ and $q_0 \in Q_{\mathrm{ad}}$. First note that for any $h \in Q$ with $\|h\| < \tilde{\delta}$ ($\tilde{\delta}$ as appearing in Theorem 2) we have

$$\begin{aligned}
&\frac{d}{dt}\big[T(t;q_0+h)z_0 - T(t;q_0)z_0\big] \\
&\quad = A(q_0+h)T(t;q_0+h)z_0 - A(q_0)T(t;q_0)z_0 \\
&\quad = A(q_0+h)\big[T(t;q_0+h)z_0 - T(t;q_0)z_0\big] + \big(A(q_0+h) - A(q_0)\big)T(t;q_0)z_0.
\end{aligned}$$

From Theorem 2 we have $(A(q_0+h) - A(q_0))T(\cdot;q_0)z_0 \in L^1(0,\infty;Z)$. It follows (see [13, Corollary 2.2]) that

$$\begin{aligned}
&T(t;q_0+h)z_0 - T(t;q_0)z_0 \\
&\quad = \int_0^t T(t-s;q_0+h)\big(A(q_0+h) - A(q_0)\big)T(s;q_0)z_0\,ds.
\end{aligned} \tag{3}$$

Therefore, for all $h \in Q$ with $\|h\| < \tilde{\delta}$, we have

$$\begin{aligned}
&\big\|T(t;q_0+h)z_0 - T(t;q_0)z_0\big\|_Z \\
&\quad \leqslant \int_0^t M_{q_0+h}e^{-\varepsilon_0(t-s)}\big\|\big(A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0)\big)z_0\big\|_Z\,ds \\
&\quad \leqslant C\big\|\big(A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0)\big)z_0\big\|_{L^1(0,\infty;Z)} \\
&\quad \leqslant C\varepsilon\|z_0\|_Z,
\end{aligned}$$

where the last inequality holds by virtue of H5. Thus for $t > 0$,

$$\big\|T(t;q_0+h) - T(t;q_0)\big\|_{\mathcal{L}(Z)} \leqslant C\varepsilon \quad \text{for } \|h\| < \tilde{\delta}, \tag{4}$$

and, since the constant C above does not depend on t,

$$\big\|T(\cdot;q_0+h) - T(\cdot;q_0)\big\|_{L^\infty(0,\infty;\mathcal{L}(Z))} \leqslant C\varepsilon \quad \text{for } \|h\| < \tilde{\delta}.$$

Hence we have the estimate

$$\begin{aligned}
&\left\|T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0)\big[D_qA(q)T(s;q_0)|_{q=q_0}\big]h\,ds\right\|_{\mathcal{L}(Z)} \\
&\quad = \bigg\|\int_0^t \big\{T(t-s;q_0+h)\big(A(q_0+h) - A(q_0)\big)T(s;q_0) \\
&\qquad - T(t-s;q_0)\big[D_qA(q)T(s;q_0)|_{q=q_0}\big]h\big\}\,ds\bigg\|_{\mathcal{L}(Z)}
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \int_0^t \left[T(t-s;q_0+h) - T(t-s;q_0)\right]\left(A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0)\right) ds \right. \\
&\quad + \int_0^t T(t-s;q_0)\left[A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0) \right. \\
&\quad \left. \left. - \left[D_q A(q)T(s;q_0)|_{q=q_0}\right]h\right] ds \right\|_{\mathcal{L}(Z)} \\
&\leqslant \int_0^t \left\|T(t-s;q_0+h) - T(t-s;q_0)\right\|_{\mathcal{L}(Z)} \\
&\quad \times \left\|A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0)\right\|_{\mathcal{L}(Z)} ds \\
&\quad + \int_0^t \left\|T(t-s;q_0)\right\|_{\mathcal{L}(Z)} \left\|A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(s;q_0)|_{q=q_0}\right]h\right\|_{\mathcal{L}(Z)} ds \\
&\leqslant \varepsilon C \int_0^t \left\|A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0)\right\|_{\mathcal{L}(Z)} ds \\
&\quad + C \int_0^t \left\|A(q_0+h)T(s;q_0) - A(q_0)T(s;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(s;q_0)|_{q=q_0}\right]h\right\|_{\mathcal{L}(Z)} ds \\
&= \varepsilon C \left\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0)\right\|_{L^1(0,t;\mathcal{L}(Z))} \\
&\quad + C \left\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,t;\mathcal{L}(Z))} \\
&\leqslant \varepsilon C \left\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,\infty;\mathcal{L}(Z))} \\
&\quad + \varepsilon C \left\|\left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,\infty;\mathcal{L}(Z))} \\
&\quad + C \left\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,\infty;\mathcal{L}(Z))} \\
&= (\varepsilon+1)C \left\|A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) \right. \\
&\quad \left. - \left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,\infty;\mathcal{L}(Z))} \\
&\quad + \varepsilon C \left\|\left[D_q A(q)T(\cdot;q_0)|_{q=q_0}\right]h\right\|_{L^1(0,\infty;\mathcal{L}(Z))}.
\end{aligned} \tag{5}
$$

Now by hypothesis H5 for the given $\varepsilon > 0$ there exists $\xi > 0$ such that

$$\big\| A(q_0+h)T(\cdot;q_0) - A(q_0)T(\cdot;q_0) - \big[D_q A(q)T(\cdot;q_0)|_{q=q_0}\big]h\big\|_{L^1(0,\infty;\mathcal{L}(Z))} \leqslant \varepsilon\|h\| \tag{6}$$

for $\|h\| < \xi$.

Also, since $D_q A(q)T(\cdot;q_0)|_{q=q_0} \in \mathcal{L}(Q, L^1(0,\infty;\mathcal{L}(Z)))$, there exists M, $0 < M < \infty$, such that

$$\big\| D_q A(q)T(\cdot,q_0)|_{q=q_0}\big\|_{\mathcal{L}(Q,L^1(0,\infty,\mathcal{L}(Z)))} \leqslant M. \tag{7}$$

Finally, employing (6) and (7) in (5), we get that for $\|h\| < \min(\tilde{\delta},\xi)$,

$$\left\| T(t;q_0+h) - T(t;q_0) - \int_0^t T(t-s;q_0)\big[D_q A(q)T(s;q_0)|_{q=q_0}\big]h\,ds \right\|_{\mathcal{L}(Z)} \leqslant (\varepsilon+1)C\varepsilon\|h\| + \varepsilon CM\|h\| \leqslant K\varepsilon\|h\|.$$

Here, the constant K depends on q_0 (and also on $\tilde{\delta}$ and ξ), but not on t. Hence the mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Z))$ defined by

$$q \to T(\cdot;q)$$

is Fréchet q-differentiable at q_0 and

$$\big[D_q T(t;q_0)\big](\cdot) = \int_0^t T(t-s;q_0)\big[D_q A(q)T(s;q_0)|_{q=q_0}\big](\cdot)\,ds. \tag{8}$$

It is therefore clear that for every $h \in Q$, the Fréchet derivative $[D_q T(t;q_0)]h$ is in fact the solution $v_h(t)$ of the IVP (S_1) in $\mathcal{L}(Z)$. Since $q_0 \in Q_{\mathrm{ad}}$ is arbitrary, part (i) of the theorem follows.

To prove (ii) we first note that by H5, for $q_0 \in Q_{\mathrm{ad}}$, $D_q A(q)T(\cdot;q_0)|_{q=q_0} \in \mathcal{L}(Q; L^1(0,\infty;\mathcal{L}(Z)))$ and thus there exists $C = C(q_0)$ such that for $h \in Q$,

$$\big\| D_q A(q)T(\cdot;q_0)|_{q=q_0}h\big\|_{L^1(0,\infty;\mathcal{L}(Z))} \leqslant C(q_0)\|h\|. \tag{9}$$

Now, it follows from (8) that for $t > 0$, $q_0 \in Q_{\mathrm{ad}}$ and $h \in Q$,

$$\begin{aligned}\big\|\big[D_q T(t;q_0)\big]h\big\|_{\mathcal{L}(Z)} &\leqslant M_{q_0}\int_0^t \big\|D_q A(q)T(s;q_0)|_{q=q_0}h\big\|_{\mathcal{L}(Z)}\,ds \\ &\leqslant M_{q_0}\big\|D_q A(q)T(\cdot;q_0)|_{q=q_0}h\big\|_{L^1(0,\infty;\mathcal{L}(Z))} \\ &\leqslant M_{q_0}C(q_0)\|h\| \\ &= \tilde{C}(q_0)\|h\|.\end{aligned}$$

Thus

$$\big\|D_q T(t;q_0)\big\|_{\mathcal{L}(Q;\mathcal{L}(Z))} \leqslant \tilde{C}(q_0),$$

and since $\tilde{C}(q_0)$ does not depend on $t > 0$, it follows that $D_q T(\cdot;q_0) \in L^\infty(0,\infty; \mathcal{L}(Q;\mathcal{L}(Z)))$. □

The next two theorems show that under slightly stronger assumptions on the mapping $q \to A(q)T(\cdot; q_0)$, it is possible to obtain the Lipschitz continuity of the mapping $q \to D_q T(\cdot; q_0)$ as a mapping from Q into $L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ and from Q into $L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z, Z_\delta)))$. More precisely, consider the following hypothesis:

H6: The mapping $q \to D_q A(q)T(\cdot; q_0)$ from Q into $L^1(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{\mathrm{ad}}$.

Theorem 4. *Let $q_0 \in Q_{\mathrm{ad}}$ and assume hypotheses* H1–H6 *hold. Then the mapping $q \to D_q T(\cdot; q_0)$ from Q into $L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$ is locally Lipschitz continuous at q_0.*

Proof. First of all, note that hypotheses H1–H5 imply, by virtue of Theorem 3(ii), that $D_q T(\cdot; q_0) \in L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$. Now, let $t > 0$, $q_0 \in Q_{\mathrm{ad}}$, select $h \in Q$ such that $\|h\| < \tilde{\delta}$ ($\tilde{\delta}$ as appearing in Theorem 2) and let us denote $G_q(t; q_0)(\cdot) = D_q A(q)T(t; q_0)|_{q=q_0}(\cdot) \in \mathcal{L}(Q, \mathcal{L}(Z))$. Theorem 3 together with the appropriate choice of $\alpha(h)$, $0 \leqslant |\alpha(h)| \leqslant 1$, yield

$$
\begin{aligned}
&\big\| D_q T(t; q_0 + h)(\cdot) - D_q T(t; q_0)(\cdot) \big\|_{\mathcal{L}(Q;\mathcal{L}(Z))} \\
&\quad = \bigg\| \int_0^t \big[T(t-s; q_0+h) G_q(s; q_0+h)(\cdot) - T(t-s; q_0) G_q(s; q_0)(\cdot) \big]\, ds \bigg\|_{\mathcal{L}(Q,\mathcal{L}(Z))} \\
&\quad \leqslant \int_0^t \big\| T(t-s; q_0+h) \big[G_q(s; q_0+h) - G_q(s; q_0) \big](\cdot) \big\|_{\mathcal{L}(Q;\mathcal{L}(Z))}\, ds \\
&\qquad + \int_0^t \big\| \big(T(t-s; q_0+h) - T(t-s; q_0) \big) G_q(s; q_0)(\cdot) \big\|_{\mathcal{L}(Q;\mathcal{L}(Z))}\, ds \\
&\quad \leqslant M_{q_0+h} \int_0^t e^{-\varepsilon_0(t-s)} \big\| G_q(s; q_0+h) - G_q(s; q_0) \big\|_{\mathcal{L}(Q,\mathcal{L}(Z))}\, ds \\
&\qquad + \int_0^t \big\| D_q T\big(t-s; q_0 + \alpha(h)h\big) G_q(s; q_0)(\cdot) h \big\|_{\mathcal{L}(Q,Z)}\, ds \\
&\quad \leqslant M_{q_0+h} \big\| G_q(\cdot; q_0+h) - G_q(\cdot; q_0) \big\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))} \\
&\qquad + \big\| D_q T\big(\cdot; q_0 + \alpha(h)h\big) \big\|_{L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))} \big\| G_q(\cdot; q_0) \big\|_{L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z)))} \|h\| \\
&\quad \leqslant C\|h\|,
\end{aligned}
$$

where the last inequality follows from H6, by the fact that $D_q T(\cdot, q) \in L^\infty(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$, which is provided by Theorem 3(ii), and by the fact that $G_q(\cdot, q_0) \in L^1(0,\infty; \mathcal{L}(Q; \mathcal{L}(Z)))$, which is a result of H6. Here the constant C depends on q_0 and h but it can be chosen independent of them on Q-bounded sets. We then have the desired result. □

As we will see later on, in order to obtain the q-Fréchet differentiability of $S(\cdot;q)$ we will need stronger regularity results for the mapping $q \to D_q T(\cdot;q_0)$ than the one just obtained in Theorem 4. In particular, we will need the local Lipschitz continuity of this mapping when viewed as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$. This can be achieved by requiring slightly stronger assumptions on the mapping $q \to D_q A(q)T(\cdot;q_0)$ than that found in H6. More precisely, consider the following hypothesis:

H7: For every $q_0 \in Q_{\mathrm{ad}}$, $D_q A(q)T(\cdot;q_0)|_{q=q_0} \in L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$ and the mapping $q \to D_q A(q)T(\cdot;q_0)$ from Q into $L^1(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$ is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{\mathrm{ad}}$.

Clearly H7 implies H6 (since the Z_δ-norm is stronger that the Z-norm).

Theorem 5. *Assume* H1–H5 *and* H7 *hold. Then, for all $q_0 \in Q_{\mathrm{ad}}$, we have that $D_q T(\cdot;q_0) \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$ and, moreover, the mapping $q \to D_q T(\cdot;q)$ from Q into the space $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$ is locally Lipschitz continuous at q_0.*

Proof. Let $t>0$, $z\in Z$, $h\in Q$. Then it follows that

$$
\begin{aligned}
&\big\|\big[D_q T(t;q_0)h\big]z\big\|_{Z_\delta}\\
&\quad= \big\|\big(-A(q_0)\big)^\delta\big(\big[D_q T(t;q_0)\big]h\big)z\big\|_Z\\
&\quad= \Bigg\|\big(-A(q_0)\big)^\delta\int_0^t T(t-s;q_0)\big\{\big[D_q A(q)T(s;q_0)|_{q=q_0}\big]h\big\}z\,ds\Bigg\|_Z\\
&\quad= \Bigg\|\int_0^t T(t-s;q_0)\big(-A(q_0)\big)^\delta\big[D_q A(q)T(s;q_0)|_{q=q_0}\big]hz\,ds\Bigg\|_Z\\
&\quad\leqslant \int_0^t \Big\|T(t-s;q_0)\big(-A(q_0)\big)^\delta\big[D_q A(q)T(s;q_0)|_{q=q_0}\big]hz\Big\|_Z\,ds\\
&\quad\leqslant M_{q_0}\int_0^t e^{-\varepsilon_0(t-s)}\big\|\big(-A(q_0)\big)^\delta D_q A(q)T(s;q_0)|_{q=q_0}\big\|_{\mathcal{L}(Q;\mathcal{L}(Z))}\|h\|\,\|z\|_Z\,ds\\
&\quad\leqslant M_{q_0}\|h\|\,\|z\|_Z\int_0^t \big\|\big(-A(q_0)\big)^\delta D_q A(q)T(s;q_0)|_{q=q_0}\big\|_{\mathcal{L}(Q;\mathcal{L}(Z))}\,ds\\
&\quad= M_{q_0}\|h\|\,\|z\|_Z\int_0^t \big\|D_q A(q)T(s;q_0)|_{q=q_0}\big\|_{\mathcal{L}(Q;\mathcal{L}(Z,Z_\delta))}\,ds\\
&\quad= M_{q_0}\|h\|\,\|z\|_Z\big\|D_q A(q)T(\cdot;q_0)|_{q=q_0}\big\|_{L^1(0,t;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))}\\
&\quad\leqslant C(q_0)\|h\|\,\|z\|_Z.
\end{aligned}
$$

The last inequality above follows from H7.

Hence,

$$\left\| D_q T(t;q_0)h \right\|_{\mathcal{L}(Z;Z_\delta)} \leqslant C(q_0)\|h\|,$$

and we have

$$\left\| D_q T(t;q_0) \right\|_{\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta))} \leqslant C(q_0).$$

Since the constant $C(q_0)$ does not depend on $t > 0$, it follows that $D_q T(\cdot;q_0) \in L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$. The Lipschitz continuity of this mapping is obtained immediately by following exactly the same steps as in Theorem 4. □

We will see next that this result implies that $q \to T(\cdot;q)$ is Fréchet differentiable as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))$. In fact we have the following:

Theorem 6. *Under the same hypotheses of Theorem* 5*, $T(\cdot;q)$ is Fréchet differentiable at q_0, for each $q_0 \in Q_{\mathrm{ad}}$, when viewed as a mapping from Q into $L^\infty(0,\infty;\mathcal{L}(Z;Z_\delta))$.*

Proof. Let $q_0 \in Q_{\mathrm{ad}}$. Then for $h \in Q$ with $\|h\| < \tilde{\delta}$ so that $q_0 + \alpha h \in Q_{\mathrm{ad}}$, for all α satisfying $|\alpha| \leqslant 1$, $\beta(h)$ appropriately chosen, $0 \leqslant |\beta(h)| \leqslant 1$, and any $t > 0$ we can write

$$\begin{aligned}
&\left\| T(t;q_0+h) - T(t;q_0) - \left[D_q T(t;q_0)\right]h \right\|_{\mathcal{L}(Z;Z_\delta)} \\
&\quad = \left\| \left[D_q T\left(t;q_0+\beta(h)h\right)\right]h - \left[D_q T(t;q_0)\right]h \right\|_{\mathcal{L}(Z;Z_\delta)} \\
&\quad \leqslant \left\| D_q T\left(t;q_0+\beta(h)h\right) - D_q T(t;q_0) \right\|_{\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta))} \|h\| \\
&\quad \leqslant \left\| D_q T\left(\cdot;q_0+\beta(h)h\right) - D_q T(\cdot;q_0) \right\|_{L^\infty(0,\infty;\mathcal{L}(Q;\mathcal{L}(Z;Z_\delta)))} \|h\| \\
&\quad \leqslant C(q_0)\left\|\beta(h)h\right\| \|h\| \\
&\quad \leqslant C(q_0)\|h\|^2 \\
&\quad \leqslant C(q_0)\epsilon\|h\| \quad \text{for } \|h\| < \epsilon, \text{ for all } \epsilon \text{ such that } 0 < \epsilon \leqslant \tilde{\delta}.
\end{aligned}$$

Here we have applied Theorem 5 to obtain the above estimate. The desired differentiability is established. □

It is important to note that Theorems 3 and 6 imply that the solution $z_h(t;q)$ of the linear homogeneous problem associated to $(\mathcal{P})_q$ is Fréchet differentiable with respect to q, both as a mapping into Z and into Z_δ, respectively. Theorems 4 and 5 imply, moreover, that the corresponding Fréchet derivatives are locally Lipschitz continuous.

We state below a generalization of Growall's lemma for singular kernels. The proof can be found in [6, Lemma 7.1.1]. This lemma will be of fundamental importance for our next results.

Lemma 7. *Let L, T, δ be positive constants, $\delta < 1$, $a(t)$ a real valued, nonnegative, locally integrable function on $[0,T]$ and $\mu(t)$ a real-valued function on $[0,T]$ satisfying*

$$\mu(t) \leqslant a(t) + L\int_0^t \frac{\mu(s)}{(t-s)^\delta}\,ds, \quad t \in [0,T].$$

Then, there exists a constant K depending only on δ such that

$$\mu(t) \leqslant a(t) + KL \int_0^t \frac{a(s)}{(t-s)^\delta}\, ds, \quad t \in [0, T].$$

Observation 1. From this point on $\|\cdot\|_\delta$ shall denote the norm $\|\cdot\|_{q_0,\delta} = \|(-A(q_0))^\delta(\cdot)\|_Z$. Recall that for $q_0 \in Q$, all these norms are equivalent. Moreover, it can be easily shown that all these norms are uniformly equivalent for q_0 in any subset of Q for which the constant C in H4 can be chosen independent of q_1 and q_2. For example, if C in H4 can be chosen independent of q_1 and q_2 on compact subsets of Q, then all these norms are uniformly equivalent for q_0 in compact subsets of Q.

Observation 2. Since $(-A(q_0))^\delta T(t; q_0+h) = (-A(q_0))^\delta(-A(q_0+h))^{-\delta} T(t; q_0+h) \times (-A(q_0+h))^\delta$ on D_δ, it follows from H4 that for all $t > 0$,

$$\left\| \left(-A(q_0)\right)^\delta T(t; q_0+h) \right\| \leqslant \frac{Ce^{-\varepsilon_0 t}}{t^\delta}.$$

Here the constant C depends on q_0 and h, but it can be chosen independent of h and q_0 on any subset of Q where the constant C in H4 can be chosen independent of q_1 and q_2.

Recall now that the solution $z(t; q)$ of $(\mathcal{P})_q$ satisfies the integral equation $z(t; q) = T(t; q)z_0 + S(t; q)$ where $S(t; q) \doteq \int_0^t T(t-s; q)F(q, s, z(s; q))\, ds$. Before proving the Fréchet differentiability of the mapping $q \to S(\cdot; q)$ from $Q \to L^\infty(0, T; Z_\delta)$, we will show that if $F(q, t, z)$ satisfies appropriate regularity properties, such a mapping is locally Lipschitz continuous at q_0, for all $q_0 \in Q_{\mathrm{ad}}$. We will need this result later.

Consider the following hypothesis:

H8: The mapping $q \to F(q, \cdot; z)$ from Q into $L^\infty(0, T; Z)$ is locally Lipschitz continuous for all $z \in Z_\delta$ with Lipschitz constant independent of z on Z_δ-bounded sets.

Theorem 8. *Let $q_0 \in Q_{\mathrm{ad}}$, $z_0 \in D_\delta$ and assume* H1–H5, H7 *and* H8 *hold. Then the mapping $q \to S(\cdot; q)$ from $Q \to L^\infty(0, T; Z_\delta)$ is locally Lipschitz continuous at q_0.*

Proof. Let $t \in [0, T]$, $q_0 \in Q_{\mathrm{ad}}$. Since Q_{ad} is open, there exists a constant $\gamma_1 > 0$ such that $q_0 + h \in Q_{\mathrm{ad}}$ for all $\|h\| < \gamma_1$. Then, from Theorem 3 we have

$$\begin{aligned} S(t; q_0+h) - S(t; q_0) &= \int_0^t \big[T(t-s; q_0+h)F\big(q_0+h, s, z(s; q_0+h)\big) \\ &\quad - T(t-s; q_0)F\big(q_0, s, z(s; q_0)\big)\big]\, ds \\ &= \int_0^t T(t-s; q_0+h)\big[F\big(q_0+h, s, z(s; q_0+h)\big) - F\big(q_0, s, z(s; q_0+h)\big)\big]\, ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t T(t-s; q_0+h)\big[F\big(q_0, s, z(s; q_0+h)\big) - F\big(q_0, s, z(s; q_0)\big)\big]\,ds \\
&+ \int_0^t \big[T(t-s; q_0+h) - T(t-s; q_0)\big]F\big(q_0, s, z(s; q_0)\big)\,ds \\
= &\int_0^t T(t-s; q_0+h)\big[F\big(q_0+h, s, z(s; q_0+h)\big) - F\big(q_0, s, z(s; q_0+h)\big)\big]\,ds \\
&+ \int_0^t T(t-s; q_0+h)\big[F\big(q_0, s, z(s; q_0+h)\big) - F\big(q_0, s, z(s; q_0)\big)\big]\,ds \\
&+ \int_0^t D_q T\big(t-s; q_0+\beta(h)h\big)hF\big(q_0, s, z(s; q_0)\big)\,ds,
\end{aligned}
$$

provided $\|h\| \leqslant \gamma_1$, where $\beta(h)$ is an appropriately selected constant satisfying $0 \leqslant |\beta(h)| \leqslant 1$.

The above identity, together with H8, H3 and Theorem 5, provides the estimate

$$
\begin{aligned}
&\big\|S(t; q_0+h) - S(t; q_0)\big\|_\delta \\
&\leqslant \int_0^t \big\|T(t-s; q_0+h)\big\|_{\mathcal{L}(Z;Z_\delta)} \big\|F\big(q_0+h, s, z(s; q_0+h)\big) \\
&\quad - F\big(q_0, s, z(s; q_0+h)\big)\big\|_Z\,ds \\
&\quad + \int_0^t \big\|T(t-s; q_0+h)\big\|_{\mathcal{L}(Z;Z_\delta)} \big\|F\big(q_0, s, z(s; q_0+h)\big) - F\big(q_0, s, z(s; q_0)\big)\big\|_Z\,ds \\
&\quad + \big\|D_q T\big(\cdot; q_0+\beta(h)h\big)\big\|_{L^\infty(0,t;\mathcal{L}(Q;\mathcal{L}(Z,Z_\delta)))} \|h\| \int_0^t \big\|F\big(q_0, s, z(s; q_0)\big)\big\|_Z\,ds \\
&\leqslant \int_0^t \frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta} C_1\|h\|\,ds \\
&\quad + \int_0^t \frac{M_{q_0+h}e^{-\varepsilon_0(t-s)}}{(t-s)^\delta} L\big\|z(s; q_0+h) - z(s; q_0)\big\|_\delta + C_2\|h\| \\
&\leqslant C_3\|h\| + C_4 \int_0^t \frac{\|z(s; q_0+h) - z(s; q_0)\|_\delta}{(t-s)^\delta}\,ds
\end{aligned}
$$

$$= C_3\|h\| + C_4 \int_0^t \frac{\|T(s;q_0+h)z_0 - T(s;q_0)z_0 + S(s;q_0+h) - S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds$$

$$= C_3\|h\| + C_4 \int_0^t \frac{\|[D_qT(s;q_0+\beta(h)h)h]z_0 + S(s;q_0+h) - S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds$$

$$\leqslant C_5\|h\| + C_4 \int_0^t \frac{\|S(s;q_0+h) - S(s;q_0)\|_\delta}{(t-s)^\delta}\,ds.$$

The constants C_1 and C_2 exist and are independent of $t \in [0,T]$ since $z(s;q_0)$ is bounded for $s \in [0,T]$ and $F(q,s,z)$ is continuous in s and z. The constants C_3, C_4 and C_5 represent particular linear combinations of C_1 and C_2.

Hence, by Lemma 7, there exist a constant K such that

$$\begin{aligned}&\big\|S(t;q_0+h) - S(t;q_0)\big\|_\delta \\ &\quad\leqslant C_5\|h\| + KC_4C_5\|h\| \int_0^T \frac{1}{(t-s)^\delta}\,ds \doteq C_6\|h\|, \quad t \in [0,T],\end{aligned}$$

provided $\|h\| \leqslant \gamma_1$. The theorem follows. □

Observation. Note that this result together with Theorem 6 imply that the mapping $q \to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ is locally Lipschitz continuous at q_0.

We proceed now to prove the Fréchet differentiability of the mapping $q \to S(t;q)$, corresponding to the nonlinear part of problem $(\mathcal{P})_q$.

Consider the following hypothesis:

H9: The mapping $(q,z(\cdot)) \to F(q,\cdot,z(\cdot))$ from $Q_{\mathrm{ad}} \times L^1(0,T;Z_\delta)$ into $L^\infty(0,T;Z)$ is Fréchet differentiable in both variables, the mapping $(q,z(\cdot)) \to F_q(q,\cdot,z(\cdot))$ from $Q \times L^\infty(0,T;Z_\delta)$ into $L^\infty(0,T:\mathcal{L}(Q;Z_\delta))$ is locally Lipschitz continuous with respect to q and z, with Lipschitz constant independent of z on Z_δ-bounded sets and $F_z(q,\cdot,z(\cdot;q)) \in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$.

Theorem 9. *Let* $q_0 \in Q_{\mathrm{ad}}$, $z_0 \in D_\delta$ *and suppose* H1–H5, H7 *and* H9 *hold. Then the mapping* $q \to S(t;q) = \int_0^t T(t-s;q)F(q,s,z(s;q))\,ds$ *from* $Q \to L^\infty(0,T;Z_\delta)$ *is Fréchet differentiable at* q_0. *Moreover, for any* $t \in [0,T]$, *and any* $h \in Q_{\mathrm{ad}}$, $[D_qS(t;q_0)]h \doteq w_h(t)$ *satisfies the integral equation*

$$\begin{aligned}w_h(t) = \int_0^t \big\{&T(t-s;q_0)\big[F_q\big(q_0,s,z(s;q_0)\big)h + F_z\big(q_0,s,z(s;q_0)\big)\big[D_qT(s;q_0)z_0\big]h \\ &+ F_z\big(q_0,s,z(s;q_0)\big)w_h(s)\big] + \big[D_qT(t-s;q_0)F\big(q_0,s,z(s;q_0)\big)\big]h\big\}\,ds,\end{aligned} \tag{10}$$

and $w_h(t)$ is the solution of the following nonhomogeneous linear IVP, the so-called "sensitivity equation" for $S(t;q)$, in Z:

$$(S_2)\quad \begin{cases} \frac{d}{dt}w_h(t) = (A(q_0) + F_z(q_0,t,z(t;q_0)))w_h(t)F_q(q_0,t,z(t;q_0))h \\ \qquad\qquad + F_z(q_0,t,z(t;q_0))[D_qT(t;q_0)z_0]h \\ \qquad\qquad + \int_0^t D_qA(q)T(t-s;q_0)|_{q=q_0}hF(q_0,s,z(s;q_0))\,ds, \\ W_h(0) = 0. \end{cases}$$

Observation. Clearly hypothesis H9 is stronger than H8. This observation is important because in order to prove this theorem we will need to make use of the results in Theorem 8 for which H8 must hold.

Proof. Using the well-known variation of constants formula from semigroup theory, the sensitivity equation (S_1) for $T(t;q)$ given in Theorem 3 and recalling that $[D_qT(0;q_0)z]h = 0$ for $z \in Z$ and $h \in Q$, it follows immediately that the solution $w_h(t)$ of the (IVP) (S_2) satisfies the integral equation (10).

For $t \in (0,T]$ we write

$$\begin{aligned}
&S(t;q_0+h) - S(t;q_0) - w_h(t) \\
&\quad = \int_0^t \bigl\{T(t-s;q_0+h)F\bigl(q_0+h,s,z(s;q_0+h)\bigr) - T(t-s;q_0)F\bigl(q_0,s,z(s;q_0)\bigr) \\
&\qquad - T(t-s;q_0)\bigl[F_q\bigl(q_0,s,z(s;q_0)\bigr)h + F_z\bigl(q_0,s,z(s;q_0)\bigr)\bigl[T_q(s;q_0)z_0\bigr]h \\
&\qquad + F_z\bigl(q_0,s,z(s;q_0)\bigr)w_h(s)\bigr] - D_qT(t-s;q_0)F\bigl(q_0,s,z(s;q_0)\bigr)h\bigr\}\,ds \\
&\quad = \int_0^t T(t-s;q_0)\bigl[F\bigl(q_0+h,s,z(s;q_0)\bigr) - F\bigl(q_0,s,z(s;q_0)\bigr) \\
&\qquad - F_q\bigl(q_0,s,z(s;q_0)\bigr)h\bigr]\,ds \\
&\qquad + \int_0^t T(t-s;q_0)\bigl[F\bigl(q_0,s,z(s;q_0+h)\bigr) - F\bigl(q_0,s,z(s;q_0)\bigr) \\
&\qquad - F_z\bigl(q_0,s,z(s;q_0)\bigr)\bigl(z(s;q_0+h) - z(s;q_0)\bigr)\bigr]\,ds \\
&\qquad + \int_0^t T(t-s;q_0)F_z\bigl(q_0,s,z(s;q_0)\bigr)\bigl[S(s;q_0+h) - S(s;q_0) - w_h(s)\bigr]\,ds \\
&\qquad + \int_0^t T(t-s;q_0)F_z\bigl(q_0,z(s;q_0)\bigr)\bigl[\bigl[D_qT\bigl(s;q_0+\alpha(h)h\bigr)z_0\bigr]h \\
&\qquad - \bigl[D_qT(s;q_0)z_0\bigr]h\bigr]\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \bigl\{T(t-s;q_0+h)F\bigl(q_0,s,z(s;q_0)\bigr)-T(t-s;q_0)F\bigl(q_0,s,z(s;q_0)\bigr)\\
&-\bigl[D_qT(t-s;q_0)F\bigl(q_0,s,z(s;q_0)\bigr)\bigr]h\bigr\}\,ds\\
&+\int_0^t T(t-s;q_0+h)\bigl[F\bigl(q_0+h,s,z(s;q_0+h)\bigr)-F\bigl(q_0,s,z(s;q_0)\bigr)\bigr]\,ds\\
&-\int_0^t T(t-s;q_0)\bigl[F\bigl(q_0+h,s,z(s;q_0)\bigr)-2F\bigl(q_0,s,z(s;q_0)\bigr)\\
&+F\bigl(q_0,s,z(s;q_0+h)\bigr)\bigr]\,ds\\
\doteq{}&\sum_{i=1}^{7} I_i,
\end{aligned}
$$

where I_i is the ith term in the expression written above. In I_2, I_3 and I_4 we have made use of the fact that $z(s;q_0+h)-z(s;q_0)=[D_qT(s;q_0+\alpha(h)h)z_0]h+S(s;q_0+h)-S(s;q_0)$, for some appropriately chosen constant $\alpha(h)$ satisfying $0\leqslant|\alpha(h)|\leqslant 1$.

In what follows, C_i will denote a generic finite positive constant depending on q_0.

Let $\gamma_1>0$ be such that $q_0+\eta\in Q_{\mathrm{ad}}$ for all $\eta\in Q$ satisfying $\|\eta\|<\gamma_1$. Then for any $h\in Q_{\mathrm{ad}}$ with $\|h\|<\gamma_1$ we can write

$$
\begin{aligned}
I_6+I_7={}&\int_0^t T(t-s;q_0+h)\bigl[F\bigl(q_0+h,s,z(s;q_0+h)\bigr)\\
&-F\bigl(q_0,s,z(s;q_0+h)\bigr)\bigr]\,ds+\int_0^t \bigl[T(t-s;q_0+h)-T(t-s;q_0)\bigr]\\
&\times\bigl[F\bigl(q_0,s,z(s;q_0+h)\bigr)-F\bigl(q_0,s,z(s;q_0)\bigr)\bigr]\,ds\\
&-\int_0^t T(t-s;q_0)\bigl[F\bigl(q_0+h,s,z(s;q_0)\bigr)-F\bigl(q_0,s,z(s;q_0)\bigr)\bigr]\,ds\\
={}&\int_0^t T(t-s;q_0+h)F_q\bigl(q_0+\alpha_1(h)h,s,z(s;q_0+h)\bigr)h\,ds\\
&+\int_0^t \bigl[D_qT\bigl(t-s;q_0+\alpha_2(h)h\bigr)F_z\bigl(q_0,s,z_h^*(q_0)\bigr)\\
&\times\bigl(z(s;q_0+h)-z(s;q_0)\bigr)\bigr]h\,ds
\end{aligned}
$$

$$- \int_0^t T(t-s;q_0) F_q\big(q_0 + \alpha_3(h)h, s, z(s;q_0)\big) h\, ds,$$

where $0 \leqslant |\alpha_i(h)| \leqslant 1$, $i = 1, 2, 3$, and $z_h^*(q_0) \doteq z(s;q_0) + \beta(z(s;q_0+h) - z(s;q_0))$ for some $0 \leqslant |\beta| \leqslant 1$.

Continuing, we have

$$\begin{aligned}
I_6 + I_7 = {} & \int_0^t \big[T(t-s;q_0+h) - T(t-s;q_0)\big] F_q\big(q_0 + \alpha_1(h)h, s, z(s;q_0+h)\big) h \\
& + \int_0^t T(t-s;q_0)\big[F_q\big(q_0 + \alpha_1(h)h, s, z(s;q_0+h)\big) h \\
& - F_q\big(q_0 + \alpha_3(h)h, s, z(s;q_0)\big) h\big]\, ds \\
& + \int_0^t \big[D_q T\big(t-s;q_0+\alpha_2(h)h\big) F_z\big(q_0, s, z_h^*(q_0)\big)\big(z(s;q_0+h) \\
& - z(s;q_0)\big)\big] h\, ds \\
= {} & \int_0^t \big[D_q T\big(t-s;q_0+\alpha_2(h)h\big) F_q\big(q_0 + \alpha_1(h)h, s, z(s;q_0+h)\big) h\big] h\, ds \\
& + \int_0^t T(t-s;q_0)\big[F_q\big(q_0 + \alpha_1(h)h, s, z(s;q_0+h)\big) h \\
& - F_q\big(q_0 + \alpha_3(h)h, s, z(s;q_0)\big) h\big]\, ds \\
& + \int_0^t \big[D_q T\big(t-s;q_0+\alpha_2(h)h\big) F_z\big(q_0, s, z_h^*(q_0)\big)\big(z(s;q_0+h) \\
& - z(s;q_0)\big)\big] h\, ds.
\end{aligned}$$

Hence, by virtue of Theorem 8 and hypothesis H9, it follows that there exist positive constants C_1, C_2 and L, such that

$$\begin{aligned}
\|I_6 + I_7\|_\delta \leqslant {} & C_1 \|h\|^2 + \int_0^t \frac{L}{(t-s)^\delta} \big(\big|\alpha_1(h) - \alpha_3(h)\big| \|h\| \\
& + \big\|z(s;q_0+h) - z(s;q_0)\big\|_\delta\big) \|h\|\, ds \\
& + \int_0^t \frac{C_2}{(t-s)^\delta} \big\|z(s;q_0+h) - z(s;q_0)\big\|_\delta \|h\|\, ds \\
\leqslant {} & C_3 \|h\|^2, \quad \text{provided } \|h\| \leqslant \gamma_1,
\end{aligned} \tag{11}$$

where the last inequality follows from the Lipschitz continuity of the mapping $q \to z(\cdot;q)$ from Q into $L^\infty(0,T;Z_\delta)$ at q_0 (note the observation after Theorem 8).

Now let ε be a fixed positive constant. It follows from hypothesis H9 that there exist $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$\|I_1\|_\delta \leqslant \int_0^t \frac{C_4}{(t-s)^\delta}, \varepsilon\|h\|\,ds \leqslant C_5\varepsilon\|h\|, \tag{12}$$

provided $\|h\| \leqslant \gamma_2$, and

$$\begin{aligned}\|I_2\|_\delta &\leqslant \int_0^t \frac{C_6\varepsilon}{(t-s)^\delta}\big\|z(s;q_0+h)-z(s;q_0)\big\|_Z\,ds \\ &\leqslant C_7\varepsilon\|h\|,\end{aligned} \tag{13}$$

provided $\|h\| \leqslant \gamma_3$. The last inequality follows by virtue of the observation following Theorem 8 and the fact that $\|\cdot\|_Z \leqslant \|\cdot\|_\delta$.

With respect to I_3, since by H9 $F_z(q_0,\cdot,z(\cdot;q_0)) \in L^\infty(0,T;\mathcal{L}(Z;Z_\delta))$, we have that there exists a constant C_8 such that

$$\|I_3\|_\delta \leqslant C_8\int_0^t \frac{\|S(s;q_0+h)-S(s;q_0)-w_h(s)\|_\delta}{(t-s)^\delta}\,ds \tag{14}$$

where we have also used the fact that $\|\cdot\|_Z \leqslant \|\cdot\|_\delta$.

Similarly, by virtue of the local Lipschitz continuity of $D_qT(\cdot;q_0)$ (Theorem 4), there exist finite positive constants C_9 and γ_4 such that

$$\|I_4\|_\delta \leqslant \int_0^t \frac{C_9}{(t-s)^\delta}\big|\alpha(h)\big|\|h\|^2\,ds \leqslant C_{10}\|h\|^2, \quad \text{provided } \|h\| \leqslant \gamma_4. \tag{15}$$

Finally, from Theorem 6, there exist finite positive constants C_{10} and γ_5 such that

$$\begin{aligned}\|I_5\|_\delta &= \Bigg\|\int_0^t \big[T(t-s;q_0+h)-T(t-s;q_0)-D_qT(t-s;q_0)h\big] \\ &\quad\times F\big(q_0,s,z(s;q_0)\big)\,ds\Bigg\|_\delta \\ &\leqslant \big\|T(\cdot;q_0+h)-T(\cdot;q_0)-D_qT(\cdot;q_0)h\big\|_{L^\infty(0,t;\mathcal{L}(Z;Z_\delta))} \\ &\quad\times\int_0^t \big\|F\big(q_0,s,z(s;q_0)\big)\big\|_Z\,ds \\ &\leqslant C(q_0)\varepsilon\|h\|\int_0^t \big\|F\big(q_0,s,z(s;q_0)\big)\big\|_Z\,ds \leqslant C_{10}\varepsilon\|h\|,\end{aligned} \tag{16}$$

provided $\|h\| \leqslant \gamma_5$. Here we used H9 to obtain our final estimate.

From (11)–(16) we conclude that there exist finite positive constants C_{11}, C_{12}, and γ such that for $t \in [0, T]$ and $h \in Q_{\mathrm{ad}}$ with $\|h\| \leqslant \gamma$,

$$\begin{aligned}&\left\| S(t; q_0 + h) - S(t; q_0) - w_h(t) \right\|_\delta \\ &\quad \leqslant C_{11}\|h\| + C_{12} \int_0^t \frac{\|S(s; q_0 + h) - S(s; q_0) - w_h(s)\|_\delta}{(t - s)^\delta}\, ds.\end{aligned}$$

Hence, Lemma 7 provides

$$\begin{aligned}\left\| S(t; q_0 + h) - S(t; q_0) - w_h(t) \right\|_\delta &\leqslant C_{11}\varepsilon\|h\| + K C_{12} C_{11}\varepsilon\|h\| \int_0^t \frac{1}{(t - s)^\delta}\, ds \\ &\leqslant C_{13}\varepsilon\|h\|, \quad t \in [0, T],\ \|h\| \leqslant \gamma.\end{aligned}$$

We conclude that the mapping $q \to S(\cdot; q)$ from $Q \to L^\infty(0, T; Z_\delta)$ is Fréchet differentiable at q_0 and $w_h(t)$ is the Fréchet derivative of $S(t; q)$ at q_0, i.e., $D_q S(t; q_0) = w_h(t)$. □

Theorem 10. *Under the same hypotheses of Theorem* 9, *the mapping $q \to z(\cdot; q)$ from the admissible parameter set Q_{ad} into the solution space $L^\infty(0, T; Z_\delta)$, is Fréchet differentiable at q_0. Moreover, for any $h \in Q$, $t \in [0, T]$, the q-Fréchet derivative of $z(t; q)$ evaluated at q_0 and applied to h, i.e., $[D_q z(t; q_0)]h$ is the solution $v_h(t)$ of the following linear nonhomogeneous initial value problem in Z, the sensitivity equation for $z(t; q)$*:

$$(S) \quad \begin{cases} \frac{d}{dt} v_h(t) = (A(q_0) + F_z(q_0, t, z(t; q_0)))v_h(t) + F_q(q_0, t, z(t; q_0))h \\ \qquad\qquad + D_q A(q) T(t; q_0) z_0|_{q=q_0} h \\ \qquad\qquad + \int_0^t D_q A(q) T(t - s; q_0)|_{q=q_0} h F(q_0, s, z(s; q_0))\, ds, \\ v_h(0) = 0. \end{cases}$$

Proof. The Fréchet differentiability of $z(t; q) = T(t; q)z_0 + S(t; q)$ follows immediately from Theorems 6 and 9 and the sensitivity equation is readily obtained by combining the sensitivity equations (S_1) and (S_2). □

4. Conclusions and final remarks

In this article we have obtained sufficient conditions that guarantee that the solutions of the abstract semilinear Cauchy problem

$$(\mathcal{P})_q \quad \begin{cases} \frac{d}{dt} z(t) = A(q)z(t) + F(q, t, z(t)), & z(t) \in Z, \\ z(0) = z_0, & t \in [0, T] \end{cases}$$

are Fréchet differentiable with respect to the parameter q. This type of regularity results are needed for the implementation of direct methods for parameter identification like quasilinearization.

Some remarks are in order. In Theorems 1–6 all spaces were considered over the entire interval $[0, \infty)$. This was the case since the solution $T(t; q_0)z_0$ of the associated linear homogeneous initial value problem exists for all $t \in [0, \infty)$.

It is interesting to see in Theorems 2–4 how the L^∞ q-regularity of the solution of the associated linear problem is entirely driven by the L^1 q-regularity of the time derivative operator of the associated C_0-semigroup, namely of $A(q)T(\cdot; q_0)$.

For the q-regularity of the term in the solution corresponding to the nonlinear part of the equation, namely of $S(t; q_0)$, not only are smoothness conditions required on the nonlinear term $F(q, t, z)$ (H8 and H9) of $(\mathcal{P})_q$ but also stronger q-regularity conditions are required on $A(q)T(\cdot; q_0)$ (namely H6 and H7). These conditions guarantee the Fréchet differentiability of $T(\cdot; q_0)$ when viewed as a mapping from the parameter space Q into the space $L^\infty(0, \infty; \mathcal{L}(Q; \mathcal{L}(Z; Z_\delta)))$, where the stronger Z_δ-norm is needed.

It is possible to allow q-dependence on the norms in the state space Z if needed. However, the domains of the operators $A(q)$ cannot depend on q. No results are yet known for this varying domain case.

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