Finite $p$-groups with few minimal nonabelian subgroups

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1. Introduction

1. A $p$-group $G$ is said to be minimal nonabelian (for brevity, $A_1$-group), if $G$ is nonabelian but all its proper subgroups are abelian. Generalizing this notion, we call a $p$-group $G$ an $A_n$-group, $n \in \mathbb{N}$, if $G$ possesses a nonabelian subgroup of index $p^{n-1}$ but all its subgroups of index $p^n$ are abelian. Given a nonabelian $p$-group $G$, there is $n \in \mathbb{N}$ such that $G$ is an $A_n$-group. For example, a $p$-group $G$ of maximal class and order $p^m$ is an $A_{m-2}$-group; this result, following from Blackburn’s theory of $p$-groups of maximal class (see, for example, [3, §9]), is not trivial. (It follows that if all subgroups of order $p^3$ in a $p$-group $G$ of order $p^m$ are abelian, then the class of $G$ is at most $m - 2$.)

$A_1$-groups of prime power order were classified by L. Redei (see Lemma 1, below). $A_2$-groups of prime power order were classified by L. Kazarin in his unpublished thesis (note that his exposition did not give a full proof; for an elementary treatment, see [5, §§3–5]). I consider this paper as a nontrivial application of the above classification and...
Blackburn’s and Janko’s results on $p$-groups all of whose maximal subgroups are two-generator. For other applications of the above results, see [5, §§4–7].

We use the standard notation as in [2]. Only finite $p$-groups are considered; $p$ is a prime, $G$ is a $p$-group. Let $K_n(G)$ denote the $n$th member of the lower central series of $G$ so that $\text{cl}(G/K_n(G)) < n$.

Let $d(G)$ be the minimal number of generators of $G$ (= the rank of $G$); then $|G : \Phi(G)| = p^{d(G)}$, where $\Phi(G)$ is the Frattini subgroup of $G$. For $i = 1, \ldots, d(G)$, let $\Gamma_i = \{H < G : \Phi(G) \leq H, |G : H| = p^i\}$. In particular, $\Gamma_1$ is the set of maximal subgroups of $G$. If $H < G$, then $\Gamma_i(H)$ is the set of $i$th maximal subgroups of $H$ and $\Gamma_i^H$ is the set of those members of the set $\Gamma_i$ that contain $H$. Clearly, $\Gamma_i^H = \Gamma_1$ if and only if $H \leq \Phi(G)$.

Let $G'$ and $Z(G)$ be the derived subgroup and the center of $G$, respectively. Next, $C_G(M)$ and $N_G(M)$ is the centralizer and the normalizer of a subset $M$ in $G$, respectively.

Let $D_{2n}$, $Q_{2n}$, $SD_{2n}$ be the dihedral, generalized quaternion and semidihedral group of order $2^n$, respectively. A 2-group of maximal class is one of the above groups. Next, $C_{p^n}$ and $E_{p^n}$ is the cyclic and elementary abelian group of order $p^n$, respectively. Let $M_{p^n} = \langle x, y \mid x^{p^{n-1}} = y^p = 1, x^y = x^{1+p^{n-2}} \rangle$, where $n \geq 3$ if $p > 2$ and $n > 3$ is $p = 2$.

Let $\alpha_n(G)$ denote the number of $A_n$-subgroups in a $p$-group $G$ (for example, if $G \cong D_{32}$, then $\alpha_1(G) = 4$, $\alpha_2(G) = 2$, $\alpha_3(G) = 1$ and $\alpha_4(G) = 0$ so $G$ is an $A_3$-group). If $G$ is an $A_n$-group, then $\alpha_n(G) > 0$ for $i = 1, \ldots, n - 1$, $\alpha_n(G) = 1$ and $\alpha_j(G) = 0$ for $j > n$. For $H < G$, we set $\beta_1(G, H) = \alpha_1(G) - \alpha_1(H)$, i.e., $\beta_1(G, H)$ is the number of $A_1$-subgroups of $G$ not contained in $H$. For example, if $H < G$ is abelian, then $\beta_1(G, H) = \alpha_1(G)$.

2. It is fairly difficult to compute $\alpha_1(G)$ even for groups with not very complicated structure (to justify this assertion, I offer to compute $\alpha_1(G)$ for $G = M_1 \times M_2$, where $M_1$ and $M_2$ are 2-groups of maximal class). Below we compute $\alpha_1(G)$ for three families of groups.

Let $X$ be a (finite) group and let $\varphi_2(X)$ be the number of noncommuting ordered pairs $x, y \in X$ such that $(x, y) = X$ (it follows that $\varphi_2(X) > 0$ if and only if $X$ is nonabelian and two-generator). Let $k(X)$ be the class number $X$. We claim that (see [8])

$$\sum_{H \leq X} \varphi_2(H) = |X|(|X| - k(X)).$$

(1)

Indeed, let $\{K_1, \ldots, K_r\}$, where $r = k(X)$, be the set of conjugacy classes of $X$. Then the number of commuting ordered pairs of $G$ equals

$$\sum_{i=1}^{r} \left( \sum_{x \in K_i} |C_X(x)| \right) = \sum_{i=1}^{r} |K_i| \frac{|X|}{|K_i|} = r|X| = |X|k(X)$$

(2)

so the number of noncommuting ordered pairs of elements of $G$ equals $|X|^2 - |X|k(X)$. On the other hand, that number also equals $\sum_{H \leq X} \varphi_2(H)$, and identity (1) is proved.

**Examples.** 1. Let us find $\alpha_1(G)$, where $G$ is an extraspecial group of order $p^{2m+1}$ (for each $m \in \mathbb{N}$, there are exactly two nonisomorphic extraspecial groups of order $p^{2m+1}$). We have
k(G) = |Z(G)| + |G - Z(G)|/p = p^{2m} + p - 1. Let E \leq G be an \mathcal{A}_1\text{-subgroup. Then } E' = G' and E/E' \cong E_{p^2} since E/E' \leq G/G' = G/G' \cong E_{p^{2m}} (see Lemma 1, below). It follows that |E| = p^3 so that

\varphi_2(E) = \left(|E - \Phi(E)|\right)(|E| - p|\Phi(E)|) = (p^3 - p)(p^3 - p^2) = p^3(p^2 - 1)(p - 1).

If A \leq G is a nonabelian two-generator subgroup, then A' = G' and A/A' \cong E_{p^2} so A is an \mathcal{A}_1\text{-group so it follows from (1) that}

\alpha_1(G)\varphi_2(E) = \alpha_1(G)p^3(p^2 - 1)(p - 1) = p^{2m+1}(p^{2m+1} - p^{2m} - p + 1)

= p^{2m+1}(p - 1)(p^{2m} - 1),

and we get \alpha_1(G) = p^{2m-2}(p^{2m} - 1)/(p^2 - 1). In particular, if G is extraspecial of order p^3, then \alpha_1(G) = p^2(p^2 + 1).

2. Let G be a nonabelian p\text{-group of order } p^m such that all nonabelian two-generator subgroups of G have the same order p^3 (this is the case if exp(G) = p). Then, using (1), we get

\alpha_1(G) = \frac{p^{m-3}(p^m - k(G))}{(p - 1)(p^2 - 1)}.

3. Let G = S \times E_{p^n}, where S is nonabelian of order p^3. Then |G| = p^{n+3}, k(G) = p^n(p^2 + p - 1). If A is a minimal nonabelian subgroup of G, then A' = G' so A/G' as a 2-generator subgroup of the elementary abelian group G/G', has order p^2, and we get |A| = p^3. Therefore, by the displayed formula in Remark 2, we get

\alpha_1(G) = \frac{p^n[p^{n+3} - p^n(p^2 + p - 1)]}{(p^2 - 1)(p - 1)} = p^{2n}.

Any nonabelian p\text{-group G satisfies } \alpha_1(G) \geq 1 and this was the unique lower estimate known up to now even for p\text{-groups G with complicated subgroup structure. However, if G is neither abelian nor an } \mathcal{A}_1\text{-group, then } \alpha_1(G) \geq p (see Remark 1, below). Moreover, that remark shows that if a nonabelian } H \in \Gamma_1, then \beta_1(G, H) \geq p - 1. Therefore, it is natural to expect that, for complicated G, the number \alpha_1(G) must be large (see also Lemma J(c)). If G is an \mathcal{A}_2\text{-group, then } \alpha_1(G) \leq p^2 + p + 1. However, there exist \mathcal{A}_3\text{-groups with } \alpha_1(G) \leq p^2 + p + 1 (for example, a 2\text{-group G of maximal class and order } 2^5, which is an } \mathcal{A}_3\text{-group, satisfies } \alpha_1(G) = 4).

Epimorphic images of an \mathcal{A}_n\text{-group are } \mathcal{A}_k\text{-groups with } k \leq n. Indeed, if } N \triangleleft G, \text{ where } G \text{ is an } \mathcal{A}_n\text{-group, then all subgroups of index } p^n \text{ in } G/N \text{ are abelian.}

The main result of this paper is the following

**Theorem A.** Let G be a nonabelian p\text{-group. If } \alpha_1(G) \leq p^2 + p + 1, then G is an \mathcal{A}_n\text{-group, } n \in \{1, 2, 3\}.
This is not an ‘Anzahlsatz’ in its usual sense.

It follows from Theorem A and Lemma 1, below, that if $G$ is a nonabelian $p$-group with $\alpha_1(G) \leq p^2 + p + 1$, then $d(G) \leq 4$. It follows from [5, Theorem 6.2] that for any group $G$ of Theorem A we also have $|G'| \leq p^4$ and $G'$ is abelian.

Some related results are presented in Sections 3–6.

In Section 2 we prove a number of intermediate results which lead to the proof of Theorem A. Appendix which is due to Z. Janko, allowed us to simplify the original statement of Theorem A.

In Section 3 we show that if a $p$-group $G$ is neither abelian nor an $A_1$-group, then $\alpha_1(G) \geq p^{d(G)} - 1$ (I do not know if this estimate is attained for $G$ with large $d(G)$).

In Section 4 we show that a nonabelian group $G$ of exponent $p$ and order $p^m$ contains at least $p^{m-3}$ minimal nonabelian subgroups. We also show that if such group contains exactly $p^{m-3}$ minimal nonabelian subgroups, then $m \leq p$ and $G$ is of maximal class with abelian subgroup of index $p$.

In Section 5 results of Section 4 are extended to some $p$-groups of arbitrary exponent.

In Section 6 we study the $p$-groups with few conjugate classes of $A_1$-subgroups.

In Section 7 we state a number of related open questions.

We gathered together, in Lemma J, some known results.

Lemma J. Let $G$ be a nonabelian $p$-group.

(a) [6, Lemma 12.12] If $A \in \Gamma_1$ is abelian, then $|G'| = \frac{1}{p}|G : Z(G)|$.

(b) [3, Lemma 10.3] The number of abelian members in the set $\Gamma_1$ is $0$, $1$ or $p + 1$.

(c) [3, Proposition 10.28] The group $G$ is generated by $A_1$-subgroups.

(d) (Mann) If $A, B \in \Gamma_1$ are distinct, then $|G' : A'B'| \leq p^1$.

(e) (Blackburn, see [2, Theorem 3.3]) If $p = 2$ and $G$ and all members of the set $\Gamma_1$ are two-generator, then $G$ is metacyclic.

(f) (Blackburn, see [2, Theorem 3.6]) If $p > 2$ and $G$ and all members of the set $\Gamma_1$ are two-generator, then either $G$ is metacyclic or $G/K_3(G)$ is nonabelian of order $p^3$ and exponent $p$; in the second case, $|G : G'| = p^2$.

(g) (Mann, see [3, Proposition 2.3]) If $|G| > p^4$, then the number of nonabelian subgroups of order $p^3$ in $G$ is a multiple of $p^2$.

(h) (Blackburn, see [5, Theorem 7.1]) If $p > 2$, then the order of minimal nonmetacyclic $p$-group is less than $p^5$.

(i) [5, Corollary 3.3, Theorems 3.7(d) and 6.1] Let $n > 1$. If $G$ is an $A_n$-group with $G' \cong C_{p^n}$, then $G$ is metacyclic. If $G$ is metacyclic with $G' \cong C_{p^n}$, then $G$ is an $A_n$-group. If $G$ is a nonmetacyclic $A_2$-group, then $\exp(G') = p$.

(j) [1, Proposition 19(a)] If $B \leq G$ is nonabelian of order $p^3$ and $C_G(B) < B$, then $G$ is of maximal class. (According to Blackburn, a $p$-group $G$ of maximal class possesses such subgroup as $B$; see, for example, [3, §9].)

(k) If $G$ is metacyclic, if $F, H \in \Gamma_1$ are (distinct) $A_{r-}$, $A_s$-groups respectively, and $r \leq s$, then $G$ is an $A_{s+1}$-group.

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1 This assertion is also contained in Kazarin’s unpublished thesis.
(1) (Huppert, see [3, Theorem 9.11]) If \( p > 2 \) and \( G \) is nonmetacyclic, then \( |G/U_1(G)| \geq p^3 \).

(m) (Blackburn, see [3, Theorem 9.5]) If \( G \) is a \( p \)-group of maximal class, then \( |G| \leq p^p \).

(n) [5, §3] Let \( p = 2 \) and \( G \) is an \( A_2 \)-group. Then \( G/Z(G) \in \{D_8, C_4 \times C_4\} \) if \( G \) is metacyclic. If \( G \) is nonmetacyclic of rank 2, then \( G/Z(G) \cong D_8 \).

Note that Lemma J(c) follows easily from Lemma J(b). Indeed, suppose that \( G \) is a counterexample of minimal order. Then the set \( \Gamma_1 \) possesses a nonabelian member \( M \). By induction, \( M \) is generated by \( A_1 \)-subgroups so it is generated by all \( A_1 \)-subgroups of \( G \). It follows that \( M \) is the unique nonabelian maximal subgroup of \( G \). Then the set \( \Gamma_1 \) has exactly \( |\Gamma_1| - 1 \) abelian members. Since \( |\Gamma_1| - 1 > 1 \), it follows from Lemma J(b) that \( |\Gamma_1| - 1 = p + 1 \). Then \( |\Gamma_1| = p + 2 \not\equiv 1 \mod p \), a contradiction.

It follows from Lemma J(c): If \( H \in \Gamma_1 \) is nonabelian, then \( \beta_1(G, H) > 0 \). Lemma J(c) has also the following unexpected consequence: A nonabelian regular \( p \)-group \( G \) possesses an \( A_1 \)-subgroup \( H \) such that \( \exp(H) = \exp(G) \).

Lemma J(g) follows easily from (1) (such proof was given by Mann [8]; it is possible to give another proof using Hall’s enumeration principle).

Let us prove Lemma J(k). Since \( G' \) is cyclic, we get \( F' \leq H' \). It follows from the above, (d) and (i) that \( |G'| = p|H'| = p^{s+1} \), and we conclude that \( G \) is an \( A_{s+1} \)-group, by (i).

2. Proof of Theorem A

To understand Remark 1, it is not necessarily to know the structure of \( A_1 \)-groups.

Remark. 1. Let a \( p \)-group \( G \) be neither abelian nor an \( A_1 \)-group and let \( F_1 \in \Gamma_1 \) be nonabelian. We claim that \( \beta(G, F_1) \geq p - 1 \). By Lemma J(b), the set \( \Gamma_1 \) has at least \( p \) nonabelian members. Let \( F_2 \in \Gamma_1 - \{F_1\} \) be nonabelian and set \( D = F_1 \cap F_2 \). Since \( D \not\cong Z(G) \), at least \( p \) members of the set \( \Gamma_1 \), say \( F_1, \ldots, F_p \), are nonabelian. Let \( U_i \leq F_i \) be an \( A_1 \)-subgroup such that \( U_i \not\leq D, i = 1, \ldots, p \) (Lemma J(c)). Then \( U_1, \ldots, U_p \) are pairwise distinct \( A_1 \)-subgroups of \( G \) and \( U_2, \ldots, U_p \not\leq F_1 \) so \( \beta_1(G, F_1) \geq p - 1 \).

Lemma 1 (L. Redei). Let \( G \) be a minimal nonabelian \( p \)-group. Then \( G = \langle a, b \rangle \) and one of the following holds:

(a) \( a^p = b^p = c^p = 1, [a, b] = c, [a, c] = [b, c] = 1, |G| = p^{m+n+1} \), \( G = \langle b \rangle \cdot \langle b \rangle \cdot \langle c \rangle \) (semidirect products with kernels in brackets) is nonmetacyclic. Here \( G' = \langle c \rangle \), \( Z(G) = \Phi(G) = \langle a^p \rangle \times \langle b^p \rangle \times \langle c \rangle \). If \( m + n > 2 \), then \( \Omega_1(G) = \langle \Omega_1(a), \Omega_1(b), \Omega_1(c) \rangle \cong E_{p^3} \).

(b) \( a^p = b^p = 1, m > 1, a^b = a^{1+p^{m-1}}, |G| = p^{m+n} \) and \( G = \langle b \rangle \cdot \langle a \rangle \) is metacyclic. Here \( G' = \langle a^{p^{m-1}} \rangle \), \( Z(G) = \Phi(G) = \langle a^p \rangle \times \langle b^p \rangle \), \( \Omega_1(G) = \langle a^{p^{m-1}}, b^{p^{m-1}} \rangle \cong E_{p^2} \).

(c) \( a^4 = 1, a^2 = b^2, a^b = a^{-1}, G \cong Q_8 \).

If \( |\Omega_1(G)| \leq p^2 \), then \( G \) is metacyclic. The group \( G \) is nonmetacyclic if and only if \( G' \) is a maximal cyclic subgroup of \( G \). Next, \( |G/\Omega_1(G)| \leq p^3 \) with equality if and only if \( G \)
is from (a) and \( p > 2 \). If, in (a), \( u \in G - \Phi(G) \), then \( \langle u \rangle \nsubseteq G \). All members of the set \( \Gamma_1 \) have ranks at most 3. If \( N \) is normal in \( G \) and \( G/N \) is not cyclic, then \( N \leq Z(G) \).

Let us prove that if, in Lemma 1, \( G' \) is not a maximal cyclic subgroup of \( G \), then \( G \) is metacyclic. Let \( G' < L < G \), where \( L \) is cyclic of order \( p^2 \). By [3, Theorem 6.1], \( L/G' \leq C/G' \), where \( C/G' \) is a cyclic direct factor of \( G/G' \); in particular, \( G/C \) is cyclic. It remains to show that \( C \) is cyclic. We get \( G' = \Phi(L) \leq \Phi(C) \). It follows that \( C/\Phi(C) \) as an epimorphic image of a cyclic group \( C/G' \), is cyclic. In that case, \( C \) is also cyclic, as was to be shown.

**Lemma 2.** Let \( G \) be a nonabelian \( p \)-group.

(a) If \( |G'| > p \), then the set \( \Gamma_1 \) contains at most one abelian member.
(b) If \( G' \leq Z(G) \), exp\( (G') = p \) and \( d(G) = 2 \), then \( G \) is an \( A_1 \)-group.
(c) Suppose that \( G \) is not an \( A_1 \)-group and \( \Delta_1 = \{ H_1, \ldots, H_p, A \} \), where \( A \) is abelian.

Then \( H_1' = \cdots = H_p' \) and \( G/H_1' \) is an \( A_1 \)-group so \( |G'| = p|H_1'| \). In particular, \( d(H_i) \leq 3 \) for all \( i \). If, in addition, \( d(H_i) = 2 \) for \( i = 1, \ldots, p \), and \( L \) is a \( G \)-invariant subgroup of index \( p \) in \( H_1' \), then \( G/L \) is an \( A_2 \)-group.

**Proof.** (a) If distinct \( A, B \in \Gamma_1 \) are abelian, then \( A \cap B = Z(G) \) and \( |G'| = \frac{1}{p} |G : Z(G)| = p \) (Lemma J(a)), contrary to the hypothesis.

(b) If \( x, y \in G \), then \( 1 = [x, y]^p = [x, y^p] \) so \( \Phi(G) = G'\hat{\cup}_1(G) \leq Z(G) \) and all members of the set \( \Gamma_1 \) are abelian; then \( G \) is an \( A_1 \)-group.

(c) By Lemma J(b), \( H_1, \ldots, H_p \) are nonabelian. Let \( |H_1'| \leq \cdots \leq |H_p'| \). By Lemma J(b), all maximal subgroups of \( G/H_1' \) must be abelian so \( H_1' = \cdots = H_p' \). Assume that \( G/H_1' \) is abelian; then \( G' = H_1' \). Let \( L \) be a \( G \)-invariant subgroup of index \( p \) in \( G' \). Then, by (b), \( G/L \) is an \( A_1 \)-group so \( H_1' \leq L < G' = H_1' \). Assume that \( G/H_1' \) is an \( A_1 \)-group and \( |G'| = p|H_1'| \). By Lemma 1, \( d(H_i) = d(H_i/L) \leq 3 \). Next suppose that \( d(H_i) = 2 \) for \( i = 1, \ldots, p \) and \( L \) is taken as above; then \( H_i/L \) is an \( A_1 \)-group, by (b), so \( G/L \) is an \( A_2 \)-group. □

**Lemma 3.** Let \( G \) be a nonabelian \( p \)-group. If all members of the set \( \Gamma_2 \) are abelian, then \( d(G) \leq 3 \). If, in addition, \( d(G) = 3 \), then \( \Phi(G) \leq Z(G) \).

**Proof.** Suppose that \( d(G) > 3 \). Let \( F \in \Gamma_1 \); then \( F \) contains at least (abelian) \( p^2 + p + 1 \) members of the set \( \Gamma_2 \) so it is abelian (Lemma J(b)). In that case, since \( G \) is not two-generator, it is abelian (Lemma 1), contrary to the hypothesis. Now let \( d(G) = 3 \). Then \( C_G(\Phi(G)) \geq \langle H \mid H \in \Gamma_2 \rangle = G \), completing the proof. □

In what follows we make use of the following fact: If \( N \) is a normal subgroup of a \( p \)-group \( G \), \( |G/N| > p^2 \), and \( G/N \) is generated by subgroups of index \( p^2 \), whose inverse images in \( G \) are abelian, then \( N \leq Z(G) \). Indeed, \( C_G(N) \geq \langle H < G \mid N < H, |G : H| = p^2 \rangle = G \).
Lemma 4. Let $G$ be a nonmetacyclic $A_2$-group of order $p^m > p^4$. Then

(a) $d(G) \leq 3$, $|G'| \leq p^3$, $\exp(G') = p$, $G/Z(G)$ is either $\cong D_8$ or of order $p^3$ and exponent $p$.
(b) $\alpha_1(G) \in \{ p, p + 1, p^2, p^2 + p, p^2 + p + 1 \}$.
(c) If $\alpha_1(G) < p^2$, then $p > 2$, $d(G) = 2$, and $\cl(G) = 3$.
(1c) If $\alpha_1(G) = p$, then $G/(G' \cap Z(G))$ is an $A_1$-group, $G' \cong E_{p^2}$.
(2c) If $\alpha_1(G) = p + 1$, then $|G| = p^5$, $G' = \Omega_1(G) \cong E_{p^2}$, $Z(G) = K_3(G) = \bar{U}_1(G) \cong E_8$.
(d) If $\alpha_1(G) = p^2$, then $d(G) = 3$, $G/Z(G) \cong E_{p^2}$, $|G'| = p$.
(e) If $\alpha_1(G) = p^2 + p$, then $d(G) = 3$, $G' \cong E_{p^2}$, $G' \leq Z(G) = \Phi(G)$.
(f) If $\alpha_1(G) = p^2 + p + 1$, then $p = 2$, $|G| = 2^6$, $G$ is special with $G' = Z(G) = \Phi(G) = \Omega_1(G) \cong E_8$.

Proof. (a) Two inequalities follow from Lemmas 1 and J(d). Next, $\exp(G') = p$ (Lemma J(i)). If $d(G) = 3$, then $\Phi(G) \leq Z(G)$, by the paragraph preceding the lemma. If $p > 2$, then $|G/\bar{U}_1(G)| > p^2$ (Lemma J(i)) so $G/\bar{U}_1(G)$ is generated by subgroups of index $p^2$, and we get $\bar{U}_1(G) \leq Z(G)$ again. Hence, the last assertion of (a) holds for $p > 2$. Now let $p = 2$. By the above, one may assume that $d(G) = 2$. However, this case does not occur according to [5, Propositions 5.1–5.5]. Indeed, the results of [5, §5] imply that an $A_2$-group of order $2^m > 2^4$ with $d(G) = 2$ must be metacyclic (this is also true for $m \leq 4$, by Lemma J(i)).

(b) to (f) is a direct consequence of [5, §5]. Indeed, if $G$ is an $A_2$-group of [5, Proposition 5.1], then $\alpha_1(G) = p^2$, $d(G) = 3$, $G/Z(G) \cong E_{p^2}$, and $|G'| = p$. If $G$ is an $A_2$-group of [5, Proposition 5.3], then $\alpha_1(G) = p$, $p > 2$, $d(G) = 2$, $\cl(G) = 3$, $G' \cong E_{p^2}$, and $\bar{G} = G/(G' \cap Z(G))$ is an $A_1$-group since $d(G) = 2$ and $|G'| = p$. If $G$ is an $A_2$-group of [5, Proposition 5.4], then $\alpha_1(G) = p^2 + p$, $d(G) = 3$, $G' \cong E_{p^2}$, and $G' \leq Z(G) = \Phi(G)$. If $G$ is an $A_2$-group of [5, Proposition 5.5(a)], then $\alpha_1(G) = p^2 + p + 1$, $p = 2$, $d(G) = 3$, $|G| = 2^6$, $G$ is special with $G' = Z(G) = \Phi(G) = \Omega_1(G) \cong E_8$; moreover, $G$ is isomorphic to a Sylow 2-subgroup of the Suzuki simple group $Sz(8)$. If $G$ is an $A_2$-group of [5, Proposition 5.5(b)], then $p > 2$, $d(G) = 2$, $\cl(G) = 3$, $\alpha_1(G) = p + 1$, $|G| = p^5$, $G' = \Omega_1(G) \cong E_{p^3}$, $Z(G) = K_3(G) = \bar{U}_1(G) \cong E_{p^2}$. Our proof is complete since the $A_2$-groups of [5, Proposition 5.2] are metacyclic. $\square$

Let us give a proof, independent of [5, §5], that if $G$ is a nonmetacyclic $A_2$-group with $\alpha_1(G) < p^2$, then $\cl(G) = 3$. It follows from Lemma J(b) that $d(G) = 2$. Assume that $\cl(G) = 2$. Then $G' \leq Z(G)$. Since $G$ is nonmetacyclic, we get $\exp(G') = p$, by Lemma 4(a). In that case, $G$ is an $A_1$-group (Lemma 2(b)), a contradiction. It remains to prove that $\cl(G) = 3$. This is the case if $|G'| = p^2$. Now let $|G'| = p^3$ (see Lemma J(d)); then $\alpha_1(G) = p + 1$ and $G$ is nonmetacyclic with two-generator members of the set $\Gamma_1$ since the latter has no abelian member (Lemma J(d)). It follows that $G/\bar{U}_1(G)$ is nonabelian of order $p^3$ and exponent $p$ and $\bar{U}_1(G) = K_3(G)$ so $|G : G'| = p^2$ (Lemma J(f)), and we get $|G| = |G : G'||G'| = p^5$. Since all subgroups of order $p^3$ that contain $K_3(G)$ are
abelian (G is an $A_2$-group!) and generate $G$, we get $K_3(G) = Z(G)$ since $|G : Z(G)| > p^2$, and so $\text{cl}(G) = 3$.

**Remarks.** Suppose that $A, B \in \Gamma_1$ are distinct.

2. If $A$ is abelian and $B$ an $A_1$-group, then $|G/Z(G)| \leq p^3$. If, in addition, $A$ is the unique abelian member of the set $\Gamma_1$, then $G/Z(G)$ is either of order $p^3$ and exponent $p$ or $G/Z(G) \cong D_8$. Indeed, $|G'| \leq p^2$ (Lemmas J(d) and 1) so $|G : Z(G)| \leq p^3$ (Lemma J(a)). If $A$ is the unique abelian member of the set $\Gamma_1$, then $|G : Z(G)| = p^3$ so $G/Z(G)$ has at most one cyclic subgroup of index $p$, and the last assertion follows.

3. If $A, B \in \Gamma_1$ are distinct $A_1$-subgroups, then $|G/Z(G)| \leq p^4$. Indeed, put $D = A \cap B$. Then $Z(A) = \Phi(A) \leq \Phi(G) \leq A \cap B = D$ and similarly $Z(B) < D$ so, comparing orders, we conclude that $Z(A)$ and $Z(B)$ are maximal in $D$ (Lemma 1). It follows that $U = Z(A) \cap Z(B)$ has index at most $p^2$ in $D$ and $U \leq Z(G)$ since $C_G(U) \supseteq AB = G$, and we get $|G/Z(G)| \leq |G/U| \leq |G/D||D/U| \leq p^4$.

The following lemma is the key result.

**Lemma 5.** Suppose that $H$ is a nonabelian maximal subgroup of a $p$-group $G$. Then $\beta_1(G, H) \geq p - 1$. If $\beta_1(G, H) = p - 1$, then the following holds:

(a) $d(G) = 2$, $\Gamma_1 = \{H_1 = H, H_2, \ldots, H_p, A = H_{p+1}\}$, where $A$ is abelian and all members of the set $\Gamma_1 - \{H, A\}$ are $A_1$-groups.

(b) $H'_1 = \cdots = H'_p$ is of order $p$.

(c) $G/H'_i$ is an $A_1$-group so $|G'| = p^2$. We also have $d(H_1) \leq 3$.

(d) $G/Z(G)$ is either nonabelian of order $p^3$ and exponent $p$ or $\cong D_8$, $Z(H_i) = Z(G)$ and if $G' \not\cong Z(G)$, then $H_i / G'$ is cyclic, $i = 2, \ldots, p$.

(e) If $G' \cong C_p^2$, then $H_2, \ldots, H_p, A$ are metacyclic and $G$ has no normal subgroup of order $p^3$ and exponent $p$. If $p > 2$, then $G$ is metacyclic (in that case, $G$ is an $A_2$-group).

(f) If $G' \cong C_p$, then $\text{cl}(G) = 3$.

**Proof.** The set $\Gamma_1(H) \cap \Gamma_2$ contains a member $N$; then $N \not\cong Z(G)$ since $H$ is nonabelian. Set $\Gamma_1^N = \{H_1 = H, \ldots, H_p, H_{p+1}\}$. Since at most one member of the set $\Gamma_1^N$ is abelian, one may assume that $H_1, \ldots, H_p$ are nonabelian. By Remark 1, $\beta_1(G, H) \geq p - 1$. Next we suppose that $\beta_1(G, H) = p - 1$.

(a)–(c) If $U_i \subseteq H_i$ is an $A_1$-subgroup not contained in $N$, then $U_2, \ldots, U_p$ are pairwise distinct. It follows that $U_{p+1}$ does not exist so $H_{p+1} = A$ is abelian (Remark 1). Then $N = H \cap A$ is also abelian, and we get $p - 1 = \beta_1(G, H) \geq \sum_{i=2}^p \alpha_1(H_i)$. It follows that $\alpha_1(H_i) = 1$ (i = 2, \ldots, $p$) so $H_2, \ldots, H_p$ are $A_1$-groups and the subgroups $H_1, \ldots, H_p$ together contain all $A_1$-subgroups of $G$. Assume that $F \in \Gamma_1 - \Gamma_1^N$. The intersection $F \cap H_i$ is abelian, since $H_i$ is an $A_1$-subgroup, $i = 2, \ldots, p$. Therefore, if $F$ is nonabelian, then all $A_1$-subgroups of $F$ are contained in $H_1$. In that case, $F \subseteq H_1$ (Lemma J(c)), a contradiction. Thus, all members of the set $\Gamma_1 - \Gamma_1^N$ are abelian. Assuming that $d(G) > 2$, we see that the set $\Gamma_1$ has at least $p^2 + 1 > p + 1$ abelian members, contrary to Lemma J(b).
Thus, $\Gamma_1 = \Gamma_1^N$ so $d(G) = 2$ and $\Phi(G) = N$, completing the proof of (a). Now (b) follows from (a) and Lemma 2(c). By Lemma 2(c) again, $G/H'_1$ is an $A_1$-group and hence $|G'| = p|H_1| = p^2$. In that case, $d(H_1) = d(H_1/H'_1) \leq 3$ (Lemma 1).

(d) It follows from $d(G) = 2$ that $|G : \Phi(G)| > p^2$. By Remark 2, $G/\Phi(G)$ is either nonabelian of order $p^3$ and exponent $p$ or $G/\Phi(G) \cong D_8$. Now let $i \in \{2, \ldots, p\}$. In view of $d(G) = 2$, we get $Z(G) \leq \Phi(G) < H_i$ so $Z(G) \leq Z(H_i)$, and the equality $|Z(G)| = |Z(H_i)|$ implies $Z(G) = Z(H_i) (= \Phi(H_i))$. Therefore, if $G' \not< Z(G)$, then $H_i/G'$ is cyclic.

(e) Suppose that $G' \cong C_{p^2}$. Then $H_i$ is metacyclic since $H'_i$ is not a maximal cyclic subgroup in $H_i$ in view of $H'_i < G'$, $i = 2, \ldots, p$ (Lemma 1). Assume that $G$ has a normal subgroup $R$ of order $p^3$ and exponent $p$. For $i = 2, \ldots, p$, we get $H_iR = G$ so $G/(R \cap H_i) = (H_i/(R \cap H_i)) \times (R/(R \cap H_i))$, and we conclude that $H_i/(R \cap H_i)$ is cyclic; then $G/(R \cap H_i)$ is abelian. On the other hand, $G/(R \cap H_i)$ is nonabelian since $C_{p^2} \cong G' \not< R \cap H_i \cong E_{p^2}$, and this is a contradiction. Thus, $R$ does not exist so $A$ is metacyclic. Let $p > 2$. Since $G'$ is cyclic, $G$ is regular so $p^2 = |\Omega_1(G)| = |G/\Phi(G)|$ whence $G$ is metacyclic, by Lemma J(l).

(f) If $G' \cong E_{p^2}$, then $G' \not< Z(G)$ ((a) and Lemma 2(b)) so $cl(G) = 3$. □

Remarks. 4. Let $G$ be a $p$-group and let members $H_1, \ldots, H_k$, $k > 1$, of the set $\Gamma_1$ contain together all $A_1$-subgroups of $G$. Suppose that there exists $A \in \Gamma_1 - \{H_1, \ldots, H_k\}$ such that $A \cap H_i$ is abelian for all $i = 2, \ldots, k$. Then $A$ is abelian. Assuming that this is false, we can take an $A_1$-subgroup $U \leq A$ such that $U \not< H_1$ (Lemma J(c)). Since, for $i = 2, \ldots, k$, also $U \not< H_i$ (recall that $A \cap H_i$ is abelian, by hypothesis), we get a contradiction.

5. For a nonabelian $p$-group $G$, the following two conditions are equivalent: (a) $G$ is of maximal class with abelian subgroup $A$ of index $p$; (b) Whenever $H \leq G$ is nonabelian of order $p^k$, then $\alpha_1(H) = p^{k-3}$. We are working by induction on $|G|$. Let us prove that (a) $\Rightarrow$ (b). Let $H < G$ be nonabelian. Then $|H \cap A| > p = |Z(G)|$ and, since $Z(H)/(H \cap A)$ is abelian and $A = C_G(H \cap A) \supseteq AZ(H)$, we get $Z(H) \leq H \cap A$ and $C_G(Z(H)) \supseteq HA = G$ so $Z(H) = Z(G)$ is of order $p$; in particular, $Z(H) \leq H'$. We claim that $H$ is of maximal class. Indeed, by Lemma J(a), we get $|H : H'| = p|Z(H)| = p^2$ since $A \cap H$ is abelian of index $p$ in $H$. If $|H| = p^3$, it is nothing to prove. If $|H| > p^3$, then $H/Z(H)$ is of maximal class, by induction since

$$|Z(H/Z(H))| = \frac{1}{p} \cdot |(H/Z(H)) : (H'/Z(H))| = \frac{1}{p} \cdot |H : H'| = p$$

(Lemma J(a)); in that case $H$ itself is of maximal class since $|Z(H)| = p$. Now, using induction and the enumeration principle and setting $|G| = p^m$, we get

$$\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) = p \cdot p^{(m-1)-3} = p^{m-3},$$

proving (b). Let us prove that (b) $\Rightarrow$ (a). In that case, all proper nonabelian subgroups of $G$ are of maximal class, by induction. Take $T \leq G$, where $T$ is an $A_1$-subgroup of $G$. Setting $|T| = p^k$, we get $1 = \alpha_1(T) = p^{k-3}$ so $k = 3$, i.e., all $A_1$-subgroups of $G$ have the same order $p^3$. Assume that $G$ is not of maximal class. Then we get $C_H(T) \not< T$ (Lemma J(j)).
Let $F$ be a subgroup of order $p^2$ in $C_H(T)$ such that $Z(T) < F$. Then $|FT| = p^4$ and $\alpha_1(FT) = p^2 \neq p = p^{4-3}$, a contradiction. Thus, $G$ is of maximal class. It remains to show that the set $\Gamma_1$ has an abelian member. Let $R < G$ be of order $p^2$ and set $U = C_G(R)$; then $|G : U| = p$ and, by induction, $U$ is abelian since it is not of maximal class.

6. Let $G$ be a $p$-group with $d(G) = 3$ and $\Phi(G) \notin Z(G)$; then the set $\Gamma_2$ possesses a nonabelian member $N$ (Lemma 3). We claim that the set $\Gamma_2$ contains at least $p^2$ nonabelian members. Let $A_1, A_2 \in \Gamma_2$ be distinct abelian; then $A = A_1 A_2 \in \Gamma_1$, by the product formula, $A$ contains exactly $p + 1$ members of the set $\Gamma_2$ and all of them are abelian since $\Phi(G) = A_1 \cap A_2 \leq Z(A)$. Assume that $B \in \Gamma_2 - \Gamma_1(A)$ is abelian. Then $C_G(\Phi(G)) \supseteq AB = G$ so $\Phi(G) \leq Z(G)$, contrary to the hypothesis. Thus, $\Gamma_1(A) \cap \Gamma_2$ is the set of all abelian members of the set $\Gamma_2$ so the last set contains exactly $|\Gamma_2| - (p + 1) = p^2$ nonabelian members. Thus, if $d(G) = 3$ and $\Phi(G) \notin Z(G)$, then the set $\Gamma_2$ has $p^2$, $p^2 + p$ or $p^2 + p + 1$ nonabelian members.

7. Let $G$ be a nonabelian $p$-group with $d = d(G) > 3$. Let $r, s$ be the numbers of nonabelian members in the sets $\Gamma_2, \Gamma_1$, respectively. We claim that $r > (p - 1)s$. Indeed, let $\Sigma = \{T_1, \ldots, T_r\} \subseteq \Gamma_2$ and $\mathcal{M} = \{M_1, \ldots, M_s\} \subseteq \Gamma_1$ be the sets of nonabelian members of the sets $\Gamma_2, \Gamma_1$, respectively ($\Sigma \neq \emptyset$, by Lemma 3, $\mathcal{M} \neq \emptyset$, by Lemma J(b)). Let $\mu_i$ be the number of members of the set $\mathcal{M}$, containing $T_i, i = 1, \ldots, r$, and let $\gamma_j$ be the number of members of the set $\Sigma$ contained in $M_j, j = 1, \ldots, s$. Then, by double counting,

$$\mu_1 + \cdots + \mu_r = \gamma_1 + \cdots + \gamma_s,$$

and, in view of $\mu_i = p + 1$ for all $i$, the displayed formula can be rewritten as follows:

$$(p + 1)r = \gamma_1 + \cdots + \gamma_s.$$  

Note that $M_j$ contains exactly $p^{d-2} + p^{d-3} + \cdots + p + 1$ members of the set $\Gamma_2$. Since, by Lemma J(b), the set $\Gamma_1(M_j)$ contains at most $p + 1$ abelian members, we get $\gamma_j \geq p^{d-2} + \cdots + p^2$ for $j = 1, \ldots, s$, and we deduce from (3) the following inequality: $(p + 1)r \geq (p^{d-2} + \cdots + p^2)s$. We get

$$r \geq \frac{p^{d-2} + \cdots + p^2}{p + 1} s \geq \frac{p^2}{p + 1} s = \frac{(p^2 - 1) + 1}{p + 1} s > (p - 1)s,$$

as was to be shown. By Lemma J(b), we have $s \geq |\Gamma_1| - (p + 1) = p^{d-1} + \cdots + p^2$.

8. Given a set $\mathcal{M}$ of subgroups of a group $G$, let $\alpha_1(\mathcal{M})$ denote the number of $A_1$-subgroups that contain together members of the set $\mathcal{M}$. Suppose that a $p$-group $G$ is neither abelian nor an $A_1$-group. We claim that if, for each $K \in \Gamma_2$, we have $\alpha_1(\Gamma_1^K) \leq p + 1$, then $G$ is an $A_2$-group. It follows from the hypothesis that there is $K \in \Gamma_2$ which is not contained in $Z(G)$. Set $\Gamma_1^K = \{H_1, \ldots, H_{p+1}\}$ and let $\alpha_1(H_1) \geq \cdots \geq \alpha_1(H_{p+1})$. Since at most one member of the set $\Gamma_1^K$ is abelian, $H_1, \ldots, H_p$ are nonabelian. Assume that $\alpha_1(H_1) > 1$; then $\alpha_1(H_1) = p$, by Remark 1 and hypothesis. We get

$$p + 1 \geq \alpha_1(\Gamma_1^K) = \alpha_1(H_1) + \sum_{i=2}^{p+1} \beta_1(H_i, K) = p + \sum_{i=2}^{p+1} \beta_1(H_i, K)$$
so $\beta_1(H_2, K) = 1$ and we get $p = 2$ and $\beta_1(H_3, K) = 0$ (Remark 1) whence $H_3$ is abelian, and we conclude that $K$ is abelian; in that case, $H_2$ is an $A_1$-group. Thus, all members of the set $\Gamma_2$ are abelian so $d(G) \leq 3$ (Lemma 3). Assume that $G$ is not an $A_2$-group. If $d(G) = 2$, then $\Gamma_1 = \Gamma_1^K = \{H_1, H_2, A\}$, where $\alpha_1(H_1) = 2$, $\alpha_1(H_2) = 1$ and $A$ is abelian. In this case, $\beta_1(G, H_1) = 1$ so $|H'_1| = 2$ (Lemma 5), and the equality $\alpha_1(H_1) = 2$ is impossible (Lemma 4(c)), a contradiction. Now assume that $d(G) = 3$; then $|\Gamma_1(H_1) \cap \Gamma_2| = p + 1 > 1$ so $|H'_1| = 2$ (Lemma J(a)), and again, as in the preceding sentence, we get a contradiction.

**Lemma 6.** Suppose that $G$ is a $p$-group with $\alpha_1(G) > 1$. Then $\alpha_1(G) \geq p$.

(a) If $\alpha_1(G) = p$, then $G$ is an $A_2$-group with $d(G) = 2$ and $|G'| = p^2$. If $G$ is nonmetacyclic, it is of class 3.

(b) If $\alpha_1(G) = p + 1$, then $G$ is an $A_2$-group with $d(G) = 2$. If $G$ is not metacyclic, then $p > 2$, $G$ is of order $p^5$ with $G' \cong E_p$ and of class $> 2$.

**Proof.** The inequality $\alpha_1(G) \geq p$ follows from Remark 1. By Remark 8, $G$ is an $A_2$-group. All remaining assertions follow from Lemma 4(c).

**Lemma 7.** Let $G$ be a nonabelian $p$-group, $N \in \Gamma_2$ and $\Gamma_1^N = \{H_1, \ldots, H_{p+1}\}$.

(a) $\alpha_1(G) \geq \alpha_1(N) + \sum_{i=1}^{p+1} \beta_1(H_i, N)$.

(b) If $N$ is nonabelian, then $\beta_1(G, N) \geq p^2 - 1$.

(c) If $\alpha_1(G) < p^2$, then $N_G(K)/K$ has only one subgroup of order $p$ for each nonabelian $K < G$.

**Proof.** (a) is obvious, (b) follows from (a) and Remark 1. It remains to prove (c). Assume that $N_G(K)/K$ contains an abelian subgroup $L/K$ of type $(p, p)$; then $\alpha_1(L) \leq \alpha_1(G) \leq p^2 - 1$ so $\beta_1(L, K) < p^2 - 1$, contrary to (b). Thus, $L/K$ does not exist so $N_G(K)/K$ has only one subgroup of order $p$.

**Lemma 8.** Suppose that a $p$-group $G$ is an $A_n$-group, $n > 2$.

(a) If $\alpha_1(G) < p^2$, then $d(G) = 2$.

(b) If $d(G) = 3$ and $\Phi(G) = Z(G)$, then $\alpha_1(G) \geq 2p^2 + p - 1$.

(c) If $d(G) = 3$ and $|G : Z(G)| = p^2$, then $\alpha_1(G) \geq 2p^2 - 1$.

**Proof.** (a) Assume that $d(G) > 2$. Then $N \not\leq Z(G)$ for some $N \in \Gamma_2$ since $\langle N \mid N \in \Gamma_2 \rangle = G$. By Lemma 7(b), (a), $N$ is abelian. Since all members of the set $\Gamma_2$ are abelian, $\alpha_1(G) = \sum_{H \in \Gamma_2} \alpha_1(H) \geq |\Gamma_2| - (p + 1) \geq p^2$, by Lemma J(b), a contradiction.

(b) If $L \in \Gamma_1$ is nonabelian, then $|L : Z(L)| = |L : Z(G)| = p^2$ so $|L'| = p$ (Lemma J(a)). Suppose that $L$ is neither abelian nor an $A_1$-group (such an $L$ exists since $n > 2$). Then, by Lemma 4, $\alpha_1(L) \geq \alpha_1(H) \geq p^2$, where $H \leq L$ is an $A_2$-subgroup. In view of $|G : Z(G)| = p^3$, the set $\Gamma_1$ has at most one abelian member (Lemma 2(a)) so this set
has at least \( p^2 + p \) nonabelian members, and we have (by hypothesis, all members of the set \( \Gamma_2 \) are abelian)

\[
\alpha_1(G) = \alpha_1(L) + \sum_{M \in \Gamma_1 \setminus \{L\}} \alpha_1(M) \geq p^2 + p^2 + p - 1 = 2p^2 + p - 1,
\]
as was to be shown. (If \( \alpha_1(G) = 2p^2 + p - 1 \), then \( \Gamma_1 = \{H_1, \ldots, H_{p^2+p}, A\} \), where \( \alpha_1(H_i) = p^2, \alpha_1(H_i) = 1 \) for \( i = 2, \ldots, p^2 + p \) and \( A \) is abelian.)

(c) In our case, \( |G'| = p \) and all members of the set \( \Gamma_2 \) are abelian, the set \( \Gamma_1 \) has exactly \( p^2 \) nonabelian members. If \( L \in \Gamma_1 \) is neither abelian nor an \( A_1 \)-group (see the proof of (b)), then \( \alpha_1(L) \geq p^2 \) so we get

\[
\alpha_1(G) = \alpha_1(L) + \sum_{M \in \Gamma_1 \setminus \{L\}} \alpha_1(M) \geq p^2 + (p^2 - 1) = 2p^2 - 1,
\]
completing the proof. (If \( \alpha_1(G) = 2p^2 - 1 \), then \( \Gamma_1 = \{H_1, \ldots, H_{p^2}, A_1, \ldots, A_{p+1}\} \), where \( \alpha_1(H_i) = p^2, \alpha_1(H_i) = 1 \) for \( i = 2, \ldots, p^2 \) and \( A_1, \ldots, A_{p+1} \) are abelian.) \( \square \)

**Theorem 9.** A \( p \)-group \( G \) with \( 1 < \alpha_1(G) < p^2 \) is a two-generator \( A_2 \)-group. In particular, \( \alpha_1(G) \in \{p, p + 1\} \).

**Proof.** Suppose that \( G \) is a counterexample of minimal order; then \( \alpha_1(G) > p + 1 \) (Lemma 6(a), (b)). In particular, \( p > 2 \).

By Lemma 8(a), \( d(G) = 2 \) so \( \Phi(G) \neq Z(G) \). By Lemma 7(c), \( \Phi(G) \) is abelian. By Lemma J(c) and induction, \( G \) is an \( A_3 \)-group. Then there is \( H \in \Gamma_1 \), which is an \( A_2 \)-group, and so, by Lemmas J(c), 4 and 6, \( d(H) = 2, |H'| > p \) and \( \alpha_1(H) \in \{p, p + 1\} \).

(i) Assume that \( A, U \in \Gamma_1 \), where \( A \) is abelian and \( U \) is an \( A_1 \)-group. Then \( G/Z(G) \) isirst order \( p^3 \) and exponent \( p \) (Remark 2), and \( Z(G) < \Phi(G) < H \) since \( d(G) = 2 \). Then \( |H : Z(G)| = p^2 \) so \( |H'| = p \) (Lemma J(a)), contrary to what has been said in the previous paragraph. Thus, a pair \( (A, U) \) does not exist.

Let \( F_1, \ldots, F_s, V_1, \ldots, V_u \) be all nonabelian members of the set \( \Gamma_1 \), where \( \alpha_1(F_i) = p \) and \( \alpha_1(V_k) = 1 \) (since \( \Phi(G) \) is abelian, the set \( \Gamma_1 \) has no member \( H \) with \( \alpha_1(H) = p + 1 \), by Lemma 4). Since the set \( \Gamma_1 \) has at most one abelian member (Lemma 2(a)), we get \( s + u \geq |\Gamma_1| - 1 = p \).

(ii) Assume that \( A \in \Gamma_1 \) is abelian. Then \( u = 0 \), by (i), so \( s = p \) and, since \( \Phi(G) \) is abelian, \( p^2 > \alpha_1(G) = p \cdot p = p^2 \), a contradiction.

Thus, all members of the set \( \Gamma_1 \) are nonabelian, i.e., \( s + u = p + 1 \). As in the previous paragraph, \( s < p \) so \( u \geq 2 \). Then \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is either metacyclic or \( G/K_3(G) \) is nonabelian of order \( p^3 \) and exponent \( p \) and then \( |G : G'| = p^2 \) (Lemma J(f)).

(iii) Assume that \( G \) is metacyclic. Then \( V_1, V_2 \in \Gamma_1 \) are distinct \( A_1 \)-subgroups so \( G \) is an \( A_2 \)-group (Lemma J(k)), a contradiction. Thus, \( G \) is nonmetacyclic.

(iv) Since \( u > 1 \), then \( |G'| \leq p|V_1|V_2| = p^3 \) so \( |G| = |G : G'||G'| \leq p^5 \). Since \( G \) is not an \( A_2 \)-group, we get \( |G| = p^5 \). In that case, \( G \) has a nonabelian subgroup of order \( p^3 \). Since
all nonabelian groups of order \( p^3 \) are \( A_1 \)-groups, we get by Lemma J(g), \( \alpha_1(G) \geq p^2 \), a final contradiction. \( \square \)

**Remark.** 9. Let a \( p \)-group \( G \) be neither abelian nor an \( A_1 \)-group. If, for each \( K \in \Gamma_2 \), we have \( \alpha_1(\Gamma_1^K) \leq p^2 \), then \( G \) is either an \( A_2 \)- or \( A_3 \)-group. Indeed, let \( H_1 \in \Gamma_1 \) be nonabelian and take \( K \in \Gamma_1(H_1) \cap \Gamma_2 \). Set \( \Gamma_1^K = \{H_1, \ldots, H_{p+1}\} \). Since at most one member of the set \( \Gamma_1^K \) is abelian, we get \( \alpha_1(H_1) < p^2 \), i.e., \( H_1 \) is either \( A_1 \)- or \( A_2 \)-group (Theorem 9) so \( G \) is an \( A_n \)-group, \( n \in \{2, 3\} \).

**Theorem 10.** If \( G \) is a \( p \)-group with \( \alpha_1(G) = p^2 \), then it is an \( A_n \)-group, \( n \in \{2, 3\} \). Next suppose that \( G \) is an \( A_3 \)-group.

(a) If the set \( \Gamma_2 \) has a nonabelian member \( N \), then \( N \) is an \( A_1 \)-group and one of the following holds:

1. \( G \) is metacyclic with \( \alpha_1(H) = p \) for all \( H \in \Gamma_1 \), i.e., all members of the set \( \Gamma_1 \) are \( A_2 \)-groups.
2. \( \alpha_1(H) = p \) for all \( i \) and \( A \) is abelian, \( \Gamma_1 = \{F_1, \ldots, F_{p^2 + p}, A\} \), where \( \alpha_1(F_i) = p \).
3. \( \Gamma_1 = \{F_1, \ldots, F_{p^2 + p}, A\} \), where \( \alpha_1(F_i) = p \) for all \( i \), \( F_1' = \cdots = F_p' \), \( A \) is abelian.

(b) If all members of the set \( \Gamma_2 \) are abelian, then \( d(G) = 2 \) and one of the following holds:

1. \( G \) is metacyclic, \( \Gamma_1 = \{H_1, \ldots, H_p, A\} \), where \( \alpha_1(H_i) = p \) for all \( i \), \( H_1' = \cdots = H_p' \), \( A \) is abelian.
2. \( \Gamma_1 = \{H_1, \ldots, H_p, A\} \), where \( \alpha_1(H_i) = p \), \( H_1' = \cdots = H_p' \), \( A \) is abelian, \( G' \) is noncyclic, \( G/H_1' \) is an \( A_1 \)-group.

In all cases, where \( G \) is an \( A_3 \)-group, \( |G'| = p^3 \) and \( \alpha_1(H) \in \{1, p\} \) for every nonabelian \( H < G \).

**Proof.** Suppose that \( G \) is not an \( A_2 \)-group; then \( G \) is an \( A_3 \)-group (Lemma J(c) and Theorem 9).

(a) Suppose that the set \( \Gamma_2 \) has a nonabelian member \( N \). Let \( \Gamma_1^N = \{H_1, H_2, \ldots, H_{p+1}\} \); then all members of the set \( \Gamma_1^N \) are neither abelian nor \( A_1 \)-groups. Using Remark 1, we get

\[
p^2 = \alpha_1(G) \geq \alpha_1(N) + \sum_{i=1}^{p+1} \beta_1(H_i, N) \geq 1 + (p-1)(p+1) = p^2,
\]

and hence \( \alpha_1(N) = 1 \), \( \beta_1(H_i, N) = p - 1 \), and so \( \alpha_1(H_i) = p \), \( d(H_i) = 2 \) and \( |H_i'| = p^2 \) for all \( i \), \( H_1, \ldots, H_{p+1} \) are \( A_2 \)-groups (Remark 1 and Lemma 6(a)) and these \( p+1 \) subgroups contain together all \( A_1 \)-subgroups of \( G \). It follows that all nonabelian members of the set \( \Gamma_1 \) are \( A_2 \)-groups. If \( G \) is metacyclic, it is a group from part (1a). Next we assume that \( G \) is nonmetacyclic. We also have \( d(G) \leq d(H_1) + 1 = 3 \).

Suppose, in addition, that \( d(G) = 2 \); then \( N = \Phi(G) \). In that case, \( G \) and all members of the set \( \Gamma_1 \) are two-generator. Since \( G \) is nonmetacyclic, we get \( p > 2 \) and \( |G : G'| = p^2 \).
(Lemma J(f)). By the previous paragraph and Lemma 6(a), \( H_i \) has the unique abelian maximal subgroup \( A_i \), all \( i \). Then the subgroups \( A_1 \neq A_2 \) are normal in \( G \); therefore, \( A = A_1A_2 \) is of class at most 2 (Fitting’s Lemma). By Lemma J(h), \( G \) is not minimal nonmetacyclic so we may assume that \( H_1 \) is not metacyclic. Since \( A > A_1 \) and \( \text{cl}(G) \geq \text{cl}(H_1) > 2 \) (Lemma 4(c)), we get \( A \in \Gamma_1 \). Then \( A_1 = H_1 \cap A = \Phi(G) = N \), a contradiction since \( A_1 \) is abelian and \( N \) is nonabelian.

Thus, \( d(G) = 3 \). Then, by Remark 6, the set \( \Gamma_2 \) has at least \( p^2 \) nonabelian members, say \( L_1, \ldots, L_{p^2} \), and all \( L_i \) are \( A_1 \)-groups since \( G \) is an \( A_3 \)-group. Since \( \alpha_1(G) = p^2 \), all \( A_1 \)-subgroups of \( G \) are members of the set \( \Gamma_2 \). In particular, each nonabelian \( H \in \Gamma_1 \) is an \( A_2 \)-group. If \( U_1, \ldots, U_{p+1} \) are all abelian members of the set \( \Gamma_2 \), then \( A = U_1U_2 \in \Gamma_1 \), by the product formula, and \( |A'| \leq p \) (Lemma J(a)) so \( A \) is not an \( A_2 \)-group since \( \alpha_1(A) < \alpha_1(G) = p^2 \) (Lemma 4). It follows that \( A \) is abelian; moreover, \( A \) is the unique abelian member of the set \( \Gamma_1 \) (Lemma 2(a)). Thus, \( \Gamma_1 = \{ F_1, \ldots, F_{p^2} + p, A \} \), where \( \alpha_1(F_i) = p \) and \( |F_i'| = p^2 \) (Lemma 6). Next, in view of \( d(F_i) = 2 \), we get \( Z(F_i) < \Phi(F_i) = \Phi(G) < A \) so \( Z(F_i) \leq Z(G) \) and, by Lemma J(a), \( |F_i : Z(F_i)| = p|F_i'| = p^3 \). Thus, \( |G : Z(G)| \leq p^4 \). By Lemma J(a), \( |G : Z(G)| = p|G'| > p|F_i'| \geq p^3 \). Since \( G/Z(G) \) is noncyclic and \( A \) is the unique abelian member of the set \( \Gamma_1 \), it follows that \( Z(G) \leq F_i \) for some \( i \). By the above, \( Z(F_i) \leq Z(G) \) so \( Z(G) = Z(F_i) \), and we get \( |G : Z(G)| = |G : F_i||F_i : Z(G)| = p^4 \) so \( |G'| = \frac{1}{p^3}|G : Z(G)| = p^3 \) and \( G \) is as stated in (2a) since, if \( N < F_i \in \Gamma_1 \), then \( Z(N) = Z(F_i) = Z(G) \) in view \( Z(N) = \Phi(N) < \Phi(G) < A \) so \( C_G(Z(N)) \geq AN = G \) so \( Z(N) \leq Z(G) \) and \( |G : Z(N)| = p^4 = |G : Z(G)| \).

(b) Now suppose that all members of the set \( \Gamma_2 \) are abelian. Then \( p^2 = \alpha_1(G) = \sum_{M \in \Gamma_1} \alpha_1(M) \). It follows from Remark 6 and Lemma 8(b), (c) that \( d(G) = 2 \) (indeed, if \( d(G) = 3 \), then \( \Phi(G) \leq Z(G) \), by Lemma 3, and, by Lemma 8(b), (c) again, \( \alpha_1(G) > 2p^2 - 1 > p^2 \), which is not the case).

Let \( \{ F_1, \ldots, F_s, V_1, \ldots, V_u \} \) be the set of nonabelian members of the set \( \Gamma_1 \), where \( \alpha_1(F_i) = p \), \( \alpha_1(V_i) = 1 \) (since \( \Phi(G) \in \Gamma_2 \) is abelian, the set \( \Gamma_1 \) has no member \( H \) with \( \alpha_1(H) = p + 1 \); see Lemma 6(b)). By assumption, \( s > 0 \). Next, \( |G'| > |F_i'| = p^2 \) so the set \( \Gamma_1 \) has at most one abelian member (Lemma 2(a)).

(i) Suppose that \( G \) is metacyclic. If the set \( \Gamma_1 = \{ H_1, \ldots, H_{p+1} \} \) has no abelian member, then, in view of \( u \leq 1 \) (Lemma J(k)), we have \( s \geq p \) so \( \alpha_1(G) = ps + u > p^2 \), a contradiction. Now suppose that \( H_{p+1} \in \Gamma_1 \) is abelian. Then \( |G : Z(G)| = p|G'| = p^4 \) (Lemma J(a)) and, by Lemma J(d), \( u = 0 \) so \( \Gamma_1 = \{ H_1, \ldots, H_p, A \} \), where \( A \) is abelian, and \( G \) is as stated in (1b).

(ii) Suppose that \( G \) is not metacyclic. By Lemma 2(a), the set \( \Gamma_1 \) has at most one abelian member so \( p \leq s + u \leq p + 1 \).

Suppose that the set \( \Gamma_1 \) has (the unique) abelian member. Then, by Lemma 2(c), the derived subgroups of all nonabelian members of the set \( \Gamma_1 \) have the same order \( p^2 \) so \( u = 0 \) (Lemma 1) and \( G/F_i' \) is an \( A_1 \)-group for all \( i \) (Lemma 2(c) again) whence \( |G'| = p|F_i'| = p^3 \). By [5, Proposition 6.1(b)], \( G' \) is noncyclic. The group \( G \) is as stated in (2b).

Now suppose that the set \( \Gamma_1 \) has no abelian member. Then \( s + u = p + 1 \) and

\[ p^2 = \alpha_1(G) = ps + u = p(p + 1 - u) + u = p^2 + p - pu + u \]
so \( pu = p + u \). It follows that \( p = u = 2 \) so \( s = 1 \). In this case, the 2-group \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is metacyclic (Lemma J(e)), contrary to the assumption. \( \square \)

**Theorem 11.** Suppose that \( G \) is a \( p \)-group with \( p^2 < \alpha_1(G) \leq p^2 + p - 1 \). Then \( G \) is a two-generator \( \mathcal{A}_3 \)-group with \( |G'| > p \) and one of the following holds:

(a) \( d(G) = 2, \Gamma_1 = \{H_1, H_2, \ldots, H_p, A\} \), where \( A \) is abelian, \( H_1 \) is an \( \mathcal{A}_2 \)-group with \( d(H_1) = 3, \alpha_1(H_1) = p^2, \alpha_1(H_2) = 1, i = 2, \ldots, p, H_i' = \cdots = H_p' \) is of order \( p \), \( G/H_1' \) is an \( \mathcal{A}_1 \)-group so \( |G'| = p^2, \alpha_1(G) = p^2 + p - 1 \). In particular, \( \beta_1(G, H_1) = p - 1 \).

(b) \( d(G) = 2, \Gamma_1 = \{H_1, \ldots, H_p, B\} \), where \( \alpha_1(H_1) = p, i = 1, \ldots, p, \alpha_1(B) = 1, \alpha_1(G) = p^2 + 1 \). If \( G \) is nonmetacyclic, then \( p > 2, |G : G'| = p^2 \) and \( |G| \in \{p^5, p^6\} \).

(c) \( G \) is metacyclic, \( \Gamma_1 = \{F_1, \ldots, F_s, H_1, \ldots, H_u\} \), where \( s + u = p + 1, \alpha_1(F_i) = p, s \geq 2, \alpha_1(H_j) = p + 1, \alpha_1(\Phi(G)) = 1 \). (All members of the set \( \Gamma_1 \) are \( \mathcal{A}_2 \)-groups and \( |G'| = p^3 \).)

In all cases, \( |G'| \geq p^2 \).

**Proof.** By Lemma 4(b), \( G \) is not an \( \mathcal{A}_2 \)-group.

Assume that \( G \) is not an \( \mathcal{A}_3 \)-group. Then there exists a nonabelian \( H \in \Gamma_1 \) which is an \( \mathcal{A}_n \)-group, \( n \geq 3 \). By Theorem 9, \( \alpha_1(H) \geq p^2 \). Then, by Lemma 5,

\[
p^2 + p - 1 \geq \alpha_1(G) = \alpha_1(H) + \beta_1(G, H) \geq p^2 + p - 1
\]

so \( \alpha_1(G) = p^2 + p - 1 \) and \( \alpha_1(H) = p^2, \beta_1(G, H) = p - 1 \) so \( |H'| = p \) (Lemma 5). Let \( L < H \) be an \( \mathcal{A}_2 \)-subgroup. Since \( |L'| = p \), we get, by Lemma 4, \( \alpha_1(L) \geq p^2 = \alpha_1(H) \), contrary to Lemma J(c). Thus, \( G \) is an \( \mathcal{A}_3 \)-group. This argument also shows that the set \( \Gamma_1 \) has no member \( U \) with \( \alpha_1(U) > p^2 \).

Suppose that \( U \in \Gamma_1 \) with \( \alpha_1(U) = p^2 \). Then \( \beta_1(G, U) = p = 1 \) so, by Lemma 5, \( G \) is as stated in (a).

Next we assume that \( \alpha_1(U) < p^2 \) for all \( U \in \Gamma_1 \); then \( d(G) \leq d(U) + 1 = 3 \) (Lemma 1 and Theorem 9). Let

\[
\Gamma_1 = \{F_1, \ldots, F_s, H_1, \ldots, H_t, B_1, \ldots, B_u, A_1, \ldots, A_v\}.
\]

where \( \alpha_1(F_i) = p, \alpha_1(H_j) = p + 1 \) (see Lemma 6), \( \alpha_1(B_k) = 1 \) and \( A_i \) are abelian. Since \( s + t > 0 \) and \( |F_i'|, |H_j'| > p \), we get \( v \leq 1 \) (Lemma 2(a)). We have

(*) If \( t > 0 \), then \( u + v = 0 \) since the set \( \Gamma_1(H_1) \) has no abelian member.

(**) If \( d(G) = 2 \), then \( uv = 0 \). Indeed, if \( uv > 0 \), then all maximal subgroups of \( G/B_1' \) are abelian, by Lemma J(b), a contradiction since \( |T'| \geq p^2 \) for \( T \in \{F_1, \ldots, F_s, H_1, \ldots, H_t\} \).

(i) Suppose that \( d(G) = 2 \). Then \( uv = 0 \), by (**). Put \( N = \Phi(G) \).
(1i) Suppose, in addition, that \( N \) is nonabelian so it is an \( A_1 \)-group since \( |G : N| = p^2 \). Then \( u = v = 0 \) so \( s + t = |\Gamma_1| = p + 1 \),

\[
p^2 + p - 2 \geq \beta_1(G, N) = \sum_{M \in \Gamma_1} \beta_1(M, N) = s(p - 1) + tp
\]

and we get \( s \geq 2 \) and \( t \leq p - 1 \). Let \( U_i \) be (the unique so normal in \( G \)) abelian maximal subgroup of \( F_i, i = 1, 2 \), and set \( A = U_1U_2 \). The subgroup \( A(> U_1) \) is of class at most 2 (Fitting). If \( A \in \Gamma_1 \), then \( \Phi(G) = A \cap F_1 = A_1 \neq N \), a contradiction. Thus, \( A = G \), so \( G \) is of class 2. In that case, all members of the set \( \Gamma_1 \) are metacyclic (Lemma 4) so \( G \) is also metacyclic (Lemma J(e), (f), (h)), and \( G \) is as stated in (c).

(2i) In what follows we assume that \( N = \Phi(G) \) is abelian; then \( t = 0 \). If \( s = p + 1 \), then \( \alpha_1(G) = sp = (p + 1)p > p^2 + p - 1 \), a contradiction. We conclude that \( u + v > 0 \). Since \( |G'| \geq |F_1'| = p^2 \), we get \( v \leq 1 \) (Lemma 2(a)).

If \( v = 1 \), then \( u = 0 \), by (**); then \( s = p \) and we get \( p^2 < \alpha_1(G) = sp = p^2 \), a contradiction. Thus, \( v = 0 \) so \( s + u = p + 1 \).

As above, \( u > 0 \). We have

\[
p^2 + 1 \leq \alpha_1(G) = \sum_{M \in \Gamma_1} \alpha_1(M) = ps + u = (p - 1)s + p + 1 \leq p^2 + p - 1.
\]

It follows that \( p \leq s \leq p + 1 - 1/(p - 1) \), and so \( s = p \) and \( u = 1 \). In that case, \( \alpha_1(G) = p^2 + 1, \Gamma_1 = \{F_1, \ldots, F_p, B\} \), where \( \alpha_1(F_i) = p, \alpha_1(B) = 1 \). We see that \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is either metacyclic or \( p > 2 \) and \( |G : G'| = p^2 \) (Lemma J(e), (f)).

Suppose that \( G \) is not metacyclic; then \( p > 2 \) and \( |G : G'| = p^2 \). In that case, \( |G'| \leq p|F_1'B'| \leq p^4 \) so \( |G| = |G : G'||G'| \leq p^6 \). Since \( G \) is an \( A_3 \)-group, we get \( |G| \in \{p^5, p^6\} \) so \( G \) is as stated in (b).

(ii) Now let \( d(G) = 3 \). By Lemma 8(b), (c), \( \Phi(G) \notin \mathbb{Z}(G) \). Then, by Lemma 3 and the last sentence of Remark 6, the set \( \Gamma_2 \) has exactly \( p^2 \) nonabelian members which are \( A_1 \)-groups so this set has at least two abelian members, say \( U_1 \) and \( U_2 \). Then \( M = U_1U_2 \in \Gamma_1 \), by the product formula and \( |\Gamma_1(M) \cap \Gamma_2| = p + 1 \). It follows from \( U_1 \cap U_2 \leq \mathbb{Z}(M) \) that all members of the set \( \Gamma_1(M) \cap \Gamma_2 \) are abelian so the set \( \Gamma_2 \) has exactly \( p + 1 \) abelian members, say \( U_1, \ldots, U_{p+1} \). We have \( |M'| \leq p \) (Lemma 2(a)) and \( \Gamma_2 = \{L_1, \ldots, L_p, U_1, \ldots, U_{p+1}\} \), where all \( L_i \) are \( A_1 \)-groups. Since \( L_i \cap M = \Phi(G) \) is abelian for all \( i \), we get \( \alpha_1(M) + p^2 \leq \alpha_1(G) \leq p^2 + p - 1 \). We conclude that \( \alpha_1(M) \leq p - 1 \) so \( M \) is either abelian or \( A_1 \)-group (Remark 1).

(1ii) Suppose that \( M \) is abelian. Then \( v = 1 \) (Lemma 2(a)) and so \( t = 0 \). Also, \( u = 0 \) since all \( p + 1 \) abelian members of the set \( \Gamma_2 \) lie in \( M \). Hence, \( s = |\Gamma_1| - v = p^2 + p \). By the enumeration principle and hypothesis,

\[
\alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H) = ps - p \cdot p^2 = p(p^2 + p) - p^3 = p^2,
\]

a contradiction. Thus, \( v = 0 \).
(2ii) It remains to consider the case where $M$ is an $\mathcal{A}_1$-group; then $u = 1$ since, by what has been said already, all $p + 1$ abelian members of the set $\Gamma_2$ lie in $M$. We get $t = 0$, by (*), so $s = |\Gamma_1| - u = p^2 + p$. Therefore,

$$p^2 + 1 \leq \alpha_1(G) = \sum_{H \in \Gamma_1} \alpha_1(H) - p \sum_{H \in \Gamma_2} \alpha_1(H)
= 1 + sp - p \cdot p^2 = 1 + (p^2 + p)p - p^3 = p^2 + 1.$$  

Note that $d(G) = 3$ but all members of the set $\Gamma_1$ are two-generator. If $G$ is of class 2, it is an $\mathcal{A}_2$-group, by [5, Theorem 4.1], a contradiction. Hence $G$ is of class $> 2$. Then $p = 2$ and $G/K_4(G)$ is of order $2^7$ or $2^8$, by [5, Theorem 4.4]. However, using the remarks following [5, Theorem 4.5], we see that the above groups of order $2^7$ and $2^8$ are $\mathcal{A}_4$-groups and $\mathcal{A}_5$-groups, respectively. This is a contradiction since $G$ is an $\mathcal{A}_3$-group. $\Box$

**Lemma 12.** Let $G$ be an $\mathcal{A}_4$-group. Suppose that $R \in \Gamma_2$ is an $\mathcal{A}_2$-group. Then $\alpha_1(G) > p^2 + p + 1$, unless $G$ is a group satisfying the following conditions:

(a) $p = 2$, $d(G) = 3$, $\alpha_1(G) = 2^2 + 2 + 1 = 7$.

(b) $\Gamma_2 = \{R, R_1, R_2, R_3, A_1, A_2, A_3\}$, where $\alpha_1(R) = 4$, $|R'| = 2$, $d(R) = 3$, $R_1, R_2, R_3$ are $\mathcal{A}_1$-groups and $A_1, A_2, A_3$ are abelian.

(c) $\Gamma_1 = \{F_1, F_2, F_3, H_1, H_2, H_3, A\}$, where $A$ is abelian and

(i) $\Gamma_1(F_i) = \{R, R_i, A_i\}$, $\alpha_1(F_i) = 2^2 + 1 = 5$, $|F_i'| = 4$, $R' = R_i'$, $F_i/R_i'$ is an $\mathcal{A}_1$-group, i.e., $F_i$ is a group of Theorem 11(a), $i = 1, 2, 3$.

(ii) $\Gamma_1(H_1) = \{A_1, R_2, R_3\}$, $\Gamma_1(H_2) = \{A_2, R_3, R_1\}$, $\Gamma_1(H_3) = \{A_3, R_1, R_2\}$ so $\alpha_1(H_i) = 2$ and $|H_i'| = 4$ hence $H_i$ is an $\mathcal{A}_2$-group, $i = 1, 2, 3$.

(d) $Z(G) < \Phi(G)$, $|G'| = 8$, $|G/Z(G)| = 16$, $Z(G) = Z(R_i)$.

All nonabelian members of the set $\Gamma_1$ are two-generator groups.

**Proof.** Let $\Gamma_1^R = \{F_1, \ldots, F_{p+1}\}$; then $F_1, \ldots, F_{p+1}$ are $\mathcal{A}_3$-groups. By Theorem 9, $\alpha_1(F_i) \geq p^2$ for all $i$. By Lemma 6, $\alpha_1(R) \geq p$.

If $\alpha_1(R) = p$, then $\beta_1(F_i, R) \geq p^2 - p$ for all $i$, and we are done since

$$\alpha_1(G) \geq \alpha_1(R) + \sum_{i=1}^{p+1} \beta_1(F_i, R) \geq p + (p^2 - p)(p + 1) = p^3 > p^2 + p + 1.$$  

Now let $\alpha_1(R) = p + 1$. Then $\alpha_1(F_i) \geq p^2 + p$, by Theorems 10 and 11, so $\beta_1(F_i, R) \geq p^2 - 1$ for all $i$, and we are done since

$$\alpha_1(G) \geq \alpha_1(R) + \sum_{i=1}^{p+1} \beta_1(F_i, R) \geq p + 1 + (p^2 - 1)(p + 1) = p^3 + p^2 > p^2 + p + 1.$$
If \( \alpha_1(R) > p + 1 \), then \( \alpha_1(R) \geq p^2 \) (Lemma J(i) and Theorem 4). Since \( F_i \) is an \( A_3 \)-group with \( R \in \Gamma_1(F_i) \), we get \( \beta_1(F_i, R) \geq p - 1 \) (Lemma 5), and so

\[
\alpha_1(G) \geq \alpha_1(R) + \sum_{i=1}^{p+1} \beta_1(F_i, R) \geq p^2 + (p - 1)(p + 1) = 2p^2 - 1 \geq p^2 + p + 1. \tag{4}
\]

If \( p > 2 \), then \( \alpha_1(G) \geq 2p^2 - 1 > p^2 + p + 1 \), and we are done in this case.

In what follows, we suppose that \( \alpha_1(G) = p^2 + p + 1 \); then \( p = 2 \) and \( \alpha_1(G) = 7 \). In that case, as it follows from (4), \( \alpha_1(R) = 4 \) and \( \beta_1(F_i, R) = 1 \) so \( \alpha_1(F_i) = 5 \) for \( i = 1, 2, 3 \). By Lemma 5 applied to the pair \( R < F_i \), we get \( |R'| = 2 \), \( d(F_i) = 2 \) and \( \Gamma_1(F_i) = \{ R, R_i, A_i \} \), where \( R_i \) is an \( A_1 \)-group with \( R' = R_i' \) and \( A_i \) is abelian, \( |F_i'| = 4 \), i.e., \( F_i \) is a group of Theorem 11(a), \( i = 1, 2, 3 \). Since \( d(F_i) = 2 \) (Lemma 5), we get \( d(G) \leq 1 + d(F_i) = 3 \).

Assume that \( d(G) = 2 \). Since all members of the set \( \Gamma_1 \) are also two-generator, \( G \) is metacyclic (Lemma J(e)). In that case, \( |G'| = 2^4 \) so \( |F_i'| = 2^3 \) (Lemma J(i)), contrary to what has been said in the previous paragraph.

Thus, \( d(G) = 3 \). Then \( \Phi(F_i) = \Phi(G) \) and so \( R_1, R_2, R_3, A_1, A_2, A_3 \) all contain \( \Phi(G) \) and they are distinct members in the set \( \Gamma_2 \) which implies \( \Gamma_2 = \{ R, R_1, R_2, R_3, A_1, A_2, A_3 \} \) and \( \Phi(G) \) is abelian.

Let \( A = A_1A_2 \); then \( A \in \Gamma_1 \), by the product formula, and \( A_3 < A \) (if not, \( C_G(\Phi(G)) \supseteq AA_3 = G \), a contradiction since \( \Phi(G) \not\subseteq \text{Z}(G) \) in view of \( |R : \Phi(G)| = 2 \)). Since \( F_1, F_2 \) and \( F_3 \) together contain all \( A_1 \)-subgroups of \( G \) and \( A \cap F_i \) is abelian for \( i = 1, 2 \), it follows that \( A \) is also abelian (Remark 4).

Set \( \Gamma_1 - \Gamma_1^R = \{ H_1, H_2, H_3, A \} \). One may assume that \( A_i < H_i \), \( i = 1, 2, 3 \). Since \( H_i \cap A > \Phi(G) \) is an abelian maximal subgroup of \( H_i \) and so \( H_i \cap A \in \Gamma_2 \), then \( R_i \not\subseteq H_i \) since \( R_iA_i = F_i \neq H_i \), \( i = 1, 2, 3 \). Since \( R \not\subseteq H_i \) and \( |\Gamma_1(H_i) \cap \Gamma_2| = 3 \), \( i = 1, 2, 3 \), we get \( H_1 = \{ A_1, R_2, R_3 \}, H_2 = \{ A_2, R_1, R_3 \} \) and \( H_3 = \{ A_3, R_1, R_2 \} \). Now, \( R, R_1, R_2, R_3 \) together contain all \( A_1 \)-subgroups of \( G \). Each of the four \( A_1 \)-subgroups \( S_1, S_2, S_3, S_4 \) of \( R \) satisfies \( S_j \Phi(G) = R, j = 1, 2, 3, 4 \), since \( \Phi(G) \) is an abelian maximal subgroup of \( R \), and so, in view of \( R \not\subseteq H_i \), no one of \( S_1, S_2, S_3, S_4 \) is contained in \( H_i, i = 1, 2, 3 \). Thus, \( \alpha_1(H_i) = 2 \), \( i = 1, 2, 3 \). By Lemma 6(a), \( H_i \) is an \( A_2 \)-group with \( d(H_i) = 2 \) and \( |H_i'| = 4 \), \( i = 1, 2, 3 \). Hence, all nonabelian members of the set \( \Gamma_1 \) are two-generator groups.

By Lemma J(d), \( |G'| \leq 2|H_i'A'| = 8 \) and so Lemma J(a) implies that \( |G' : \text{Z}(G)| = 2|G'| \leq 2^4 \). On the other hand, \( G = R_iA \) with \( A \cap R_i = \Phi(G) \). We have \( \text{Z}(R_i) = \Phi(R_i) < \Phi(G) \), \( |\Phi(G) : \text{Z}(R_i)| = 2 \), by the product formula, and \( C_G(\text{Z}(R_i)) \supseteq R_iA = G \). Thus, \( \text{Z}(R_i) \subseteq \text{Z}(G), i = 1, 2, 3 \). Assume that \( |G : \text{Z}(G)| < 2^4 \); then \( |G : \text{Z}(G)| = 2^3 \). In that case, all members of the set \( \Gamma_1 \), containing \( \text{Z}(G) \), must be abelian or \( A_1 \)-subgroups since nonabelian members of the set \( \Gamma_1 \) are two-generator. Since \( A \) is the unique abelian member of the set \( \Gamma_1 \), this set also contains an \( A_1 \)-group, contrary to what has been said about \( \Gamma_1 \). Thus, \( \text{Z}(G) = \text{Z}(R_i) \) so \( |G : \text{Z}(G)| = 2^4 \) and \( |G'| = 8 \) (Lemma J(a)). The proof is complete. \( \Box \)

**Definition 1.** A 2-group \( G \) of Lemma 12 is said to be a 2-group of type \( \Phi_{7,1} \).
It will be proved in the appendix following Theorem 15 that groups of type $\mathfrak{G}_{7,1}$ do not exist.

The following lemma is the first step in the classification of the pairs $H < G$ of $p$-groups such that $\beta_1(G, H) = p$.

**Lemma 13.** Let a $p$-group $G$ be an $\mathcal{A}_n$-group, $n > 2$, let $H_1 \in \Gamma_1$ be nonabelian\(^2\) and let $\beta_1(G, H_1) = p$. Then $d(G) = 2$ so $\Gamma_1 = \{H_1, \ldots, H_{p+1}\}$, and one of the following holds:

(a) If $H_2$ is neither abelian nor an $\mathcal{A}_1$-group, then $p = 2$, $\alpha_1(H_2) = 2$, $H_3$ is abelian, $H_1' = H_2'$ is of order 4, $G/H_2'$ is an $\mathcal{A}_1$-group so $|G'| = 8$.

(b) All $p$ members of the set $\Gamma_1 - \{H_1\}$ are $\mathcal{A}_1$-groups, $H_1' = H_2' = \cdots = H_{p+1}'$ and $G/H_1'$ is an $\mathcal{A}_1$-group so $|G'| = p^2$.

(c) All $p$ members of the set $\Gamma_1 - \{H_1\}$ are $\mathcal{A}_1$-groups and $H_2', \ldots, H_{p+1}'$ are pairwise distinct. Set $Q = H_2' \cdots H_{p+1}'$; then $Q \cong E_{p^2}$, $G/Q$ is an $\mathcal{A}_1$-group so $H_1' \subseteq Q$ and $|G'| = p^3$.

In all cases, $|H^i| \leq p^2$ and $p^2 \leq |G'| \leq p^3$.

**Proof.** Let $R \in \Gamma_1(H_1) \cap \Gamma_2$ and set $\Gamma_1^R = \{H_1, H_2, \ldots, H_{p+1}\}$. Since $H_1$ is nonabelian, we have $R \not\subseteq Z(G)$ so the set $\Gamma_1^R$ has at most one abelian member.

(i) Suppose that $R$ is nonabelian; then all subgroups $H_i$ are nonabelian. In that case, $\beta_1(H_i, R) \geq p - 1$ for $i > 1$ (Remark 1), and we get

$$p = \beta_1(G, H_1) \geq \sum_{i=2}^{p+1} \beta_1(H_i, R) \geq p(p - 1)$$

so that $p = 2$ and $\beta_1(H_i, R) = 1$ for $i = 2, 3$. Therefore, by Lemma 5, applied to the pair $R < H_i$, $i = 2, 3$, we have

$$|R'| = 2, \quad d(H_i) = 2, \quad |H_i'| = 4,$$

$$\Gamma_1(H_2) = \{R, L_2, A_2\}, \quad \Gamma_1(H_3) = \{R, L_3, A_3\},$$

where $\alpha_1(L_i) = 1$, $A_i$ is abelian and normal in $G$. Since $H_1, H_2, H_3$ contain together all $\mathcal{A}_1$-subgroups of $G$, the members of the set $\Gamma_1$ are not $\mathcal{A}_1$-groups. Set $A = A_2A_3$; then $\text{cl}(A) \leq 2$ (Fitting). We have $|G : A| \leq 2$ so either $A \in \Gamma_1$ or $A = G$.

(ii) Assume that $A \in \Gamma_1$; then $A$ is abelian since $A \cap H_i = A_i$ is abelian for $i = 2, 3$ (Remark 4). Therefore, in view of $|G'| \geq |H_2'| = 4$, $A$ is the unique abelian member of the set $\Gamma_1$ (Lemma 2(a)).

Since $R \not\subseteq A$, $d(G) > 2$ and, in view of $d(G) \leq d(H_2) + 1 = 3$, we get $d(G) = 3$, $|\Gamma_1(H_i) \cap \Gamma_2| = 2 + 1 = 3$, $i = 1, 2, 3$, so $\Gamma_1(H_2) \cup \Gamma_1(H_3) \subset \Gamma_2$. We have $H_i \cap A =

\(^2\) If $H_1$ is abelian with $\beta_1(G, H_1) = p$, then $\alpha_1(G) = p$ and $G$ is an $\mathcal{A}_2$-group (Lemma 6).
\( A_i \in \Gamma_2 \) and \( A_i \) is the unique abelian member of the set \( \Gamma_i(H_i), i = 2, 3 \). Also put \( A_1 = H_1 \cap A(\in \Gamma_2) \) and \( \Gamma_i^{A_i} = \{H_i, F_i, A_i\}, i = 1, 2, 3 \). We claim that

\[
\Gamma_1 = \{H_1, H_2, H_3, F_1, F_2, F_3, A_1\}.
\]

Indeed, \( F_i \not= H_j \) for all \( i, j \). This is the case for \( i = j \). Next, for \( i > 1 \), \( F_i \not= H_i \) since \( F_i \cap H_i = A_i \not= R = H_i \cap H_1 \). Similarly, \( F_2 \not= H_i \) for \( i \neq 2 \) since \( A_2 \not= R \) and \( F_3 \not= H_i \) for \( i \neq 3 \) since \( A_3 \not= R \). Since \( A_i \) is the unique abelian member of the set \( F_i \) contained in \( F_i, i = 1, 2, 3 \), it follows that \( F_1, F_2, F_3 \) are pairwise distinct. Our claim is justified.

Since \( F_1 \cap H_1 = A_1 \) is abelian and \( a_1(F_1) = 2 = \beta_1(G, H_1) \), the subgroup \( F_1 \) contains all \( \mathcal{A}_1 \)-subgroups that are not contained in \( H_1 \). Set \( S_i = F_1 \cap H_i, i = 2, 3 \). Since \( H_i \not\in \Gamma_1^{A_1} \), it follows that \( S_i \not= A_1 \), and we have \( S_i \in F_2, i = 2, 3 \). Since \( H_2 \cap H_3 = R \), we get \( S_i \not= R \). It follows that \( S_2 \not= S_3 \) so \( \Gamma_1(F_1) \cap \Gamma_2 = \{S_2, S_3, A_1\} \), \( \Gamma_i^{S_i} = \{F_1, H_2, T\} \) for some \( T \in \Gamma_2 \).

Let \( \Gamma_i(H_3) \cap \Gamma_2 = \{R, U, A_1\} \). Then \( H_1 \cap F_i = U \) for \( i = 2, 3 \) so \( \Gamma_i^U = \{H_1, F_2, F_3\} \).

By the above,

\[
A_1 = H_1 \cap F_1 \cap A, \quad A_2 = H_2 \cap F_2 \cap A_2, \quad A_3 = H_3 \cap F_3 \cap A, \\
R = H_1 \cap H_2 \cap H_3, \quad U = H_1 \cap F_2 \cap F_3, \\
S_2 = H_2 \cap F_1 \cap T, \quad S_3 = H_3 \cap F_1 \cap W.
\]

It follows that \( \{T, W\} = \{F_2, F_3\} \). To fix ideas, set \( T = F_2 \); then \( W = F_3 \). By the above, \( R, S_2, S_3, U \) are \( \mathcal{A}_1 \)-subgroups.

By Lemma J(d), \( |G'| \leq 2|H_i'A_i| \). Assume that \( |G'| = 4 \). Then all nonabelian maximal subgroups of \( G \) have the same derived subgroup \( G' \). By Lemma J(a), \( |G : Z(G)| = 8 \). If \( K/Z(G) \) is a maximal subgroup of \( G/Z(G) \) such that \( K \not= A \), then \( |K'| = 2 \) (Lemma J(a)), a contradiction. Thus, \( |G'| = 8 \) so \( |G'/Z(G)| = 16 \) (Lemma J(a)). If \( Z(G) \not= H \in \Gamma_1 \), then \( G = HZ(G) \) so \( |G'| = |H'| \), which is a contradiction. Thus, \( Z(G) < \Phi(G) \). The same argument shows that \( Z(G) = Z(R) = Z(S_2) = Z(S_3) = Z(U) \).

Let us consider the quotient group \( \tilde{G} = G/Z(G) \) of rank 3 and order 16. The subgroups \( \tilde{R}, \tilde{S}_2, \tilde{S}_3 \) and \( \tilde{U} \) are four-groups so \( \tilde{G} \) has at least 9 involutions. Since \( \exp(\tilde{G}) = 4 \), it follows that \( \tilde{G} \) is nonabelian. Let \( \tilde{D} \) be a minimal nonabelian subgroup of \( \tilde{G} \). Then, by Lemma J(j), \( \tilde{G} = \tilde{D}Z(\tilde{G}) \), and now it is easy to see that \( \tilde{G} \cong D_8 \times C_2 \) (see in [3, §10] the description of groups of order 16). Then \( \tilde{G} \) contains exactly two distinct subgroups \( \cong E_8 \). Let \( \tilde{K} \cong E_8 \) be such that \( \tilde{K} \not= \tilde{A} \). Then \( K \in \Gamma_1 \) is nonabelian of rank \( \geq 3 \), contrary to what has been proved already.

(2i) It follows that the set \( \Gamma_1 \) has no abelian member. Then \( G = A_2A_3 \) so \( \text{cl}(G) = 2 \) (Fitting). Since \( d(H_i) = 2 \), \( H_i' \leq Z(H) \) and \( |H_i'| = 4 \), it follows from Lemma 2(b) that \( H_i' \cong C_4, i = 2, 3 \) (otherwise, \( H_i \) is an \( \mathcal{A}_1 \)-group). By Lemma J(i), \( H_2 \) and \( H_3 \) are metacyclic \( \mathcal{A}_2 \)-subgroups.

We have \( Z(G) \leq A_2 \cap A_3 \) (otherwise, the set \( \Gamma_1 \) has an abelian member which coincides with one of maximal in \( G \) subgroups \( A_2Z(G), A_3Z(G) \)). On the other hand, \( C_G(A_2 \cap A_3) \geq A_2A_3 \geq G \) so \( A_2 \cap A_3 \leq Z(G) \). Thus, \( Z(G) = A_2 \cap A_3 \). Then, by the product formula, \( |G : Z(G)| = 2^4 \). We have \( Z(G) < A_i < H_i, i = 2, 3 \). Applying Lemma J(a) to the pair \( A_i < H_i \) and taking into account that \( |H_i'| = 4 \) and \( A_i \in \Gamma_1(H_i) \), we get
\[ |H_i'| : Z(H_i) | = 2 |H_i'| = 8 \] so, comparing the orders, we get \( Z(G) = Z(H_i) \), \( i = 2, 3 \). The quotient group \( G/Z(G) = (A_2/Z(G)) \times (A_3/Z(G)) \) is abelian. Since \( d(H_2) = 2 \), we get \( Z(G) = Z(H_2) < \Phi(H_2) \) and \( H_2/Z(G) \) is abelian of type \((4, 2)\), and we conclude that \( H_2 \) is an \( A_1 \)-group, a contradiction.

Thus, all members of the set \( \Gamma_1(H_1) \cap \Gamma_2 \) are abelian.

(ii) Taking into account that \( R \) is abelian, by (i), we have

\[
p = \beta_1(G, H_1) \geq \sum_{i=2}^{p+1} \beta_1(H_i, R) = \sum_{i=2}^{p+1} \alpha_1(H_i).
\]

It follows from the displayed formula that a nonabelian member \( H \) of the set \( \Gamma_1^R - \{H_1\} \) satisfies \( \alpha_1(H) \in \{1, \ldots, p\} \) so \( d(H) = 2 \) (Lemmas 1 and 6), and we conclude that \( d(G) \leq 1 + d(H) = 3 \).

(iii) Assume that \( H_2 \) is neither abelian nor an \( A_1 \)-group. Then, by what has just been said, \( \alpha_1(H_2) = p = \beta_1(G, H_1) \) so \( H_1 \) is abelian for \( i > 2 \). Next, \( H_2 \) is an \( A_2 \)-subgroup and \( |G'| = |H'_2| = p^2 \) (Lemma 6) so the set \( \Gamma_1 \) has at most one abelian member (Lemmas J(b), (a) and 2(a)). It follows that \( p = 2 \) and \( \Gamma_1^R = \{H_1, H_2, H_3 = A\} \); here \( A \) is abelian, \( \alpha_1(H_2) = 2 \) and \( |H'_2| = 4 \) (Lemmas 6 and 4).

Assume that \( \Gamma_1^R = \Gamma_1 \). Then \( H'_1 = H'_2 \) has order 4 and \( G/H'_2 \) is an \( A_1 \)-group (Lemmas 6 and 2(c)) so \( |G'| = 8 \). In that case, \( G \) is as stated in part (a).

Now assume that \( d(G) = 3 \). In that case, \( |\Gamma_2 \cap \Gamma_1(H_1)| = 2 + 1 = 3 \). Take \( S \in (\Gamma_2 \cap \Gamma_1(H_1)) - \{R\} \); then \( S \) is abelian, by (i). Let \( \Gamma_1^S = \{H_1, F_2, F_3\} \). Since \( H_1 \) and \( H_2 \) together contain all \( A_1 \)-subgroups of \( G \), \( F_2 \) and \( F_3 \) are not \( A_1 \)-subgroups. Since \( F_1 \cap H_1 = S (\in \Gamma_1(H_1)) \) is abelian, we get \( 1 < \alpha_1(F_i) \leq \beta_1(G, H_1) = 2 \) so \( \alpha_1(F_i) = 2 \) and \( F_i \) is an \( A_2 \)-group, \( i = 2, 3 \) (Lemma 6(a)). In that case, \( F_2 \) and \( F_3 \) together contain 4 distinct \( A_1 \)-subgroups and all of them are not contained in \( H_1 \) (indeed, \( F_2 \cap H_1 = S = F_3 \cap H_1 \) is abelian), i.e., \( \beta_1(G, H_1) \geq 4 > 2 \), contrary to the hypothesis.

Thus, all nonabelian members of the set \( \Gamma_1^R - \{H_1\} \) are \( A_1 \)-groups.

We claim: If \( U \in \Gamma_2 \), then \( U < H_i \) for some \( i \in \{1, 2, 3\} \). Indeed, the subgroups \( H_1, H_2, H_3 \) contain together exactly

\[
1 + \sum_{i=1}^{3} (|\Gamma_1(H_i) \cap \Gamma_2| - 1) \cdot 3 = 1 + p(p + 1) = p^2 + p + 1 = |\Gamma_2|
\]

distinct members of the set \( \Gamma_2 \), and our claim is proved. Thus, \( \Gamma_2 = (\Gamma_2 \cap \Gamma_1(H_1)) \cup (\bigcup_{i=2}^{p} \Gamma_1(H_i)) \) (recall that, in view of \( d(H_i) = 2 \) and \( d(G) = 3 \), we have \( \Gamma_1(H_i) \subset \Gamma_2 \), for \( i = 2, \ldots, p + 1 \)). Since \( A_1 \)-subgroups \( H_2 \) and \( H_3 \) contain together 5 members of the set \( \Gamma_2 \) which are abelian, it follows from Remark 6 that all members of the set \( \Gamma_2 \) are abelian.

(ii) Assume that \( d(G) = 3 \). Then

\[
p = \beta_1(G, H_1) = \sum_{F \in \Gamma_1 \setminus \{H_1\}} \alpha_1(F) \geq p^2 - 1
\]
since the set $\Gamma_1 \setminus \{H_1\}$ has at least $p^2 - 1$ nonabelian members (Lemma J(a)) which are $\mathcal{A}_1$-subgroups. However, $p^2 - 1 > p$, and we get a contradiction.

Thus, $d(G) = 2$. Then $H_2, \ldots, H_{p+1}$ are $\mathcal{A}_1$-groups, by (1ii).

(3ii) Let $H'_2 = H'_1$. Then all maximal subgroups of the quotient group $G/H'_2$ are abelian (Lemma J(b)). It follows from Lemma 2(b), that $G/H'_2$ is nonabelian so an $\mathcal{A}_1$-group. We get $H'_1 = \cdots = H'_{p+1}$, $|G'| = |(G/H'_2)| |H'_2| = p \cdot p = p^2$, and $G$ is as stated in part (b) of the lemma.

(4ii) Now suppose that $H'_2, \ldots, H'_{p+1}$ are pairwise distinct. Set $Q = H'_2H'_3$. Then, as above, all maximal subgroups of the quotient group $G/Q$ are abelian so $H'_2 \cdots H'_{p+1} = Q \cong \mathbb{E}_{p^2}$ and $H'_1 \leq Q$. To find ideas, assume that $H'_2 \neq H'_1$. Then $H_1/H'_2$ is nonabelian and $\beta_1(G/H'_2, H_1/H'_2) = p - 1$ so, by Lemma 5, $|G'| = |(G/H'_2)| |H'_2| = p^2 \cdot p = p^3$ and $G/Q$ is an $\mathcal{A}_1$-group. Thus, $G$ is as stated in part (c) of the lemma. □

Now we are ready to complete the proof of Theorem A for odd $p$.

**Theorem 14.** Let $G$ be a $p$-group, $p > 2$. Suppose that $\alpha_1(G) \in \{p^2 + p, p^2 + p + 1\}$. Then $G$ is an $\mathcal{A}_n$-group, $n \in \{2, 3\}$.

**Proof.** Assume that $G$ is not an $\mathcal{A}_n$-group with $n = 2, 3$; then some $M \in \Gamma_1$ is an $\mathcal{A}_k$-group with $k > 2$. In that case, $\alpha_1(M) \geq p^2$ (Theorem 9) so $\beta_1(G, M) \leq p + 1$, by hypothesis.

Let $|M'| = p$; then $d(M) > 2$ (Lemma 2(b)). Let $S < M$ be an $\mathcal{A}_2$-subgroup. Then, by Lemma 4, $\alpha_1(S) \geq p^2$ so, by Lemma 5, applied to the pair $S < M$, we obtain $\alpha_1(M) \geq \alpha_1(S) + p \geq p^2 + p$ since $|M'| < p^2$. Again, by Lemma 5,

$$\alpha_1(G) \geq \alpha_1(M) + p - 1 \geq p^2 + 2p - 1 > p^2 + p + 1 = \alpha_1(G)$$

since $p > 2$, a contradiction. It follows that

$$(A^*) \quad |M'| \geq p^2 \text{ so } \beta_1(G, M) \geq p \text{ (Lemma 5)}.$$

Therefore,

$$p^2 + p + 1 \geq \alpha_1(G) = \alpha_1(M) + \beta_1(G, M) \geq \alpha_1(M) + p,$$

hence $p^2 \leq \alpha_1(M) \leq p^2 + 1$ and so $M$ is an $\mathcal{A}_3$-group of Theorems 10 and 11; then $G$ is an $\mathcal{A}_4$-group.

By Lemma 12, since $p > 2$, an $\mathcal{A}_2$-group cannot be a member of the set $\Gamma_2$ so we have

$$(B^*) \quad \text{Any member of the set } \Gamma_2 \text{ is either abelian or an } \mathcal{A}_1\text{-group.}$$

(i) Let $\alpha_1(M) = p^2$; then $M$ is a group of Theorem 10. In that case, $\beta_1(G, M) \in \{p, p + 1\}$, by hypothesis. Next, by Theorem 10, $|M'| = p^3$. Since the set $\Gamma_1(M) \cap \Gamma_2$ has no member which is an $\mathcal{A}_2$-group, by (B*), it follows that $M$ is not a group of Theorem 10(1a) since all maximal subgroups of that group are $\mathcal{A}_2$-groups.
Lemma 13. By that lemma, we must have $\beta_1(G, M) = p$ so the pair $M < G$ satisfies the hypothesis of Lemma 13. By that lemma, we must have $|M'| \leq p^2$, contrary to the first paragraph of (i).

Thus, we must have $\alpha_1(G) = p^2 + p + 1$. In view of $\beta_1(G, M) = p + 1 > p = \left|\{M_2, \ldots, M_{p+1}\}\right|$, one may assume that $M_2$ is neither abelian nor an $A_1$-group. Since $\Phi(G) = A \in \Gamma_1(M_2)$ is abelian, $M_2$ is an $A_2$-group and $p \leq \alpha_1(M_2) \leq \beta_1(G, M) = p + 1$, we must have $\alpha_1(M_2) = p$ (Lemma 6). It follows from

$$p + 1 = \beta_1(G, M) = \alpha_1(M_2) + \sum_{i=3}^{p+1} \alpha_1(M_i) = p + \sum_{i=3}^{p+1} \alpha_1(M_i)$$

that $|\Gamma_1| - 3$ members of the set $\Gamma_1$ are abelian and exactly one its member, say $M_3$, is an $A_1$-group. Since $|G'| \geq |M'| = p^3 > p$, the set $\Gamma_1$ has at most one abelian member (Lemma 2(a)) so we get $p = 3$ (by hypothesis, $p > 2$). Then $\Gamma_1 = \{M_1 = M, M_2, M_3, M_4 = A\}$, where $A$ is abelian. However, by Lemma 2(c), we get $M_4' = M'$, a contradiction since $|M'| = p^3 > p = |M_3'|$.

(2i) Let $M$ be a group of Theorem 10(1b), (2b), i.e., $d(M) = 2$, $\Gamma_1(M) = \{F_1, \ldots, F_p, A\}$, where $\alpha_1(F_i) = p$, $F_i' = \cdots = F_p'$ has order $p^2$, $A$ is abelian. It follows from (B*) that $F_i \notin \Gamma_1(M) \cap \Gamma_2$ for all $i$ so $A = \Phi(G)$ and $d(G) = 2$. Let $\Gamma_1 = \{M_1 = M, \ldots, M_{p+1}\}$. As in (1i), one has to consider two possibilities: $\alpha_1(G) \in \{p^2 + p, p^2 + p + 1\}$.

Let $\alpha_1(G) = p^2 + p$; then $\beta_1(G, M) = p$ so one can use Lemma 13. By that lemma, we must have $|M'| \leq p^2$, contrary to the first paragraph of (i).

Now let $\alpha_1(G) = p^2 + p + 1$. In that case, $\beta_1(G, M) = p + 1$. Therefore, as in (1i), one may assume that $M_2$ is neither abelian nor an $A_1$-group (otherwise, $\beta_1(G, M) < p + 1$). In that case, as in (1i), we get $p = 3$ and $\Gamma_1 = \{M_1 = M, M_2, M_3, M_4 = A\}$, where $A$ is abelian, $\alpha_1(M_3) = 1$. By Lemma 2(c), we get $M_4' = M'$, a contradiction since $|M'| \geq p^2 > p = |M_3'|$.

All the above, together with (A*) and (B*), yields

$$\alpha_1(M) = p^2 + 1,$$

i.e., $M$ is a group of Theorem 11(b).

(ii) In view of (C*), Theorem 11(b) and Lemma 5, we must have $\beta_1(G, M) \geq p$ so that $\alpha_1(G) = p^2 + p + 1$; then $\beta_1(G, M) = p$.

By Theorem 11(b), $d(M) = 2$, $\Gamma_1(M) = \{F_1, \ldots, F_p, B\}$, where $\alpha_1(F_i) = p$, $i = 1, \ldots, p$, $\alpha_1(B) = 1$. By (B*), $\Phi(G) = B$, which is of index $p^2$ in $G$, so $d(G) = 2$. Let $\Gamma_1 = \{M = M_1, \ldots, M_{p+1}\}$; then $\beta_1(M_i, B) \geq p - 1, i > 1$ (Lemma 5). We have, in view of $p > 2$: 
Let \( H \) be a group of type \( G \). Note that

\[
\alpha_1(G) = \alpha_1(M) + \sum_{i=2}^{p+1} \beta_1(M_i, B) \geq p^2 + 1 + (p - 1) p > p^2 + p + 1,
\]

which is a contradiction since \( p > 2 \). The proof is complete. \( \blacksquare \)

Thus, Theorem A is proved for all odd \( p \). The case \( p = 2 \) is essentially more difficult.

**Theorem 15.** A 2-group \( G \) satisfying \( \alpha_1(G) = 6 \) is an \( \mathcal{A}_n \)-group with \( n \in \{2, 3\} \).

**Proof.** Assume that the theorem is false. By Theorems 9–11, \( G \) is an \( \mathcal{A}_4 \)-group. Then there exists \( H \in \Gamma_1 \) which is an \( \mathcal{A}_3 \)-group. Since \( \alpha_1(H) \geq 4 \) (Theorem 9) and \( \beta_1(G, H) > 0 \) (Lemma J(c)), we get \( \alpha_1(H) \in \{4, 5\} \) so \( \beta_1(G, H) = \{2, 1\} \), respectively.

(i) Let \( \alpha_1(H) = 4 \); then \( \beta_1(G, H) = 2 \) so we can use Lemma 13; in that case, \( |H'| \leq 4 \). On the other hand, \( H \) is a group of Theorem 10, and so \( |H'| = 8 \), a contradiction.

(ii) Let \( \alpha_1(H) = 5 \). Then \( H \) is a group of Theorem 11 so \( |H'| \geq 4 \). On the other hand, \( \beta_1(G, H) = 1 \) so \( |H'| = 2 \) (Lemma 5), a final contradiction. \( \blacksquare \)

**Appendix. Nonexistence of groups of types \( \mathcal{G}_{7,1} \) and \( \mathcal{G}_{7,2} \)**

**Theorem A.1.** 2-groups of type \( \mathcal{G}_{7,1} \) do not exist.

**Proof.** Let \( G \) be a 2-group of type \( \mathcal{G}_{7,1} \), i.e., \( G \) is a 2-group of Lemma 12 with \( \alpha_1(G) = 7 \). We shall use freely the notation and the results of Lemma 12.

By Lemma 4(c), \( H_i \) is metacyclic for \( i = 1, 2, 3 \) since \( p = 2 \) and \( \alpha_1(H_i) = 2 \). Here we note that \( |G| \geq 2^6 \) and so \( |H_i| \geq 2^5 \) since \( R \in \Gamma_2 \) and \( R \) is an \( \mathcal{A}_2 \)-group and so \( |R| \geq 2^4 \). In particular, \( R_1, R_2, R_3, A_1, A_2, A_3 \) are also metacyclic. Next, \( R \) is nonmetacyclic since \( \text{d}(R) = 3 \).

Since \( A_1 \) is a maximal subgroup of \( A \) and \( \text{d}(A_1) = 2 \), we have \( \text{d}(A) \leq 3 \). Assume that \( \text{d}(A) = 2 \). In that case, all members of the set \( \Gamma_1 \) are two-generator so we can use the results of [5, §4]. By [5, Theorem 4.1], \( c\mathcal{c}(G) > 2 \). Then [5, Theorem 4.2] implies that \( (G/K_3(G))' \cong E_8 \). But \( K_3(G) \neq \{1\} \) and so \( |G'| > 8 \), contrary to Lemma 12(d). Hence we have \( \text{d}(A) = 3 \) and so \( \Omega_1(A) \cong E_8 \).

Since \( H_i \) is metacyclic and \( |H_i| \geq 2^5 \) (\( i = 1, 2, 3 \)), we can use [5, Proposition 5.2]. It follows that \( H_i' \cong C_4 \) (Lemma J(k)) and \( H_i \) is isomorphic to one of the following groups:

\[
H_i = \langle a, b \mid a^{2^m} = 1, m \geq 4, b^{2^n} = a^{4^{2^m-1}}, n \geq 2, \epsilon = 0, 1, a^b = a^{1+2^{m-2}} \rangle, \quad (5)
\]

\[
H_i = \langle a, b \mid a^8 = 1, b^{2^n} = a^{4^\epsilon}, n \geq 2, \epsilon = 0, 1, a^b = a^{-1+4^\eta}, \eta = 0, 1 \rangle. \quad (6)
\]

However, in case (5) we get \( \mathcal{U}_1(H_i) = \langle a^2, b^2 \rangle \) (this subgroup is abelian) and we see that \( \langle a, b^2 \rangle, \langle a^2, b \rangle, \langle a^2, b^2 \rangle \langle ab \rangle \) are three maximal subgroups of \( H_i \) and they are all \( \mathcal{A}_1 \)-groups, contrary to \( \alpha_1(H_i) = 2 \) (Lemma 12). (Indeed, to check that a nonabelian metacyclic

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$p$-group is an $A_1$-group, is suffices, in view of Lemma 2(b), to show that its derived subgroup has order $p$. For example, if, in (5), $K = \langle a, b^2 \rangle$, then $a^{b^2} = a^{(1+2^{m-2})2} = a^{1+2^{m-1}}$ so $K' = \langle a^{2^{m-1}} \rangle$ has order 2 and $K$ is an $A_1$-group.) We have proved that $H_1$ must be isomorphic to group given in (6). Here we distinguish two essentially different cases $\epsilon = 0$ (splitting case) and $\epsilon = 1$ (nonsplitting case).

(i) We consider first the splitting case $\epsilon = 0$ so that

$$H_1 = \langle a, b \mid a^8 = b^{2^n} = 1, n \geq 2, a^b = a^{-1}z^\eta, z = a^4, \eta = 0, 1 \rangle$$

and we also set $u = b^{2^n-1}$ and $v = a^2$ so that $v^2 = z$ and $u$ are involutions. Since $|H_1| = 2^{n+3}$, we have $|G| = 2^{n+4}, n \geq 2$. But $A_1 = \langle b^2 \rangle \times \langle a \rangle$ is the unique abelian maximal subgroup of $H_1$ and so $A \cap H_1 = A_1$. The fact that $d(A) = 3$ implies that there is an involution $t \in A - A_1$ so that $\Omega_1(A) = \langle u, z, t \rangle$ because $\Omega_1(H_1) = \langle u, z \rangle < \Omega_1(A)$. We have $Z(H_1) = \langle b^2, z \rangle = Z(G)$ (indeed, $Z(G) \leq H_1$ and $|Z(G)| = |Z(H_1)|$) and $\Phi(H_1) = \langle b^2 \rangle \times \langle v \rangle = \Phi(G)$ since $\Phi(H_1) \subseteq \Phi(G)$ and $d(G) = 3$; in particular, $u = b^{2^{n-1}} \in Z(G)$. Also, $H'_1 = \langle v \rangle \cong C_4$ is normal in $G$ and $G'/\langle v \rangle = \langle b \rangle$ (Lemma 12). Since $\Phi(G) = \langle b^2 \rangle \times \langle v \rangle$ and $H_1 > G' > \langle v \rangle$, it follows, by the modular law, that $G' = \langle u \rangle \times \langle v \rangle$ is abelian of type $(2, 4)$. Note that $R_1, R_2, R_3$ are metacyclic of order $2^{n+2}$ because $\Gamma_1(H_1) = \{A_1, R_2, R_3\}$, $\Gamma_1(H_2) = \{A_2, R_1, R_3\}$ and $H_1$ and $H_2$ are metacyclic of order $2^{n+3}$.

We have $G = \langle H_1, t \rangle = \langle a, b, t \rangle$, $\langle [a, b] \rangle = \langle v \rangle$, $[a, t] = 1$ (recall that $a, t \in A$) and, since $G'/\langle v \rangle$ is nonabelian, we get $[b, t] \in G' - \langle v \rangle$. On the other hand, $\Omega_1(A) = \langle u, z, t \rangle$ is normal in $G$ and so $[b, t] = t^i b t^{-1} \in \Omega_1(A)$.

It follows that $[b, t] = u z^\delta$ with $\delta \in \{0, 1\}$ since $[b, t] = b^{-1} b \in H_1 \cap \Omega_1(A) = \Omega_1(H_1)$ and $[b, t] \in G' - \langle v \rangle$ (recall that $z = v^2$). Also note that $R \in \Gamma_2$ is an $A_2$-group of order $2^{n+2}$ and therefore four $A_1$-subgroups $S_1, S_2, S_3, S_4$ which are contained in $R$ are of order $2^{n+1}$. Since $\alpha_1(G) = 7 \{S_1, S_2, S_3, S_4, R_1, R_2, R_3\}$ is the set of all $A_1$-subgroups of $G$ in particular, all $A_1$-subgroups of $G$ of order $< 2^{n+2}$ are contained in $R$.

Suppose that $\delta = 1$ so that $[b, t] = u z$. Since $u z \in \langle b^2, z \rangle = Z(G)$, $\langle u z \rangle$ is normal in $G$ and so $\langle b, t \rangle/\langle u z \rangle$ is abelian which implies that $\langle b, t \rangle' = \langle u z \rangle \cong C_2$. It follows that $\langle b, t \rangle$ is an $A_1$-subgroup (Lemma 2(b)) containing $\langle b^{2^{n-1}} = u, z, t \rangle \cong E_8$ and so $\langle b, t \rangle$ is nonmetacyclic. Also, $\langle b, t \rangle$ is of order $\geq 2^{n+2}$ (since $\langle u, z, t \rangle \cap \langle b \rangle = \langle u \rangle$) and so $\langle b, t \rangle$ must be of order $2^{n+2}$ and therefore $\langle b, t \rangle$ is one of $R_1, R_2, R_3$ (see the last sentence of the previous paragraph). This is a contradiction since $R_1, R_2, R_3$ are metacyclic.

Suppose that $\delta = 0$; then $[b, t] = u = b^{2^n-1}$ is an involution, and therefore we have either $\langle b, t \rangle \cong M_{2^{n+1}} (n > 2)$ or $\langle b, t \rangle \cong D_{2^n} (n = 2)$. It follows that $\langle b, t \rangle$ is an $A_1$-subgroup of order $2^{n+1}$ and so $\langle b, t \rangle < R$. On the other hand, $R > \Phi(G)$ and so $R \geq \Phi(G) \langle b, t \rangle$. But $\Phi(G) \langle b, t \rangle \in \Gamma_1$. This is a contradiction since $R \in \Gamma_2$.

(ii) We consider now the nonsplitting case $\epsilon = 1$ so that

$$H_1 = \langle a, b \mid a^8 = 1, b^{2^n} = a^4 = z, n \geq 2, a^b = a^{-1}z^\eta, \eta = 0, 1 \rangle$$

and we set $v = a^2$, $w = b^{2^n-1}$ and $u = vw$. We see again that $A_1 = \langle b^2, a \rangle = A \cap H_1$ is the unique abelian maximal subgroup in $H_1$. Since $d(A) = 3$ and $\Omega_1(A_1) = \Omega_1(H_1) = \langle u, z \rangle$, there is an involution $t \in A - A_1$ such that $\Omega_1(A) = \langle u, z, t \rangle \cong E_8$. Also, $|H_1| =
$2^{n+3}$, $|G| = 2^{n+4}$ and the metacyclic $A_1$-subgroups $R_1$, $R_2$, $R_3$ are of order $2^{n+2}$. Since $R \in \Gamma_2$ and $R$ is an $A_2$-group with $\alpha_1(R) = 4$, four $A_1$-subgroups $S_1$, $S_2$, $S_3$, $S_4$, which are contained in $R$, are of order $2^{n+1}$.

We have

$$Z(H_1) = \langle b^2 \rangle \cong C_{2^n}, \quad Z(H_1) = Z(G), \quad \Phi(G) = \Phi(H_1) = \langle b^2 \rangle \langle v \rangle$$

since $\Phi(H_1) \leq\Phi(G)$ and $d(G) = 3$. Also, $H_1' = \langle v \rangle \cong C_4$ is normal in $G$, $|G'| = 8$ (Lemma 12) and so $G' = \langle v, w \rangle = \langle u \rangle \times \langle v \rangle$ is abelian of type $(4,2)$.

We have $G = \langle H_1, t \rangle = \langle a, b, t \rangle, \langle [a, b] \rangle = \langle v \rangle$, where $t \in \Omega_1(A) - \Omega_1(H_1)$, $[a, t] = 1$ and, since $G/\langle v \rangle$ is nonabelian (recall that $o(v) = 4 < 8 = |G'|$), we get $[b, t] \in G' - \langle v \rangle$.

On the other hand, $\Omega_1(A) = \langle u, z, t \rangle$ is normal in $G$ and so

$$b^{-1}b' = [b, t] = t^b t \in \Omega_1(A) \cap H_1 = \Omega_1(H_1) = \Omega_1(G') = \langle u \rangle \times \langle z \rangle.$$

Since $z = v^2 \in \langle v \rangle$, it follows that $[b, t] = u$ or $uz$. Note that $o(b) = 2^{n+1} \geq 8$ and so, taking into account that $u^b = (vw)^b = v^{-1}w = vwz = uz$, we have $\Phi(G) = \langle b^2, v \rangle < \langle b, u \rangle \cong M_{2n+2}$ noting that $b^{2^{-1}} = w, v = uw^{-1}$ and so $v \in \langle b, u \rangle$. Therefore $\langle b, u \rangle$ is an $A_1$-subgroup of order $2^{n+2}$ contained in $H_1$ and consequently $\langle b, u \rangle = R_2$ or $R_3$. Since $t$ normalizes $\langle b, u \rangle$, we have $|\langle b, t \rangle : \langle b, u \rangle| = 2$ and so $\langle b, t \rangle$ is of order $2^{n+3}$ which implies that $\langle b, t \rangle$ is a maximal subgroup of $G$ with $\langle b, t \rangle' \supseteq \langle z, u \rangle$ and so $\langle b, t \rangle' = \langle z, u \rangle \cong E_4$. It follows that $\langle b, t \rangle$ is a nonmetacyclic member of the set $\Gamma_1$. Hence $\langle b, t \rangle = F_2$ or $F_3$ and so $\langle b, t \rangle$ is an $A_2$-group. On the other hand, $\tilde{A} = \langle b^2 \rangle \times \langle u \rangle \times \langle t \rangle$ is an abelian nonmetacyclic subgroup of order $2^{n+2}$ in $\langle b, t \rangle$. Hence $\tilde{A}$ is the unique abelian maximal subgroup of $\langle b, t \rangle$ and so $\tilde{A} = A_2$ or $A_3$ (see, in Lemma 12, lists of the members of the sets $\Gamma_1(F_2)$ and $\Gamma_1(F_3)$). This is a contradiction since $A_1$, $A_2$, $A_3$ are metacyclic. □

**Definition 2.** A 2-group $G$ is said to be a group of type $\mathfrak{S}_{7,2}$, if it satisfies the following conditions:

1. $\mathfrak{S}_{7,2}$ $d(G) = 2, \Gamma_1 = \{H_1, H_2, H_3\}, \alpha_1(H_1) = 4, \alpha_1(H_2) = 2, \alpha_1(H_3) = 1.$

2. $\mathfrak{S}_{7,2}$ $\Gamma_1(H_1) = \{F_1, \ldots, F_6, A\}$, where $\alpha_1(F_i) = 2, i \leq 6, A = \Phi(G)$ is abelian, $|H_1'| = 8$. We have $\beta_1(G, H_1) = 3$ so $\alpha_1(G) = 7$.

3. $\mathfrak{S}_{7,2}$ $|G : Z(G)| = 2^5, Z(G) = Z(H_1) < \Phi(H_1)$.

4. $\mathfrak{S}_{7,2}$ $H_1' = H_2' \times H_3'$, $G/H_1'$ is an $A_1$-group so $|G'| = 2^4, \beta_1(G/H_2', H_1/H_2') = 1, \beta_1(G/H_3', H_1/H_3') = 2$.

It follows from Definition 2 that a 2-group $G$ of type $\mathfrak{S}_{7,2}$, if it exists, is an $A_4$-group.

**Theorem A.2.** Groups of type $\mathfrak{S}_{7,2}$ do not exist.

**Proof.** We prove first that a two-generator 2-group $X$ which is an $A_2$-group is metacyclic. Indeed, if $|X| > 2^4$, then the result follows from Lemma 4(c). So let $|X| = 2^4$ and let $Y$ be a nonabelian subgroup of order $2^3$. Since $d(X) = 2$, we have $C_X(Y) \leq Y$. It follows from Lemma J(j) that $X$ is of maximal class and so $X$ is metacyclic and we are done.
Let $G$ be a 2-group of type $\mathfrak{S}_{7,2}$, where we use the notation from Definition 2. By Lemma 6(a), $H_2, F_1, \ldots, F_6$ are all two-generator $A_2$-groups. Therefore, $H_2, F_1, \ldots, F_6$ are all metacyclic. But $A = \Phi(G) < H_2$ and so $A$ is also metacyclic. Hence $H_1$ is minimal nonmetacyclic and so by [7, Theorem 7.1] $H_1$ is an $A_2$-group of order $2^5$. This is a contradiction since $H_1$ contains a proper subgroup $F_1$ which is an $A_2$-group. \[\Box\]

Now we are ready to prove

**Theorem 16.** If a 2-group $G$ satisfies $\alpha_1(G) = 7$, then $G$ is an $A_n$-group, $n \in \{2, 3\}$.

**Proof.** Suppose that $G$ is a counterexample of minimal order. If $H \in \Gamma_1$, then $\alpha_1(H) < \alpha_1(G) = 7$, so, by Lemma 6 and Theorems 9–11 and 15, $H$ is either abelian or an $A_n$-group, $n \leq 3$. Therefore, $G$ is an $A_4$-group so one can choose $H \in \Gamma_1$ so that $H$ is an $A_3$-group. In that case, $4 \leq \alpha_1(H) \leq 6$ (Theorem 9 and Lemma J(c)) so $H$ is one of the $A_3$-groups of Theorems 10, 11 or 15. Since $\mathfrak{S}_{7,1}$-groups do not exist (appendix), it follows from Lemma 12 that:

(i) The set $\Gamma_2$ has no member which is an $A_2$-group. It follows that $H$ is not a group of Theorems 10(a) and 11(c) since all maximal subgroups of these groups are $A_2$-groups.

(ii) Let $H \in \Gamma_1$ be a group of Theorem 10(2a), (1b), (2b); in that case, $\alpha_1(H) = 4$, $|H'| = 8$, the set $\Gamma_1(H)$ has exactly one abelian member $A$ and all other its members are $A_2$-groups. It follows from (i) that $A = \Phi(G)$ so, in view of $|G : A| = 2^2$, we get $d(G) = 2$.

We conclude that $Z(G) \leq \Phi(G) = A$. Let $\Gamma_1 = \{H_1, H_2, H_3\}$; then $A \in \Gamma_1(H_i)$, $i = 1, 2, 3$. In that case,

$$\alpha_1(H_2) + \alpha_1(H_3) = \beta_1(G, H) = \alpha_1(G) - \alpha_1(H) = 7 - 4 = 3$$

so one may assume that $\alpha_1(H_2) = 2$ and $\alpha_1(H_3) = 1$ (indeed, if $\alpha_1(X) = 3$, then the set $\Gamma_1(X)$ has no abelian member; see Lemmas 6(b) and 4). Thus, $H_2$ is an $A_2$-group (Lemma 6(a)) and $H_3$ is an $A_1$-group. We have $d(H_2) = 2 = d(H_3)$ (Lemmas 1 and 6(a)).

Assume that $H$ is a group from Theorem 10(1b), (2b); then $d(H) = 2$. In that case, by the above, the 2-group $G$ and all members of the set $\Gamma_1$ are two-generator so $G$ is metacyclic (Lemma J(e)). Since $H_2$ is an $A_2$-group and $H_3$ is an $A_1$-group, we deduce from Lemma J(k) that $G$ is an $A_3$-group, contrary to the assumption.

Thus, $H$ is a group from Theorem 10(2a). Then $d(H) = 3$, $|H'| = 8$, $\Gamma_1(H) = \{F_1, \ldots, F_6, A\}$, where $\alpha_1(F_i) = 2$ and $d(F_i) = 2$ for $i \leq 6$ and $A$ is abelian, $|H : Z(H)| = 2|H'| = 16$ (Lemma J(a)). By Lemma J(n), $H_2/Z(H_2) \in \{D_8, C_4 \times C_4\}$ since $H_2$ is a 2-group of rank 2.

Suppose that $H_2/Z(H_2) \cong D_8$ (see Lemma J(n)). In view of $d(H_2) = 2$, we have $Z(H_2) \leq \Phi(H_2) \leq \Phi(G) = A$ so $|A : Z(H_2)| = 4$. Since $H_3$ is an $A_1$-group, we get $Z(H_3) = \Phi(H_3) < \Phi(G) = A$ and $|A : \Phi(H_3)| = 2$. We conclude that $Z(H_2) \cap Z(H_3) \leq Z(G)$ and, by the product formula, $|A : Z(G)| \leq 8$ so $|G : Z(G)| = |G : A||A : Z(G)| \leq 2^5$. In particular, $|H : Z(G)| \leq 2^4$. In fact, by the previous paragraph, we have there equality. Thus, $Z(G) = Z(H)$ has index $2^5$ in $G$. 

**\* \* \* \**
Now let \( H_2/Z(H_2) \) be abelian of type \((4, 4)\). We have \( Z(H_2) < \Phi(H_2) < \Phi(G) < H_3 \). Assume that \( Z(H_2) \notin Z(H_3) \). Then \( H_3/Z(H_2) \) is cyclic since \( Z(H_3) = \Phi(H_3) \) (Lemma 1). Thus, \( G/Z(H_2) \) has the cyclic subgroup \( H_3/Z(H_2) \) of index 2 and the abelian subgroup \( H_2/Z(H_2) \) of type \((4, 4)\) and index 2, which is impossible. It follows that \( Z(H_2) \leq Z(H_3) \) so \( Z(H_2) \leq Z(G) \), and we get \( |G : Z(G)| = |G : H_2||H_2 : Z(H_2)| \leq 2 \cdot 2^4 = 2^5 \). Since \( Z(G) < \Phi(G) < H_2 \), we get \( Z(H_2) = Z(G) \) and so \( |G : Z(G)| = 2^5 \). In particular, comparing the orders, we get \( Z(G) = Z(H) \).

Since \( |H'| = 8 \) and \( |F'_i| = 4 \), the equality \( F_iZ(G) = H \) is impossible for \( i \leq 6 \). It follows that \( Z(G) < F_i \) for all \( i \). In particular, \( Z(G) < \Phi(H) \). We see that \( G \) is a group of type \( \mathfrak{G}_{7,2} \). By the appendix, groups of type \( \mathfrak{G}_{7,2} \) do not exist, and this is a contradiction.

(iii) Now let \( \alpha_1(H) = 2^2 + 1 \), i.e., \( H \) is a group of Theorem 11. As we have noticed in (i), \( H \) is not a group of Theorem 11(c). We have \( |H'| > 2 \) (Theorem 11) and \( \beta_1(G, H) = 2 \).

(1iii) Let \( H \in \Gamma_1 \) be a group of Theorem 11(a), i.e.,

\[
\alpha_1(H) = 2^2 + 1 = 5, \quad d(H) = 2, \quad |H'| = 4, \quad \Gamma_1(H) = \{F_1, F_2, A\},
\]

where \( d(F_1) = 3 \), \( \alpha_1(F_1) = 4 \), \( |F'_1| = 2 \), \( \alpha_1(F_2) = 1 \) and \( A \) is abelian. If \( d(G) = 3 \), then \( F_1 \in \Gamma_1(H) \subset \Gamma_2 \), contrary to (i). Thus, \( d(G) = 2 \) so \( \Gamma_1 = \{H_1 = H, H_2, H_3\} \). We have \( \Phi(G) \in \{F_2, A\} \), by (i).

Assume that \( F_2 = \Phi(G) \). Then

\[
7 = \alpha_1(G) = \alpha_1(H_1) + \sum_{i=2}^{3} \beta_1(H_i, F_2) = 5 + \sum_{i=2}^{3} \beta_1(H_i, F_2)
\]

so \( \beta_1(H_2, F_2) + \beta_1(H_3, F_2) = 2 \). By Remark 1, we get \( \beta_1(H_1, F_2) = 1 \), \( i = 2, 3 \). In that case, \( \alpha_1(H_i) = 2 \) so \( |H'| = 4 \), \( d(H_i) = 2 \), \( i = 2, 3 \) (Lemma 6(a)). Thus, \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is metacyclic (Lemma J(e)). Then, by Lemma J(k), applied to \( H_2 \) and \( H_3 \), \( G \) is an \( \mathcal{A}_3 \)-group so \( G \) is not a counterexample.

Therefore, we must have \( \Phi(G) = A \). Then we have \( \alpha_1(H_2) + \alpha_1(H_3) = 2 \). If \( \alpha_1(H_i) = 1 \), \( i = 2, 3 \), then \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is metacyclic (Lemma J(e)). In that case, by Lemma J(k), \( G \) must be an \( \mathcal{A}_2 \)-group so it is not a counterexample.

Thus, \( \alpha_1(H_2) = 2 \) so \( |H'_2| = 4 \) and \( H_3 \) is abelian. By Lemma 2(c), \( H' = H'_2 \) and \( G/H' \) is an \( \mathcal{A}_1 \)-group. It follows that \( |G'| = 2|H'| = 8 \) so \( |G : Z(G)| = 16 \) (Lemma J(a)). Since \( d(G) = 2 \), we get \( Z(G) < \Phi(G) < H_i \) so \( |H_i : Z(G)| = 8 \), \( i = 1, 2, 3 \). Since \( H_1/Z(G) \) is noncyclic and two-generator, we get \( Z(G) < \Phi(H_i) < F_i \), \( i = 1, 2 \). Then \( F_i : Z(G)| = 4 \), \( i = 1, 2 \). Since \( d(F_i) = 3 \) and \( Z(F_i) > \Phi(F_i) \), we get, by Lemma 8(c), \( \alpha_1(F_i) \geq 2 \cdot 2^2 - 1 = \alpha_1(G) \), contrary to Lemma J(c).

(2iii) Let \( H \in \Gamma_1 \) be a group from Theorem 11(b). Retaining the notation of Theorem 11, we get, in view of (i), \( \Phi(G) = B \) so \( d(G) = 2 \). Let \( \Gamma_1 = \{H_1 = H, H_2, H_3\} \). Then, as above, \( \beta_1(H_i, R) = 1 \), and we conclude that so \( \alpha_1(H_i) = 2 \) and \( d(H_i) = 2 \) for \( i = 2, 3 \) (Lemma 6(a)). In that case, \( G \) and all members of the set \( \Gamma_1 \) are two-generator so \( G \) is metacyclic (Lemma J(e)). Then, by Lemma J(k), applied to \( H_2 \) and \( H_3 \), \( G \) is an \( \mathcal{A}_3 \)-group, contrary to the assumption.
(iv) It remains to consider the case where $H \in \Gamma_1$ is a group of Theorem 15, i.e., $\alpha_1(H) = 6$. In that case, $\beta_1(G, H) = 1$ so $|H'| = 2$ and $|G'| = 4$, $d(G) = 2$, $\Gamma_1 = \{H_1 = H, H_2, A\}$, where $H_2$ is an $A_1$-group and $A$ is abelian (Lemma 5). By Lemma J(a), $|G : Z(G)| = 8$ and, since $Z(G) < \Phi(G) < H$, we get $|H : Z(G)| = 4$ so $|H : Z(H)| = 4$. Since $H$ is not an $A_1$-group, we get $d(H) = 3$. Then, by Lemma 8(c), $\alpha_1(H) > 2 \cdot 2^2 - 1 > 6$, a final contradiction.

Theorem A follows immediately from Lemma 6, Theorems 9–11, 14–16 and the appendix.

If $G$ is a 2-group of maximal class and order $2^6$, then $\alpha_1(G) = 8$ and $G$ is an $A_4$-group. I do not know if, for odd $p$, there exists an $A_4$-group $G$ with $\alpha_1(G) = (p^2 + p + 1)$.

If $G$ is a nonabelian $p$-group such that for each $K \in \Gamma_2$, we have $\alpha_1(K) < p^2 + 2p + 1$, then $G$ is an $A_n$-group, $n < 4$. Assume that this is false. Then there is $H \in \Gamma_1$ such that $H$ is an $A_k$-group with $k \geq 4$. By Theorem A, $\alpha_1(H) \geq p^2 + p + 2$. If $K \in \Gamma_1(H)$ is abelian, then, since the set $\Gamma_1^K$ has at most one abelian member, we have

$$\alpha_1(\Gamma_1^K) = \alpha_1(H) + \sum_{M \in \Gamma_1^K \setminus \{H\}} \alpha_1(M) \geq (p^2 + p + 2) + (p - 1) = p^2 + 2p + 1,$$

a contradiction. If $K$ is nonabelian, then, by Remark 1,

$$\alpha_1(\Gamma_1^K) = \alpha_1(H) + \sum_{M \in \Gamma_1^K \setminus \{H\}} \beta_1(M, K) \geq (p^2 + p + 2) + p(p - 1)$$

$$\geq 2p^2 + 2 > p^2 + 2p + 1,$$

a final contradiction.

3. A lower estimate of $\alpha_1(G)$ in terms of $d(G)$

To facilitate the proof of Theorem B, we begin with two remarks.

Remarks. 10. Let $G$ be a nonabelian $p$-group, $d(G) = 3$ and let $H \in \Gamma_1$. Let us prove that $\beta_1(G, H) \geq p(p - 1)$. This is the case if $H$ is abelian since then $\beta_1(G, H) = \alpha_1(G) \geq p^2$ (Theorem 9). In what follows we assume that $H$ is nonabelian. If $\alpha_1(G) = p^2$, then $\alpha_1(H) \leq p$ (Theorem 10) so $\beta_1(G, H) \geq p^2 - p = p(p - 1)$. Next we let $\alpha_1(G) \geq p^2 + 1$. The result is true if $\alpha_1(H) \leq p + 1$. So, in view of Lemma 6 and Theorem 9, one may assume that $\alpha_1(H) \geq p^2$. Take $R \in \Gamma_1(H) \cap \Gamma_2$. Assuming that $R$ is nonabelian, we get, using Remark 1,

$$\beta_1(G, H) \geq \sum_{F \in \Gamma_1^K \setminus \{H\}} \beta_1(F, R) \geq p(p - 1)$$
since all $p + 1$ members of the set $\Gamma_1^R$ are nonabelian. If $R$ is abelian, then the set $\Gamma_1^R$ has at least $p$ nonabelian members so, by Lemma 6, $\beta_1(G, H) \geq \sum_{F \in \Gamma_1^R\setminus\{H\}} \alpha_1(F) \geq (p - 1)p$, completing the proof.

11. Suppose that $G$ is a $p$-group such $d = d(G) > 3$. We claim that if $H \in \Gamma_1$ is nonabelian, then $\beta_1(G, H) \geq p^{d - 2}(p - 1)$. We are working by induction on $d$. Remark 10 is the basis of induction. If $X \in \Gamma_1$, then $d(X) \geq d - 1 > 3$. Since $|\Gamma_1(H) \cap \Gamma_2| = p^{d - 2} + \cdots + p + 1 > p + 1$, the set $\Gamma_1(H) \cap \Gamma_2$ has a nonabelian member $R$ (Lemma J(b)). Then all members of the set $\Gamma_1^R = \{H_1 = H, \ldots, H_{p+1}\}$ are nonabelian. Let $d(H_i) = d_i$, where $d_i \geq d - 1$. By induction, $\beta_1(H_i, R) \geq p^{d_i - 2}(p - 1) \geq p^{d - 3}(p - 1), i > 1$. Therefore,

$$\beta_1(G, H) \geq \sum_{i=2}^{p+1} \beta_1(H_i, R) \geq p^{d - 3}(p - 1) \cdot p = p^{d - 2}(p - 1),$$

as was to be shown.

**Theorem B.** Suppose that a $p$-group $G$ is neither abelian nor an $A_1$-group and $d = d(G) > 3$. Then $\alpha_1(G) \geq p^{d - 1}$.

**Proof.** By Lemma 6 and Theorem 9, the theorem is true for $d \leq 3$ so we may assume in what follows that $d > 3$. If $X \in \Gamma_1$, then $d(X) \geq d - 1$. We proceed by induction on $d$. If $H \in \Gamma_1$ is nonabelian and such that $d(H) \geq d$, then, by induction, $\alpha_1(H) \geq p^{d - 1}$, and we are done. Therefore, one may assume that, for all nonabelian $H \in \Gamma_1$, we have $d(H) < d$; then, for such $H$ we have $d(H) = d - 1 > 3$. It follows from Lemma 3 that there is a nonabelian $R \in \Gamma_2$. Let $\Gamma_1^R = \{H_1, \ldots, H_{p+1}\}$. By induction, $\alpha_1(H_i) \geq p^{d - 2}$ and, by Remark 11, $\beta_1(H_i, R) \geq p^{d - 3}(p - 1)$ for all $i$. Therefore,

$$\alpha_1(G) \geq \alpha_1(H_1) + \sum_{i=2}^{p+1} \beta_1(H_i, R) \geq p^{d - 2} + p^{d - 3}(p - 1) \cdot p = p^{d - 1},$$

completing the proof. □

As $p$-groups of Lemma 6(a) and Theorem 10(2a) show, the estimate of Theorem B is best possible for $d \in \{2, 3\}$.

It follows from the proof of Theorem B that, if $d(G) = d > 3$ and $\alpha_1(G) = p^{d - 1}$, then $d(H) = d - 1$ and $\alpha_1(H) = p^{d - 2}$ for all nonabelian $H \in \Gamma_1$ so that $G' = \Phi(G)$.

### 4. Groups of exponent $p$

We assume that all groups, considered in this section, have exponent $p$.

If $G$ is an $A_1$-group of exponent $p$, then $G/G' \cong E_{p^2}$ so $|G| = p^3$. It follows that the order of any $A_n$-group of exponent $p$ equals $p^{n+2}$.
Let $G$ be an $A_2$-group of exponent $p$; then $|G| = p^4$. The set $\Gamma_1$ has an abelian member $A$. If $A$ is the unique abelian member of the set $\Gamma_1$, then $|Z(G)| = p$ and $d(G) = 2$ (Lemma J(a)), and we have $\alpha_1(G) = p$. Now suppose that $\Gamma_1$ has $p + 1$ distinct abelian members (see Lemma J(b)). Then $\alpha_1(G) = p^2$ and $G = S \times C$, where $S$ is nonabelian of order $p^3$.

**Remarks.**

12. Suppose that $G$ is a group of exponent $p$ and order $> p^5$. Let us prove that if $H \in \Gamma_1$, then $\beta_1(G, H) \equiv 0 \pmod{p^2}$. One may assume that $G$ is nonabelian. By Lemma J(g), the numbers $\alpha_1(H)$ and $\alpha_1(G)$ are multiples of $p^2$, and the claim follows: $\beta_1(G, H) = \alpha_1(G) - \alpha_1(H) \equiv 0 \pmod{p^2}$.

13. Let us prove that if $G$ is a nonabelian group of order $p^6$ and exponent $p$, then $\alpha_1(G) \geq p^3$ (the equality is possible as Remark 5 shows). Let $R \in \Gamma_2$ be such that $R \neq Z(G)$. Set $\Gamma_1^R = \{H_1, \ldots, H_{p+1}\}$. If $R$ is nonabelian, then we have, by Remark 12 and Lemma J(c),

$$\alpha_1(G) \geq \alpha_1(R) + \sum_{i=1}^{p+1} \beta_1(H_i, R) > p^2(p + 1) > p^3.$$  

If $R$ is abelian, one may assume, without loss of generality, that $H_1, \ldots, H_p$ are nonabelian so $\alpha_1(H_i) \geq p^2$ ($i = 1, \ldots, p$), by Lemma J(g). In that case, $\alpha_1(G) \geq \sum_{i=1}^{p} \alpha_1(H_i) \geq p^2 \cdot p = p^3$, completing the proof.

**Theorem C.** If $G$ is a nonabelian group of order $p^m$ and exponent $p$, $m > 3$, then

(a) If $H \in \Gamma_1$, then $\beta_1(G, H) \geq p^{m-4}(p - 1)$.

(b) $\alpha_1(G) \geq p^{m-3}$.

(c) If $\alpha_1(G) = p^{m-3}$, then $G$ is of maximal class with abelian subgroup of index $p$ and $m \leq p$.\footnote{Every $p$-group $G$ of maximal class and order $p^m$ with abelian subgroup of index $p$ satisfies $\alpha_1(G) = p^{m-3}$, independent of $\exp(G)$, by Remark 5.}

**Proof.** In all three cases we proceed by induction on $m$.

(a) If $m = 4$, the assertion is true (see the paragraph, preceding Remark 12). So one may assume that $m > 4$.

Let $H \in \Gamma_1$ be nonabelian and let $R \in \Gamma_1(H) \cap \Gamma_2$. Set $\Gamma_1^R = \{H_1 = H, \ldots, H_{p+1}\}$; the set $\Gamma_1^R$ has at most one abelian member.

Suppose that $H_{p+1}$ is abelian; then $R$ is also abelian. In that case, by induction,

$$\beta_1(G, H) \geq \sum_{i=2}^{p} \alpha_1(H_i) \geq p^{(m-1)-3} \cdot (p - 1) = p^{m-4}(p - 1).$$
Now suppose that the set $\Gamma_1^R$ has no abelian member. Then, by induction,

$$\beta_1(G, H) \geq \sum_{i=2}^{p+1} \beta_1(H_i, R) \geq p^{(m-1)-4} (p - 1) \cdot p = p^{m-4} (p - 1),$$

completing the proof of (a).

(b) As in (a), one may assume that $m > 4$. Take in $G$ a normal subgroup $R$ of order $p^2$. Since $|G : C_G(R)| \leq p$, the subgroup $C_G(R)$ contains a subgroup $H \in \Gamma_1$ such that $R < H$. Since $H$ is not of maximal class, it is either abelian or $\alpha_1(H) > p^{(m-1)-3} = p^{m-4}$, by induction. Suppose that $H$ is nonabelian. Take $T \in \Gamma_1(H) \cap \Gamma_2$ and set $\Gamma_1^T = \{H_1 = H, \ldots, H_{p+1}\}$; then the set $\Gamma_1^T$ has at most one abelian member. Suppose that $H_{p+1}$ is abelian; then $T$ is also abelian. In that case, by (a),

$$\alpha_1(G) \geq \alpha_1(H) + \sum_{i=2}^{p} \alpha_1(H_i) > p^{m-4} + p^{m-4} \cdot (p - 1) = p^{m-3}.$$  

Next suppose that the set $\Gamma_1^T$ has no abelian member. Then, by (a),

$$\beta_1(H_i, T) \geq p^{m-5} (p - 1) \quad \text{for } i > 1$$

so we get since $H$ is not of maximal class

$$\alpha_1(G) \geq \alpha_1(H) + \sum_{i=2}^{p+1} \beta_1(H_i, T) > p^{m-4} + p^{m-5} \cdot (p - 1) \cdot p = p^{m-3}.$$  

Now suppose that $H$ is abelian. Suppose that subgroups $H_2, \ldots, H_{p+1}$ are nonabelian; then $\alpha_1(H_i) \geq p^{m-4}$, by induction, and we have

$$\alpha_1(G) \geq \sum_{i=2}^{p+1} \alpha_1(H_i) \geq p^{m-4} \cdot p = p^{m-3}.$$  

Now suppose that $H_i$ is abelian for some $i > 1$; then all members of the set $\Gamma_1^T$ are abelian so that $T = Z(G)$. Since $m > 4$, $G$ is not an $A_1$-group so there exists a nonabelian $F \in \Gamma_1$. Take $U \in \Gamma_1(F) \cap \Gamma_2$ and set $\Gamma_1^U = \{F_1 = F, \ldots, F_{p+1}\}$. Suppose for definiteness, that $F_1, \ldots, F_p$ are nonabelian. If $U$ is nonabelian, then, as above, (b) is true. Now let $U$ be abelian. Since $m > 4$ and $|G : Z(G)| = p^2$, the subgroup $F_i$ is not of maximal class so, by induction, $\alpha_1(F_i) > p^{m-4}$, $i = 1, \ldots, p$, and we get

$$\alpha_1(G) \geq \sum_{i=1}^{p} \alpha_1(F_i) > p \cdot p^{m-4} = p^{m-3},$$

and the proof of (b) is complete.
(c) Now suppose that $\alpha_1(G) = p^{m-3}$. Arguing as in (b), we prove that $\Gamma_1$ has an abelian member.

(i) Assume that $|Z(G)| = p$. Then $G$ is of maximal class (Remark 5) and, by Lemma J(m), since $\exp(G) = p$, we must have $m \leq p$. In what follows we assume that $|Z(G)| > p$.

(ii) Assume that $Z(G) < H_1 \in \Gamma_1$, where $H_1$ is nonabelian. Take $Z(G) < R \in \Gamma_1(H_1) \cap \Gamma_1$ and set $\Gamma_{1R} = \{H_1, \ldots, H_p\}$. One may assume that $H_1, \ldots, H_p$ are nonabelian. It follows from $Z(G) < H_i$ and $|Z(G)| > p$ that $H_i$ is not of maximal class so, by induction, $\alpha_1(H_i) > p^{m-4}$. If $H_{p+1}$ is abelian, then $R$ is also abelian, and we have $\alpha_1(G) \geq \sum_{i=1}^{p} \alpha_1(H_i) > p^{m-4} \cdot p = p^{m-3}$, contrary to the hypothesis. If $H_{p+1}$ is nonabelian, then, by (a),

$$\alpha_1(G) \geq \alpha_1(H_1) + \sum_{i=2}^{p+1} \beta_1(H_i, R) > p^{m-4} + p^{m-5} (p - 1) \cdot p = p^{m-3},$$

contrary to the hypothesis. It remains to consider the case where $|G : Z(G)| = p^2$. Since $m > 3$, $G$ is not an $A_1$-group. Then, arguing as in (b), we get a contradiction. The proof is complete. □

Let $G$ be a nonabelian group of order $p^m$ and exponent $p$, $m > p$. It follows from Theorem C that then $\alpha_1(G) > p^{m-3}$ since $G$ is not of maximal class (Lemma J(m)). It is interesting obtain for such $G$ a better lower estimate.

5. $M_3$-Groups

A $p$-group $G$ is said to be an $M_3$-group, if all its $A_1$-subgroups have the same order $p^3$. The following $p$-groups are $M_3$-groups: groups of exponent $p$, extraspecial groups, groups of maximal class with abelian subgroup of index $p$ (Remark 5). We do not assert that epimorphic images of $M_3$-groups are $M_3$-groups.

If an $A_2$-group $G$ is also an $M_3$-group, then $|G| = p^4$ and $\alpha_1(G) \in \{p, p^2\}$.

Supplement 1 to Theorem C. Let $G$ be a nonabelian $M_3$-group of order $p^m$ and exponent $p$, $m > 3$. Then

(a) If $H \in \Gamma_1$, then $\beta_1(G, H) \geq p^{m-4}(p - 1)$.

(b) $\alpha_1(G) \geq p^{m-3}$.

(c) If, in addition, $\alpha_1(G) = p^{m-3}$, then $G$ is of maximal class with abelian subgroup of index $p$.

The proof is omitted since it is the repetition of the proof of Theorem C. Indeed, in the proof of Theorem C we used only the fact that $G$ is an $M_3$-group.

Part (b) of the above supplement may be generalized as follows.
Supplement 2 to Theorem C. Let $G$ be a nonabelian group of order $p^m$. Suppose that $G$ has an $A_1$-subgroup of order $p^a$ but has no an $A_1$-subgroup of order $> p^a$. Then $\alpha_1(G) \geq p^{m-a}$. If, in addition, $\alpha_1(G) = p^{m-a}$, then all $A_1$-subgroups of $G$ have the same order $p^a$.

6. $p$-Groups with few conjugate classes of $A_1$-subgroups

Given a $p$-group $G$, let $\kappa_1(G)$ denote the number of conjugate classes of $A_1$-subgroups in $G$.

Given $H \in \Gamma_1$, define (i) $\bar{\beta}_1(G,H)$ is the number of conjugate $G$-classes of $A_1$-subgroups not contained in $H$, (ii) $\bar{\kappa}_1(H)$ is the number of $G$-classes contained in $H$. We have $\bar{\kappa}_1(H) \leq \kappa_1(H)$ and, as a rule, the strong inequality holds.

Theorem 17. (Compare with Lemma 5.) Let $G$ be $p$-group and let $H \in \Gamma_1$ be nonabelian. Then $\bar{\beta}_1(G,H) \geq p - 1$. If $\bar{\beta}_1(G,H) = p - 1$, then

(a) $d(G) = 2$ and $\Gamma_1 = \{H_1 = H, \ldots, H_p, A\}$, where $A$ is the unique abelian member of the set $\Gamma_1$.
(b) $H'_1 = \cdots = H'_p$, $G/H'_1$ is an $A_1$-group so $|G'| = p|H'_1|$ and $d(H_1) \leq 3$.
(c) If $|H'_i| = p$ for some $i > 1$, then $H_2, \ldots, H_p$ are $A_1$-groups and $\bar{\beta}_1(G,H) = p - 1$.

In particular, if $\kappa_1(G) = p$, then $d(G) = 2$ and $G$ has an abelian subgroup of index $p$.

Lemma 18. Let $G$ be a $p$-group and let $H \in \Gamma_1$ be neither abelian nor an $A_1$-group. Suppose that all $A_1$-subgroups of $H$ are conjugate in $G$ or, what is the same, $\bar{\kappa}_1(H) = 1$. Then

(a) The set $\Gamma_1(H)$ has exactly one abelian member and $H$ has no proper nonabelian $G$-invariant subgroup.
(b) $d(G) = 2$.
(c) $|H'| > p$.

Proof. Let a nonabelian $A \in \Gamma_1(H)$ be $G$-invariant and let $U \leq A$ be an $A_1$-subgroup. By Lemma J(c), $H$ has an $A_1$-subgroup $V$ that is not contained in $A$. Since $U^G \leq A$, $U$ and $V$ are not conjugate in $G$, a contradiction. Thus, each $G$-invariant proper subgroup of $H$ is abelian. In particular, all members of the set $\Gamma_1(H) \cap \Gamma_2$ and $\Phi(G)$, $\Phi(H)$ are abelian.

Assume that $|H'| = p$. Since $H$ is not an $A_1$-group, we get $d(H) > 2$ (Lemma 2(b)). Let $U < H$ be an $A_1$-subgroup. Since $|H'| = p = |U'|$, we get $U' = H'$ so $U$ is normal in $H$. It follows that $H \leq N_G(U)$ so $|G : N_G(U)| \leq p$. In that case, $H$ contains at most $p$ subgroups conjugate with $U$ in $G$ so, by hypothesis, $\alpha_1(H) \leq p$. By Lemma 6 and Theorem 9 (see also Theorem B), $\alpha_1(H) \geq p^2$, contrary to what has just been said. Thus, $|H'| > p$, proving (c).

It follows from two previous paragraphs that the set $\Gamma_1(H)$ has exactly one abelian member (otherwise, $|H'| = p$), and this completes the proof of (a). Therefore, since all
members of the set $\Gamma_1(H) \cap \Gamma_2$ are normal in $G$, they must be abelian, so $|\Gamma_1(H) \cap \Gamma_2| = 1$ (Lemma J(a)), and we get $d(G) = 2$, completing the proof of (b) and thereby the lemma. □

**Proof of Theorem 17.** Let $N \in \Gamma_2 \cap \Gamma_1(H)$; then $N \not\subseteq Z(G)$ since $H$ is nonabelian. Let $\Gamma_1^N = \{H_1 = H, \ldots, H_{p+1}\}$. Since the set $\Gamma_1^N$ has at most one abelian member, one may assume that $H_1, \ldots, H_p$ are nonabelian. By Lemma J(c), there exists an $A_1$-subgroup $B_i \leq H_i$ such that $B_i \not\subseteq N$, $i = 2, \ldots, p$. For $i \neq j$, we have

$$B_i^G N = H_i \neq H_j = B_j^G N$$

since $N \in \Gamma_1(H_i) \cap \Gamma_1(H_j)$, so $B_i^G \neq B_j^G$, and we conclude that $B_i$ and $B_j$ are not conjugate in $G$. It follows that

$$\tilde{\beta}_1(G, H) \geq |\{B_2, \ldots, B_p\}| = p - 1.$$

Next we assume that $\tilde{\beta}_1(G, H) = p - 1$. For $i = 2, \ldots, p$, all $A_1$-subgroups of $H_i$ that are not contained in $N$, are conjugate in $G$ with $B_i$ so $H_{p+1}$ must be abelian (otherwise, $\tilde{\beta}_1(G, H) > p - 1$); then $N$ is also abelian. In that case, $\tilde{k}_1(H_i) = 1$ for $i = 2, \ldots, p$ so, by Lemma 18(b), $d(G) = 2$. It follows from $\Gamma_1 = \Gamma_1^N$ that $N = \Phi(G)$. Then $H_1^1 = \cdots = H_p^1$ and $G/H_1^1$ is an $A_1$-subgroup (Lemma 2(c)) so $|G'| = p|H_1'|$ and, since $H_1^1 < \Phi(H_1)$, $d(H_1) = d(H_1^1) \leq 3$ (Lemma 1). □

Given a $p$-group $G$ with $\kappa_1(G) = p$, a pair $H < G$ satisfies the hypothesis of Theorem 17 provided $H \in \Gamma_1$ is nonabelian (compare with Lemma 6(a)). If $G$ is a $p$-group of maximal class and order $p^n > p^3$ with abelian subgroup of index $p$ and $H \in \Gamma_1$ is nonabelian, then $\tilde{\beta}(G, H) = p - 1$ and $|H'| = p^{n-2} > p$ (compare with Lemma 5). The group $G$ of the previous sentence satisfies also $\kappa_1(G) = p$.

**Corollary 19.** (Compare with Lemma 6(b).) Let $G$ be a $p$-group. If $\kappa_1(G) = p + 1$, then $d(G) = 2$ and $\Phi(G)$ is abelian.

**Proof.** Since $G$ is neither abelian nor an $A_1$-group, there exists $R \in \Gamma_2$ such that $R \not\subseteq Z(G)$ (Lemma 3). Let $\Gamma_1^R = \{H_1, \ldots, H_{p+1}\}$; then at most one member of the set $\Gamma_1^R$ is abelian. Suppose that $H_1$ is nonabelian; then $\tilde{\beta}(G, H_1) \leq p$, by hypothesis. If $\tilde{\beta}(G, H_1) = p - 1$, then $d(G) = 2$ and $R = \Phi(G)$ is abelian (Theorem 17). Now we let $\tilde{k}(H_1) = 1$ (otherwise, $\kappa_1(G) > p + 1$). In that case, either (i) $d(G) = 2$ with abelian $\Phi(G)$ or (ii) $H_1$ is an $A_1$-group (Lemma 18). It remains to consider possibility (ii) for all nonabelian $H_1 \in \Gamma_1$. Since $H_1$ is an arbitrary nonabelian member of the set $\Gamma_1$, it follows that $G$ is an $A_2$-group with $\alpha_1(G) = p + 1$. In that case, $d(G) = 2$ and $\Phi(G)$ is abelian (Lemma 6(b)). □

**Proposition 20.** Suppose that a $p$-group $G$ is neither abelian nor an $A_1$-group. Let all proper nonabelian normal subgroups of $G$ be members of the set $\Gamma_1$. Then $d(G) \leq 3$ and one of the following holds:

[Continue the text with the remaining content of the page.]
(a) $d(G) = 2$ and $G$ is either of class 2 or $G/G'$ has a cyclic subgroup of index $p$.
(b) $d(G) = 3$. If $R$ is a $G$-invariant subgroup of index $p$ in $G'$, then $G/R$ is an $A_2$-group.

**Proof.** Suppose that $d(G) > 2$. By hypothesis, all members of the set $\Gamma_2$ are abelian so $\Phi(G) \leq Z(G)$ and $d(G) = 3$ (Lemma 3). Let $R$ be a $G$-invariant subgroup of index $p$ in $G'$. We claim that $G/R$ is an $A_2$-group. Without loss of generality, one may assume that $R = \{1\}$; then $|G'| = p$. It follows from $d(G) = 3$ that $G$ is not an $A_1$-group (Lemma 1). Let $B < G$ be an $A_1$-subgroup. In view of $B' = G'$ we get $B \trianglelefteq G$ so $B \in \Gamma_1$, by hypothesis, and we conclude that $G$ is an $A_2$-group.

Now let $d(G) = 2$. Let $M/G'$ be a subgroup of index $p^2$ in $G/G'$. Then, by hypothesis, $M$ is abelian. It follows that $M \leq C_G(G')$. If $G$ is not of class 2, then $G/G'$ is not generated by subgroups of index $p^2$ so it has a cyclic subgroup of index $p$. $\square$

**Proposition 21.** Let $G$ be a nonabelian $p$-group.

(a) (Compare with Theorem 9) If $d(G) > 2$, then $\kappa_1(G) \geq p^2$.
(b) If $d(G) > 3$, then $\kappa_1(G) > p^2 + p + 1$.

**Proof.** (a) Suppose that all members of the set $\Gamma_2$ are abelian. Then $d(G) = 3$ and $\Phi(G) \leq Z(G)$ (Lemma 3). Let $\{M_1, \ldots, M_s\}$ be the set of all nonabelian members of the set $\Gamma_1$; then $s \geq p^2$ (Lemma J(b)). It remains to show that $\kappa_1(G) \geq s$. Let $A_i \leq M_i$ be an $A_1$-subgroup; then $M_i = A_i\Phi(G)$ is the unique member of the set $\Gamma_i$, containing $A_i$, $i = 1, \ldots, s$. It follows that $A_1, \ldots, A_s$ are pairwise not conjugate in $G$, and we are done in this case.

Now suppose that the set $\Gamma_2$ has a nonabelian member $N$. Let $T \in \Gamma_1(N) \cap \Gamma_3$; then $T \not\leq Z(G)$. Since $G/T$ is generated by any $p + 2$ subgroups of order $p$, it follows that $T$ is contained in at most $p + 1$ abelian members of the set $\Gamma_2$. In that case, there are $p^2$ distinct subgroups $L_1/T, \ldots, L_{p^2}/T$ of order $p$ in $G/T$ such that $L_1, \ldots, L_{p^2}$ are nonabelian; obviously, $L_i \in \Gamma_2$ for all $i$. Let $A_i \leq L_i$ be an $A_1$-subgroup not contained in $T$ (Lemma J(c)), $i = 1, \ldots, p^2$; then $L_i = A_iT$. Since $A_1, \ldots, A_{p^2}$ are pairwise not conjugate in $G$, we get $\kappa_1(G) \geq p^2$.

(b) Let $d(G) > 3$. Suppose that all members of the set $\Gamma_3$ are abelian. Then all members of the set $\Gamma_4$ are contained in $Z(G)$ so $d(G) = 4$ (otherwise, $G = \langle H \mid H \in \Gamma_4 \rangle$ is abelian). If $|G : Z(G)| = p^2$, then $G = BZ(G)$ for each $A_1$-subgroup $B < G$ so all $A_1$-subgroups are normal in $G$. Therefore, $\kappa_1(G) = \alpha_1(G) \geq p^3$ (Theorem B). Since $p^3 > p^2 + p + 1$, we are done in this case. Thus, $|G : Z(G)| \in \{p^3, p^4\}$. It follows that at least $|\Gamma_3| - 1 = p^3 + p^2 + p$ members of the set $\Gamma_3$ are contained in $Z(G)$. Let $T_1, T_2$ and $T_3$ be those distinct members of the set $\Gamma_3$ that are not contained in $Z(G)$. Since any $p + 2$ distinct subgroups of order $p$ generate $G/T_i$, it follows that at least $p^2$ members of the set $\Gamma_2^{T_i}$ are nonabelian, $i = 1, 2, 3$. Let $i \neq j, i, j \neq 3$ and let $K \in \Gamma_2^{T_i} \cap \Gamma_2^{T_j}$.

Then $K = T_iT_j$ is determined uniquely. It follows that the set $\mathfrak{M} = \bigcup_{i=1}^{3} \Gamma_2^{T_i}$ contains at least $3p^2 - 3$ distinct nonabelian members. Let $\mathfrak{M}' = \{H_1, \ldots, H_k\}$ be the set of all nonabelian members in the set $\mathfrak{M}$, where $k \geq 3p^2 - 3$, and let $L_s$ be an $A_1$-subgroup
of $H_s$, $s \leq k$. Since the intersection of any two distinct members of the set $\mathcal{M}$ is abelian, $H_s$ in the unique member of the set $\mathcal{M}$ containing $L_s$, $s \leq k$. It follows that $L_1, \ldots, L_k$ are not pairwise $G$-conjugate so $\kappa_1(G) \geq k \geq 3p^2 - 3 > p^2 + p + 1$.

Now suppose that the set $\Gamma_3$ has a nonabelian member $T$. Set $\Gamma_3^T = \{H_1, \ldots, H_k\}$, $k = p^2 + p + 1$. Let $L_i \leq H_i$ be an $A_1$-subgroup not contained in $T$, $i = 1, \ldots, k$ ($L_i$ exists, by Lemma J(c)), and let $L \leq T$ be an $A_1$-subgroup. We have $L_iT = H_i$ for $i \leq k$. Since $L, L_1, \ldots, L_k$ are pairwise not conjugate in $G$, we get $\kappa_1(G) \geq k + 1 > k = p^2 + p + 1$. $\square$

It follows from Proposition 21(a) that if $G$ is a nonabelian $p$-group with $\kappa_1(G) < p^2$, then $d(G) = 2$. This generalizes Proposition 19 and a part of Theorem 9.

Let $G$ be a nonabelian $p$-group. Given $\mathcal{M} \subseteq \Gamma_1$, let $\kappa_1(\mathcal{M})$ be the number of conjugate $G$-classes of $A_1$-subgroups contained in the members of the set $\mathcal{M}$ (obviously, $\kappa_1(\Gamma_1) = \kappa_1(G)$, unless $G$ is an $A_1$-group).

Remarks. 14. (Compare with [4, Theorem 14.2] due to Janko.) Suppose that every $A_1$-subgroup is contained in the unique maximal subgroup of a $p$-group $G$ which is neither abelian nor $A_1$-group. Then $\Phi(G)$ is abelian and one and only one of the following holds: (i) $d(G) = 2$, (ii) $d(G) = 3$, $\Phi(G) \leq Z(G)$. Indeed, the groups from (i) and (ii) satisfy the hypothesis (if $H < G$ is an $A_1$-subgroup, then, in both cases, $H\Phi(G)$ is the unique maximal subgroup of $G$ containing $H$). Now let $G$ satisfies the hypothesis. Then all members of the set $\Gamma_2$ are abelian. By Lemma 3, $d(G) \leq 3$ and, if $d(G) = 3$, then $\Phi(G) \leq Z(G)$.

15. Suppose that $G$ is a nonabelian $p$-group with $d(G) > 2$. We claim that the following assertions are equivalent: (a) $\kappa_1(\Gamma_1^K) \leq p$ for all $K \in \Gamma_2$ and (b) $G$ is an $A_2$-group with $\alpha_1(G) = p^2$. Indeed, (b) $\Rightarrow$ (a) since in this case $|G : Z(G)| = p^2$. It remains to prove the reverse implication. There exists $K \in \Gamma_2$ such that $K \not\subseteq Z(G)$. Then $\Gamma_1^K = \{H_1, \ldots, H_p, A\}$, where $A$ is abelian and $\tilde{\kappa}_1(H_i) = 1$, $i = 1, \ldots, p$, so $H_i$ is an $A_1$-subgroup for all $i$ and $d(G) = 3$ (Lemma 18). Let $K = \langle (H_1) \cap \Gamma_2 \rangle - \{K\}$; then $\Gamma_1^K = \{F_1 = H_1, F_2, \ldots, F_p, A\}$, where $A_1$ is abelian and $A_1 \neq A$. It follows that the set $\Gamma_1$ has exactly $p + 1$ abelian members and $|G| = p$ so all $A_1$-subgroups are normal in $G$. Then $\kappa_1(\Gamma_1^K) = \alpha_1(\Gamma_1^K)$ for all $K \in \Gamma_2$, and (b) follows from Remark 8.

Proposition 22. (Compare with Remark 8.) Let $G$ be a $p$-group with $\kappa_1(G) > p + 1$. Suppose that for each $K \in \Gamma_2$ we have $\kappa_1(\Gamma_1^K) \leq p + 1$ with equality for at least one $K$. Then $d(G) = 3$, all members of the set $\Gamma_2$ are abelian so $\Phi(G) \leq Z(G)$, and one of the following holds:

(a) $G$ is an $A_2$-group.
(b) $p = 2$, $|G : Z(G)| = 4$, $\Gamma_1 = \{H_1, H_2, H_3, H_4, A_1, A_2, A_3\}$, where $A_1, A_2$ and $A_3$ are abelian, $H_1$ has exactly two conjugate $G$-classes of $A_1$-subgroups of size 2, and $\kappa_1(H_i) = 1$, $i = 2, 3, 4$. We have $\kappa_1(G) = 5$.

Proof. Since $\kappa_1(\Gamma_1^{\Phi(G)}) = \kappa_1(G) > p + 1$, we get $\Phi(G) \notin \Gamma_2$ so $d(G) > 2$. If $L \in \Gamma_2$ is nonabelian, then $\kappa_1(\Gamma_1^{L}) > p + 1$ (here we use Lemma J(c)), which is not the case. Thus, all members of the set $\Gamma_2$ are abelian. It follows that $\Phi(G) \leq Z(G)$ and $d(G) = 3$.
7. Open questions

Below we formulate some related open questions.

1. Study the $p$-groups in which all members of the set $\Gamma_2$ are two-generator.
2. Find all values of the function $\alpha_1(\ast)$ at $A_3$-groups.
3. Study the pairs of $p$-groups $H < G$, where $H \in \Gamma_1$ is nonabelian and $\beta_1(G, H) = p + 1$.
4. Classify the $p$-groups $G$ with (i) $\kappa_1(G) = p$, (ii) $\kappa_1(G) = p + 1$.
5. Study the $p$-groups $G$ with $\bar{\beta}_1(G, H) = p$ for some nonabelian $H \in \Gamma_1$ (see Lemma 13).
6. Let $G$ run over all $p$-groups and $H \in \Gamma_1$. Is it true that

$$
\begin{align*}
\text{(i) } & \lim_{|H'| \to \infty} \beta_1(G, H) = \infty, \\
\text{(ii) } & \lim_{|G'| \to \infty} \beta_1(G, H) = \infty?
\end{align*}
$$

7. Classify the two-generator $p$-groups, $p > 2$, all of whose nonabelian maximal subgroups are two-generator.
8. Let $G$ be a nonabelian $p$-group with $d(G) = 3$ and let a fixed $H \in \Gamma_2$. Describe the structure of $G$ provided all members of the set $\Gamma_1^H$ are either abelian or $A_3$-groups.
9. Let $A$ and $B$ be $p$-groups. Express $\alpha_1(A \times B)$ in terms of $A$ and $B$. The case where $B$ is abelian (even cyclic) is of special interest.
10. Is it true that, for each $n \in \mathbb{N}$, there exists a 2-group $G$ such that $\alpha_1(G) = n$?
11. Is it true that for each $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $\alpha_1(G) \notin \{k + 1, \ldots, k + n\}$ for all $p$-groups $G$?
12. Classify the $A_3$-groups of class 2. (This will be the first step to surprisingly difficult classification of $A_3$-groups.)
13. Find all $n \in \mathbb{N}$ such that there exists an $A_n$-group $G$ such that $\alpha_1(G) \leq p^3$.
14. Let $m \in \mathbb{N}$ and suppose that there exists a $p$-group $X$ with $\alpha_1(X) = m$. Find the minimal $t = t(m) \in \mathbb{N}$ such that there exists an $A_t$-group $G$ with $\alpha_1(G) = m$. 

(Lemma 3). Therefore, if $H \in \Gamma_1$, then $|\Gamma_1(H) \cap \Gamma_2| = p + 1$ so, using Lemma J(a), we conclude that $|H'| \leq p$.

Suppose that the set $\Gamma_1$ has no abelian members. Then $\kappa_1(H) = 1$ for all $H \in \Gamma_1$ (otherwise, if $K \in \Gamma_1(H) \cap \Gamma_2$, then $\kappa_1(\bar{\Gamma}_K) > p + 1$) so $H$ is an $A_1$-group since $d(G) > 2$ (Lemma 18), and hence $G$ is an $A_2$-group.

Now suppose that the set $\Gamma_1$ has an abelian member and $G$ is not an $A_2$-group. In that case there exists $H \in \Gamma_1$ such that $H$ is neither abelian nor an $A_1$-group; then, in view of $|H'| = p$, we get $d(H) > 2$ (Lemma 2(b)). It follows from Theorem 9 that $\alpha_1(H) \geq p^2$. Let $B < H$ be an $A_1$-subgroup. Then $B' = H'$ implies that $N_B(B) \leq H$ so there are at most $p$ subgroups conjugate with $B$ is $G$. It follows that $\kappa_1(H) \geq p$. Let $K \in \Gamma_1(H) \cap \Gamma_2$. Since the set $\Gamma_1^K$ has at most one abelian member and $\kappa_1(\bar{\Gamma}_K) \leq p + 1$, we have there equality and also $p = 2$. Moreover, $\Gamma_1^K = \{H_1 = H, H_2, A_1\}$, where $A_1$ is abelian so $K$ is also abelian and $\kappa_1(H_2) = 1$. In view of $d(G) = 3$, we have $\Gamma_1(H) \cap \Gamma_2 = \{K, R, S\}$. We also have $\Gamma_1^R = \{H, H_3, A_2\}$, $\Gamma_1^S = \{H, H_4, A_3\}$, where $A_2, A_3$ are abelian and $\kappa_1(\{H_i\}) = 1$, $i = 3, 4$. Then $\Gamma_1 = \{H_1 = H, H_2, H_3, H_4, A_1, A_2, A_3\}$. Since all members of the set $\Gamma_2$ are abelian, we get $\kappa_1(G) = \sum_{i=1}^4 \kappa_1(H_i) = 2 + 1 + 1 + 1 = 5$. The proof is complete. 

\[\square\]
15. Classify the $p$-groups $G$ satisfying $\alpha_1(G) = p^{d(G)-1}$ (see Theorem B).
16. Let $G$ be a $p$-group and let $H \in \Gamma_1$ be nonabelian. Study the structure of $G$ provided all members of the set $\Gamma_1 - \{H\}$ are either abelian or $A_1$-groups (see Lemma 5).
17. Suppose that a $p$-group $G$ is neither abelian nor an $A_1$-group and suppose that there is $R \in \Gamma_2$ such that $\alpha_1(\Gamma_1^R) = \alpha_1(G)$. Is it true that $d(G) \leq 3$?
18. Classify the $p$-groups $G$ satisfying $\alpha_2(G) = 1$.
19. Let $G$ be a $p$-group and let $H < G$ but $H \not\leq \Phi(G)$. Suppose that $d(G) > 2$ and all members of the set $\Gamma - \Gamma_1^H$ are either abelian or $A_1$-groups. Study the structure of $G$.
20. Find $\max\{\alpha_1(G) \mid G$ is a group of order $p^m$ and exponent $p^e\}$. Even the case $e = 1$ is very difficult.
21. Let $G$ be a nonabelian group of order $p^m$. Suppose that all $A_1$-subgroups of $G$ have the same order $p^k$. We have $\alpha_1(G) \geq p^{m-k}$ (Supplement 2 to Theorem C). Describe the structure of $G$ in the case $\alpha_1(G) = p^{m-k}$ (for $k = 3$, see Supplement 1 to Theorem C).
22. Let $p^\mu(X)$ be the maximal order of minimal nonabelian subgroups of a $p$-group $X$ (we set $\mu(\{1\}) = 1$). Is it possible to express $\mu(A \times B)$ in terms of $A$ and $B$? (I think that, in general case, the answer is ‘no’. However, if $B$ is elementary abelian, the answer is ‘yes’. To prove this, it suffices to assume that $|B| = p$. Let $M$ be a minimal nonabelian subgroup of $G$. By the modular law, $B \not\leq M$ so $M$ is isomorphic to a subgroup of $G/B \cong A$, and we conclude that $|M| \leq p^{\mu(A)}$, i.e., $\mu(G) = \mu(A)$.)
23. Classify the $p$-groups $G$ such that $d(G) > 2$ and $\alpha_1(\Gamma_1^K) = \alpha_1(G)$ for all $K \in \Gamma_2$ (see also #17).
24. Classify the $p$-groups $G$ with $d(G) = 3$ such that all members of the set $\Gamma_2$, but one, are metacyclic (see the proof of Theorem A.1).
25. Find $\alpha_{m,1} = \max\{\alpha_1(G) \mid |G| = p^m, \exp(G) = p\}$ and $\alpha_{m,e} = \max\{\alpha_1(G) \mid |G| = p^m, \exp(G) = p^e\}$. Is it true that $\alpha_{m,1} \geq \alpha_{m,e}^e$?

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