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Mappings of Domains by Components of Solutions of Differential Systems

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1. SUMMARY AND PRELIMINARIES

Let D be a domain in the z-plane bounded by a piecewise smooth Jordan curve C. We assume that the n^2 elements $a_{ik}(z)$ of the matrix $A(z) = (a_{ik}(z))_1^n$ are analytic functions which are regular in $\bar{D}$ and we consider there the vector differential equation

$$w'(z) = A(z)\, w(z). \tag{1.1}$$

Here $w(z)$ is the column vector $(w_1(z),..., w_n(z))$. Each component $w_i(z)$, $i = 1,..., n$, maps D onto a domain $D_i = w_i(D)$ and we denote the diameter of D_i by d_i. Theorem 1 states that the norm $\| d \|$ of the diameter vector $d = (d_1 ,..., d_n)$ is not larger than half the product of the line integral $\int_C \| A(\zeta)\| \, | d\zeta |$ and $\max \| w(\zeta)\|$, $\zeta \in C$. This follows from an inequality of Nehari, which we state as Lemma 1, and from the basic properties of absolute vector norms and matrix norms consistent with them. If the above line integral is not larger than 2, then the system (1.1) is disconjugate in D (Theorem 2). We conclude with a modification of Theorem 1 dealing only with the maps $C_i = w_i(C)$ of a piecewise smooth Jordan curve C (Theorem 3).

It seems to be convenient for our applications to quote Nehari's result [4, formula (16)] in two forms:

LEMMA 1. (a) *Let the analytic function $f(z)$ be regular in the closed region $\bar{D}$ whose boundary is a piecewise smooth Jordan curve C. Then*

$$| f(\alpha) - f(\beta)| \leqslant \tfrac{1}{2} \int_C | f'(\zeta)| \, | d\zeta | \tag{1.2}$$

holds for all $\alpha, \beta \in \bar{D}$.

(b) *Let the analytic function $f(z)$ be regular and single valued on the piecewise smooth Jordan curve C. Then* (1.2) *holds for all $\alpha, \beta \in C$.*

For the proof [4] one notes that if α and β lie on C then $f(\alpha)$ and $f(\beta)$ divide the image $f(C)$ of C into two arcs. The length of either of these arcs cannot be smaller than the distance $|f(\alpha) - f(\beta)|$ and their combined length is $\int_C |f'(\zeta)| \, |d\zeta|$. In case (a) the maximum principle shows that $|f(\alpha) - f(\beta)|$, attains its largest value for $\alpha, \beta \in \bar{D}$ if both α and β are on C.

For completeness we now quote the definitions of vector and matrix norms and give also the basic properties of the absolute vector norms [1, 3]. A vector norm $\| w \|$ is a real valued function defined for all column vectors $w = (w_1, \ldots, w_n)$ with complex elements w_i satisfying

$$\begin{aligned} w \neq 0 \quad &\text{implies} \quad \| w \| > 0, \\ \| cw \| = | c | \, \| w \|, \quad &c \text{ scalar}, \\ \| w + v \| \leqslant \| w \| + \| v \|. & \end{aligned} \tag{1.3}$$

If $w = (w_1, \ldots, w_n)$ then $| w | = (| w_1 |, \ldots, | w_n |)$; $| w | \leqslant | v |$ means $| w_i | \leqslant | v_i |$ for all i. A vector norm is called *absolute* or *monotonic* if the following two properties hold:

$$\| w \| = \| \, | w | \, \|, \tag{1.4}$$

and

$$| w | \leqslant | v | \quad \text{implies} \quad \| w \| \leqslant \| v \|; \tag{1.4'}$$

moreover, these two properties are equivalent. The Hölder norms

$$\| w \|_p = \left(\sum_{i=1}^{n} | w_i |^p \right)^{1/p}, \qquad 1 \leqslant p \leqslant \infty, \tag{1.5}$$

are absolute norms and the cases $p = 1$, 2 and ∞, i.e.,

$$\| w \|_1 = \sum_{i=1}^{n} | w_i |, \qquad \| w \|_2 = \left(\sum_{i=1}^{n} | w_i |^2 \right)^{1/2}, \qquad \| w \|_\infty = \max_i | w_i | \tag{1.6}$$

are the most frequently used norms.

A matrix norm $\| A \|$ is a real valued function defined for all $n \times n$ matrices $A = (a_{ik})_1^n$ with complex elements a_{ik} satisfying

$$\begin{aligned} A \neq 0 \quad &\text{implies} \quad \| A \| > 0, \\ \| cA \| = | c | \, \| A \|, \quad &c \text{ scalar}, \\ \| A + B \| \leqslant \| A \| + \| B \|, & \\ \| AB \| \leqslant \| A \| \, \| B \|. & \end{aligned} \tag{1.7}$$

If a matrix norm $\| A \|$ and a vector norm $\| w \|$ are related in such a way that always the condition

$$\| Aw \| \leqslant \| A \| \| w \| \tag{1.8}$$

holds, then the two norms are said to be *consistent.*

From a given vector norm $\| w \|$ a matrix norm $\| A \|$ can be obtained by defining

$$\| A \| = \sup_{w \neq 0} \frac{\| Aw \|}{\| w \|}. \tag{1.9}$$

This *induced norm* (*bound*) $\| A \|$ is consistent with the given vector norm $\| w \|$; moreover, if $\| A \|'$ is any matrix norm consistent with $\| w \|$ and $\| A \|$ is the norm induced by $\| w \|$ then, for any matrix A, $\| A \| \leqslant \| A \|'$. We denote the norms induced by the Hölder norms $\| w \|_p$ by $\| A \|_p$; i.e., we set

$$\| A \|_p = \sup_{w \neq 0} \frac{\| Aw \|_p}{\| w \|_p}, \qquad 1 \leqslant p \leqslant \infty. \tag{1.5'}$$

For $p = 1$, 2 and ∞, these norms are explicitly given by

$$\| A \|_1 = \max_k \sum_{i=1}^{n} | a_{ik} |, \qquad \| A \|_2 = \lambda^{1/2}(AA^*), \qquad \| A \|_\infty = \max_i \sum_{k=1}^{n} | a_{ik} |. \tag{1.6'}$$

(Here $A^* = (\bar{a}_{ki})_1^n$, and $\lambda(AA^*)$ is the maximal characteristic value of AA^*.)

If the analytic functions $w_i(z)$, $i = 1,..., n$, are regular in a domain D, then every norm $\| w(z) \|$ of the vector $w(z) = (w_1(z),..., w_n(z))$ is a continuous subharmonic function in D. The analogous statement holds for every norm $\| A(z) \|$ of a matrix $A(z)$ with regular analytic elements [6, Lemma 2].

2. A Bound for the Norm of the Diameter Vector

Theorem 1. *Let the analytic functions $a_{ik}(z)$, $i, k = 1,..., n$, be regular in the closed region $\bar{D}$ whose boundary is a piecewise smooth Jordan curve C. Let the matrix norm $\| A \|$ be consistent with the absolute vector norm $\| w \|$. Let $w(z) = (w_1(z),..., w_n(z))$ be a solution of the system*

$$w'(z) = A(z)\, w(z) \tag{1.1}$$

$(A(z) = (a_{ik}(z))_1^n)$ in $\bar{D}$ and denote the diameter of the image region $\bar{D}_i = w_i(\bar{D})$ by d_i, $i = 1,..., n$. Let d be the diameter vector $(d_1, ..., d_n)$. Then

$$\| d \| \leqslant \tfrac{1}{2} \max_{\zeta \in C} \| w(\zeta) \| \int_C \| A(\zeta) \| \, | d\zeta |. \tag{2.1}$$

As $\| w(z) \|$ is subharmonic it follows that $\max_{\zeta \in C} \| w(\zeta) \| = \max_{z \in \bar{D}} \| w(z) \|$. Furthermore, as the matrix norm induced by $\| w \|$ is never larger than any matrix norm consistent with $\| w \|$, it would suffice to state this theorem for absolute vector norms and the matrix norms induced by them.

Proof. Let α_i and β_i, $i = 1,..., n$, be points on C such that

$$d_i = | w_i(\alpha_i) - w_i(\beta_i) |.$$

Lemma 1 (part (a)) implies that

$$d_i \leqslant \tfrac{1}{2} \int_C | w_i'(\zeta) | \, | d\zeta |, \qquad i = 1,..., n. \tag{2.2}$$

We denote

$$l_i = \int_C | w_i'(\zeta) | \, | d\zeta |, \qquad i = 1,..., n. \tag{2.3}$$

For the two nonnegative vectors $d = (d_1, ..., d_n)$ and $l = (l_1, ..., l_n)$ it thus follows that

$$d \leqslant \tfrac{1}{2} l. \tag{2.4}$$

Applying successively (2.3), (1.3), (1.4), (1.1), and (1.8), we obtain

$$\begin{aligned} \| l \| &= \left\| \int_C | w'(\zeta) | \, | d\zeta | \right\| \leqslant \int_C \| \, | w'(\zeta) | \, \| \, | d\zeta | = \int_C \| w'(\zeta) \| \, | d\zeta | \\ &= \int_C \| A(\zeta) \, w(\zeta) \| \, | d\zeta | \leqslant \int_C \| A(\zeta) \| \, \| w(\zeta) \| \, | d\zeta |. \end{aligned} \tag{2.5}$$

Equations (2.4), (1.4′) and (2.5) give

$$\| d \| \leqslant \tfrac{1}{2} \int_C \| A(\zeta) \| \, \| w(\zeta) \| \, | d\zeta |, \tag{2.6}$$

which implies the assertion (2.1) of the theorem.

We do not know whether the constant 1/2 on the right side of (2.1) is the best possible and this constant may, perhaps, be slightly lowered. If we consider specific cases—i.e., specify the norms, the order n of the differential systems, and the region D—then each case will have its specific best possible constant which thus depends on the choice of the norms and on n (but, as we shall see, not on the choice of D). With respect to the Hölder norms $\| w \|_p$ and the matrix norms $\| A \|_p$ induced by them the following statement holds. *For any given p, $1 \leqslant p \leqslant 2$, for any given n, $n \geqslant 2$, and for any given region D, the constant $1/2$ in the assertion*

$$\| d \|_p \leqslant \tfrac{1}{2} \max_{\zeta \in C} \| w(\zeta) \|_p \int_C \| A(\zeta) \|_p \, | d\zeta | \tag{2.1$_p$}$$

cannot be replaced by any constant smaller than $\sqrt{2}/\pi = 0.450$. For p satisfying $2 < p \leqslant \infty$, the constant $\frac{1}{2}$ in $(2.1)_p$ cannot be replaced by any constant smaller than $2^{1/p}/\pi$.

To prove this we first assume that n is even, $n = 2m$, and we consider the constant coefficient matrix $A = (a_{ik})_1^{2m}$ whose elements are defined as follows: $a_{i,i+1} = 1$, $i = 1,\ldots, 2m - 1$, $a_{2m,1} = (-1)^m$; all other elements of A are zero [5, p. 343]. Equations (1.5) and (1.5′) imply

$$\| A \|_p = 1, \qquad 1 \leqslant p \leqslant \infty. \tag{2.7}$$

The system $w'(z) = Aw(z)$ is equivalent to the differential equation $y^{(2m)} + (-1)^{m+1} y = 0$, and has thus the solution $w(z)$ with the components $w_1(z) = \sin(z - \pi/4)$, $w_2(z) = \cos(z - \pi/4),\ldots, w_{2m}(z) = (-1)^{m+1}\cos(z - \pi/4)$. For any $\epsilon > 0$ we consider the region $\bar{D}_\epsilon$ bounded by the ellipse C_ϵ having its vertices at the points $\pm((\pi/4) + \epsilon)$ and $\pm i\epsilon$. As $\epsilon \to 0$, C_ϵ tends to the segment of the real axis bounded by $\pm\pi/4$. It follows that $d_i = 1 + O(\epsilon)$, $i = 1,\ldots, 2m$. Hence,

$$\| d \|_p = (2m)^{1/p} + O(\epsilon), \qquad 1 \leqslant p \leqslant \infty. \tag{2.8}$$

As $\| w(\zeta) \|_p = (m(|\sin^p(\zeta - \pi/4)| + |\cos^p(\zeta - \pi/4)|))^{1/p}$, it follows that

$$\max_{\zeta \in C_\epsilon} \| w(\zeta) \|_p = (2m)^{1/p} \frac{1}{\sqrt{2}} + O(\epsilon), \qquad 1 \leqslant p \leqslant 2. \tag{2.9}$$

Equation (2.7) gives

$$\int_{C_\epsilon} \| A \|_p \, | d\zeta | = \pi + O(\epsilon), \qquad 1 \leqslant p \leqslant \infty. \tag{2.10}$$

Inserting (2.8)–(2.10) into

$$\| d \|_p \leqslant c \max_{\zeta \in C} \| w(\zeta) \|_p \int_C \| A(\zeta) \|_p \, | d\zeta |, \tag{2.11}$$

we obtain, for any p, $1 \leqslant p \leqslant 2$,

$$(2m)^{1/p} + O(\epsilon) \leqslant c(2m)^{1/p} \frac{\pi}{\sqrt{2}} + O(\epsilon). \tag{2.12}$$

It follows that if, for given p, $1 \leqslant p \leqslant 2$, we choose $c < \sqrt{2}/\pi$, then we can find a small enough ϵ, $\epsilon = \epsilon(p, c)$, such that (2.11) is not valid for the region $\bar{D}_\epsilon$. But this implies that—keeping p and c fixed—(2.11) is not valid for any region $\bar{D}$. This follows from the invariance of each term in (2.11), or in (2.1), under conformal mapping. (See [6, p. 556]; let $z = \phi(z^*)$ map the region $\bar{D}^*$ of the z^*-plane bounded by C^* onto $\bar{D}$. Equation (1.1) transforms into

$$v'(z^*) = B(z^*)\, v(z^*), \tag{1.1*}$$

where $v(z^*) = w(\phi(z^*))$ and $B(z^*) = \phi'(z^*)\, A(\phi(z^*))$. Clearly, $d = d^*$, $\max_{\zeta\in C} | w(\zeta)| = \max_{\zeta^*\in C^*} | w(\zeta^*)|$ and $\int_C \| A(\zeta)\| \, | d\zeta | = \int_{C^*} \| B(\zeta^*)\| \, | d\zeta^* |$.)

For odd $n(\geqslant 3)$, $n = 2m + 1$, we add a last zero row and zero column to the above matrix A of order $2m$ and we add the component $w_{2m+1}(z) = 0$ to the above solution vector $(w_1(z),..., w_{2m}(z))$. Equations (2.8)–(2.10) remain valid and we have thus proved the part of the italicized statement referring to the case $1 \leqslant p \leqslant 2$. For $2 \leqslant p \leqslant \infty$, (2.9) has to be replaced by

$$\max_{\zeta\in C_\epsilon} \| w(\zeta)\|_p = m^{1/p} + O(\epsilon), \qquad 2 < p \leqslant \infty. \tag{2.9$'$}$$

3. Norm Conditions for Disconjugacy

Let D be a simply connected domain in the z-plane not containing $z = \infty$ and let the n^2 analytic functions $a_{ik}(z)$ be regular in D. The system (1.1) is called *disconjugate in D* if, for every choice of n (not necessarily distinct) points z_1,..., z_n in D, the only solution $w(z) = (w_1(z),..., w_n(z))$ of (1.1) satisfying $w_i(z_i) = 0$, $i = 1,..., n$, is the trivial one $w(z) \equiv 0$. Using the maps $D_i = w_i(D)$, $i = 1,..., n$, disconjugacy of (1.1) in D means that, for every nontrivial solution $w(z)$, $0 \notin \bigcap_{i=1}^n D_i$. Theorem 1 now yields the following result.

Theorem 2. *Let the analytic functions $a_{ik}(z)$, $i, k = 1,..., n$, be regular in a simply connected domain D not containing $z = \infty$. Let C be the boundary of D and let the matrix norm $\| A \|$ be consistent with an absolute vector norm $\| w \|$. If*

$$\int_C \| A(\zeta)\| \, | d\zeta | \leqslant 2 \tag{3.1}$$

$(A(z) = (a_{ik}(z))_1^n)$, *then the system*

$$w'(z) = A(z)\, w(z) \tag{1.1}$$

is disconjugate in D.

We remark again that it would suffice to state this theorem for matrix norms induced by absolute vector norms.

Proof. We prove the theorem first in the special case where the boundary C of D is a piecewise smooth curve and the n^2 analytic functions $a_{ik}(z)$ are regular in $\bar{D}$. Assume, to the contrary, that there exist points $z_i \in D$, $i = 1,..., n$, and a nontrivial solution $w(z) = (w_1(z),..., w_n(z))$ such that $w_i(z_i) = 0$, $i = 1,..., n$. Let ζ^* be a point on C such that

$$\| w(\zeta^*) \| = \max_{\zeta \in C} \| w(\zeta) \|. \tag{3.2}$$

If $d_i > 0$, then

$$| w_i(\zeta^*) | = | w_i(\zeta^*) - w_i(z_i) | < d_i . \tag{3.3}$$

There exists thus a constant c, $c > 1$, such that

$$c \, | w_i(\zeta^*) | \leqslant d_i \tag{3.4}$$

holds for all i (i.e., also if $d_i = 0$). As $\| w(\zeta^*) \| > 0$, equations (3.2), (3.4) with $c > 1$, (1.3), and (1.4′) give

$$\max_{\zeta \in C} \| w(\zeta) \| < \| d \|. \tag{3.5}$$

Equation (3.5) and the assertion (2.1) of Theorem 1 imply

$$2 < \int_C \| A(\zeta) \| \, | d\zeta |,$$

which contradicts (3.1). We thus proved the validity of Theorem 2 in the special case.

For the general case, of arbitrary boundary C and regularity of the functions $a_{ik}(z)$ only in D, the integral $\int_C \| A(\zeta) \| \, | d\zeta |$ has to be interpreted as the limit, for $r \to 1$, of integrals taken along the level lines C_r, $0 < r < 1$, of the function $\phi(z)$ which maps D onto $| \phi | < 1$. To prove the general case from the special one, it is convenient to establish first the validity of the theorem for the case of the open unit disk, and then to obtain the general result for an arbitrary domain by conformal mapping [6, pp. 557–560].

Theorem 2 improves a former result stating that

$$\int_C \| A(\zeta)\|_\infty \,| d\zeta | \leqslant 2 \log 2$$

implies disconjugacy of (1.1) in D [6, Theorem 2′]; it also improves a result due to Kim [2, Theorem 2.5] and the author [6, p. 561] stating that

$$\int_C \| A(\zeta)\|_2 \,| d\zeta | \leqslant 1$$

implies disconjugacy.

With regard to the best possible constant on the right side of (3.1), the following statement holds for the norms $\| A \|_p$, induced by the Hölder norms $\| w \|_p$: *For any given* p, $1 \leqslant p \leqslant \infty$, *for any given* n, $n \geqslant 2$, *and for any given domain* D, *the constant* 2 *on the right side of the sufficient norm condition*

$$\int_C \| A(\zeta)\|_p \,| d\zeta | \leqslant 2 \tag{3.1$_p$}$$

cannot be replaced by any constant larger than π. This follows from the example of the last section (see (2.10)). For systems defined on an interval, Nehari obtained, for $\| A(x)\|_2$, a sharp theorem [5, Theorem 3.3].

4. Mappings of Curves by Components

We now consider solutions of (1.1) along a piecewise smooth curve C and thus assume that the elements of the coefficient matrix $A(z)$ and of the solution $w(z)$ are regular and single valued on C. Equation (2.1) remains valid in this case and we also add some of its immediate consequences to the statement of our final result.

Theorem 3. *Let the analytic functions* $a_{ik}(z)$, $i, k = 1,..., n$, *and* $w_i(z)$, $i = 1,..., n$, *be regular and single valued on the piecewise smooth Jordan curve* C. *Let the matrix norm* $\| A \|$ *be consistent with the absolute vector norm* $\| w \|$ *and let* $w(z) = (w_1(z),..., w_n(z))$ *be a solution of the system*

$$w'(z) = A(z)\, w(z) \tag{1.1}$$

($A(z) = (a_{ik}(z))_1^n$) *on* C. *Let* d_i *be the diameter of the image* $C_i = w_i(C)$, $i = 1,..., n$, *and set* $d = (d_1 ,..., d_n)$. *Then* (a)

$$\| d \| \leqslant \tfrac{1}{2} \max_{\zeta\in C} \| w(\zeta)\| \int_C \| A(\zeta)\| \,| d\zeta |. \tag{2.1}$$

(b) *If*

$$\int_C \|A(\zeta)\| \, |d\zeta| < 2c, \qquad 0 < c \leq 2, \tag{4.1}$$

and if $w(z) \not\equiv 0$, *then there exists at least one component* $w_i(z)$ *such that*

$$d_i < c \max_{\zeta \in C} |w_i(\zeta)|. \tag{4.2}$$

(c) *If* $w(z) \not\equiv 0$ *and if for each component* $w_i(z)$, $i = 1,..., n$, *exist two points* ζ_i' *and* ζ_i'' *on* C *such that*

$$|w_i(\zeta_i')| = |w_i(\zeta_i'')| = \max_{\zeta \in C} |w_i(\zeta)| \tag{4.3}$$

and

$$w_i(\zeta_i'') = -w_i(\zeta_i'), \tag{4.4}$$

then

$$\int_C \|A(\zeta)\| \, |d\zeta| \geq 4. \tag{4.5}$$

Proof. The proof of part (a) is the same as the proof of Theorem 1, but we now use part (b) of Lemma 1. If $d_i \geq c \max_{\zeta \in C} |w_i(\zeta)|$ holds for all components, then $\|d\| \geq c \max_{\zeta \in C} \|w(\zeta)\| > 0$. This and (2.1) contradict (4.1) and part (b) is thus proved. (4.3) and (4.4) imply $d_i = 2 \max_{\zeta \in C} |w_i(\zeta)|$, $i = 1,..., n$, and part (c) follows from the case $c = 2$ of part (b).

The statement on best possible constants in the assertion $(2.1)_p$ of Section 2 remains valid. The constant 2, appearing on the right side of (4.1), cannot be replaced by any constant larger than π, but this may be possible for specific norms and specific values of c. We note that the assumptions (4.3) and (4.4) of part (c) are satisfied if each image $C_i = w_i(C)$ is symmetric with regard to the origin. If C is a circle with center at the origin, then this symmetry of the images C_i is assured if each component $w_i(z)$ is an odd function of z or is of the form z^n, $n = \pm 1, \pm 2,...$. The differential system

$$w'(z) = \left(\frac{1}{z} I\right) w(z), \qquad I = (\delta_{ik})_1^n ,$$

has $w(z) = (z,..., z)$ as one of its solutions. For any matrix norm $\|A\|$, induced by an absolute vector norm,

$$\left\|\frac{1}{z} I\right\| = \frac{1}{|z|},$$

and it follows that $\int \| (1/\zeta)I \| \, | d\zeta | = 2\pi$ if the integral is taken along a circle with center at the origin. This proves that *for any given curve C and for any given matrix norm $\| A \|$, induced by an absolute vector norm, the constant* 4 *in* (4.5) *cannot be replaced by any constant larger than* 2π.

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