Higher-Order Modeling of Continua by Finite-Element, Boundary-Element, Meshless, and Wavelet Methods

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Abstract—This paper is an overview of the higher-order modeling schemes for the continua based on finite-element, boundary-element, meshless, and wavelet methods. The first discrete numerical method is well established, so discussions are mainly focused on the last three methods. Apart from critically discussing the main theoretical, algorithmic, and implementation characteristics of the methods, the paper expounds on the relative merits and demerits of the methods. The results of a number of smooth and nonsmooth problems are presented. Finally, the future potential of the wavelet method in discrete numerical modeling of continua is explored. © 2003 Elsevier Science Ltd. All rights reserved.

Keywords—μ- and hμ-version, Finite-element, Boundary-element, Meshless, Wavelet.

1. INTRODUCTION

Over the past few decades, computational simulation has gained significant importance in many branches of science and engineering, as well as in the research and development work carried out at major U.S. corporations and federal agencies like DOD and DOE research laboratories. As a result, scientists and engineers are routinely undertaking reliable modeling and simulation of complex physical systems. In addition, there is an increasing demand for handling problems of ever-increasing size and complexities. Scientists, mathematicians, and engineers are effectively responding to it through interdisciplinary teamwork. Typical examples of such effort, to name a few, are those at Barna Szabo's Center for Computational Mechanics at Washington University.
Tinsley Oden's Texas Institute for Computational and Applied Mathematics at University of Texas, Austin, and O.C. Zienkiewicz's Institute for Numerical Methods in Engineering at the University of Wales-Swansea and Technical University of Catalunya, Barcelona. Such teamwork has led to rapid advancements in modeling and computational schemes at all levels, namely, conceptual, discrete modeling, algorithmic, and software.

Moore's law shows that computer speed (FLOPS) and memory increase approximately by a factor of 2 per 1.5 years. The boundary between the power of personal computer and workstation, and between network of workstations and powerful supercomputer, has become blurred. At the end of year 2002, in the list of the top 500 world's most powerful computer systems, the machine named MCR occupied the fifth position. This is an assembly of 2304 Xeon 2.4 GHz computers at the Lawrence Livermore National Laboratory. This system solved the standard Linpack benchmark problem at a speed of 5.69 TF/s. Such explosion of power of a large number of networked workstations is not matched by corresponding development of software to utilize this power. This has created an ongoing challenge to engineers and scientists to effectively utilize this power to solve increasingly difficult challenging problems of the time. Some examples of such problems are: modeling the evolution of universe, multiscale modeling of materials for enhanced performance, developing intelligent synthesis environment to reduce design and development time of new products, reducing testing requirements, and predicting life-cycle costs within 10%, etc.

Discrete numerical methods are almost exclusively used in the numerical solution of mathematical models of the continua expressed as initial and/or boundary value problems of differential calculus. The finite difference and Ritz type methods of the precomputer era have largely been replaced in the computer era by finite-element method, boundary-element method, meshless method, and in the near future it may be the turn for wavelet method. Other methods, like differential quadrature method, that are of limited significance have also been proposed. The first two of the four methods were available to the general users in the sixties and seventies, whereas, the meshless method became known during the last decade or so. The wavelet method has come into the scene very recently. In all these methods the problem is reduced into one of finding numerical values at a discrete number of points covering the problem domain. An essential difference between the first two methods is that in the finite-element method the points are selected in the volume of the domain, whereas, in the boundary-element method the discrete points need to be selected at the boundary of the domain only. This fact introduces the possibility of higher computational efficiency in the boundary-element method, especially, when the volume of the problem domain becomes significantly larger than the enclosing surface. The first method may be termed as the volume integral method and the second one as the boundary-integral method; see Figure 1. The meshless and wavelet methods can be considered either in volume integral sense or in boundary-integral sense. In all the methods, the unknown function(s) is suitably approximated in terms of its values and, sometimes, its derivatives at the specified discrete points (or nodes). The nodes are selected in both volume and boundary of the problem domain in volume integral methods. The nodes are located only in the boundary of the problem domain in boundary-integral methods. In addition, the node-based expression(s) can be appended by the one based on additional coefficients (or nodeless variables). In the time domain, the function values at the discrete points are obtained at discrete instants of time covering the time domain of interest. In the time domain also, the temporal variation of the unknown function at any discrete point is approximated locally in terms of values in the neighboring instants of time. Thus, in general, discrete numerical methods involve

(a) global discretization in space or in the boundary only;
(b) global discretization in time;
(c) piecewise spatial approximation of an unknown function defined in terms of values at discrete points in a subregion or in the neighborhood of a point of interest;
(d) piecewise temporal approximation of an unknown function defined in terms of values at
discrete instants of time in a small temporal window.

(a) Finite-element scheme. (b) Boundary-element scheme. (c) Meshless (volumetric) scheme. (d) Meshless (boundary) scheme.

Figure 1. Geometric discretization schemes.

The primary advantage of these methods over analytical solutions lies in the ability to handle problems with irregularities of different kinds with varying degrees of success measured in terms of accuracy of the results obtained and degree of computational efficiency determined by both manpower needs and computational time. The irregularities may be due to nonlinearities in the characteristic parameters of a problem and also those due to different sources of singularity. In the case of problems of continuum mechanics, singularity may result from

(i) geometric sources like notches, cracks, re-entrant corners, etc.;
(ii) material sources resulting from inclusions of dissimilar material, discontinuity in material properties, etc.;
(iii) external effects like sharp discontinuity in loading, such as point or line load; and
(iv) boundary conditions, namely, mixed boundary conditions.

2. DISCRETE NUMERICAL METHODS

The primary difference between the methods lies in the way the approximating function is defined. In the case of finite-element method, the function is a polynomial defined over individual subdomains that cover the problem domain. In the case of boundary-element method, the situation is the same as the finite-element method except that the modeled region is of one dimension less than the dimension of the problem. For instance, in the case of three-dimensional problems, the boundary is a surface, and hence, the subregions selected on this bounding surface are two-dimensional. In the case of this method, there is the additional requirement for appending the fundamental solution for the problem, which is essentially the homogeneous solution of the differential equation due to a source term. In the case of meshless method the approximating function is not confined within a subregion. For each integration point, it is defined over a predefined region of influence for that point. The wavelet method can be viewed as a method in which the approximating function is defined using a multiresolution technique based on the use of wavelets, similar to those used in signal and image processing [1]. For simple domains, the function can be defined for the whole region. In the case of complex domains, the function can be defined individually over a few subdomains covering the problem domain. The wavelet method can also be implemented following the strategy of the meshless method.

The first three methods use polynomials to approximate the functional space, which create difficulty in representing the singular behavior in a problem, say, when the unknown function or its first derivative is singular in the case of a problem governed by, say, a second-order differential equation. The singular behavior can, however, be accounted for by the use of special functions near a singular point. In the solution phase, all the methods lead to a set of linear or nonlinear simultaneous equations. The condition number of the matrix of coefficients affects the quality of solution obtained by solving these equations, due to round-off errors. Although the weighted residual method can be the starting point with all the four methods, there are other possibilities such as variational methods, least squares method, Somigliana's identity (for boundary-element method), etc. Another area of concern is the ability to predict the quality of solution based on a posteriori analysis and the ability to control the solution quality by refining the model locally as well as globally, as needed. In the finite-element method, significant advancements have been made in this area. The other methods are, however, lagging behind in this respect.
Although the finite-element method has become the dominant tool for discrete numerical analysis, the other three methods may demonstrate superior performance for certain problems. Sometimes, a combination of more than one method will lead to an optimal modeling scheme, namely, the combination of volume integral and boundary-integral methods. The main objective of this paper is to evaluate these discrete methods in the context of one-dimensional and two-dimensional problems. Equation (1) gives the governing differential equation for the one-dimensional problems to be considered.

Given \( f(x, t) : \Omega \rightarrow \mathbb{R} \).

Coefficients \( a, b, c, \) and \( d \). Parameters \( u_0 \) and \( g(t) \).

Required to find \( u(x, t) : \Omega \rightarrow \mathbb{R} \), so that

\[
\frac{\partial}{\partial x} \left( a \frac{\partial u}{\partial x} \right) + bu + c \frac{\partial u}{\partial t} + d \frac{\partial^2 u}{\partial x^2} + f = 0, \quad \text{on } \Omega = [0, \ell]
\] (1)

along with the conditions

\[
u(0, t) = u_0, \]
\[-a \frac{\partial u}{\partial t} = g(t).\]

In this equation, the constants \( a, b, c, \) etc. and \( f \) may not be smooth or continuous functions of \( x \). In addition, \( a, b, c, \) etc. may be functions of \( x \) and/or \( f \) and can be a singular or Dirac delta function. This will give rise to difficulties that will require special treatment, affecting solution quality as well as computational efficiency. A physical example of such a problem is the one-dimensional problem of mechanics of solid continua illustrated in Figure 2. In this example, \( a \) is discontinuous at \( x = L_1 \), and \( u_0 = 0 \). In addition, the function \( f(x) \) can be defined as

\[
f(x) = 0, \quad \text{for } 0 \leq x < \left( b - \frac{\varepsilon}{2} \right) \quad \text{and} \quad \left( b + \frac{\varepsilon}{2} \right) < x \leq L, \\
f(x) = \frac{P_b}{\varepsilon}, \quad \text{for } \left( b - \frac{\varepsilon}{2} \right) \leq x \leq \left( b + \frac{\varepsilon}{2} \right).
\]

![Figure 2. One-dimensional continua problem.](image)

### 2.1. Finite-Element Method (FEM)

This most widely used method has numerous commercial software available in the marketplace. In this method, the problem domain is partitioned into a number of subdomains (or finite elements) with connectivity between the elements provided through common nodal points. Piecewise approximating functions for the unknown variables are then selected for each finite element satisfying certain minimum continuity requirements at the interelement boundaries depending upon the choice of type of nodal variables as well as nodeless coefficients, if used. The continuity requirements will depend upon the assumptions made in the basic mathematical model. The set of simultaneous equations, normally, with symmetric coefficients, is arrived at by using the weak form of the differential equation obtained by the weighted residual method, or by applying the stationary condition to the functional for the problem, if available.

The quality of the resulting approximate solution depends upon the closeness of the finite-element model to that represented by the mathematical model. The closeness is controlled by
the attributes of the problem, the numerical algorithm used, distribution and nature of degrees of freedom, number of digits of accuracy used in computations, and the type of algorithms used to evaluate the integrals and to solve the simultaneous equations. In the case of elliptic problems, the whole spatial domain needs refinement because the boundary conditions affect the whole region. In the case of initial value problems, model refinement requires different strategies than that for elliptic problems in the sense that the initial value problems are open ended, the future does not affect the present, and the refinement can be varied from one stage to another depending upon the local behavior of the solution.

In the case of elliptic problems, the quality of a solution is dependent upon the model used, namely, the sizes and orders of elements as well as their distribution. The use of weak form of the differential equation (or, minimization of the functional in the variational sense) leads to results in which the derivatives of the function are less accurate than the function values, unless derivatives are also approximated, as in the less convenient mixed method. Solution quality can be improved upon by refining the finite-element model, either by using more elements of smaller size (known as h-extension), or by increasing the order of the piecewise approximating functions over a fixed mesh (known as p-extension), or by an optimal combination of the two (known as hp-extension). Szabo and his colleagues at Washington University were not only the initial proponents of p- and hp-extensions, but were responsible for identifying the sets of basis functions based on Legendre polynomials (or their integrals) as a better choice for higher-order finite elements due to orthogonal property and hierarchical nature [2,3]. Another alternative for model improvement is to redistribute the degrees of freedom by strategically changing the size of the elements, reducing the number of elements, and increasing the order of approximating functions so that the total number of degrees of freedom remains unchanged (known as r-extension). In elements bordering a singular point, instead of using a regular polynomial (say, \( u = \sum_{j=1}^{p+1} u_j N_j(\xi) \) with \(-1 \leq \xi \leq 1\) and \( N_j \)s as basis functions based on, say, integrals of Legendre polynomials, Chebyshev polynomials, or Jacobi polynomials, the function can be approximated by a series [4] of the form

\[
  u = \sum_{j=1}^{n_0} c_j M_j(\xi, \alpha) + \sum_{j=2}^{p+1} u_j N_j(\xi),
\]

where the exponent \( 0 < \alpha < 1 \). The value of exponent \( \alpha \) depends on the strength of singularity selected automatically in the solution process.

Solution quality can be ascertained in the a posteriori sense using information on nonsatisfaction of, say, natural boundary conditions, and the violation of continuity requirements of functional derivatives. Another indirect indicator can be the magnitude of residuum obtained when the approximate solution is substituted into the governing differential equations. Methods have been proposed to estimate the error in different norms (say, energy norm) at elemental and global levels using improved estimates of solution through postprocessing of the current solution. Point-wise estimates of error in function values and derivatives can also be obtained by comparing the current solution with improved estimates of the solution. After local and global estimates of error are determined, the finite-element model can be optimally improved upon by using one of the stated extensions. The example problems solved by this method will be presented with the results by other methods.

2.2. Boundary-Element Method (BEM)

The boundary-element method consists of transforming the governing partial differential equations of a domain to a set of integral equations, which relate the boundary variables (known and unknown), followed by the determination of the approximate solution of the unknown boundary variables by means of numerical integration [5,6]. In the process, the dimension of the problem is reduced by one—that is, from three-dimensional to two-dimensional, or from two-dimensional
to one-dimensional. For numerical solution, the discretization is done in the boundary of the domain for the BEM, while in the FEM it is done in the entire domain. The number of equations associated with BEM is always smaller than in the FEM approach, for the same degree of accuracy. The larger the volume of the region enclosed by a certain boundary, the larger is the ratio of the number of equations required in FEM to those in BEM.

The choice for a specific method when a particular problem or model is to be studied, for example, related to potential or elasticity theories might depend on such computational efficiency, or on the applicability of the methods, or on the desired accuracy of results. For example, problems with singularities, such as in linear elastic fracture mechanics, are well handled with BEM, with accurate pointwise results close to the singular points, while FEM gives a damped global result valid in the sense of energy norm. Also, infinite domain problems like half-plane or far-field problems can be better handled with BEM. On the other hand, material nonlinearities in the domain or complex geometric shapes are better handled by FEM.

The system of equations in BEM arises from the discretization of an integral identity, which is an exact representation of the original boundary-value problem. Examples of such integrals are Green’s second identity, for potential theory, and Somigliana’s stress identity, for elasticity theory. For instance in the case of potential problems, the integral equation takes the form

\[ C(\xi_s)u(\xi_s) + \int_\Gamma Q(\xi_s, \xi_f)u(\xi_f) \, d\Gamma = \int_\Gamma U(\xi_s, \xi_f)q(\xi_f) \, d\Gamma. \]  \hspace{1cm} (2)

Here, \( \xi_s \) and \( \xi_f \) are the source (or collocation) and field points on the boundary; \( u \) and \( q \) denote potential and flux values at a point; \( Q \) and \( U \) are the kernel functions representing the flux and potential values at \( \xi_f \) due to a unit source at \( \xi_s \); and \( C(\xi_s) \) is the free term dependent upon the geometry of the boundary point at \( \xi_s \) with a value of 1/2 for smooth boundaries. For an isotropic two-dimensional domain, the potential kernel function is the fundamental solution to Laplace’s equation and can be expressed as \( U(\xi_s, \xi_f) = -\ln(r)/2\pi \) and the flux kernel function which is the outward normal derivative of the fundamental solution can be expressed as \( Q(\xi_s, \xi_f) = -1/2\pi r \), with \( r \) as the distance between a pair of source and field points.

The integral formulation can also be seen arising out of the weighted residuals approach in which the weight functions are now fundamental solutions of an associated homogeneous problem. This implies that the fundamental solution for a specific problem has to be known prior to applying BEM to the problem. The solution of these integral equations is exactly the same as for the original BVP, provided the required degree of continuity of the solution is satisfied. Weight functions are not only admissible functions, but also exact solutions of the associated problem. By discretizing the boundary, the solution is only as good as the boundary approximation is. If the discretized and the original boundary coincide, the solution is exact, affected only by the errors arising from numerical integration, and round-off. The fundamental solution refers to the whole problem domain. This solution appearing in the integral equation as being multiplied by the nodal natural and essential values are functions of two boundary points, namely, collocation and the field points. Also, the integration, for every collocation point, is performed for all the field points using approximation of the function in the boundary. This means that the matrices associated with this approach will be fully populated and, also, be nonsymmetric.

All the advantages of using piecewise approximation apply to BEM, with the advantage that the possible modifications or refinements to improve the solution only affect the boundary. It has to be pointed out, though, that a refinement in a small region or in an element will have consequences in many elements of the fully populated matrices. But as the fundamental solutions usually decrease in magnitude with the distance between the collocation and the field points, the influence of this local refinement will be more important in a boundary region in the neighborhood of the boundary elements being refined. The functions that represent essential and natural boundary conditions appear independently in the integral equations, and consequently are approximated in an independent form, having the same degree of accuracy. Only the quantities that are dependent
on postprocessing, such as the surface traction in elasticity that partly depends on the tangential
derivatives of the displacements, may have a lower degree of accuracy.

In the case of adaptivity, FEM and BEM have different characteristics. FEM has a global
character in the sense that it requires the modeling of the whole domain, but it has a local
character in the sense that modifications in the discretization have only local effect, in the region
of the domain where these modifications are performed. BEM has a global character in the sense
that it requires the modeling of the entire boundary, but a local modification or refinement affects
all the equations, due to the global character of the fundamental solution. All the elements of
the system matrices will have to be re-evaluated, even for a local refinement of the mesh.

2.2.1. Example problems

For the sake of outlining the features of the different approaches for numerical modeling, simple
elasticity problems are considered, consisting of bars under axial load, modeled by equation (1).
The first problem is presented with no singularities. In the second problem, the cross-section
area is made to change smoothly so that a singularity appears in the axial stress. The third is a
modification on the second one, in which the cross-section area is made to change in a way such
that both the longitudinal displacement and in the axial stress are singular.

EXAMPLE B1. Bar under axial load, with no singularities: a bar under axial load is presented
in Figure 3. Let the cross-sectional area be A and the elastic modulus be E. The boundary
conditions on axial displacement are: \( u(0)=u(L)=0 \). Let the total length be \( L=2 \) and \( P/AE=1 \).

![Figure 3. A bar under axial load P.](image)

The differential equation corresponding to this problem is

\[
-\frac{d}{dx} \left( AE \frac{du}{dx} \right) = \psi(x) = P\delta(x, x_P),
\]

where \( \delta(x, x_P) \) is the Dirac Delta function. The exact solution is \( u = 2x/3 \) for \( 0 \leq x \leq 2/3 \) and
equal to \( -x/3 + 2/3 \) for \( 2/3 \leq x \leq 2 \).

In the BEM approach, the integral equation for this boundary-value problem is

\[
I = \int_0^2 w \left( -\frac{d^2u}{dx^2} \right) dx + \int_0^2 \psi(x) dx = 0,
\]

where \( w \) is the fundamental solution for the associated problem: \( \frac{d^2w}{dx^2} = -\delta(x, x_0) \) with \( x \) as the
field point and \( x_0 \) as the collocation point. Integrating by parts and considering the boundary
conditions, the boundary equation is written with respect to the two end points as

\[
u(x_0) = -w(0, x_0) \frac{du}{dx}
\]

\[
+ w(2, x_0) \frac{du}{dx}
\]

\[
+ w(x_0, x_0).
\]

Here the fundamental solution for this problem is found by embedding the bar in an infinite
domain

\[
w(x, x_0) = \frac{L - (x - x_0)}{2}, \quad 0 < x_0 < x \quad \text{and} \quad \frac{L - (x_0 - x)}{2}, \quad 0 < x < x_0.
\]
By collocation of equation (5) at the end points: $\xi_0 = 0$ and $\xi_0 = 2$, the end reactions are obtained as

$$EA \left. \frac{du}{d\xi} \right|_{\xi=0} = \frac{2}{3} P \quad \text{and} \quad EA \left. \frac{du}{d\xi} \right|_{\xi=2} = -\frac{1}{3} P,$$

substitution of which into equation (5) gives $u(\xi_0)$ for any collocation point $\xi_0$ in the domain. The solution obtained is the same as the exact solution, shown above.

**EXAMPLE B2.** Bar with stress and displacement singularities: a bar under an axial load at left end is shown in Figure 4. The cross-sectional area $A = \xi$, so that the longitudinal displacement and the axial stress are both singular at $\xi = 0$. The boundary conditions are

$$-\xi \left. \frac{du}{d\xi} \right|_{\xi=0} = p = 1 \quad \text{and} \quad u|_{\xi=2} = 0.$$

![Figure 4: Singular bar problem](image)

The differential equation corresponding to this problem with $E = 1$ is

$$-\frac{d}{d\xi} \left( \xi \frac{du}{d\xi} \right) = \psi(\xi) = P\delta(\xi, \xi_0), (7)$$

with $\delta(\xi, \xi_0)$ as the Dirac Delta function. The exact solution for this boundary-value problem is $u(\xi) = \ln 2 - \ln(\xi)$ and the integral equation is

$$I = \int_0^2 w - \frac{d}{d\xi} \left( \xi \frac{du}{d\xi} \right) d\xi - \int_0^2 w(\xi_0, \xi_0') d\xi = 0. \quad (8)$$

Here $w$ is the fundamental solution for the associated problem: $\frac{d}{d\xi} (-\xi \frac{dw(\xi, \xi_0)}{d\xi}) = -\delta(\xi, \xi_0)$, with $\xi$ is the field point and $\xi_0$ is the collocation point. Integrating by parts and considering the boundary conditions, the boundary equation is written with respect to the two end points as

$$u(\xi_0) = - \left( u(\xi_0) \xi \frac{du(\xi_0)}{d\xi} \right) \bigg|_{\xi=0} + w(2, \xi_0) - w(0, \xi_0). \quad (9)$$

The fundamental solution for this problem is found by embedding the bar in a semi-infinite domain, $\xi > 0$. For a half-plane type problem it is obtained by adding the fundamental solutions for an infinite domain at collocation points $\xi_0$ and $-\xi_0$, as $w(\xi, \xi_0) = 0.5 \ln \xi - \ln \xi_0 + \ln 2$ for $0 < \xi_0 < \xi$ and equal to $-0.5 \ln \xi + \ln 2$ for $0 < \xi < \xi_0$. By collocation of equation (8) at $0 \leq \xi_0 \leq 2$, the exact solution for $u(\xi)$ is obtained.

**EXAMPLE B3.** Rectangular domain with boundary singularity: a $14 \times 7$ rectangular problem domain with singular Point 0 is shown in Figure 5a. The expression for the value of the potential near the singular point is given by [7]

$$\phi(r, \theta) = 500 + a_1r^{0.5}\cos \left( \frac{\theta}{2} \right) + a_2r^{1.5}\cos \left( \frac{3\theta}{2} \right) + \ldots \quad (10)$$

In this expression, if the constant $a_1$ is not equal to zero, $\phi'$ tends to infinity as $r$ tends to zero, where the singular point is located. Three meshes used in the analysis are shown in Figures 5b–5d. The convergence characteristics of potential at a point located at $(0, 1)$ are shown in
Figure 5. Potential problem with singularity at O.

Figure 6. Mesh A has five elements. Mesh B has seven elements with two of length 1.75 next to Point O. Mesh C has nine elements with four of length 0.875 near O.

The results for the three meshes are shown in Figure 6, based on a symmetric nonhierarchical choice of collocation points as $p$-level is increased. All the meshes show monotonic convergence to published results [5]. This behavior is contrary to what was found with hierarchic choice of collocation points. The $p$-version results compared favorably with those computed by the $h$-version of BEM, which uses special formulation to account for singularity.

2.3. Meshless Method

Like the finite-difference method, the meshless method is based on a grid of nodes covering the continuum. In the first method, the governing differential equation(s) is directly satisfied at each node by defining it in terms of discrete difference approximations based on the nodal values of the unknown function(s). On the other hand, in the meshless method, the weak form of the governing differential equation is satisfied at selected sampling (or quadrature) points based on weighed nodal values within a predefined subdomain of influence associated with the sampling point. The number and distribution of nodes within the subdomain of influence govern the order of the chosen polynomial approximation. The conventional meshless method, which is based on moving least square interpolant by Lancaster and Salkaukas, requires only a mesh of nodes and boundary description to develop the Galerkin equations for the system. The method is claimed [8] to be particularly promising for the treatment of crack propagation, large deformation, and plastic flow problems because it does not require any element connectivity data and does not suffer much degradation in accuracy when nodal arrangements are very irregular. This method
can be used for volumetric discretization as in the case of the finite-element method, and also for surface discretization as in the case of the boundary-element method; see Figure 7. For the purpose of defining a polynomial through a finite number of points (or nodes), the quadrature scheme with cell structure, as shown in Figure 7, is chosen in the problem domain. The cells are independent of the nodes and are arranged in a regular pattern. All the \( n \) nodes lying inside the domain of influence of a quadrature point are identified and a polynomial of order \( p \) (say, for 2-D, \( p = \sqrt{2n + 0.25} - 1.5 \)) is passed through these nodes. Based on this polynomial, a set of equations is defined for the nodes in its zone of influence. This procedure is repeated for all quadrature points, and a set of equations for all the nodes in the problem domain which when solved gives the values of the unknown parameter at these discrete points. Next, the nodes in the zone of influence of the midpoint of a cell are identified. These nodes are used to define the polynomial for any output point lying in the cell. With the discrete values of the parameters known, the desired parametric value is evaluated at all the output points in a cell. The same procedure is followed to evaluate the unknown functions at all the desired output points.

![Figure 7. Discretization elements of meshless method.](image)

For the sake of simplicity, the method will be studied in the context of equation (1) for three problem types: smooth, singular, and those with discontinuities. In the meshless method, the discretization process requires covering the problem domain with a gridwork of nodes, dividing the problem domain into integration cells containing quadrature points, and estimating the maximum number of nodes associated with a cell based on the defined size of subdomain of influence. The approximation of the unknown function for a quadrature point can be expressed in terms of suitably weighted

(a) modified Lagrangian type polynomials, or
(b) integrals of Legendre polynomials.

Both choices essentially lead to higher-order approximation.

**LAGRANGIAN TYPE INTERPOLATION.** Consider a point \( x \) for which the unknown parameter \( u(x) \) is approximated as

\[
u_a = \sum_{j=1}^{m} N_j(x)C_j(\bar{x}) \equiv [N] \{C\} \equiv [N]^T \{C\}.
\]

Here, \( u_a \) is the approximate value of the unknown parameter at any point \( x \) obtained by polynomial interpolation with respect to a sampling point at \( \bar{x} \), \( \{N\}^T = \{1, x, x^2, \ldots, x^m\} \), the interpolants, and \( C_j(\bar{x}) \) is the \( j \)th coefficient, which is a function of \( \bar{x} \). If \( u_l \) is the actual value at a discrete point \( x = x_l \), then the error in the value of \( u \) at the discrete point is equal to \( \left[ (N(x_l))^T \{C(\bar{x})\} - u_l \right]^2 \). After weighting,

\[
I = w(x - x_l) \left[ (N(x_l))^T \{C(\bar{x})\} - u_l \right]^2.
\]
Summation of the error over all the nodes in the zone of influence of sampling point \( x = \bar{x} \) gives error
\[
E = \sum_{I=1}^{n} w(x - x_I) \left[ \{N(x_I)\}^T \{C(\bar{x})\} - u_I \right]^2.
\]
Applying the stationary condition to \( E \) with respect to \( C_j(z') \), it follows
\[
\frac{\partial E}{\partial C_j(z')} = \sum_{I=1}^{n} w(x - x_I) \left[ \{N(x_I)\}^T \{C(\bar{x})\} - u_I \right] \times N_j(x_I) = 0, \quad \text{for } j = 1, 2, 3, \ldots, m;
\]
i.e.,
\[
w(x - x_I) \left[ \{N(x_I)\}^T \{C(\bar{x})\} - u_I \right] N_j(x_I) + w(x - x_2) \left[ \{N(x_2)\}^T \{C(\bar{x})\} - u_2 \right] N_j(x_2)
+ \cdots w(x - x_n) \left[ \{N(x_n)\}^T \{C(\bar{x})\} - u_n \right] N_j(x_n) = 0;
\]
\[
\left[ w(x - x_1) \{N(x_1)\}^T N_j(x_1) + \cdots + w(x - x_n) \{N(x_n)\}^T N_j(x_n) \right] C_j(\bar{x})
- \left[ w(x - x_1)u_1N_j(x_1) + \cdots + w(x - x_n)u_nN_j(x_n) \right] = 0;
\]
or
\[
\sum_{I=1}^{n} w(x - x_I) \{N(x_I)\}^T N_j(x_I) \{C(\bar{x})\} = \sum_{I=1}^{n} w(x - x_I)N_j(x_I)u_I, \quad \text{for } j = 1, 2, \ldots, m,
\]
Therefore,
\[
u_\alpha(\bar{x}) = \{N(\bar{x})\}^T \{C(\bar{x})\} = \{N(x)\}^T [A(\bar{x})] [B(\bar{x})] \{u\} = \{N(x)\}^T [\bar{M}] \{u\} = \sum_{I=1}^{n} M_I u_I,
\]
where \( M_I \) are the shape functions for discrete points \( I \).

The weighting function used by Belytschko et al. [8] \( w \) is the Gaussian weight function of the form
\[
w(x - x_I) = \begin{cases} 
eq \left( \frac{d}{d_{\text{max}} I/c} \right)^{2k}, & \text{for } d \leq d_{\text{max}} I, \\ 0, & \text{for } d > d_{\text{max}} I. \end{cases}
\]
Here, \( d = ||x - x_I||; d_{\text{max}} I \)-domain of influence for the point \( x; c = \sigma c_I, \text{ with } c_I = \max_{J \in S_J} ||x_J - x_I||; 1 \leq \alpha \leq 2, \alpha \text{ is near } 1 \text{ for problems with singularities and high gradients; and } S_J \text{ is the minimum set of neighboring points of } x_I \text{ that are used to construct a polynomial.}

**Legendre Type Interpolation.** The choice is similar to the ones used in the \( p \)-version of the finite-element method [3]. In this approach, only the end discrete points in the zone of influence are identified and the interpolation is done using integrals of Legendre polynomials and no least square interpolation is used. This interpolation is then transformed into a matrix form that contains contributions from various discrete points. Thereafter it is used to obtain a set of equations for the continuum. The solution of these equations is transformed back in terms of the interpolation, which is then used to obtain the values of the unknown variable(s) at the discrete points. In the meshless approach, the problem domain is divided into cells as shown in Figure 8. For each cell the quadrature points and the nodes are as shown in Figure 9.

---

**Figure 8.** Cell structure for axial bar problem.  
**Figure 9.** Nodes and quadrature points.
In the case of smooth problems, the method is applied as discussed in the case of volumetric discretization approach. In the case of singular problems, the polynomial solution obtained by least squares approximation is appended with singular functions to account for the singular behavior of the derivative of unknown function \( u \), so that

\[
u = \sum_{I=1}^{n} M_I u_I + \sum_{J=1}^{n} M'_J u'_J.
\]

Here \( M'_J \) are constant coefficients and \( u'_J \) are of the form: \( u'_J = (1 - x/L)x^{(1-\lambda)} \). This choice of singular function serves two purposes. The first is that it involves function \( x^{(1-\lambda)} \). The second derivative of this function gives \( -\lambda(1-\lambda)/x^{1+\lambda} \), which is proportional to, say, a singular loading function \( c_1/x^{1+\lambda} \). At the same time it satisfies the boundary conditions \( u|_{x=0} = 0 \) and \( u|_{x=L} = 0 \), if the ends of the bar are constrained. As cell matrix and vector are generated by Gauss quadrature, the contributions of all the cells are assembled into the global matrix and vector and solved for the unknown nodal variables.

2.3.1. Numerical examples

The size of the subdomain of influence was based on the following considerations:

(a) in the case of small domains, all the sampling points in every cell refer to all the nodes in the problem domain;

(b) only a few nodes in the neighborhood of a sampling point are used to generate the approximating polynomial.

It was found that depending upon geometric and loading complexities, the quality of results was controlled by model attributes like: cell structure, number of nodes, domain size \( d \), number of sampling points, method of interpolation during postprocessing, and the choice of weight function in least squares interpolation.

**EXAMPLE M1. AXIAL BAR WITH GEOMETRIC SINGULARITY.** This problem is similar to Example B2 with the exception that the length is unity, area \( A = 1/x^2 \), \( E = 5890.5 \), and the end load \( p = 2 \). The value of parameter \( a(x) = AE = 5890.5/x^2 \). The meshless model with typical nodal and sampling point locations and cells is shown in Figure 10. The solutions for a 21 node and 20 cell model with six sampling points in each cell are shown in Figures 11a and 11b for displacement and axial stress. It may be noted that both displacement and axial stress are singular at the loaded end \( (x = 0) \). The exact solutions are \( u = (1 - x)/(2945.25x) \) and \( \bar{E}u' = -2/x^2 \). In the meshless solution no weight function is used. To account for singularity, singular function of the form \( \sum c_j/x^j \) was appended to the approximating polynomial. Use of one singular term was found to be adequate.

**EXAMPLE M2. AXIAL POINT LOAD AT CENTER OF BAR.** This problem shown in Figure 12 was solved for a point load \( p = 5 \), \( a = 3000 \), and \( L = 1 \). The problem was treated in two ways:

(a) the subdomain of influence was assumed to include all the nodes \( (d = 2h_c) \), and

(b) the subdomain of influence did not include nodes beyond the load point \( (d = h_c) \).

It can be seen from Figures 13a and 13b that the choice given by (a) is very poor. So, the subdomain of influence cannot extend beyond a point of discontinuity.

**EXAMPLE M3. BAR WITH GEOMETRIC AND/OR MATERIAL DISCONTINUITY.** In this example a bar of unit length, with \( a \) for the left and right halves of the bar assigned the values 3000 and 6000,
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![Figure 11. Tapered bar problem.](image1)

![Figure 12. Axial bar problem.](image2)

![Figure 13. Axial force at center of a bar with constrained ends.](image3)

is subjected to a distributed axial force of 2 units/unit length. As in the case of Example M2, two cases of size of domain of influence were considered leading to similar conclusions, as evidenced in Figures 14a and 14b. When p-version formulation was used, very good results were obtained even for problems with singularity, without the need for using special singular functions. However, some minor oscillations in stress values formulation based on integrals of Legendre polynomials gave superior performance.

2.4. Wavelet Method

Wavelet theory has evolved over the last two decades. Similar to Fourier series expansion but without some of its restrictions, wavelets are used in the series expansion of signals and can be so used in the case of arbitrary functions as well, enabling localization in both space and time. In other words, wavelets can analyze different spatial and temporal parts of a signal or function at
different scales. Therefore, functions with rapid oscillations, or, possibly, with discontinuities, in localized regions are amenable to an accurate representation by a linear combination of a modest number of wavelets.

So far wavelets have found primary application in signal and image processing, but its unique properties make it a worthwhile candidate for the numerical solution of the partial differential equations of the continua having different kinds of irregularities. Wavelets-based discrete numerical method for the solution of such equations may offer some unique advantages. In the aforementioned methods, the presence of irregularities (or discontinuities) requires proper attention in spatial discretization, and sometimes requires the use of special basis functions. But it is expected that the use of basis functions based on wavelets will eliminate or relax such restrictions. This may be possible because this approach is so designed that it adapts itself to the local nature of the problem by retaining the finer wavelet components in the vicinity of an irregular point (say, near a singular point), whereas the coarser components are retained in smooth regions. Basically, the wavelet theory is based on the idea that any signal can be broken down into a series of local basis functions called “wavelets”. Any particular local feature of a signal can be analyzed based on the translation and scaling characteristics of wavelets. The translation characteristics determine the location of a wavelet in space or time. The concept of scale is similar to the scale used in cartography. As in the case of maps, high scale (or low frequency) gives a zoomed out or global view of a signal, and low scale (or high frequency) gives a zoomed in or detailed view. Thus, scaling either compresses or dilates a signal.

**Basic Approximation Functions Based on Wavelets.** In a one-dimensional problem domain of the type defined by equation (1), the unknown function \( u \) can be approximated by a series expansion and the Galerkin method can be applied to the weighted residual of the equation. In general, the series expansion can be defined in terms of wavelets as

\[
    u = \sum_{k=-\infty}^{\infty} c(k) \varphi(x - k) + \sum_{j=j_0}^{\infty} \sum_{k=-\infty}^{\infty} d(j,k) \psi_{j,k}(x). \tag{15}
\]

Here \( \varphi(x - k) \in L^2(R) \) is the scaling function with zero starting scale (\( j_0 = 0 \)) and translation index \( k \in \mathbb{Z} \) controlling spatial location; \( \psi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \) is the wavelet function; \( c(k) \) and \( d(j,k) \) are coefficients; \( j \in \mathbb{Z} \) is the scaling parameter; and \( \mathbb{Z} \) is the space of all integers. The subspace of \( L^2(R) \) spanned by the scaling functions \( \varphi_k(x) = \varphi(x - k) \) is obtained as \( v_0 = \text{Span}_k \{ \varphi_k(x) \} \). Scaling and translating the basic scaling function, \( \varphi_k(x) \), generates a two-dimensional family of functions \( \psi_{j,k}(x) = 2^{j/2} \varphi(2^j x - k) \). Its subspace over \( k \) is \( v_j = \text{Span}_k \{ \varphi_{j,k}(x) \} \). The different subspaces are related (or nested) as \( \cdots v_{-1} \subset v_0 \subset v_1 \subset \cdots \)

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**Figure 14.** Axial bar under uniform load with stepped variation of property.

(a) Axial displacement.  
(b) Axial stress.
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$V_2 \subset V_3 \cdots \subset L^2(R)$ with $u \in V_j \Rightarrow u(2) \in V_{j+1}$. The general relationship for $(j+1)^{\text{th}}$ space is:

$v_{j+1} = v_0 + w_0 + w_1 + w_2 + \cdots + w_j$. This relationship shows that the spaces are hierarchical, and any degree of refinement can be achieved by selecting a suitable value for $j$. In order to find a numerically well-conditioned system, it is advantageous to select the functions $\varphi(x)$ and $\psi(x)$ so that the following orthogonal relationship holds well:

$$\int_{-\infty}^{\infty} \varphi_{j,k}(x) \varphi_{j,l}(x) = \delta_{lk} \quad \text{and} \quad \int_{-\infty}^{\infty} \psi_{j,k}(x) \psi_{j,l}(x) = \delta_{lk}. \quad (16)$$

Wavelet theory has profound impact on fields of signal and image processing. In order to determine its possible impact in the numerical solution of problems of the continua, a number of 1-D problems represented by the following reduced form of equation (1) are considered:

$$\frac{d}{dx} \left( a(x) \frac{du(x)}{dx} \right) + f(x) = 0, \quad \text{for } 0 \leq x \leq L. \quad (17)$$

The weighted residual method is used here to solve this differential equation enabling slackening of the continuity requirement of $u$ to $C^0$. The unknown function $u \in V_j$ can be represented in different ways. A pure scaling function expansion can be used, such as at some chosen level $j$

$$u(x) = \sum_{k=-\infty}^{\infty} c_{j,k} \varphi_{j,k}(x), \quad x \in R. \quad (18)$$

Here, the scaling functions can be viewed to replace the piecewise polynomials used in, say, finite-element analysis. The deficiency of this choice is that it is not adaptive because of the absence of the possibility of wavelet compression. Another possibility is that for any $j > j_0$, the wavelet expansion can be appended to equation (18), as shown in equation (15). This allows adaptivity to be restored. In addition, an important advantage of this representation is that the use of higher powers may lead to sparser matrices (that is, a large number of elements in the coefficient matrix are small enough to be set to zero). Furthermore, many wavelet bases are stable for families of Sobolev spaces $H^0$. This property implies that for linear systems associated with Galerkin discretization, the choice of bases in equation (15) will be uniformly well conditioned irrespective of the final refinement level $j$. In turn this leads to "scalable" algorithms that scale linearly with the number of basis functions. For convenience, initially $V_0$ will be chosen, meaning thereby $j_0 = 0$. Then equation (15) becomes

$$u(x) = \sum_{k=-\infty}^{\infty} c(k) \varphi_k(x) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d(j,k) \psi_{j,k}(x). \quad (19)$$

This equation is used for approximating signals or functions that span over $(k = -\infty \text{ to } \infty)$. For the 1-D problem, the displacement $u$ is restricted in the compact space $[0, L]$, so $k$ should be so chosen that the translated scaling functions $\varphi$ and wavelet functions $\psi$ are partly or totally located in the space $[0, L]$. In selecting the proper scaling functions $\varphi_k(x)$ and wavelet functions $\psi_{j,k}(x)$, it is necessary to satisfy the requirements of differentiability, completeness, and boundary conditions.

**SCALING AND WAVELET FUNCTIONS.** There is a wide variety of scaling and wavelet functions that are being used by researchers [9], for example, the Daubechies functions, Hermite cubic functions, Haar functions, hat functions, etc. As compared to the first three functions, Haar and hat functions are much simpler and easier to use in the case of 1-D continuum problems. Here, the use of hat functions, although not unconditionally orthogonal, will only be presented. Hat wavelets, shown in Figure 15 with support $[0, 2]$, are $C^0$ continuous and form orthonormal basis (i.e., $(\int_{-\infty}^{\infty} |\varphi(x)|^2 \, dx)^{1/2} = 1$). On mapping the weak form of the differential equation over the
standard interval $0 \leq \xi = 2a/L \leq 2$ by replacing $a$ by $\alpha = 2a/L$ and $f$ by $\beta = fL/2$, the normalized weak form of the differential equation becomes
\[
\int_0^2 \alpha \left( \frac{dw}{d\xi} \right) \left( \frac{du}{d\xi} \right) d\xi = \int_0^2 w \beta d\xi + w \left( \alpha \frac{du}{d\xi} \right)^2, \quad 0 \leq \xi \leq 2.
\]  \(20\)

**EXAMPLE W1. AXIAL BAR WITH CENTRAL POINT LOAD.** The following simple example demonstrates the use of wavelet functions. Consider an axial bar of length $L$ restrained at its ends and acted upon by a concentrated axial load $P$ at its center (Figure 12). Assume the parameter $a$ is so chosen that $\alpha = 2a/L = 1$, and $P$ and $L$ are such that $PL/2 = 1$. So, in equation (20), $\beta = fL/2 = P \cdot \delta(\xi = 1) \cdot L/2 = \delta(\xi = 1)$.

In order to construct the approximate expansion consisting of scaling functions and wavelets functions, it is necessary to consider approximation of $u$ in level $v_1$. For level $v_1$, if $v_0$ is used as the base level, then by satisfying the boundary condition $u(0) = 0$ and $u(L) = 0$, the complete set of scaling functions in compact support $[0,2]$ should include the three scaling functions: $\phi_{1,k}(x) = (1/2^k) \phi_0(x - k/2^2)$, with $k = 0, 1, 2$. As $v_1 = v_0 + v_1$, these three scaling functions can be replaced in the base level by a scaling function $\phi_{0,0}$ and two wavelet functions $\psi_{0,0}$ and $\psi_{0,1}$ which are complete in the compact support $[0,2]$, and satisfy the boundary conditions. This leads to $u(\xi) = c_1\phi_{0,0}(\xi) + d_1\psi_{0,0}(\xi) + d_2\psi_{0,1}(\xi) = \sum_{i=1}^{3} c_i u_i(\xi)$. Substituting $\alpha$, $\beta$, $u(\xi)$ into equation (20), and using the boundary conditions $u(0) = 0$ and $u(2) = 0$,
\[
\sum_i c_i \int_0^2 \frac{dw}{d\xi} \frac{du_i}{d\xi} d\xi = w|_{\xi=1}, \quad \text{with } i = 1, 2, 3.
\]  \(21\)

Now, using Galerkin’s method and setting the weight function $w$ in equation (21) equal to $u_1(\xi)$, $u_2(\xi)$, and $u_3(\xi)$, respectively, a set of three linear simultaneous equations with $c_1$, $c_2$, and $c_3$ as the unknowns are obtained. The solution for the problem is obtained as $0.5, 0, 0$. It means that the solution is defined by $c_1$ alone. It so happened in this case because the chosen scaling function corresponds to the exact solution. In order to study the generality of the method, the model was refined in level $v_2$, in which the complete scaling functions in support $[0,2]$ include seven scaling functions. Again, $v_2$ can be separated into $v_2 = v_1 + v_1$ by choosing $v_1$ as the base level. There are three scaling functions in $v_1$, and four wavelet functions in $w_1$. Then the approximation for $u$ becomes
\[
u(\xi) = c_{1,0} \phi_{1,0}(\xi) + c_{1,1} \phi_{1,1}(\xi) + c_{1,2} \phi_{1,2}(\xi) + d_{1,0} \psi_{1,0}(\xi) + d_{1,1} \psi_{1,1}(\xi) + \sum_{j=1}^{3} d_{1,j} \psi_{1,j}(\xi) = \sum_{i=1}^{7} c_i u_i.
\]
Using this approximation the following solution is obtained with contributions from scaling functions only:

\[
\begin{bmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7
\end{bmatrix} = \begin{bmatrix} 0.5 & 1 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}.
\] (22)

The solution is exact because the concentrated load acts at the support point \( (x = (1/2^m) L, m \in \text{int}) \) of the scaling and wavelet functions. Now, if \( v_0 \) is taken as the base level, then \( v_2 = v_0 + w_0 + w_1 \). In this case, \( u_1, u_2, \) and \( u_3 \) represent \( \phi_{0,0}(\xi), \phi_{0,0}(\xi), \) and \( \phi_{0,1}(\xi) \), respectively, leading to the solution

\[
\begin{bmatrix}
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7
\end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\] (23)

It was found from this study that if \( v_0 \) is selected as the basis level, then the \( m^{\text{th}} \) approximation level \( v_m = v_0 + w_0 + w_1 + \cdots + w_{m-1} \) is not only hierarchic but also leads to matrices which continue to be well conditioned as \( m \) is increased. It may be noted that at the \( m^{\text{th}} \) level the number of equations is \( 2^{m+1} - 1 \).

**EXAMPLE W2.** Same as Example W1, except that the load \( P \) is at third point: here the location of the concentrated load is not at a support point of the scaling and wavelet function. The exact solution of this problem is given under Problem B1. The plot of exact solution is the same as shown in Figure 16, corresponding to Level 8. As the support point does not coincide with the load point, it took successive refinement of solution to converge to the exact solution, which occurred monotonically as the solution was refined through levels 1 to 8. This refinement was done without any consideration of the location of the point load.

![Figure 16. Solution for Example M2 corresponding to Levels 1 and 8.](image)

From these and a number of other problems, it was found that the coefficients of system matrices of a level form a subset of those of higher levels. So, the different levels of approximation are hierarchical. Irrespective of the level of approximation, the base level should be \( v_0 \) so that the following relationship holds good:

\[
v_m = v_0 \oplus w_0 \oplus w_1 \oplus \cdots \oplus w_{m-1}.
\]

As the approximation level is increased (or refined), monotonic convergence in results is observed. The performance was more dramatic in the case of singular problems, as shown by Example W3.

**EXAMPLE W3. SINGULAR TAPERED BAR PROBLEM.** This problem is similar to Examples B2 and M2, with the exception that the area is proportional to \( \sqrt{\xi} \) instead of \( \xi \). The exact solution of the problem is

\[
u(\xi) = -2 \left( \sqrt{\xi} - \sqrt{2} \right) \quad \text{and} \quad \frac{u'}{\sqrt{\xi}} = \frac{1}{\sqrt{\xi}}, \quad 0 \leq \xi \leq 2.
\]
The wavelet solution for the problem along with the exact solution and \( h \)-version FEM solution are shown in Figure 17. In the case of \( p \)-version solution, oscillations were noticed in the stress values near the singular end. With 33 support points, the value of \( u \) at \( \xi = 0.001 \) in the wavelet solution was found to be less than 0.1%; with a 33 dof finite-element model, the error was found to be 5%. The 50-element model did not improve accuracy over the 32-element model. For all the numerical experimentation undertaken in the case of second- and fourth-order elliptic equations, and second order hyperbolic equations (namely, wave propagation and impact problems), the wavelet method provided higher accuracy with less computational effort the wavelet method was
found to be robust in handling all kinds of irregularities including stress singularities. Near a singular point, the wavelet method gave significantly higher accuracy than the finite-element and meshless methods.

3. SUMMARY AND CONCLUSIONS

Four discrete numerical methods were discussed with the help of a number of example problems. The meshless and wavelet methods were discussed in some detail. In the case of the boundary-element method, both 1-D and 2-D example problems were presented. In the case of meshless and wavelet methods, only 1-D problems were presented. In all cases, both smooth and singular problems were considered and the relative merits of the methods were discussed.

The finite-element and meshless methods use numerical solution of an approximate representation of the problem. The boundary-element method, on the other hand, involves numerical solution of otherwise exact representation of the problem offering highly accurate local solutions. The meshless method, though numerically more cumbersome, offers some advantages in the sense that it avoids model revision in the case of the finite-element method, while considering nonlinearities caused by, say, damage propagation.

Based on the performance of all four methods with the problems considered, the newly emerging wavelet method offers a lot of promise in efficient modeling of discontinuities of different kinds. This method assures convergence even if the location of the hot spots and singular points are not known a priori. It can be applied in the context of the rest of the three methods and thus is an attractive candidate for further research.

REFERENCES