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## ENGINEERING PHYSICS AND MATHEMATICS

# Optimal harvesting strategy and stochastic analysis for a two species commensaling system



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Optimal harvesting;  
Pontryagin's principle;  
Stochastic perturbation

**Abstract** In this paper, we have considered a mathematical model of commensalism between two species ( $S_1$  and  $S_2$ ) with a limited resource of food, in addition the paper also highlights how the commensal and host species are harvested. The model is characterized by a couple of first order non-linear differential equations. Here, the stable equilibrium point is identified and its stability (both local and global) criteria are discussed (both analytical and numerical). An optimal harvesting strategy is being conversed using Pontryagin's maximum principle. We have explored the stochastic stability by finding the corresponding variances. Finally numerical simulations illustrate the effectiveness of our results.

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## 1. Introduction

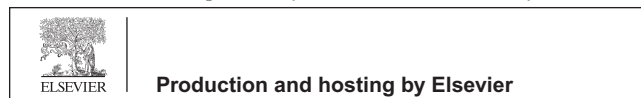
Ecology is the study of relationships between living organisms and their environment. Research in the area of theoretical ecology was started by Lotka [10] and Volterra [19]. Since then many mathematicians and ecologists have contributed to the growth of this area creating awareness as reported in the dissertations of Meyer [11], Cushing [4], Paul Colinvaux [14], Kapur [5,6], etc. The ecological interactions can be extensively classified as ammensalism, neutralism, commensalism,

competition, predation, and so forth. Srinivas [17] deliberated competitive eco-system of two species and three species with limited and unlimited resources. Later, Lakshminarayan and Pattabhiramacharyulu [8,9] premeditated prey predator ecological models with a partial cover for the prey and alternate food for the predator. In recent times stability analysis of competitive species was carried out by Archana Reddy et al. [1] and Sharma and Pattabhiramacharyulu [2], whereas Ravindra Reddy [16] investigated mutualism between two species. In 1996, Mesterton-Gibbons [12] described the skills to find the finest harvesting strategy for a Lotka–Volterra eco-system of two independent inhabitants. He also advocated that the technique may be extensively applicable in ecological modeling and other recent claims. In 2009, Phanikumar et al. [15] inspected the stability conditions for a mathematical model of commensalism between two species  $S_1$  and  $S_2$  with limited resources; the linearized disturbed equations are solved and the trajectories are illustrated. In 2005, Kar and Swarnakamal [7] proposed a prey predator model in a two patch environment: 1.

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**Nomenclature**

$x$	biomass density of commensal species	$a_{ii}, i = 1, 2$	of decrease in $S_i$ due to limitations of natural resources
$y$	biomass density of host species	$a_{12}$	commensal coefficient
$S_1$	commensal species	$q_1$	represents the catchability coefficient of $S_1$ species
$S_2$	host species	$E_1$	effort applied to harvest the $S_1$ species
$a_i, i = 1, 2$	natural growth rates of $S_i$		

Accessible to both prey and predators (patch 1) and 2. Being a refuge for the prey (patch 2). They assumed that the prey refuge (patch 2) constitutes a reserve zone of prey and fishing is not permitted, while the unreserved area is an open-access fishery zone. The existence of possible steady state points along with their local and global stability is discussed. They also examined the possibilities of the existence of bionomic equilibrium. Phanikumar et al. [15], Kar and Swarnakamal [7], and Carletti [3] inspired us to consider a commensalism model, incorporating harvesting in commensal species with a stochastic term. The present exploration is devoted to the analytical and numerical comparisons of commensalism with harvesting for commensal species. This also includes stochastic immovability. Two species commensalism is an ecological relationship between two species where one species  $S_1$  derives benefit from the other species  $S_1$  which would not get affected by it.  $S_1$  may be referred as the commensal species, while  $S_1$  is the host. Some of the examples are cattle Egrat, Anemonetish, Barnacles, etc. The host species  $S_1$  supports the commensal species  $S_1$  and has its own natural growth rate in spite of a support apart from  $S_2$ . The commensal species  $S_1$  in spite of the limitation of its natural resources flourishes drawing strength from the host species  $S_2$ . The model is characterized by a couple of first order non-linear differential equations. All the four steady state points of the system are recognized and their stability analysis is carried out. It is detected that the co-existence state is the only stable state that pertains to specified clauses. However, the other three steady states are unstable.

**2. Basic mathematical model**

$$(dx)/(dt) = x[(a_1 - q_1 E_1) - a_{11}x + a_{12}y] \tag{2.1}$$

$$(dy)/(dt) = y[a_2 - a_{22}y] \tag{2.2}$$

where  $x(t)$  represents the biomass density of commensal species  $S_1$ ,  $y(t)$  represents the biomass density of host species  $S_2$ .  $a_i, i = 1, 2$  represents the natural growth rates of  $S_i$ .  $a_{ii}, i = 1, 2$  represents the rate of decrease in  $S_i$  due to limitations of natural resources.  $a_{12}$  represents the commensal coefficient.  $q_1$  represents the catchability coefficient of  $S_1$  species.  $E_1$  represents the effort applied to harvest the  $S_1$  species. Throughout our analysis, let us assume that

$$a_1 - q_1 E_1 > 0 \tag{2.3}$$

**3. Analysis of steady states**

The possible equilibrium points are  $E_1(0, 0)$ ,  $E_2(\bar{x}, 0)$ ,  $E_3(0, \bar{y})$ , and  $E_4(x^*, y^*)$ .

Case (i):  $E_1(0, 0)$ : This equilibrium point always exist.

Case (ii):  $E_2(\bar{x}, 0)$ :

Here  $\bar{x}$ , is the positive solution of  $(dx)/(dt) = 0$ , which gives

$$\bar{x} = [1/(a_{11})](a_1 - q_1 E_1) \tag{3.1}$$

Clearly we observe that (3.1) is positive due to inequality (2.3).

Case (iii):  $E_3(0, \bar{y})$ :

Here  $\bar{y}$  is the positive solution of  $(dy)/(dt) = 0$ , which gives

$$\bar{y} = a_2/(a_{22}) \tag{3.2}$$

Case (iv):  $E_4(x^*, y^*)$  (The interior equilibrium):

Here  $x^*$  and  $y^*$  are positive solutions of  $(dx)/(dt) = 0$  and  $(dy)/(dt) = 0$ , which gives

$$y^* = a_2/(a_{22}) \tag{3.3}$$

$$x^* = [1/(a_{11})][(a_1 - q_1 E_1) + [(a_2 a_{12})/a_{22}]] \tag{3.4}$$

Clearly we have identified that (3.4) is positive due to the inequality (2.3).

**4. Local stability**

To determine the local stability character of the interior equilibrium  $E_4(x^*, y^*)$ , we compute the variational matrix about  $E_4$ .

$$J(x, y) = \begin{bmatrix} a_1 - 2a_{11}x + a_{12}y + a_{13}z - q_1 E_1 & a_{12}x \\ 0 & a_2 - 2a_{22}y \end{bmatrix} \tag{4.1}$$

The characteristic equation of (4.1) at the interior equilibrium  $E_4(x^*, y^*)$  is

$$(a_{11}x^* + \lambda)(a_{22}y^* + \lambda) = 0 \tag{4.2}$$

The roots  $\lambda_1 = -a_{11}x^*$ ;  $\lambda_2 = -a_{22}y^*$  of the Eq. (4.2) are both negative. Hence the steady state is stable. Since  $\lambda_1 + \lambda_2 = -(a_{11}x^* + a_{22}y^*) < 0$  and  $\lambda_1 \lambda_2 = -a_{11}a_{22}x^* y^* > 0$ ,  $E_4(x^*, y^*)$  is locally asymptotically stable.

**5. Global stability**

Theorem: The equilibrium point  $E_4(x^*, y^*)$  is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(x, y) = [(x - x^*) - x^* \ln(x/x^*)] + l_1 [(y - y^*) - y^* \ln(y/y^*)]$$

where  $l_1$  is the positive constant.

$$(dV)/(dt) = [(x - x^*)/x][dx/dt] + l_1 [(y - y^*)/y][dy/dt];$$

$$(dV)/(dt) = (x - x^*)[(a_{11}x^* - a_{12}y^*) + (-a_{11}x + a_{12}y)] + l_1(y - y^*)(a_{22}y^* - a_{22}y);$$

$$(dV)/(dt) = (x - x^*)[-a_{11}(x - x^*) + a_{12}(y - y^*)] + l_1(y - y^*)[-a_{22}(y - y^*)].$$

By choosing  $l_1 = 1/(a_{22})$ , we get,

$$(dV)/(dt) = -a_{11}(x - x^*)^2 + a_{12}(x - x^*)(y - y^*) - (y - y^*)^2;$$

$(dV)/(dt) = -[a_{11}(x - x^*)^2 - a_{12}(x - x^*)(y - y^*) + (y - y^*)^2]$ , which is in the form of  $-X^T AX$ , where  $X^T = (x - x^* \quad y - y^*)$ ;

$$A = \begin{pmatrix} a_{11} & -(a_{12})/2 \\ -(a_{12})/2 & 1 \end{pmatrix}.$$

The equilibrium point  $E_4 (x^*, y^*)$  is globally asymptotically stable when  $(dV)/(dt) < 0$ . This is possible only when the matrix  $A$  is positive definite, i.e. all principal minors of  $A$  should be positive. Obviously the principal minors  $M_1 = |1|$  and  $M_2 = |a_{11}|$  of  $A$  are positive. Hence the equilibrium point  $E_4$  is globally asymptotically stable.

## 6. Bionomic equilibrium

In Section (2), we have already discussed about the biological equilibrium. Now, we discuss about the bionomic equilibrium which is the combination of biological and economic equilibria. Let  $c_1$  be the constant fishing cost per unit effort and  $p_1$  be the constant price per unit biomass of commensal species. Then the economic rent (or) net revenue at any time is given by

$$R = (p_1 q_1 x - c_1) E_1 \quad (6.1)$$

The bionomic equilibrium  $((x)_\infty, (y)_\infty, (E_1)_\infty)$  is obtained from the equations

$$a_1(x)_\infty - a_{11}(x)_\infty^2 + a_{12}(x)_\infty(y)_\infty - q_1 E_1(x)_\infty = 0 \quad (6.2)$$

$$a_2(y)_\infty - a_{22}(y)_\infty^2 = 0 \quad (6.3)$$

$$R = (p_1 q_1 (x)_\infty - c_1)(E_1)_\infty = 0 \quad (6.4)$$

If  $c_1 < p_1 q_1 (x)_\infty$ , i.e. if fishing cost is less than the revenue, i.e. if the net revenue is positive, then the fishery will be in operation.

From (6.2)–(6.4) we get,

$$(x)_\infty = c_1 / (p_1 q_1) \quad (6.5)$$

$$(y)_\infty = a_2 / (a_{22}) \quad (6.6)$$

$$(E_1)_\infty = [1/q_1][a_1 - [(a_{11}c_1)/(p_1 q_1)] + [(a_{12}a_2)/a_{22}]] \quad (6.7)$$

For  $(E_1)_\infty$  to be positive, we must have

$$a_1 + a_{12}(y)_\infty > a_{11}(x)_\infty \quad (6.8)$$

The non-trivial bionomic equilibrium point  $((x)_\infty, (y)_\infty, (E_1)_\infty)$  exists if (6.8) hold. If  $(E_1) > (E_1)_\infty$ , then the total cost utilized in harvesting the fish population would exceed the total revenues obtained from the fishery. Hence some of the fisherman would be in loss and naturally they would withdraw their participation from the fishery. Hence  $(E_1) > (E_1)_\infty$  cannot be maintained indefinitely. If  $(E_1) < (E_1)_\infty$ , then the fishery is more profitable, and hence in an open access fishery, it would

attract more and more fisherman. This will have an increasing effect on the harvesting effort. Hence  $(E_1) < (E_1)_\infty$  also cannot be maintained indefinitely.

## 7. Optimal harvesting policy

In this section, we study optimal harvesting policy of the system (2.1) and (2.2). We employ the Pontryagin's maximum principle to obtain the path of optimal harvesting policy.

We consider the following present value  $J$  of a continuous time-stream

$$J = \int_0^\infty P(x, y, E_1) e^{-\delta t} dt \quad (7.1)$$

where  $P$  is the net revenue given by

$$P(x, y, E_1) = (p_1 q_1 x - c_1) E_1 \quad (7.2)$$

Here,  $\delta$  is the instantaneous annual discount rate. The aim of this section is to maximize  $J$  subject to the state Eqs. (2.1) and (2.2). Firstly, we construct the following Hamiltonian function

$$H = e^{-\delta t} (p_1 q_1 x - c_1) E_1 + \lambda_1 [a_1 x - a_{11} x^2 + a_{12} x y - q_1 E_1 x] + \lambda_2 [a_2 y - a_{22} y^2] \quad (7.3)$$

where  $\lambda_1, \lambda_2, \lambda_3$  are adjoint variables.

$E_1$  is the control variable satisfying the constraint

$$0 \leq E_1 \leq (E_1)_{\max} \quad (7.4)$$

and the switching function is given by

$$\phi(t) = e^{-\delta t} (p_1 q_1 x - c_1) - \lambda_1 q_1 x \quad (7.5)$$

Now, we aim to find an optimal equilibrium  $((x)_\infty, (y)_\infty, (E_1)_\infty)$  to maximize the Hamiltonian  $H$ . Since the Hamiltonian  $H$  is linear in the control variable  $E_1$ , the optimal control can be extreme control (or) the singular control, thus we have

$$(E_1) = (E_1)_{\max} \text{ where } \phi(t) > 0 \text{ i.e. } \lambda_1 e^{\delta t} < p_1 - (c_1)/(q_1 x) \quad (7.6)$$

$$E_1 = 0 \text{ where } \phi(t) < 0 \text{ i.e. } \lambda_1 e^{\delta t} > p_1 - (c_1)/(q_1 x) \quad (7.7)$$

when  $\phi(t) = 0$ ,

$$\lambda_1 e^{\delta t} = p_1 - (c_1)/(q_1 x) \text{ (or) } (\partial H)/(\partial E_1) = 0 \quad (7.8)$$

In this case, the optimal control is called the singular control and (7.8) is the necessary condition for the maximization of Hamiltonian  $H$ .

By Pontryagin's maximum principle, the adjoint equations are

$$\begin{aligned} (d\lambda_1)/(dt) &= -(\partial H)/(\partial x) \\ &= -[e^{-\delta t} p_1 q_1 E_1 + \lambda_1 (a_1 - 2a_{11}x + a_{12}y - q_1 E_1)] \end{aligned} \quad (7.9)$$

$$\begin{aligned} (d\lambda_2)/(dt) &= -(\partial H)/(\partial y) \\ &= -[\lambda_1 (a_{12}x) + \lambda_2 (a_2 - 2a_{22}y)] \end{aligned} \quad (7.10)$$

Now we seek to find the optimal equilibrium solution of the problem so that  $x, y$  and  $E_1$  can be treated as constants.

Eq. (7.10) can also be written as

$$(d\lambda_2)/(dt) + A_1\lambda_2 = -A_2e^{-\delta t} \tag{7.11}$$

where  $A_1 = a_2 - 2a_{22}y^*$ ;  $A_2 = a_{12}x^* (p_1 - (c_1)/(q_1x^*))$  and whose solution is given by

$$\lambda_2 = -A_2/(A_1 - \delta)e^{-\delta t} \tag{7.12}$$

Eq. (7.9) can also be written as

$$(d\lambda_1)/(dt) + A_3\lambda_1 = -A_4e^{-\delta t} \tag{7.13}$$

where  $A_3 = a_1 - 2a_{11}x^* + a_{12}y^* - q_1E_1$ ;  $A_4 = p_1q_1E_1$  and whose solution is given by

$$\lambda_1 = -A_4/(A_3 - \delta)e^{-\delta t} \tag{7.14}$$

From (7.8) and (7.14), we get the singular path

$$p_1 - (c_1)/(q_1x^*) = -A_4/(A_3 - \delta) \tag{7.15}$$

Using (3.3) and (3.4),  $A_i, i = 1, 2, 3, 4$  can also be written as follows:

$$A_1 = -a_2; \quad A_2 = \frac{p_1a_{12}}{a_{11}} \left[ (a_1 - q_1E_1) + \frac{a_2a_{12}}{a_{22}} \right] - \frac{c_1a_{12}}{q_1};$$

$$A_3 = -(a_1 - q_1E_1) - \frac{a_2a_{12}}{a_{22}}; \quad A_4 = p_1q_1E_1$$

Thus (7.15) can also be written as

$$F(x^*) = (p_1 - (c_1)/(q_1x^*)) + A_4/(A_3 - \delta) = 0 \tag{7.16}$$

There exists a unique positive root  $x^* = x_\delta$  of  $F(x^*) = 0$  in the interval  $0 < (x)_\infty < K$ , if the following inequalities hold.  $F(0) < 0; F(K) > 0; F'(x^*) > 0$ , for  $x^* > 0$ , where  $K = a_1/(a_{11})$ .

For  $x^* = x_\delta, y^* = y_\delta$ , we get  $(E_1)_\delta = 1/(q_1)[a_1 - a_{11}x_\delta + a_{12}y_\delta]$ .

Here,  $(E_1)_\delta > 0$  if  $a_1 + a_{12}y_\delta > a_{11}x_\delta$ .

From (7.12) and (7.14), we observe that  $\lambda_i e^{\delta t}, i = 1, 2$  is independent of time and is an optimum equilibrium. Hence they satisfy the transversality condition at  $\infty$ . That is, they remain bounded as  $t \rightarrow \infty$ . From (7.15), we also have  $p_1q_1x^* - c_1 = -A_4/(A_3 - \delta) \rightarrow 0$  as  $\delta \rightarrow \infty$ . Thus the net economic revenue  $R((x)_\infty, (y)_\infty, (E_1)_\infty) = 0$ . This implies that an infinite discount rate leads to the net economic revenue tending to zero and the fishery would remain closed.

**8. Stochastic model**

The foremost notion that leads us to broaden the deterministic model (2.1) and (2.2) to a stochastic matching part is that it is practical to imagine the open sea as deafening surrounding. There are a number of ways in which the located 'noise' may be included in the system (2.1) and (2.2). This reminds that the environmental noise should be discriminated from demographic (or) internal noise, for which the variation over time is due. External noise may arise either from random fluctuations of one or more model parameters around some known mean values or from stochastic fluctuations of the population densities around some constant values. In this segment, we compute the population intensities of fluctuations (variances) around the positive equilibrium  $E_4$  due to noise, according to the method introduced by Nisbet and Gurney [13] in 1982. A similar method was also successfully applied in Tapaswi and Mukhopadhyay [18] in 1999. Now we assume the presence of a randomly fluctuating driving force on the deterministic growth of the species  $S_1$  and  $S_2$  (commensal and host species) at time  $t$ , so that the system

(2.1) and (2.2) results in the stochastic system with additive noise as follows:

$$(dx)/(dt) = a_1x - a_{11}x^2 + a_{12}xy - q_1E_1x + \alpha_1\xi_1(t) \tag{8.1}$$

$$(dy)/(dt) = a_2y - a_{22}y^2 + \alpha_2\xi_2(t) \tag{8.2}$$

where  $x(t)$  stand for commensal species,  $y(t)$  stand for host species.  $\alpha_1, \alpha_2$  are real constants and  $\xi(t) = [\xi_1(t), \xi_2(t)]$  is a two dimensional Gaussian white noise process agreeable

$$E[\xi_i(t)] = 0; \quad i = 1, 2$$

$$E[\xi_i(t)\xi_j(t')] = \delta_{ij}\delta(t - t'); \quad i = j = 1, 2$$

where  $\delta_{ij}$  is the Kronecker symbol;  $\delta$  is the Dirac-delta function.

Let

$$x_1(t) = u_1(t) + S^*; \quad x_2(t) = u_2(t) + P^*;$$

$$\frac{dx_1}{dt} = \frac{du_1(t)}{dt}; \quad \frac{dx_2}{dt} = \frac{du_2(t)}{dt} \tag{8.3}$$

Using (8.3), Eq. (8.1) becomes

$$u_1'(t) = a_1u_1(t) + a_1S^* - a_{11}u_1^2(t) - a_{11}(S^*)^2$$

$$- 2a_{11}u_1(t)S^* + a_{12}u_1(t)u_2(t) + a_{12}u_1(t)P^*$$

$$+ a_{12}u_2(t)S^* + a_{12}S^*P^* - q_1E_1u_1(t) - q_1E_1S^*$$

$$+ \alpha_1\xi_1(t) \tag{8.4}$$

The linear part of (8.4) is

$$u_1'(t) = -a_{11}u_1(t)S^* + a_{12}u_2(t)S^* + \alpha_1\xi_1(t) \tag{8.5}$$

Using (8.3), Eq. (8.2) becomes

$$u_2'(t) = a_2u_2(t) + a_2P^* - a_{22}u_2^2(t) - a_{22}(P^*)^2$$

$$- 2a_{22}u_2(t)P^* + \alpha_2\xi_2(t) \tag{8.6}$$

The linear part of (8.6) is

$$u_2'(t) = -a_{22}u_2(t)P^* + \alpha_2\xi_2(t) \tag{8.7}$$

Taking the Fourier transform on both sides of (8.5) and (8.7) we get,

$$\alpha_1\tilde{\xi}_1(\omega) = (i\omega + a_{11}S^*)\tilde{u}_1(\omega) - a_{12}S^*\tilde{u}_2(\omega) \tag{8.8}$$

$$\alpha_2\tilde{\xi}_2(\omega) = (i\omega + a_{22}P^*)\tilde{u}_2(\omega) \tag{8.9}$$

The matrix form of (8.8) and (8.9) is

$$M(\omega)\tilde{u}(\omega) = \tilde{\xi}(\omega) \tag{8.10}$$

where

$$M(\omega) = \begin{pmatrix} A(\omega) & B(\omega) \\ C(\omega) & D(\omega) \end{pmatrix}; \quad \tilde{u}(\omega) = \begin{bmatrix} \tilde{u}_1(\omega) \\ \tilde{u}_2(\omega) \end{bmatrix};$$

$$\tilde{\xi}(\omega) = \begin{bmatrix} \alpha_1\tilde{\xi}_1(\omega) \\ \alpha_2\tilde{\xi}_2(\omega) \end{bmatrix};$$

$$\begin{aligned} A(\omega) &= i\omega + a_{11}S^*; & B(\omega) &= -a_{12}S^*; \\ C(\omega) &= 0; & D(\omega) &= i\omega + a_{22}P^* \end{aligned} \quad (8.11)$$

Eq. (8.10) can also be written as  $\tilde{u}(\omega) = [M(\omega)]^{-1} \tilde{\xi}(\omega)$

Let  $[M(\omega)]^{-1} = K(\omega)$ , then

$$\tilde{u}(\omega) = K(\omega) \tilde{\xi}(\omega) \quad (8.12)$$

where

$$K(\omega) = \begin{bmatrix} \frac{D(\omega)}{|M(\omega)|} & -\frac{B(\omega)}{|M(\omega)|} \\ -\frac{C(\omega)}{|M(\omega)|} & \frac{A(\omega)}{|M(\omega)|} \end{bmatrix} \quad (8.13)$$

If the function  $Y(t)$  has a zero mean value, then the fluctuation intensity (variance) of its components in the frequency interval  $[\omega, \omega + d\omega]$  is  $S_Y(\omega)d\omega$ , where  $S_Y(\omega)$  is spectral density of  $Y$  and is defined as

$$S_Y(\omega) = \lim_{T \rightarrow \infty} \frac{|\tilde{Y}(\omega)|^2}{T} \quad (8.14)$$

If  $Y$  has a zero mean value, the inverse transform of  $S_Y(\omega)$  is the auto covariance function

$$C_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) e^{i\omega\tau} d\omega \quad (8.15)$$

The related variance of fluctuations in  $Y(t)$  is given by

$$\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega) d\omega \quad (8.16)$$

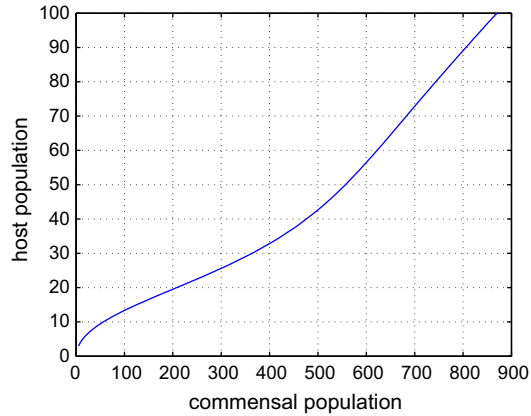
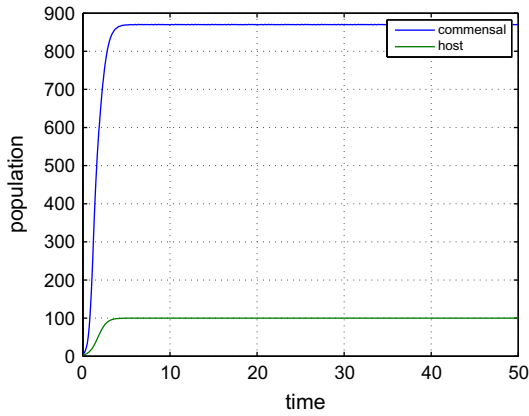
and the auto correlation function is the normalized auto covariance

$$P_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)} \quad (8.17)$$

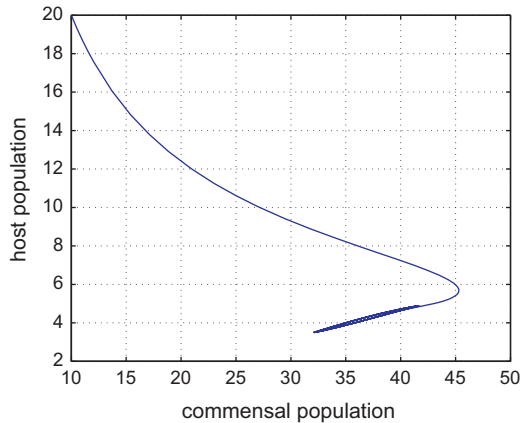
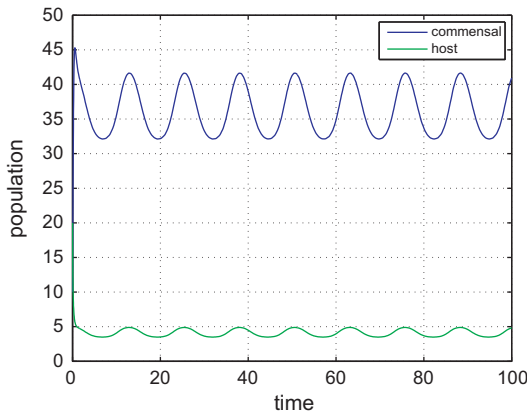
For a Gaussian white noise process, it is

$$\begin{aligned} S_{\xi_i, \xi_j}(\omega) &= \lim_{T \rightarrow +\infty} \frac{E[\tilde{\xi}_i(\omega) \tilde{\xi}_j(\omega)]}{T} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} E[\tilde{\xi}_i(t) \tilde{\xi}_j(t')] e^{-i\omega(t-t')} dt dt' = \delta_{ij} \end{aligned} \quad (8.18)$$

From (8.12), we have



**Figure 1** The variation of population against time initially with  $x = 5; y = 3$  and the variation between commensal and host populations for the parameters  $a_1 = 4; a_{11} = 0.01; a_{12} = 0.05; q_1 = 0.02; E_1 = 15; a_2 = 2; a_{22} = 0.02$ .



**Figure 2** The variation of population against time initially with  $x = 10; y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 1.5; a_{11} = 0.1; a_{12} = 0.5; q_1 = 0.02; E_1 = 5; a_2 = 1.5; a_{22} = 0.5$ .

$$\tilde{u}_i(\omega) = \sum_{j=1}^2 K_{ij}(\omega) \tilde{\xi}_j(\omega); \quad i = 1, 2 \tag{8.19}$$

From (8.14) we have

$$S_{u_i}(\omega) = \sum_{j=1}^2 \alpha_j |K_{ij}(\omega)|^2; \quad i = 1, 2 \tag{8.20}$$

Hence by (8.16) and (8.20), the intensities of fluctuations in the variable  $u_i$ ;  $i = 1, 2$  are given by

$$\sigma_{u_i}^2 = \frac{1}{2\pi} \sum_{j=1}^2 \int_{-\infty}^{\infty} \alpha_j |K_{ij}(\omega)|^2 d\omega; \quad i = 1, 2 \tag{8.21}$$

and by (8.13), we obtain

$$\sigma_{u_1}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \alpha_1 \left| \frac{D(\omega)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left| \frac{B(\omega)}{|M(\omega)|} \right|^2 d\omega \right\}$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \alpha_1 \left| \frac{A(\omega)}{|M(\omega)|} \right|^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \left| \frac{C(\omega)}{|M(\omega)|} \right|^2 d\omega \right\} \tag{8.22}$$

where

$$|M(\omega)| = R(\omega) + iI(\omega) \tag{8.23}$$

$$R(\omega) = -\omega^2 + \alpha_{11}\alpha_{22}S^*P^* \tag{8.24}$$

$$I(\omega) = \omega(\alpha_{11}S^* + \alpha_{22}P^*) \tag{8.25}$$

Finally from (8.11), we get

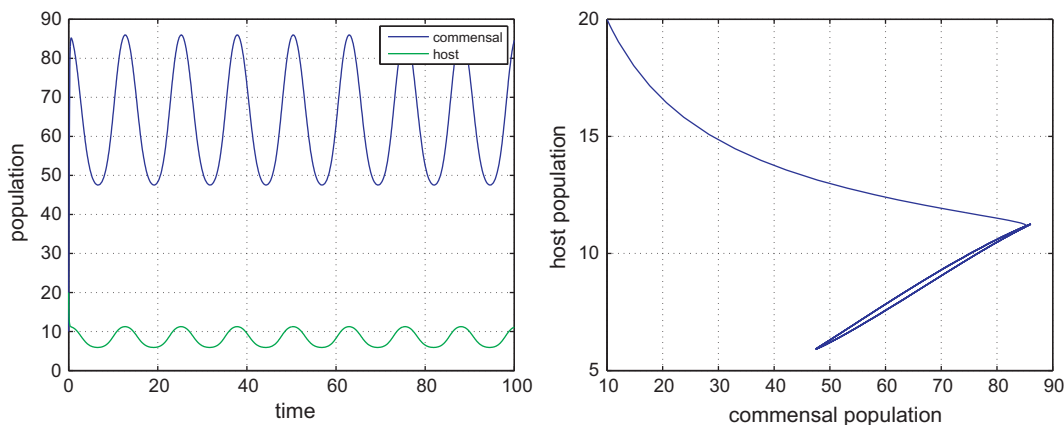
$$|A(\omega)|^2 = \omega^2 + (\alpha_{11}S^*)^2; \quad |B(\omega)|^2 = (\alpha_{12}S^*)^2;$$

$$|C(\omega)|^2 = 0; \quad |D(\omega)|^2 = \omega^2 + (\alpha_{22}P^*)^2 \tag{8.26}$$

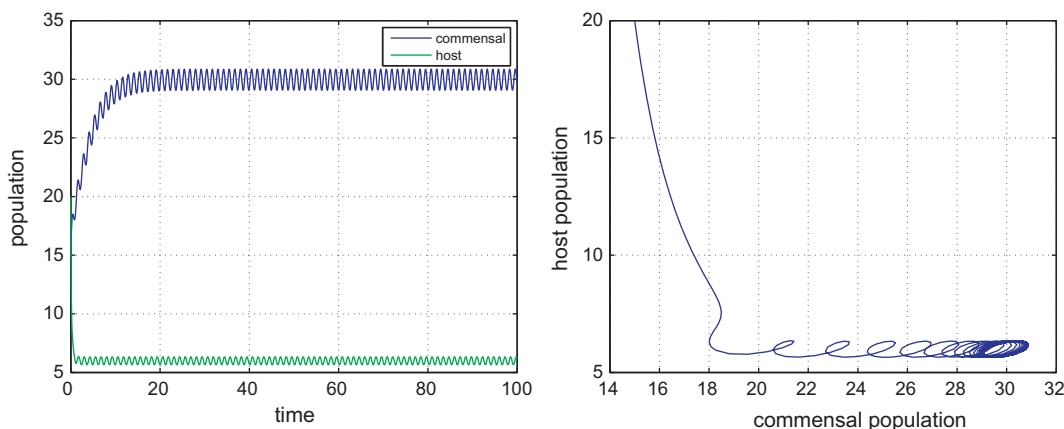
By substitution of (8.23) and (8.11) in (8.22), we get,

$$\sigma_{u_1}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[ \alpha_1 \left\{ \omega^2 + (\alpha_{22}P^*)^2 \right\} + \alpha_2 (\alpha_{12}S^*)^2 \right] d\omega \right\} \tag{8.27}$$

$$\sigma_{u_2}^2 = \frac{1}{2\pi} \left\{ \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[ \alpha_1 \left\{ \omega^2 + (\alpha_{11}S^*)^2 \right\} \right] d\omega \right\} \tag{8.28}$$

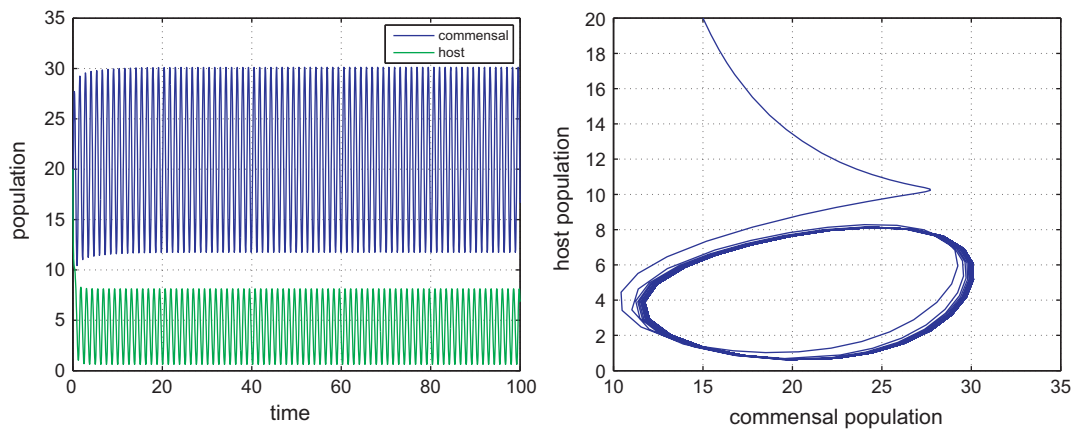


**Figure 3** The variation of population against time initially with  $x = 10$ ;  $y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 1.5$ ;  $a_{11} = 0.1$ ;  $a_{12} = 0.5$ ;  $q_1 = 0.02$ ;  $E_1 = 5$ ;  $a_2 = 1.5$ ;  $a_{22} = 0.5$ .

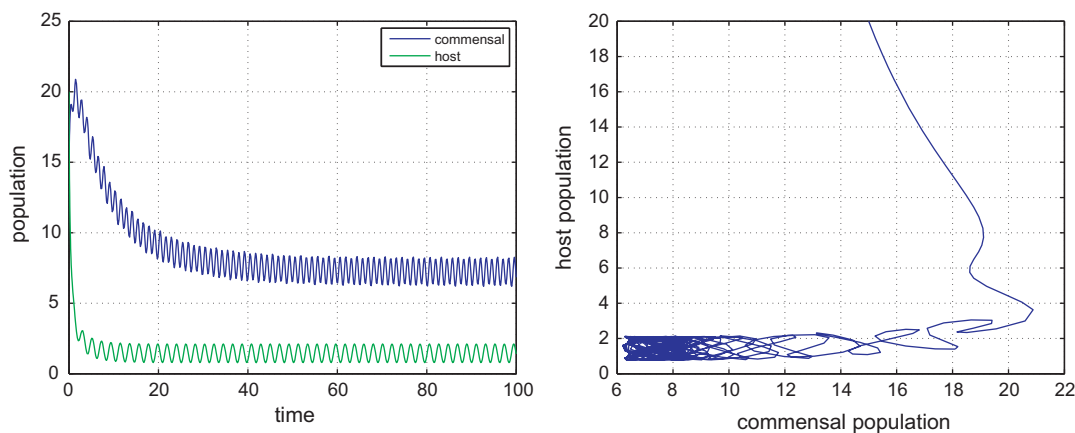


**Figure 4** The variation of population against time initially with  $x = 15$ ;  $y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 3$ ;  $a_{11} = 0.01$ ;  $a_{12} = 0.05$ ;  $q_1 = 0.2$ ;  $E_1 = 15$ ;  $a_2 = 3$ ;  $a_{22} = 0.5$ .

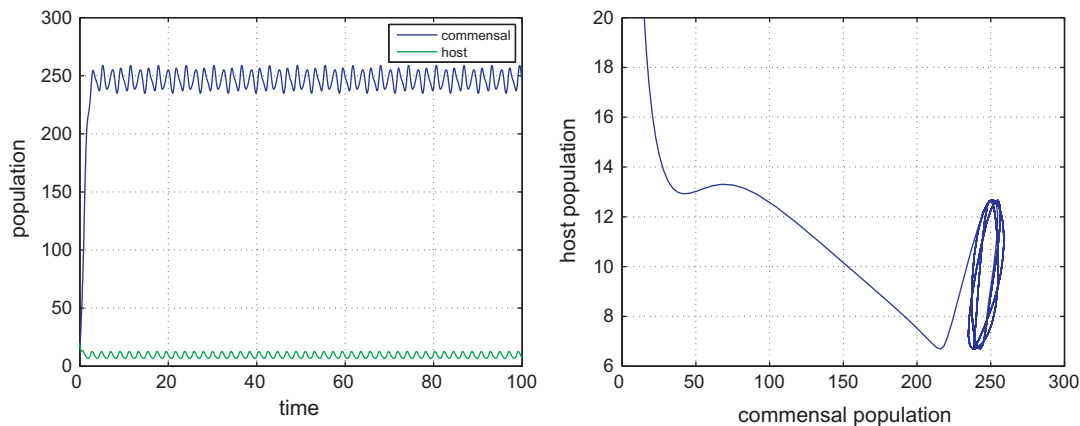




**Figure 5** The variation of population against time initially with  $x = 15; y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 3; a_{11} = 0.01; a_{12} = 0.05; q_1 = 0.2; E_1 = 15; a_2 = 3; a_{22} = 0.5$ .



**Figure 6** The variation of population against time initially with  $x = 15; y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 3; a_{11} = 0.01; a_{12} = 0.05; q_1 = 0.2; E_1 = 15; a_2 = 0.32; a_{22} = 0.2$ .



**Figure 7** The variation of population against time initially with  $x = 15; y = 20$  and the variation between commensal and host populations for the parameters  $a_1 = 3; a_{11} = 0.01; a_{12} = 0.05; q_1 = 0.2; E_1 = 5; a_2 = 0.32; a_{22} = 0.2$ .

If we are interested in the dynamics of system (8.1) and (8.2) with either  $\alpha_1 = 0$  or  $\alpha_2 = 0$ .

If  $\alpha_1 = 0$ , then

$$\sigma_{u_1}^2 = \frac{\alpha_2 (a_{12} S^*)^2}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + F^2(\omega)} d\omega \tag{8.29}$$

$$\sigma_{u_2}^2 = 0 \tag{8.30}$$

If  $\alpha_2 = 0$ , then

$$\sigma_{u_1}^2 = \frac{\alpha_1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[ \omega^2 + (a_{22}P^*)^2 \right] d\omega \quad (8.31)$$

$$\sigma_{u_2}^2 = \frac{\alpha_1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} \left[ \omega^2 + (\alpha_{11}S^*)^2 \right] d\omega \quad (8.32)$$

The expression in (8.22) gives two variances of the populations. The mergers over the real line can be judged which gives the variances of the populations.

## 9. Numerical simulations

In this section, we assign numerical values to the parameters of the model system (2.1) and (2.2) and compute some simulations using those values. For the purpose of simulation experiments we mainly used the software MATLAB (7.2).

## 10. Conclusion

In this paper, a model of a distinctive two species syn eco-system with a stochastic term was invented. Initially the model was discussed without the stochastic term. The survival of equilibrium points is discussed. The local stability by using Routh–Hurwitz criteria and conniving global stability using Lyapunov function are computed and analyzed. The idea of bionomic equilibrium and optimal harvesting strategy through Pontryagin's maximum principle are computed. Later, we introduced the stochastic term into the model and the population intensities of fluctuations (variances) around the positive equilibrium due to 'noise' are computed and analyzed for stability.

The numerical results agree with the analytical results of two species eco-system model and this shows that the deterministic two species eco-system model is stable. The stable nature of the interior equilibrium point (870, 100) is revealed in Fig. 1, where the trajectory of the system for the chosen parameters is converging to the interior equilibrium point. From the Figs. 2 and 3, it is clear that the trajectories of the model oscillate due to addition of noise and the oscillations are increases as increase in the amplitude of noise  $\alpha$ . Figs. 4 and 5 exhibit the periodic time series evolution of populations and the limit cycles of phase-portraits confirm the periodic behavior of the system due to increase in noise effect. It is also observed that when we control the effect of random noise by decreasing  $\alpha$ , the mean square fluctuations of populations are reduced and the stochastic stability with the decreased intensity is shown in Figs. 6 and 7. Hence we conclude that insertion of stochastic perturbation creates a momentous change in the intensity of our dynamical system due to amendment of the receptive parameters, which cause huge environmental fluctuations leading to chaos in realism.

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