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Bivariant cyclic cohomology and models for cyclic homology types

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Abstract

This paper concerns types of algebraic objects, such as mixed complexes and S-modules, which are used to obtain the homology and cohomology of interest in cyclic homology theory. We prove that the following five categories are equivalent: The derived category of mixed complexes. The homotopy category of free mixed complexes. The derived category of S-modules. The homotopy category of divisible S-modules. The homotopy category of special towers of supercomplexes. Thus any of these categories represents the category of cyclic homotopy types.

Introduction

This paper is concerned with various types of complexes one can use to obtain the homology and cohomology of interest in cyclic homology theory. The most familiar type employed for this purpose is a mixed complex [11]. A mixed complex gives rise first of all to Hochschild and cyclic homology linked by a Connes exact sequence, and secondly to negative cyclic and periodic cyclic homology. In addition there is bivariant cyclic cohomology [9] associated to a pair of mixed complexes. All of this homology and cohomology is invariant with respect to quasi-isomorphism, so that quasi-isomorphic mixed complexes have the same cyclic homology type, that is, they are equivalent from the viewpoint of cyclic homology theory.

The idea of cyclic homology type can be made precise by introducing the derived category of mixed complexes. Concretely this is the category consisting of mixed complexes in which a map from M to N is an element of the bivariant cyclic cohomology group $HC^0(M, N)$ of Jones and Kassel, and where composition is given by cup product. It can also be described abstractly as the category one obtains from the category of

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mixed complexes by formally adjoining inverses for the quasi-isomorphisms. We may thus interpret the derived category as the category of cyclic homology types.

Besides mixed complexes we also consider S-modules, which play an important role in the construction of bivariant cyclic cohomology [9], and special towers of supercomplexes, which arise when one studies cyclic homology for an algebra A with the aid of an extension $A = R/I$ where R is quasi-free [4]. These three types of objects are linked by functors known to be compatible with the cyclic type homology and cohomology we have mentioned [4]. Our primary aim in this paper is to show that they yield equivalent descriptions of cyclic homology types. We prove that the following five categories are equivalent:

- The derived category of mixed complexes.
- The homotopy category of free mixed complexes.
- The derived category of S-modules.
- The homotopy category of divisible S-modules.
- The homotopy category of special towers of supercomplexes.

Thus any of these categories represents the category of cyclic homology types.

In the first section we prove the equivalence of the first four categories in the above list with the aid of the Jones-Kassel theory [9]. Because we work over a field we are able to simplify their theory at certain points, for example, the key results on quasi-isomorphism invariance are derived under the assumptions of freeness and divisibility instead of the more technical hypotheses of their paper. Of the four categories the nicest for proving results about bivariant cyclic cohomology is the homotopy category of divisible S-modules, and this is discussed in Section 2.

The next two sections concern towers of supercomplexes. Section 3 contains general results about such towers, in particular the fact that any tower splits into a minimal tower and a contractible tower. Then special towers are introduced in Section 4 and shown to give rise to the kinds of homology and cohomology of interest in cyclic homology theory.

In Section 5 we complete the proof that all five categories above are equivalent by applying our results concerning minimal towers to construct an equivalence between the homotopy categories of divisible S-modules and special towers. Finally, this equivalence is illustrated in Section 6, where we examine the special tower $\mathcal{X}(R, I)$ for R quasi-free studied in [4] and explicitly construct divisible S-modules corresponding to it.

Throughout this paper we work over the complex numbers $\mathbb{C}$, however, it is easily seen that the first five sections are valid over an arbitrary groundfield and the last section holds for any groundfield of characteristic zero.

1. Derived categories of mixed complexes and S-modules

In this section we study the derived categories relevant to the bivariant cyclic cohomology of Jones and Kassel [9].

We consider $\mathbb{Z}$-graded complexes of vector spaces $C = \bigoplus C_n$ with differential d of degree -1. We say C is bounded below when $C_n = 0$ for $n \ll 0$. Let $C[k]$, the kth suspension of C, be the complex given by

$$C[k]_n = C_{n-k} \tag{1.1}$$

with d on $C[k]$ given by $(-1)^k d$ on C. If C, C' are complexes, let $\mathrm{Hom}(C, C')$ be the mapping complex such that $\mathrm{Hom}(C, C')_n$ is the space of linear operators $f : C \to C'$ of degree n, where the differential is

$$[d, f] = d \cdot f - (-1)^{|f|} f \cdot d. \tag{1.2}$$

We usually shift to upper indexing for the mapping complex:

$$\mathrm{Hom}^k(C, C') = \mathrm{Hom}(C, C')_{-k}.$$

We recall that a *mixed complex* [11] is a complex, where the differential is traditionally denoted b, which is equipped with an operator B of degree $+1$ such that $[b, B] = B^2 = 0$. A mixed complex is the same as a DG module over the DG algebra $\Lambda = \mathbb{C} \oplus \mathbb{C}B$, where B has degree one and boundary zero.

We recall also that an *S-module* [12] is a complex equipped with an operator S of degree -2 commuting with the differential. An S-module is the same as a DG module over the polynomial algebra $\mathbb{C}[S]$ regarded as a DG algebra in which the differential is zero and S has degree -2.

All mixed complexes and S-modules will be assumed bounded below unless stated otherwise. Let C_Λ be the category of mixed complexes, where maps are degree zero operators commuting with b, B. Similarly let C_S be the category of S-modules, where maps are degree zero operators commuting with d, S. We shall use the letters M, N, possibly with primes, to denote mixed complexes, and P, Q for S-modules.

Mixed complexes and S-modules are related by canonical adjoint functors which can be understood in the following manner [9, Section 1]. On one hand, there are in general canonical adjoint functors linking DG modules over a DG algebra with DG comodules over its bar construction [8]. On the other hand, $\mathbb{C}[S]$ is the dual of the bar construction $B(\Lambda)$ of Λ, so that a DG comodule over $B(\Lambda)$ gives rise to a DG module over $\mathbb{C}[S]$. In fact, one obtains in this way an equivalence between DG comodules over $B(\Lambda)$ and those DG modules over $\mathbb{C}[S]$ which are *S-torsion*, i.e. any element is killed by some power of S. This condition is automatically satisfied when the comodule is bounded below, so DG comodules over $B(\Lambda)$ which are bounded below are the same as S-modules.

These adjoint functors can be described concretely as follows.

For a mixed complex M let $\mathcal{B}M$ be the S-module given by the total complex of the b, B bicomplex of M in the right half plane $p \geq 0$, i.e.

$$(\mathcal{B}M)_n = \bigoplus_{p \geq 0} M_{n-2p},$$

where $d = b + B$ and S is the obvious projection killing the summand for $p = 0$. For an S-module P let $\Lambda \otimes P$ be the mixed complex given by the tensor product of Λ

and P equipped with the twisted differential $b = 1 \otimes d - B \otimes S$ and the obvious left multiplication by $\mathcal{B}$.

Let $\mathrm{Hom}_\Lambda(M, N)$ be the subcomplex of $\mathrm{Hom}(M, N)$ consisting of operators f such that $[B, f] = 0$, and similarly let $\mathrm{Hom}_S(P, Q)$ be the subcomplex of $\mathrm{Hom}(P, Q)$ consisting of f such that $[S, f] = 0$.

We then have the following adjunction formula [9, 1.2].

Proposition 1.1. *There is a canonical isomorphism of complexes*

$$\mathrm{Hom}_\Lambda(\Lambda \otimes P, M) = \mathrm{Hom}_S(P, \mathcal{B}M). \qquad \square$$

Notice that $Z^0 \mathrm{Hom}_\Lambda(M, N)$, where Z^0 designates the subspace of cocycles of degree zero, is the vector space of maps $M \to N$ in $\mathcal{C}_\Lambda$, and similarly for $\mathcal{C}_S$. Applying Z^0 to both sides of this adjunction formula, we conclude that $\Lambda \otimes -$, $\mathcal{B}$ are adjoint functors between the categories $\mathcal{C}_\Lambda$ and $\mathcal{C}_S$.

As a consequence there are canonical adjunction maps

$$\varepsilon_M : \Lambda \otimes \mathcal{B}M \to M, \qquad \eta_P : P \to \mathcal{B}(\Lambda \otimes P). \tag{1.3}$$

One refers to $\Lambda \otimes \mathcal{B}M$ and $\mathcal{B}(\Lambda \otimes P)$ as the *bar resolutions* of M and P, the term "resolution" being justified by

Proposition 1.2. *These adjunction maps are quasi-isomorphisms, i.e. they induce isomorphisms on homology.*

This follows from the fact that the maps of underlying complexes are homotopy equivalences in a canonical way [8]. On the other hand, it will be useful to present a proof tailored to the present situation. We first need some definitions.

Let us call a mixed complex *free* when its homology with respect to the differential B is zero. It is easily seen that this property is equivalent to being free as a Λ-module. An S-module P will be called *divisible* when the operator S on P is surjective. Since P is S-torsion by the bounded below assumption and $\mathbb{C}[S]$ is a principal ideal domain, divisible is equivalent to being injective as a module over $\mathbb{C}[S]$.

For example, $\Lambda \otimes P$ is always free and $\mathcal{B}M$ is always divisible. Thus the bar resolutions are functorial free and divisible resolutions for mixed complexes and S-modules respectively.

Lemma 1.3. (i) *If M is free, there is a canonical quasi-isomorphism $\mathcal{B}M \to M/BM$.*

(ii) *If P is divisible, there is a canonical quasi-isomorphism ${}_SP \to \Lambda \otimes P$, where ${}_SP$ is the kernel of the operator S on P.*

Proof. Since $\mathcal{B}M$ is the total complex of b, B bicomplex of M in the half plane $p \geq 0$, there is a canonical edge homomorphism from $\mathcal{B}M$ to the complex M/BM of horizontal homology along the column $p = 0$. When B is exact, this edge homomorphism is a quasi-isomorphism, proving (i).

In general, if we ignore the Λ-module structure, then the complex $\Lambda \otimes P$ can be interpreted as the homotopy "fibre" or kernel of $S : P \to P[2]$. Thus there is a canonical map from the actual fibre ${}_SP$ to $\Lambda \otimes P$. When S is surjective, it follows by comparing long exact sequences on homology that this map is a quasi-isomorphism, proving (ii). □

We now prove Proposition 1.2. If P is any S-module, we apply (i) of the lemma to the free mixed complex $M = \Lambda \otimes P$ to obtain the quasi-isomorphism q in

$$P \xrightarrow{\eta_P} \mathcal{B}(\Lambda \otimes P) = \mathcal{B}M \xrightarrow{q} M/BM = P.$$

One can check by means of the explicit description [9, 1.2] of the adjunction formula, Proposition 1.1, that this composition is the identity. Thus η_P is a quasi-isomorphism.

Similarly, if M is any mixed complex, we apply (ii) of the lemma to the divisible S-module $P = \mathcal{B}M$ to obtain the quasi-isomorphism q in

$$M = {}_SP \xrightarrow{q} \Lambda \otimes P = \Lambda \otimes \mathcal{B}M \xrightarrow{\varepsilon_M} M.$$

Again the composition is the identity, so ε_M is a quasi-isomorphism. □

Let $\mathcal{C}_\Lambda^{\mathrm{f}}$, $\mathcal{C}_S^{\mathrm{d}}$ denote the full subcategories of $\mathcal{C}_\Lambda$, $\mathcal{C}_S$ consisting of free mixed complexes and divisible S-modules respectively. Then we have a circular diagram of four categories and functors

$$\begin{array}{ccc} \mathcal{C}_\Lambda^{\mathrm{f}} & \subset & \mathcal{C}_\Lambda \\ {\scriptstyle \Lambda\otimes -}\uparrow & & \downarrow{\scriptstyle \mathcal{B}} \\ \mathcal{C}_S & \supset & \mathcal{C}_S^{\mathrm{d}} \end{array} \tag{1.4}$$

In order to get some feeling for the significance of these categories, let us consider Hochschild and cyclic homology. We recall that any mixed complex M gives rise to Hochschild and cyclic homology groups defined respectively by

$$HH_nM = H_nM, \qquad HC_nM = H_n(\mathcal{B}M). \tag{1.5}$$

Hochschild homology and cyclic homology are functors from $\mathcal{C}_\Lambda$ to vector spaces, which are related by the Connes exact sequence. We can now compose with the functors in (1.4) to obtain Hochschild and cyclic homology functors on the other categories. We observe that by Lemma 1.3(i), if cyclic homology is restricted to free mixed complexes, then it can be computed using the smaller complex M/BM instead of $\mathcal{B}M$. Taking advantage of such simplifications, we obtain the following table of formulas for Hochschild and cyclic homology which are adapted to the four categories.

$$\begin{array}{lcccc} & \mathcal{C}_\Lambda & \mathcal{C}_\Lambda^{\mathrm{f}} & \mathcal{C}_S & \mathcal{C}_S^{\mathrm{d}} \\ HH_n: & H_nM & H_nM & H_n(\Lambda\otimes P) & H_n({}_SP) \\ HC_n: & H_n(\mathcal{B}M) & H_n(M/BM) & H_nP & H_nP \end{array} \tag{1.6}$$

Let us now consider the Connes exact sequence linking Hochschild and cyclic homology. On $\mathcal{C}_\Lambda$ this is the homology long exact sequence arising from the short exact sequence of complexes

$$0 \to M \to \mathcal{B}M \xrightarrow{S} \mathcal{B}M[2] \to 0. \tag{1.7}$$

If we restrict to free mixed complexes and replace $\mathcal{B}M$ by M/BM, then (1.7) is replaced by

$$0 \to (M/BM)[1] \xrightarrow{B} M \to M/BM \to 0 \tag{1.8}$$

in the sense that (1.7) and (1.8) give rise to triangles in the homotopy category of complexes which are canonically isomorphic up to a shift. Next, (1.8) in the case $M = \Lambda \otimes P$ for any S-module P is

$$0 \to P[1] \to \Lambda \otimes P \to P \to 0. \tag{1.9}$$

If we restrict to divisible S-modules and replace $\Lambda \otimes P$ by ${}_SP$, then (1.9) is replaced by

$$0 \to {}_SP \to P \to P[2] \to 0. \tag{1.10}$$

Finally (1.10) yields (1.7) in the case $P = \mathcal{B}M$.

We thus conclude that the four exact sequences above are the appropriate means of obtaining the Connes exact sequence when using the formulas (1.6).

We next derive properties of free mixed complexes and divisible S-modules with respect to quasi-isomorphisms, which imply the uniqueness up to homotopy of free and divisible resolutions.

Lemma 1.4. *If M is free, then $\operatorname{Hom}_\Lambda(M,-)$ respects quasi-isomorphisms, i.e. for any quasi-isomorphism $N \to N'$ of mixed complexes the induced map $\operatorname{Hom}_\Lambda(M,N) \to \operatorname{Hom}_\Lambda(M,N')$ is a quasi-isomorphism.*

Proof. By introducing the mapping cone of $N \to N'$ we reduce to showing that if M free and N is acyclic, then $\operatorname{Hom}_\Lambda(M,N)$ is acyclic. Let F^nN be the subcomplex of N which is equal to N in degrees $> n$, is the space of n-cycles in degree n, and is zero in degrees $< n$. This defines a decreasing filtration of N by mixed subcomplexes such that each quotient $F^nN/F^{n+1}N$ is elementary, i.e. supported in two consecutive degrees with the b differential between them an isomorphism. Such a mixed complex has $B = 0$, and it is contractible, hence $\operatorname{Hom}_\Lambda(M, F^nN/F^{n+1}N)$ is acyclic for all n. As M is free the functor $\operatorname{Hom}_\Lambda(M,-)$ is exact, so using the fact that N is bounded below we can argue by induction that $\operatorname{Hom}_\Lambda(M, N/F^nN)$ is acyclic for all n. As this inverse system of complexes has surjective arrows, it follows from the Milnor exact sequence that $\operatorname{Hom}_\Lambda(M,N)$ is acyclic. $\square$

Lemma 1.5. *If Q is divisible then $\operatorname{Hom}_S(-,Q)$ respects quasi-isomorphisms.*

We shall deduce this from

Lemma 1.6. *If Q is divisible, then for any S-module P we have an exact sequence*

$$0 \to \mathrm{Hom}_S(P,Q) \xrightarrow{i} \mathrm{Hom}(P,Q) \xrightarrow{\mathrm{ad}(S)} \mathrm{Hom}(P,Q)[2] \to 0, \qquad (1.11)$$

where i is the inclusion and $\mathrm{ad}(S)f = Sf - fS$.

Proof. This sequence is clearly left exact, so only the surjectivity of $\mathrm{ad}(S)$ is at issue. Let us consider this sequence as a functor $E(P)$ of P. Note that $E(P)$ is exact when $S = 0$ on P, for then $\mathrm{ad}(S)f = Sf$ is surjective, as Q is assumed divisible. Let $P_{\le n}$ denote the S-submodule of P which agrees with P in degrees $\le n$ and is zero in degrees $> n$. As $S = 0$ on $P_{\le n}/P_{\le n-1}$, an application of the serpent lemma shows that if $E(P_{\le n-1})$ is exact then $E(P_{\le n})$ is exact and $E(P_{\le n}) \to E(P_{\le n-1})$ is surjective. Thus $E(P_{\le n})$ is exact for all n by induction using the fact that P is bounded below. We then have a short exact sequence of inverse systems with surjective arrows, so the inverse limit $E(P)$ is exact. □

Lemma 1.6 says that if Q is divisible, then $\underline{\mathrm{Ext}}_S(P,Q) = 0$ for all P in the notation of [9, 4.1].

The proof of Lemma 1.5 now proceeds as in [9, 4.5]. Namely, by virtue of the exact sequence it suffices to show that $\mathrm{Hom}(-,Q)$ respects quasi-isomorphisms. Since we work over a field, a quasi-isomorphism $P \to P'$ is a homotopy equivalence when the S operator is ignored, so this is clear. □

We turn next to the homotopy relation for maps of mixed complexes and S-modules and the corresponding homotopy categories.

We recall that a map $f : M \to N$ of mixed complexes is a 0-cocycle in $\mathrm{Hom}_\Lambda(M,N)$. We say f is homotopic to zero, when it is a 0-coboundary in this mapping complex, i.e. when there is an operator $h : M \to N$ of degree one such that $f = [b,h]$, $[B,h] = 0$. If we set

$$[M,N] = H^0(\mathrm{Hom}_\Lambda(M,N)),$$

then $[M,N]$ is the vector space of homotopy classes of maps $M \to N$. Let $\mathrm{Ho}\,\mathcal{C}_\Lambda$ denote the *homotopy category of mixed complexes* in which $[M,N]$ is the set of morphisms from M to N.

In parallel fashion we put

$$[P,Q] = H^0(\mathrm{Hom}_S(P,Q))$$

and define the homotopy category of S-modules $\mathrm{Ho}\,\mathcal{C}_S$. Let $\mathrm{Ho}\,\mathcal{C}_\Lambda^{\mathrm{f}}$ and $\mathrm{Ho}\,\mathcal{C}_S^{\mathrm{d}}$ be the homotopy categories of free mixed complexes and divisible S-modules; these are full subcategories of $\mathrm{Ho}\,\mathcal{C}_\Lambda$ and $\mathrm{Ho}\,\mathcal{C}_S$ respectively.

Applying H^0 to the adjunction formula, Proposition 1.1 yields

$$[\Lambda \otimes P, M] = [P, \mathcal{B}M]. \qquad (1.12)$$

This implies that $\Lambda \otimes -$ and $\mathcal{B}$ induce adjoint functors between $\mathrm{Ho}\,\mathcal{C}_\Lambda$ and $\mathrm{Ho}\,\mathcal{C}_S$.

Applying H^0 to Lemmas 1.4 and 1.5 we get

Proposition 1.7. (i) *If M is free and $N \to N'$ is a quasi-isomorphism, then*

$$[M,N] \xrightarrow{\sim} [M,N'].$$

(ii) *If Q is divisible and $P \to P'$ is a quasi-isomorphism, then*

$$[P',Q] \xrightarrow{\sim} [P,Q]. \qquad \square$$

We now deduce corollaries of this result, giving the arguments based on (i) and omitting similar arguments based on (ii).

Note that when N, N' are free, (i) implies formally that $N \to N'$ becomes an isomorphism in the homotopy category of free mixed complexes. Thus we have

Corollary 1.8. *A quasi-isomorphism between free mixed complexes (resp. divisible S-modules) is a homotopy equivalence.* □

Since the adjunction maps ε_M, η_P are always quasi-isomorphisms by Proposition 1.2, we deduce at once

Corollary 1.9. *If M is free (resp. P is divisible), then ε_M (resp. η_P) is a homotopy equivalence. Consequently $\mathcal{B}$ and $\Lambda \otimes -$ provide an equivalence between the categories* $\mathrm{Ho}\,\mathcal{C}_\Lambda^{\mathrm{f}}$ *and* $\mathrm{Ho}\,\mathcal{C}_S^{\mathrm{d}}$. □

Consider next the adjunction formula (1.12), and note that since $\Lambda \otimes P$ is free, (i) implies that the functor $[\Lambda \otimes P, -]$ sends quasi-isomorphisms to isomorphisms. Hence the functor $[P, \mathcal{B}(-)]$ sends quasi-isomorphisms to isomorphisms for any S-module P, yielding

Corollary 1.10. *The functors $\mathcal{B}$ and $\Lambda \otimes -$ send quasi-isomorphisms into homotopy equivalences.* □

This also follows from Corollary 1.8 and the fact that these functors respect quasi-isomorphisms.

Next recall from the theory of adjoint functors that the composition

$$[M,N] \xrightarrow{\varepsilon_M^*} [\Lambda \otimes \mathcal{B}M, N] = [\mathcal{B}M, \mathcal{B}N]$$

is the effect of the functor $\mathcal{B}$ on maps. Since ε_M is a homotopy equivalence for M free, we obtain

Corollary 1.11. *If M is free then $[M,N] \xrightarrow{\sim} [\mathcal{B}M, \mathcal{B}N]$, and if Q is divisible, then* $[P,Q] \xrightarrow{\sim} [\Lambda \otimes P, \Lambda \otimes Q]$. □

We are now ready to discuss the derived categories of mixed complexes and S-modules.

We define the *bivariant cyclic cohomology* of degree zero for mixed complexes by

$$HC^0(M,N) = [\mathcal{B}M, \mathcal{B}N] \tag{1.13}$$

and observe that for any triple of mixed complexes there is a cup (or Yoneda) product

$$HC^0(M',M'') \otimes HC^0(M,M') \to HC^0(M,M'')$$

given by composition in $\operatorname{Ho}\mathcal{C}_S^{\mathrm{d}}$. By [9, 2.2 and 5.1] this definition of HC^0 and its cup product is equivalent to the one given by Jones and Kassel.

We now define the *derived category of mixed complexes* $D\mathcal{C}_\Lambda$ to be the category with mixed complexes as objects, in which a map from M to N is an element of $HC^0(M,N)$, where composition is given by cup product.

We then have a functor

$$j_\Lambda : \operatorname{Ho}\mathcal{C}_\Lambda \to D\mathcal{C}_\Lambda$$

such that $j_\Lambda M = M$ for all M and such that on maps

$$j_\Lambda : [M,N] \to HC^0(M,N)$$

is the effect of the functor $\mathcal{B}$. We also have a functor

$$k_\Lambda : D\mathcal{C}_\Lambda \to \operatorname{Ho}\mathcal{C}_S^{\mathrm{d}}$$

sending M to $\mathcal{B}M$, which by the definition of maps in the derived category is fully faithful. The composition of these functors is $\mathcal{B}$. We can thus identify

$$\mathcal{B} : \operatorname{Ho}\mathcal{C}_\Lambda \xrightarrow{j_\Lambda} D\mathcal{C}_\Lambda \xrightarrow{k_\Lambda} \operatorname{Ho}\mathcal{C}_S^{\mathrm{d}} \tag{1.14}$$

with the canonical factorization of $\mathcal{B}$ into a functor which is bijective on objects followed by a fully faithful functor.

Similarly we put

$$HC^0(P,Q) = [\Lambda \otimes P, \Lambda \otimes Q] \tag{1.15}$$

and define the derived category of S-modules $D\mathcal{C}_S$ so that we have an analogous factorization

$$\Lambda \otimes - : \operatorname{Ho}\mathcal{C}_S \xrightarrow{j_S} D\mathcal{C}_S \xrightarrow{k_S} \operatorname{Ho}\mathcal{C}_S^{\mathrm{d}}. \tag{1.16}$$

Let j_Λ^{f} denote the restriction of j_Λ to $\operatorname{Ho}\mathcal{C}_\Lambda^{\mathrm{f}}$ and define j_S^{d} similarly.

Proposition 1.12. *There are equivalences of categories*

$$\begin{array}{ccc} \operatorname{Ho}\mathcal{C}_\Lambda^{\mathrm{f}} & \xrightarrow{j_\Lambda^{\mathrm{f}}} & D\mathcal{C}_\Lambda \\ {\scriptstyle k_S}\uparrow & & \downarrow{\scriptstyle k_\Lambda} \\ D\mathcal{C}_S & \xleftarrow{j_S^{\mathrm{d}}} & \operatorname{Ho}\mathcal{C}_S^{\mathrm{d}} \end{array} \tag{1.17}$$

Indeed, the composite functors $k_A j_A^{\mathrm{f}}$, $k_S j_S^{\mathrm{d}}$ are equivalences by Corollary 1.9, and the functors k_A, k_S are fully faithful, so this is clear.

These equivalences are compatible in the sense that the functor from any category to itself obtained by composing the functors is canonically isomorphic to the identity, the isomorphism being given by the appropriate adjunction map (1.3).

Let us call a map in the homotopy categories Ho $\mathcal{C}_A$, Ho $\mathcal{C}_S$ a quasi-isomorphism when it induces an isomorphism on homology. The following is immediate from Corollaries 1.10 and 1.11.

Proposition 1.13. (i) *The functors j_A, j_S send quasi-isomorphisms to isomorphisms.*
(ii) *If M is free, then $j_A : [M,N] \xrightarrow{\sim} HC^0(M,N)$.*
(iii) *If Q is divisible, then $j_S : [P,Q] \xrightarrow{\sim} HC^0(P,Q)$.*

This can be applied to characterize the derived categories by universal mapping properties.

Proposition 1.14. *Let $\mathcal{C}$ be a category and let $F : \mathrm{Ho}\,\mathcal{C}_A \to \mathcal{C}$ be a functor sending quasi-isomorphisms to isomorphisms. Then there is a unique functor $\tilde{F} : D\mathcal{C}_A \to \mathcal{C}$ such that $\tilde{F} j_A = F$.*

Proof. Set $L_M = A \otimes \mathcal{B}M$, $j = j_A$. Using (i) and (ii) above we have

$$[L_M, L_N] \xrightarrow{\sim} HC^0(L_M, L_N) \xrightarrow{\sim} HC^0(M,N),$$
$$u \mapsto j(u) \mapsto j(\varepsilon_N) j(u) j(\varepsilon_M)^{-1},$$

hence a map $g : M \to N$ in $D\mathcal{C}_A$ can be uniquely represented $g = j(\varepsilon_N)j(u)j(\varepsilon_M)^{-1}$ with $u \in [L_M, L_N]$. Thus if $\tilde{F}$ exists, we have $\tilde{F}M = \tilde{F}jM = FM$ and $\tilde{F}(g) = F(\varepsilon_N)F(u)F(\varepsilon_M)^{-1}$, proving the uniqueness of $\tilde{F}$. On the other hand if we define $\tilde{F}$ on objects and maps by these formulas, then it is easily checked that $\tilde{F}$ is the desired functor. □

We have the following variant of this proposition. Let $j_A' : \mathcal{C}_A \to D\mathcal{C}_A$ be the canonical functor from $\mathcal{C}_A$ to Ho $\mathcal{C}_A$ followed by j_A.

Corollary 1.15. *Let $F : \mathcal{C}_A \to \mathcal{C}$ be a functor sending quasi-isomorphisms to isomorphisms. Then there is a unique functor $\tilde{F} : D\mathcal{C}_A \to \mathcal{C}$ such that $\tilde{F} j_A' = F$.*

Proof. This follows from Proposition 1.14 once we show that F equalizes homotopic maps. By naturality it is enough to show F equalizes the canonical embeddings i_0, i_1 of M into the "cylinder" $I \otimes M$, where I is the complex of chains on the 1-simplex. But the canonical projection $p : I \otimes M \to M$ is a quasi-isomorphism satisfying $pi_0 = pi_1$, so we have $F(i_0) = F(i_1)$ as $F(p)$ is invertible. □

These universal mapping properties identify the derived category $D\mathcal{C}_A$ with the category obtained from either $\mathcal{C}_A$ or $\mathrm{Ho}\,\mathcal{C}_A$ by formally adjoining inverses for the quasi-isomorphisms.

The analogous versions of these universal mapping properties hold for DC_S.

2. Bivariant cyclic cohomology

So far we have treated mixed complexes and S-modules on an equal footing, but for the rest of the paper we want to focus our attention on divisible S-modules. We have seen that the derived category DC_Λ is the category obtained by formally inverting the quasi-isomorphisms in the category of mixed complexes. This result allows us to view DC_Λ as the category of cyclic homology types. On the other hand, we have by Proposition 1.12 an equivalence of categories

$$k_\Lambda : DC_\Lambda \to \mathrm{Ho}\, C_S^{\mathrm{d}}$$

sending M to $\mathcal{B}M$, so divisible S-modules up to homotopy give another model for cyclic homology types. This model is particularly nice for establishing results about bivariant cyclic cohomology, as we shall now show.

From (1.6) we know Hochschild and cyclic homology for divisible S-modules can be defined simply as

$$HH_nP = H_n({}_SP), \qquad HC_nP = H_nP \tag{2.1}$$

and when these are pulled back via $\mathcal{B}$, we obtain the corresponding homology for mixed complexes. We now present a similar picture for bivariant cyclic cohomology.

Let us define *bivariant cyclic cohomology* for any pair of divisible S-modules by

$$HC^k(P,P') = H^k(\mathrm{Hom}_S(P,P')). \tag{2.2}$$

For any triple there is a cup product

$$HC^j(P',P'') \otimes HC^k(P,P') \to HC^{j+k}(P,P'') \tag{2.3}$$

which is induced by the pairing of complexes

$$\mathrm{Hom}_S(P',P'') \otimes \mathrm{Hom}_S(P,P') \to \mathrm{Hom}_S(P,P'')$$

obtained by composing operators.

We note that $HC^0(P,P')$ is the set of maps $P \to P'$ in the homotopy category $\mathrm{Ho}\, C_S^{\mathrm{d}}$, and that cup product on HC^0 corresponds to composition in this category.

We define the *suspension* Σ on S-modules to be the operation whose kth power for any $k \in \mathbb{Z}$ is $\Sigma^k P = P[k]$, where

$$P[k]_n = P_{n-k} \tag{2.4}$$

with d, S on $P[k]$ given by $(-1)^k d$, S on P. Then

$$\mathrm{Hom}_S(P,P') = \mathrm{Hom}_S(P[k],P'[k])$$

hence

$$HC^0(P,P') = HC^0(P[k],P'[k]) \tag{2.5}$$

showing that suspension is an automorphism of the homotopy category.

We also have

$$\mathrm{Hom}_S(P,P'[k]) = \mathrm{Hom}_S(P,P')[k] \tag{2.6}$$

and hence

$$HC^0(P,P'[k]) = HC^k(P,P'). \tag{2.7}$$

Thus an element of $HC^k(P,P')$ can be identified with a map $P \to P'[k]$ in $\mathrm{Ho}\,\mathcal{C}_S^{\mathrm{d}}$. The cup product is given by

$$(g : P' \to P''[j])(f : P \to P'[k]) = (\Sigma^k g \cdot f : P \to P''[j+k]). \tag{2.8}$$

Therefore we conclude that bivariant cyclic cohomology for divisible S-modules and its cup product can be recovered from the homotopy category of divisible S-modules and the suspension automorphism.

We now use $\mathcal{B}$ to pull back (2.2) to mixed complexes. Let bivariant cyclic cohomology for a pair of mixed complexes be defined by

$$HC^k(M,M') = HC^k(\mathcal{B}M,\mathcal{B}M') \tag{2.9}$$

and similarly let the cup product in the case of a triple of mixed complexes be obtained by pulling back the cup product (2.3) via $\mathcal{B}$. These definitions are consistent with those of Jones and Kassel by [9, 2.2, 5.1].

Recall that in Section 1 we defined the derived category DC_Λ so that maps $M \to M'$ are elements of $HC^0(M,M')$ and composition is the cup product on HC^0.

We define the suspension Σ on mixed complexes so that $\Sigma^k M = M[k]$, where

$$M[k]_n = M_{n-k} \tag{2.10}$$

with b, B on $M[k]$ given by $(-1)^k b$, $(-1)^k B$ on M. Then

$$\mathcal{B}(M[k]) = (\mathcal{B}M)[k], \tag{2.11}$$

since

$$\mathcal{B}(M[k])_n = \bigoplus_{p\geq 0} M[k]_{n-2p} = \bigoplus_{p\geq 0} M_{n-k-2p} = (\mathcal{B}M)[k]_n.$$

Thus from (2.5) we have

$$HC^0(M,M') = HC^0(M[k],M'[k]) \tag{2.12}$$

showing that suspension is an automorphism of the derived category. Also from (2.7) we have

$$HC^0(M,M'[k]) = HC^k(M,M'), \tag{2.13}$$

so an element of $HC^k(M, M')$ can be identified with a map $M \to M'[k]$ in DC_A. In terms of this identification one has a description of the cup product analogous to (2.8), so we see that HC^* for mixed complexes and its cup product can be recovered from the derived category and its suspension automorphism.

In order to illustrate the utility of the divisible S-module model for cyclic homology types, we next derive some known facts about bivariant cyclic cohomology and comment on their consequences.

We first consider the bivariant Connes exact sequence [12, I.2]. Let P, Q be divisible S-modules, and recall that the exact sequence

$$0 \to {}_SP \to P \xrightarrow{S} P[2] \to 0 \tag{2.14}$$

yields on passing to homology the Connes exact sequence

$$\to HC_{n-1}P \to HH_nP \to HC_nP \xrightarrow{S} HC_{n-2}P \to \tag{2.15}$$

linking the Hochschild and cyclic homology (2.1) of P. Next we observe that, since Q is divisible, the functor $\mathrm{Hom}_S(-, Q)$ on S-modules is exact. This follows from Lemma 1.6 or from the fact already mentioned, that a divisible S-module is injective as a module over $\mathbb{C}[S]$. Applying this functor to (2.14) we obtain an exact sequence of complexes

$$0 \to \mathrm{Hom}_S(P, Q)[-2] \to \mathrm{Hom}_S(P, Q) \to \mathrm{Hom}({}_SP, {}_SQ) \to 0$$

which yields the *bivariant Connes exact sequence*

$$\to HC^{k-2}(P, Q) \to HC^k(P, Q) \to HH^k(P, Q) \to HC^{k-1}(P, Q) \to \tag{2.16}$$

where

$$HH^k(P, Q) = \mathrm{Hom}(HH_*P, HH_*Q)_{-k} = \prod_n \mathrm{Hom}(HH_nP, HH_{n-k}Q).$$

Secondly, we consider the relation between bivariant cyclic cohomology and the types of homology of interest in cyclic homology theory. Let V be an arbitrary vector space, and regard V as a mixed complex concentrated in degree zero. Then we have

$$\mathrm{Hom}_S(P, BV) = \mathrm{Hom}(P, V),$$

so we obtain a canonical isomorphism

$$HC^n(P, BV) = \mathrm{Hom}(HC_nP, V), \tag{2.17}$$

cf. [9, 2.3]. Now by Yoneda's lemma this means that HC_nP is determined up to canonical isomorphism by the functor $V \mapsto HC^n(P, BV)$. Moreover we have

$$HH^n(P, BV) = \mathrm{Hom}(HH_nP, V) \tag{2.18}$$

and it is routine to check that (2.17) and (2.18) identify the result of applying $\mathrm{Hom}(-, V)$ to the Connes exact sequence (2.15) with the bivariant Connes exact sequence (2.16) in the case $Q = BV$.

Thus cyclic homology and the Connes exact sequence can be recovered via Yoneda's lemma from bivariant cyclic cohomology and the bivariant Connes exact sequence.

On the other hand, negative cyclic homology is a special case of bivariant cyclic cohomology:

$$HC_n^- P = HC^{-n}(B\mathbb{C}, P), \tag{2.19}$$

cf. [9, 2.3], and periodic cyclic homology can be obtained from negative cyclic homology:

$$HP_{n+2\mathbb{Z}}P = HC_n^- P, \quad n \ll 0, \tag{2.20}$$

since P is assumed bounded below.

Therefore, we can conclude that the various kinds of homology of interest in cyclic homology theory are subsumed under bivariant cyclic cohomology.

Finally, we mention the universal coefficient formula [9]. Note that Lemma 1.6 yields upon passing to cohomology the *universal coefficient exact sequence*

$$0 \to \mathrm{Ext}^1_S(HC_*P, HC_*Q)_{-k-1} \to HC^k(P,Q) \to \mathrm{Hom}_S(HC_*P, HC_*Q)_{-k} \to 0$$

as in [9, 4.4]. In particular we have surjectivity of the map

$$HC^0(P,Q) \to \mathrm{Hom}^0_S(HC_*P, HC_*Q).$$

This implies that an isomorphism $HC_*P \xrightarrow{\sim} HC_*Q$ respecting the graded module structure over $\mathbb{C}[S]$ is induced by a map $P \to Q$ in $\mathrm{Ho}\,\mathcal{C}^{\mathrm{d}}_S$ which must then be an isomorphism. In other words, the graded module HC_*P over $\mathbb{C}[S]$ determines P up to noncanonical isomorphism in the homotopy category.

Therefore, a cyclic homology type is determined by its cyclic homology in much the same manner as a generalized Eilenberg–Mac Lane space is determined by its homotopy groups.

3. Towers

3.1. Supercomplexes

We recall that a *supercomplex* or $\mathbb{Z}/2$-graded complex is a $\mathbb{Z}/2$-graded vector space K equipped with an odd operator d of square zero. It will be convenient to identify the elements of $\mathbb{Z}/2$ with the sets of even and odd integers and also, depending on the context, to use the standard abbreviations $+$ and $-$ for even and odd respectively. Thus we have

$$K = K_{2\mathbb{Z}} \oplus K_{1+2\mathbb{Z}} = K_+ \oplus K_-$$

for the $\mathbb{Z}/2$-grading, and the homology of K is

$$H_{n+2\mathbb{Z}}(K) = Z_{n+2\mathbb{Z}}K \big/ d(K_{n+1+2\mathbb{Z}}),$$

where $Z_\nu K$ denotes the subspace of cycles of degree ν.

If K, L are two supercomplexes, let $\mathrm{Hom}(K,L)$ be the vector space of all linear operators from K to L not necessarily respecting the grading and differentials. This is naturally a supercomplex with the differential $f \mapsto [d,f]$. An even cycle $f \in Z_+ \mathrm{Hom}(K,L)$ is the same as a map $f : K \to L$ in the category of supercomplexes. By definition f is homotopic to zero when it is a boundary: $f = [d,h]$ for some $h \in \mathrm{Hom}(K,L)_-$. Thus $H_+(\mathrm{Hom}(K,L))$ is the space of homotopy classes of maps from K to L.

By a *special contraction* on K we mean an odd operator h such that $[d,h] = 1$ and $h^2 = 0$.

Proposition 3.1. (a) *A special contraction on K exists iff K is contractible, i.e. homotopy equivalent to zero.*

(b) *A special contraction h on K is equivalent to a splitting $K = K' \oplus K''$, where K' and K'' are subcomplexes such that in K' the differential is an isomorphism from even degree to odd, and in K'' the differential is an isomorphism from odd degree to even.*

Proof. (a) K is contractible iff there exists an odd operator h satisfying $[d,h] = 1$, so it suffices to replace such an h by a special contraction. Now the operators hd and $dh = 1 - hd$ commute and annihilate each other, hence they are idempotent. Thus

$$[d, hdh] = (dh)^2 + (hd)^2 = dh + hd = 1,$$
$$(hdh)^2 = h(dh)(hd)h = 0,$$

so hdh is a special contraction.

(b) Given a special contraction h, we have $K = hK \oplus dK$, where $d : hK \to dK$ is an isomorphism with inverse $h : dK \to hK$. Then

$$K' = (hK)_+ \oplus (dK)_-, \qquad K'' = (dK)_+ \oplus (hK)_-$$

gives a splitting of the type described in (b). Conversely given such a splitting, let h be the inverse of the isomorphism $d : K'_+ \to K'_-$ on K'_-, the inverse of the isomorphism $d : K''_- \to K''_+$ on K''_+, and zero on K'_+ and K''_-. Then h is a special contraction. These two constructions are inverse, proving (b). □

It is useful to generalize the concept of special contraction as follows. We define a *special deformation retraction* on K to be a pair (e,h), where e and h are even and odd elements of $\mathrm{Hom}(K,K)$ satisfying the conditions

$$e^2 = e, \qquad [d,h] = 1 - e, \qquad h^2 = eh = he = 0. \tag{3.1}$$

This implies $[d,e] = 0$, hence we have a splitting $K = eK \oplus e^\perp K$ into subcomplexes, where $e^\perp = 1 - e$. Moreover h may be viewed as special contraction on $e^\perp K$ extended to K so as to be zero on eK. In this way a special deformation retraction on K is equivalent to a splitting of K into the direct sum of two subcomplexes together with a special contraction on the second subcomplex, the latter being in turn equivalent to a decomposition of this subcomplex as in Proposition 3.1(b).

Proposition 3.2. *On any supercomplex K there exists a special deformation retraction (e,h) such that the subcomplex eK is minimal in the sense that $d = 0$ on eK. Equivalently, there exists a decomposition into subcomplexes*

$$K = K^{\mu} \oplus K' \oplus K''$$

such that K^{μ} is minimal, such that d in K' is an isomorphism from even degree to odd, and such that d in K'' is an isomorphism from odd degree to even.

Proof. We consider the filtration $0 \subset dK \subset ZK \subset K$ by boundaries and cycles, and choose $\mathbb{Z}/2$-graded subspaces V, W of K such that $dK \oplus V = ZK$ and $ZK \oplus W = K$. Then K is the direct sum of the subcomplex V where $d = 0$, and the subcomplex $L = W \oplus dK$ where $d : W \to dK$ is an isomorphism. Thus we obtain the desired decomposition with $K^{\mu} = V$, $K' = W^+ \oplus d(K^+)$, and $K'' = W^- \oplus d(K^-)$. □

We see immediately from this proposition that any supercomplex is homotopy equivalent to the supercomplex given by its homology equipped with the zero differential. In particular, a quasi-isomorphism of supercomplexes, i.e. a map inducing isomorphisms on homology, is necessarily a homotopy equivalence.

3.2. Towers of vector spaces

By a tower of vector spaces we will mean an inverse system $X = (X^n)$ indexed by the integers such that the maps $X^n \to X^{n-1}$ are all surjective, and such that X is bounded below in the sense that $X^n = 0$ for $n \ll 0$.

We associate to X its inverse limit equipped with the induced filtration

$$\widehat{X} = \varprojlim X^n, \qquad F^n\widehat{X} = \operatorname{Ker}(\widehat{X} \to X^n). \tag{3.2}$$

As $\widehat{X}/F^n\widehat{X} = X^n$, we see that X can be recovered from its inverse limit as filtered vector space. Moreover, one obtains in this way an equivalence between towers of vector spaces and vector spaces V equipped with a decreasing filtration (F^nV) such that $F^nV = V$ for $n \ll 0$, and such that V is complete for the topology defined by the filtration.

Let

$$\bar{X} = \bigoplus_n \bar{X}^n, \qquad \bar{X}^n = \operatorname{Ker}(X^n \to X^{n-1}) = F^{n-1}\widehat{X}/F^n\widehat{X} \tag{3.3}$$

be the *associated graded vector space* and *nth layer* of X respectively. Since we are working over a field, we can *split* X as follows. Choose a projection of X^n onto the subspace $\bar{X}^n$ for each n. Then we have $X^n \xrightarrow{\sim} X^{n-1} \times \bar{X}^n$, which can be iterated to obtain isomorphisms

$$X^n \xrightarrow{\sim} \bigoplus_{j \le n} \bar{X}^j \tag{3.4}$$

since $X^n = 0$ for $n \ll 0$. In this way we obtain an isomorphism of X with the split tower of vector spaces associated to the graded vector space $\bar{X}$.

Let X, Y be towers of vector spaces and let

$$\operatorname{Hom}_c(\widehat{X}, \widehat{Y}) = \{ f : \widehat{X} \to \widehat{Y} \mid \forall m\ \exists n,\ f(F^n \widehat{X}) \subset F^m \widehat{Y} \} \tag{3.5}$$

be the vector space of linear maps which are continuous with respect to the natural topologies on the inverse limits. Sitting inside this is the subspace

$$\operatorname{Hom}^k(X, Y) = \{ f : \widehat{X} \to \widehat{Y} \mid \forall m,\ f(F^{m+k} \widehat{X}) \subset F^m \widehat{Y} \} \tag{3.6}$$

of maps of *order* $\le k$. Thus $f : \widehat{X} \to \widehat{Y}$ has order $\le k$ iff it induces maps of the quotient spaces $X^{m+k} \to Y^m$ for all m. In this way a map of order $\le k$ can be identified with a map of inverse systems $(X^n) \to (Y^{n-k})$.

The subspaces $\operatorname{Hom}^k(X, Y)$ increase with k and we let

$$\operatorname{Hom}^\infty(X, Y) \subset \operatorname{Hom}_c(\widehat{X}, \widehat{Y}) \tag{3.7}$$

denote their union. For readers familiar with pro-objects we mention that the latter is the space of morphisms in the category of pro-objects, while the former is the space of morphisms in the category of Artin–Rees pro-objects [5].

Proposition 3.3. *One has an exact sequence*

$$0 \to \operatorname{Hom}^{k-1}(X, Y) \to \operatorname{Hom}^k(X, Y) \to \operatorname{Hom}^k(\bar{X}, \bar{Y}) \to 0, \tag{3.8}$$

where

$$\operatorname{Hom}^k(\bar{X}, \bar{Y}) = \prod_n \operatorname{Hom}(\bar{X}^n, \bar{Y}^{n-k}).$$

Proof. The maps in this sequence are canonical maps, and the sequence is evidently left exact, so only the surjectivity at the right needs proof. By choosing a splitting of X we can assume

$$X^n = \bigoplus_{j \le n} \bar{X}^j, \qquad \widehat{X} = \prod_j \bar{X}^j$$

and similarly for Y. If $f : \widehat{X} \to \widehat{Y}$ has order $\le k$, then it induces $X^{i+k} \to Y^i$, so the component of fx in $\bar{Y}^i$ depends on the components of x in $\bar{X}^j$ for $j \le i + k$. From this we see that $\operatorname{Hom}^k(X, Y)$ can be identified with the vector space of matrices (f_{ij}) with $f_{ij} : \bar{X}^j \to \bar{Y}^i$, such that $f_{ij} \ne 0$ implies $j \le i + k$. Thus

$$\operatorname{Hom}^k(X, Y) = \prod_{j \le i+k} \operatorname{Hom}(\bar{X}^j, \bar{Y}^i)$$

and the desired surjectivity is clear. □

3.3. Towers

By a tower of supercomplexes we mean a tower of vector spaces $X = (X^n)$ such that the levels X^n are given supercomplex structures compatible with the surjections $X^n \to X^{n-1}$. For the sake of brevity we shall henceforth use the term *tower* to mean tower of supercomplexes unless stated otherwise.

If X is a tower, then $\bar{X}^n$, $\widehat{X}$, and $F^n\widehat{X}$ are naturally supercomplexes, and we have a short exact sequence of supercomplexes

$$0 \to \bar{X}^n \to X^n \to X^{n-1} \to 0 \tag{3.9}$$

for each n. Moreover, for any pair X, Y of towers, $\mathrm{Hom}^k(X,Y)$ is naturally a supercomplex with the differential $f \mapsto [d,f]$, and (3.8) is an exact sequence of supercomplexes.

Let $\mathcal{T}$ denote the category of towers in which a map $X \to Y$ is a map of inverse systems respecting the supercomplex structure. Then the set of maps $X \to Y$ can be identified with $Z_+ \mathrm{Hom}^0(X,Y)$.

We call two maps $f, f' : X \to Y$ *homotopic* when $f - f' = [d,h]$ for some $h \in \mathrm{Hom}^0(X,Y)_-$, and we let $\mathrm{Ho}\,\mathcal{T}$ be the *homotopy category* of towers in which the maps are the homotopy classes of maps of towers. Then the set of maps $X \to Y$ in the homotopy category can be identified with $H_+(\mathrm{Hom}^0(X,Y))$. It should be evident what is meant for a map of towers to be a homotopy equivalence, and for a tower to be contractible.

We call a map $f : X \to Y$ a *quasi-isomorphism* when it satisfies the following equivalent conditions:

(i) $f^n : X^n \to Y^n$ is a quasi-isomorphism for all n.

(ii) $\bar{f}^n : \bar{X}^n \to \bar{Y}^n$ is a quasi-isomorphism for all n.

The equivalence of these conditions is proved as usual by applying the five lemma to the induced map of long exact homology sequences associated to the short exact sequence (3.9) and the similar one for Y, the long exact sequence being hexagonal in the case of supercomplexes. The direction (ii) $\Rightarrow$ (i) proceeds by induction, using the assumption that towers are bounded below to get started.

A homotopy equivalence of towers is evidently a quasi-isomorphism. In order to prove the converse, we now carry over to towers the concepts of special contraction and special deformation retraction in the obvious way by replacing $\mathrm{Hom}(K,K)$ by $\mathrm{Hom}^0(X,X)$. It is easily seen that Proposition 3.1 extends to towers, in particular, a special contraction $h \in \mathrm{Hom}^0(X,X)_-$ is equivalent to a decomposition $X = X' \oplus X''$ into subtowers, where the differential in X' (resp. X'') is an isomorphism from even degree to odd (resp. from odd degree to even). Moreover, a special deformation retraction on X is equivalent to a decomposition of X into two subtowers together with a special contraction on the second subtower.

The following is a variant of the basic construction in homological perturbation theory [7].

Lemma 3.4. *Any special deformation retraction* (e,h) *in* $\mathrm{Hom}^0(\bar{X},\bar{X})$ *can be lifted* *to a special deformation retraction* $(\tilde{e},\tilde{h})$ *in* $\mathrm{Hom}^0(X,X)$.

Proof. We can choose a splitting (3.4) of the tower X which respects the $\mathbb{Z}/2$-grading but not necessarily the differentials. Let $R = \mathrm{Hom}^0(X,X)$; as in the proof of Proposition 3.3 we can identify R with the algebra of triangular matrices (f_{ij}), $f_{ij} : \bar{X}^j \to \bar{X}^i$, $f_{ij} = 0$ for $i < j$, and $\mathrm{Hom}^0(\bar{X},\bar{X})$ with the subalgebra of diagonal matrices. Then e, h and the differential d of $\bar{X}$ become diagonal matrices, and the differential of X has the form $d - \theta$, where $\theta \in R_-$ is zero on the diagonal and satisfies $[d,\theta] = \theta^2$.

The elements $1 - h\theta$ and $1 - \theta h$ of R are invertible with inverses given by the usual geometric series. Define $\tilde{e},\tilde{h}$ in R by

$$\tilde{e} = \frac{1}{1-h\theta}e\frac{1}{1-\theta h}, \qquad \tilde{h} = h\frac{1}{1-\theta h} = \frac{1}{1-h\theta}h.$$

We now show $\tilde{e},\tilde{h}$ satisfy the identities for a special deformation retraction. Using $h^2 = eh = he = 0$ we have

$$e\frac{1}{1-\theta h}\frac{1}{1-h\theta}e = \sum_{i,j\geq 0} e(\theta h)^i(h\theta)^j e = e^2 = e,$$

whence $\tilde{e}^2 = \tilde{e}$. Similarly we have $\tilde{h}^2 = \tilde{e}\tilde{h} = \tilde{h}\tilde{e} = 0$. Finally,

$$\begin{aligned}
&(1-h\theta)[d-\theta,\tilde{h}](1-\theta h)\\
&\quad = (1-h\theta)(d-\theta)h + h(d-\theta)(1-\theta h)\\
&\quad = dh + hd - \theta h - h\theta + h(-\theta d - d\theta + 2\theta^2)h\\
&\quad = (1-h\theta)(1-\theta h) - e
\end{aligned}$$

yields $[d-\theta,\tilde{h}] = 1 - \tilde{e}$. □

By a *minimal tower* we mean a tower X such that $d = 0$ on $\bar{X}$.

Proposition 3.5. *On any tower* X *there exists a special deformation retraction* $(\tilde{e},\tilde{h})$ *such that* $\tilde{e}X$ *is minimal. Equivalently, there exists a decomposition into subtowers*

$$X = X^\mu \oplus X' \oplus X''$$

such that X^μ *is minimal, such that* d *in* X' *is an isomorphism from even degree to odd, and such that* d *in* X'' *is an isomorphism from odd degree to even.*

Proof. Applying Proposition 3.2 to $\bar{X}^n$ for each n we obtain a special deformation retraction (e,h) in $\mathrm{Hom}^0(\bar{X},\bar{X})$ such that $d = 0$ on $e\bar{X}$. By the perturbation lemma we can lift (e,h) to a special deformation retraction $(\tilde{e},\tilde{h})$ in $\mathrm{Hom}^0(X,X)$. Since $\overline{\tilde{e}X} = e\bar{X}$, we see that $\tilde{e}X$ is minimal as desired. □

This proposition implies that any tower is homotopy equivalent to a minimal tower. Now observe that a quasi-isomorphism $X \to Y$ between minimal towers, in particular a homotopy equivalence, is necessarily an isomorphism, because the induced map $\bar{X} \to \bar{Y}$ is an isomorphism, and hence $X^n \xrightarrow{\sim} Y^n$ by induction. From these facts we deduce

Corollary 3.6. *Any tower is homotopy equivalent to a minimal tower which is unique up to noncanonical isomorphism. Any quasi-isomorphism of towers is a homotopy equivalence.*

4. Special towers

We now restrict our attention to those towers (of supercomplexes) which are relevant for cyclic homology theory. By a *special tower* we mean a tower X satisfying

$$H_{n-1+2\mathbb{Z}}\bar{X}^n = 0, \quad \text{for all } n, \tag{4.1}$$

in other words, the homology of the nth layer is supported in degree $n + 2\mathbb{Z}$ for all n. Our aim in this section is to construct for special towers the sort of homology and cohomology occurring in cyclic homology theory.

Let us define *Hochschild, cyclic,* and *de Rham homology* for a special tower X by

$$\begin{aligned} HH_nX &= H_{n+2\mathbb{Z}}(\bar{X}^n), \\ HC_nX &= H_{n+2\mathbb{Z}}(X^n), \\ HD_{n-1}X &= H_{n-1+2\mathbb{Z}}(X^n), \end{aligned} \tag{4.2}$$

respectively. From the short exact sequence of supercomplexes

$$0 \to \bar{X}^n \to X^n \to X^{n-1} \to 0$$

we obtain a circular six-term exact sequence on passing to homology. By (4.1) this homology sequence can be written as a five-term exact sequence

$$0 \to HD_{n-1}X^n \to HC_{n-1}X \to HH_nX \to HC_nX \to HD_{n-2}X \to 0.$$

Splicing these together for different n yields the Connes exact sequence

$$\to HC_{n+1}X \xrightarrow{S} HC_{n-1}X \to HH_nX \to HC_nX \xrightarrow{S} HC_{n-2}X \to, \tag{4.3}$$

where $S : HC_nX \to HC_{n-2}X$ is the map on homology of degree $n + 2\mathbb{Z}$ induced by the canonical surjection $X^n \to X^{n-2}$. In addition we have

$$HD_nX = S(HC_{n+2}X) \subset HC_nX \tag{4.4}$$

expressing de Rham homology in terms of cyclic homology. This formula justifies the terminology "de Rham homology" by virtue of the Connes-Karoubi theorem, cf. [1, II.33], [10, 2.15] and [3, Section 6].

Let X, Y be special towers, and consider the sequence of supercomplexes

$$0 \to \mathrm{Hom}^{k-1}(X, Y) \to \mathrm{Hom}^k(X, Y) \to \mathrm{Hom}^k(\bar{X}, \bar{Y}) \to 0 \tag{4.5}$$

which is exact by Proposition 3.3. By condition (4.1) for X and Y we have

$$H_{k+2\mathbb{Z}}(\mathrm{Hom}(\bar{X}^n, \bar{Y}^{n-k})) = \mathrm{Hom}(H_{n+2\mathbb{Z}}\bar{X}^n, H_{n-k+2\mathbb{Z}}\bar{Y}^{n-k})$$

and also that the homology of degree $k+1+2\mathbb{Z}$ vanishes. Indeed, because any supercomplex is homotopy equivalent to its homology with zero differential by Proposition 3.2, this reduces to the case where $\bar{X}^n$, $\bar{Y}^{n-k}$ have zero differential, and in this case the assertion is obvious.

Let us define the bivariant Hochschild, cyclic, and de Rham cohomology of X, Y to be

$$\begin{aligned} HH^k(X,Y) &= \mathrm{Hom}^k(HH_*X, HH_*Y), \\ HC^k(X,Y) &= H_{k+2\mathbb{Z}}(\mathrm{Hom}^k(X,Y)), \\ HD^k(X,Y) &= H_{k+2\mathbb{Z}}(\mathrm{Hom}^{k-1}(X,Y)), \end{aligned} \tag{4.6}$$

respectively. Then we have

$$H_{k+2\mathbb{Z}}(\mathrm{Hom}^k(\bar{X},\bar{Y})) = HH^k(X,Y), \qquad H_{k-1+2\mathbb{Z}}(\mathrm{Hom}^k(\bar{X},\bar{Y})) = 0,$$

so the homology sequence associated to (4.5) yields the exact sequence

$$\begin{aligned} 0 \to HD^k(X,Y) \to HC^k(X,Y) \to HH^k(X,Y) \\ \to HC^{k-1}(X,Y) \to HD^{k+1}(X,Y) \to 0. \end{aligned}$$

Combining these for different k we obtain the bivariant Connes exact sequence

$$\to HC^{k-2}(X,Y) \xrightarrow{S} HC^k(X,Y) \to HH^k(X,Y) \to HC^{k-1}(X,Y) \to, \tag{4.7}$$

where S here is the map on $H_{k+2\mathbb{Z}}$ induced by the inclusion of $\mathrm{Hom}^{k-2}(X,Y)$ in $\mathrm{Hom}^k(X,Y)$ and

$$HD^k(X,Y) = S(HC^{k-2}(X,Y)) \subset HC^k(X,Y). \tag{4.8}$$

In the case of three special towers there is a cup product on bivariant cyclic cohomology

$$HC^j(X',X'') \otimes HC^k(X,X') \to HC^{j+k}(X,X'') \tag{4.9}$$

which is induced by the pairing

$$\mathrm{Hom}^j(X',X'') \otimes \mathrm{Hom}^k(X,X') \to \mathrm{Hom}^{j+k}(X,X'')$$

given by composition.

Let $\mathcal{T}^s \subset \mathcal{T}$ and $\mathrm{Ho}\,\mathcal{T}^s \subset \mathrm{Ho}\,\mathcal{T}$ denote the full subcategories consisting of special towers. Recall that for arbitrary towers X, Y the maps $X \to Y$ in $\mathcal{T}$ and $\mathrm{Ho}\,\mathcal{T}$ can be identified with elements of $Z_+ \mathrm{Hom}^0(X,Y)$ and $H_+ \mathrm{Hom}^0(X,Y)$ respectively. Thus when X, Y are special towers, an element of $HC^0(X,Y) = H_+ \mathrm{Hom}^0(X,Y)$ can be identified with a map $X \to Y$ in $\mathrm{Ho}\,\mathcal{T}^s$. Moreover, cup product on HC^0 corresponds to composition in this category.

We define the suspension Σ on special towers to be the operation whose kth power is $\Sigma^k X = X[k]$, where

$$X[k]^n_\nu = X^{n-k}_{\nu-k} \tag{4.10}$$

with d on the left given by $(-1)^k d$ on the right. We have

$$\mathrm{Hom}^j(X,Y) = \mathrm{Hom}^j(X[k],Y[k]),$$

hence

$$HC^0(X,Y) = HC^0(X[k],Y[k])$$

showing that suspension is an automorphism of $\mathrm{Ho}\,\mathcal{T}^s$. We also have

$$\mathrm{Hom}^j(X,Y[k])_\nu = \mathrm{Hom}^{j+k}(X,Y)_{\nu+k} \tag{4.11}$$

with d on the left corresponding to $(-1)^k d$ on the right. Consequently

$$HC^0(X,Y[k]) = HC^k(X,Y), \tag{4.12}$$

so an element of $HC^k(X,Y)$ can be identified with a map $X \to Y[k]$ in $\mathrm{Ho}\,\mathcal{T}^s$. In this way bivariant cyclic cohomology HC^* for special towers together with its cup product operation can be recovered from the homotopy category $\mathrm{Ho}\,\mathcal{T}^s$ and the suspension automorphism.

Given a vector space V, let $\theta V = (\theta^n V)$ be the special tower such that $\theta^n V = 0$ for $n < 0$, and such that $\theta^n V$ for $n \geq 0$ is the supercomplex given by V in even degree and zero in odd degree. Then we have canonical isomorphisms of supercomplexes

$$\begin{aligned} &\mathrm{Hom}^n(X,\theta V) = \mathrm{Hom}(X^n,V), \\ &\mathrm{Hom}^{-n}(\theta V,X) = \mathrm{Hom}(V,F^{n-1}\widehat{X}). \end{aligned} \tag{4.13}$$

The former yields

$$HC^n(X,\theta V) = \mathrm{Hom}(HC_n X, V) \tag{4.14}$$

showing via Yoneda's lemma as in Section 2 that cyclic homology is subsumed under bivariant cyclic cohomology; this can be extended to include Hochschild homology and the Connes exact sequence.

From the second formula in (4.13) we have

$$\begin{aligned} &H_{n+2\mathbb{Z}}(F^{n-1}\widehat{X}) = HC_n^- X, \\ &H_{n-1-2\mathbb{Z}}(F^{n-1}\widehat{X}) = HD_{n-1}^- X, \end{aligned} \tag{4.15}$$

where negative cyclic homology is defined in analogy with (2.19) in terms of bivariant cyclic cohomology by

$$HC_n^- X = HC^{-n}(\theta\mathbb{C},X) \tag{4.16}$$

and similarly for negative de Rham homology. The Connes exact sequence linking the Hochschild and negative cyclic homology [6]

$$\to HC^-_{n+2}X \to HC^-_n X \to HH_n X \to HC^-_{n+1} X \to \tag{4.17}$$

can be obtained either from the bivariant sequence (4.7), or by splicing the hexagonal homology sequences associated to the short exact sequences

$$0 \to F^n \widehat{X} \to F^{n-1}\widehat{X} \to \bar{X}^n \to 0$$

If periodic cyclic homology is defined for special towers in terms of negative cyclic homology as in (2.20), then we have

$$HP_\nu X = H_\nu(\widehat{X}) \tag{4.18}$$

and the exact sequence

$$\to HC^-_{n+2}X \to HP_{n+2\mathbb{Z}}X \to HC_n X \to HC^-_{n+1} X \to \tag{4.19}$$

linking negative cyclic, periodic cyclic, and cyclic homology [6] results by splicing the homology sequences associated to

$$0 \to F^n\widehat{X} \to \widehat{X} \to X^n \to 0.$$

5. Divisible S-modules and special towers

Our aim in this section is to construct a canonical equivalence between the homotopy categories $\mathrm{Ho}\,\mathcal{C}^{\mathrm{d}}_S$ and $\mathrm{Ho}\,\mathcal{T}^{\mathrm{s}}$ of divisible S-modules and special towers respectively.

Let P, Q denote divisible S-modules. Let $(\alpha P)^n$ be the supercomplex given by

$$(\alpha P)^n_{n+2\mathbb{Z}} = P_n/d(sP_{n+1}), \qquad (\alpha P)^n_{n-1+2\mathbb{Z}} = P_{n-1}, \tag{5.1}$$

where half of the differential is induced by $d : P_n \to P_{n-1}$ and the other half is given by lifting with respect to $S : P_{n+1} \to P_{n-1}$ and then applying $d : P_{n+1} \to P_n$. We have

$$\mathrm{Ker}\{d : P_n/d(sP_{n+1}) \to P_{n-1}\} = Z_nP/d(sP_{n+1}),$$

$$\mathrm{Im}\{d : P_{n-1} \to P_n/d(sP_{n+1})\} = dP_{n+1}/d(sP_{n+1}),$$

where Z_nP denotes the space of cycles of degree n. Thus

$$H_{n+2\mathbb{Z}}(\alpha P)^n = H_nP = HC_nP, \tag{5.2}$$

where we have used the definition (2.1) of cyclic homology for divisible S-modules.

For each n there is a surjection of supercomplexes

$$\begin{array}{cccc} (\alpha P)^n: & P_n/d({}_SP_{n+1}) & \leftrightarrows & P_{n-1} \\ & \downarrow & & \downarrow \\ (\alpha P)^{n-1}: & P_{n-2} & \leftrightarrows & P_{n-1}/d({}_SP_n) \end{array}$$

induced by $S : P_n \to P_{n-2}$ and the identity on P_{n-1}. We thus have a tower αP consisting of the supercomplexes $(\alpha P)^n$ and these surjections. The nth layer is

$$(\overline{\alpha P})^n : {}_SP_n/d({}_SP_{n+1}) \underset{d}{\overset{0}{\leftrightarrows}} d({}_SP_n), \tag{5.3}$$

hence

$$\begin{aligned} H_{n+2\mathbb{Z}}(\overline{\alpha P})^n &= H_n({}_SP) = HH_nP, \\ H_{n-1+2\mathbb{Z}}(\overline{\alpha P})^n &= 0. \end{aligned} \tag{5.4}$$

We see from (5.2) and (5.4) that αP is a special tower having the same Hochschild and cyclic homology as P. Clearly αP is a functor of P so we obtain

Proposition 5.1. *The construction $P \mapsto \alpha P$ gives a functor $\alpha : \mathcal{C}_S^{\mathrm{d}} \to \mathcal{T}^{\mathrm{s}}$ which is compatible with Hochschild and cyclic homology up to canonical isomorphism.*

As an example, consider the divisible S-module $\mathcal{B}M$, where M is a mixed complex. In this case the tower $\alpha\mathcal{B}M$ is the tower $\theta M = (M/F^nM)$, where M is regarded as a supercomplex with differential $b + B$, and

$$F^nM = bM_{n+1} \oplus \bigoplus_{k>n} M_k$$

is the *Hodge filtration* [3].

For any tower X put $(\beta X)_n = X^{n+1}_{n+2\mathbb{Z}}$ for all n, define $d : (\beta X)_n \to (\beta X)_{n-1}$ to be the composition

$$X^{n+1}_{n+2\mathbb{Z}} \twoheadrightarrow X^n_{n+2\mathbb{Z}} \xrightarrow{d} X^n_{n-1+2\mathbb{Z}}$$

and define $S : (\beta X)_n \to (\beta X)_{n-2}$ to be the composition

$$X^{n+1}_{n+2\mathbb{Z}} \twoheadrightarrow X^n_{n+2\mathbb{Z}} \twoheadrightarrow X^{n-1}_{n+2\mathbb{Z}}.$$

Then $\beta X = \bigoplus_n (\beta X)_n$ is a divisible S-module. We obtain in this way a functor $\beta : \mathcal{T} \to \mathcal{C}_S^{\mathrm{d}}$ such that $\beta\alpha$ is the identity.

Let us compute $\alpha\beta X$. If $P = \beta X$, then ${}_SP_{n+1} = \operatorname{Ker}\{X^{n+2}_{n+1+2\mathbb{Z}} \to X^n_{n-1+2\mathbb{Z}}\}$ maps onto $\bar{X}^{n+1}_{n+1+2\mathbb{Z}}$, hence

$$\begin{aligned} (\alpha P)^n_{n+2\mathbb{Z}} &= P_n/d({}_SP_{n+1}) = X^{n+1}_{n+2\mathbb{Z}}/d(\bar{X}^{n+1}_{n+1+2\mathbb{Z}}), \\ (\alpha P)^n_{n-1+2\mathbb{Z}} &= X^n_{n-1+2\mathbb{Z}}. \end{aligned}$$

Consequently we have a canonical surjection of towers

$$\alpha\beta X \twoheadrightarrow X \tag{5.5}$$

whose kernel at level n is $\bar{X}^{n+1}_{n+2\mathbb{Z}}/d(\bar{X}^{n+1}_{n+1+2\mathbb{Z}})$ in degree $n+2\mathbb{Z}$ and zero in the opposite degree. From this we conclude

Proposition 5.2. *The functors α, β give an equivalence between $\mathcal{C}^{\mathrm{d}}_S$ and the full subcategory of $\mathcal{T}$ consisting of towers satisfying*

$$d(\bar{X}^n_{n+2\mathbb{Z}}) = \bar{X}^n_{n-1+2\mathbb{Z}}, \quad \textit{for all } n. \tag{5.6}$$

One can check that the natural transformation (5.5) and the identity transformation $P \to \beta\alpha P$ make α, β adjoint functors between $\mathcal{C}^{\mathrm{d}}_S$ and $\mathcal{T}$ (or $\mathcal{T}^{\mathrm{s}}$), where α is left adjoint to β.

We note for later reference that it is clear from the definitions that α, β commute with suspension:

$$\alpha(Q[k]) = (\alpha Q)[k], \qquad \beta(X[k]) = (\beta X)[k] \tag{5.7}$$

on $\mathcal{C}^{\mathrm{d}}_S$ and $\mathcal{T}^{\mathrm{s}}$ as defined by (2.4) and (4.10).

We next bring in our results from Section 3 about minimal towers. By Corollary 3.6 any special tower is homotopy equivalent to a minimal tower X which is necessarily special. *Minimal special towers* (i.e. both minimal and special) are clearly characterized by the condition

$$\bar{X}^n_{n-1+2\mathbb{Z}} = 0, \quad \text{for all } n. \tag{5.8}$$

Since (5.8) implies (5.6), a minimal special tower X is isomorphic to αP for $P = \beta X$.

Let us define a *minimal S-module* to be a divisible S-module Q such that $d({}_SQ) = 0$. From (5.3) it is clear that a divisible S-module P is minimal iff αP is a minimal special tower. Thus, if we consider minimal S-modules and minimal special towers as forming full subcategories of $\mathcal{C}^{\mathrm{d}}_S$ and $\mathcal{T}^{\mathrm{s}}$ respectively, then we have established

Corollary 5.3. *The functors α, β give an equivalence between the categories of minimal S-modules and minimal special towers.*

To facilitate the understanding of this corollary and the following discussion it is helpful to observe that for a minimal S-module P the tower αP can be drawn

$$
\begin{array}{rccc}
 & \downarrow & & \| \\
\alpha^{n+1}P: & P_n & \leftrightarrows & P_{n+1} \\
 & \| & & \downarrow \\
\alpha^{n}P: & P_n & \leftrightarrows & P_{n-1} \\
 & \downarrow & & \| \\
\alpha^{n-1}P: & P_{n-2} & \leftrightarrows & P_{n-1} \\
 & \| & & \downarrow
\end{array}
\tag{5.9}
$$

where the downward arrows are surjective and the vertical equalities stand for identity maps. An arbitrary minimal special tower has a similar picture with the vertical equalities replaced by isomorphisms.

We next deduce from Proposition 3.5 the corresponding decomposition for divisible S-modules.

Proposition 5.4. *Any divisible S-module P has a decomposition*

$$P = P^{\mu} \oplus P' \oplus P'' \tag{5.10}$$

such that P^{μ} is minimal, the differential in P' from even degree to odd is an isomorphism, and the differential in P'' from odd to even is an isomorphism.

Proof. Let $X = \alpha P$ and let $X = X^{\mu} \oplus X' \oplus X''$ be as in Proposition 3.5. Each of the direct summands in this decomposition satisfies (5.6), hence applying β to this decomposition we obtain a decomposition $P = P^{\mu} \oplus P' \oplus P''$ with $\alpha P^{\mu} = X^{\mu}$, etc. Now we have already seen that P^{μ} must be minimal since X^{μ} is minimal. One can check easily that a divisible S-module Q is such that d is an isomorphism from even degree to odd iff the same is true for αQ, and similarly with odd and even reversed. Consequently this decomposition of P has the required properties. □

Since the summands P' and P'' in (5.10) are homotopy equivalent to zero, we obtain the following analogue of Corollary 3.6 by similar arguments.

Corollary 5.5. *Any divisible S-module is homotopy equivalent to a minimal S-module, which is unique up to noncanonical isomorphism. Any quasi-isomorphism of divisible S-modules is a homotopy equivalence.*

This gives an independent proof of Corollary 1.8 in the case of divisible S-modules.

We next show that α gives rise to a functor $\mathrm{Ho}\,\mathcal{C}_S^{\mathrm{d}} \to \mathrm{Ho}\,\mathcal{T}^{\mathrm{s}}$, which is an equivalence of categories.

The first point is that $\alpha : \mathcal{C}_S^{\mathrm{d}} \to \mathcal{T}^{\mathrm{s}}$ is a fully faithful functor by Proposition 5.2, hence there are induced isomorphisms

$$Z^0 \operatorname{Hom}_S(P,Q) \xrightarrow{\sim} Z_+ \operatorname{Hom}^0(\alpha P, \alpha Q), \qquad f \mapsto \alpha(f) \tag{5.11}$$

for every pair of divisible S-modules.

Lemma 5.6. *Relative to the isomorphism* (5.11) *we have*

$$[d, \operatorname{Hom}_S^{-1}(P,Q)] \xrightarrow{\sim} [d, \operatorname{Hom}^0(\alpha P, \alpha Q)_-], \tag{5.12}$$

in other words, a map $f : P \to Q$ *is nullhomotopic iff* $\alpha(f)$ *is nullhomotopic.*

Proof. We split P as in Proposition 5.4. It clearly suffices to prove the lemma in the three cases where P equals P^μ, P', or P'', and the other two summands are zero. If $P = P'$ or $P = P''$, then we know that P is a contractible S-module and that αP is a contractible tower. Hence both the complex $\operatorname{Hom}_S(P,Q)$ and the supercomplex $\operatorname{Hom}^0(\alpha P, \alpha Q)$ are contractible. Thus (5.12) is immediate from (5.11), because the cycles coincide with the boundaries. Thus we can suppose P is a minimal S-module, and similarly we can suppose Q is minimal.

In this case it is easy to see using the picture (5.9) for the α tower in the case of a minimal S-module that we have natural identifications

$$\operatorname{Hom}_S^0(P,Q) = \operatorname{Hom}^0(\alpha P, \alpha Q)_+ = \operatorname{Hom}^1(\alpha P, \alpha Q)_+ \tag{5.13}$$

and more generally

$$\operatorname{Hom}_S^k(P,Q) = \operatorname{Hom}^k(\alpha P, \alpha Q)_{k+2\mathbb{Z}} = \operatorname{Hom}^{k+1}(\alpha P, \alpha Q)_{k+2\mathbb{Z}} \tag{5.14}$$

for any integer k. The formula (5.14) follows from (5.13) with $Q[k]$ in place of Q by means of the shifting formulas (5.7), (2.6) and (4.11).

We can visualize the relations (5.14) by means of the diagram

$$\begin{array}{lccc}
 & \uparrow & & \| \\
\operatorname{Hom}^1(\alpha P, \alpha Q): & \operatorname{Hom}_S^0(P,Q) & \leftrightarrows & \operatorname{Hom}_S^1(P,Q) \\
 & \| & & \uparrow \\
\operatorname{Hom}^0(\alpha P, \alpha Q): & \operatorname{Hom}_S^0(P,Q) & \leftrightarrows & \operatorname{Hom}_S^{-1}(P,Q) \\
 & \uparrow & & \| \\
\operatorname{Hom}^{-1}(\alpha P, \alpha Q): & \operatorname{Hom}_S^{-2}(P,Q) & \leftrightarrows & \operatorname{Hom}_S^{-1}(P,Q) \\
 & \| & & \uparrow
\end{array} \tag{5.15}$$

where the upward arrows are injective. This diagram is analogous to (5.9) in a dual sense. The minimal S-module P of (5.9) is replaced by the complex $\operatorname{Hom}_S(P,Q)$ having an injective S operator, which is minimal in the sense that $d = 0$ on the cokernel of S. Moreover, the tower αP of (5.9) is replaced by the inductive system with injective arrows consisting of the supercomplexes $\mathrm{Hom}^k(\alpha P, \alpha Q)$.

From either (5.14) for $k = -1$ or from the above diagram we then obtain

$$[d, \mathrm{Hom}_S^{-1}(P, Q)] = [d, \mathrm{Hom}^0(\alpha P, \alpha Q)_-]$$

concluding the proof of the lemma. □

Proposition 5.7. *The functor α induces a functor on homotopy categories*

$$\alpha : \mathrm{Ho}\, \mathcal{C}_S^{\mathrm{d}} \to \mathrm{Ho}\, \mathcal{T}^{\mathrm{s}} \tag{5.16}$$

commuting with the suspension, which is an equivalence of categories. Consequently, we have canonical isomorphisms

$$HC^k(P, Q) \xrightarrow{\sim} HC^k(\alpha P, \alpha Q) \tag{5.17}$$

compatible with cup product.

Proof. The isomorphism (5.17) for $k = 0$ results immediately from (5.11) and (5.12), and it implies that we have an induced functor (5.16) on homotopy categories which is fully faithful. On the other hand, if X is any special tower, then we know from Corollary 3.6 that X is homotopy equivalent to a minimal tower X^μ, which is necessarily special. Then X^μ is a minimal special tower, so by Corollary 5.3 it is isomorphic to αP with P a minimal S-module. This shows that the functor (5.16) is essentially surjective, hence it is an equivalence of categories.

Finally the isomorphism (5.17) for any k reduces to $k = 0$ by means of the shifting formulas (5.7), (2.7) and (4.12). □

For another proof of the fully faithful property of (5.16) and the isomorphism (5.17) see [4].

We note that Proposition 1.12 and Proposition 5.7 together prove the equivalence of the five categories mentioned in the Introduction.

6. The tower $\mathcal{X}(R, I)$

In this section we examine the tower $\mathcal{X}(R, I)$ for R quasi-free which is studied in [4]. This tower is normally not isomorphic to a tower in the image of α, however, we shall show it can be modified simply to obtain quasi-isomorphic towers which do come from divisible S-modules. In this way we can link the towers used in [4] to study cyclic homology in the case of a quasi-free extension to the spectral sequences used in [13].

We begin with some notation from [4]. Let R be an algebra, let $\Omega^1 R_\natural = \Omega^1 R / [\Omega^1 R, R]$ be the commutator quotient space of the R-bimodule $\Omega^1 R$ of noncommutative

1-forms, and let $\natural : \Omega^1 R \to \Omega^1 R_\natural$ denote the canonical surjection. Let $X(R)$ be the supercomplex

$$R \underset{\natural d}{\overset{\bar{b}}{\leftrightarrows}} \Omega^1 R_\natural$$

where $\bar{b}$ is defined by $\bar{b}(\natural(xdy)) = [x,y]$.

Let I be an ideal in R. We have then a tower $\mathcal{X}(R,I)$ of quotient complexes of $X(R)$ given by

$$\mathcal{X}^{2n+1}(R,I): \quad R\big/I^{n+1} \leftrightarrows \Omega^1 R_\natural\big/\natural(I^{n+1}dR + I^n dI)$$

$$\mathcal{X}^{2n}(R,I): \quad R\big/I^{n+1} + [R,I^n] \leftrightarrows \Omega^1 R_\natural\big/\natural(I^n dR)$$

Assume R is quasi-free [2]. The main result 9.4 of [4] says that $\mathcal{X} = \mathcal{X}(R,I)$ is a special tower which is homotopy equivalent to the Hodge tower $\alpha\mathcal{B}(\Omega A)$ of ΩA, where A is the algebra R/I and ΩA is the mixed complex of its noncommutative differential forms equipped with the canonical operators b, B. In particular, the cyclic homology of $\mathcal{X}$ is the cyclic homology of the algebra A, and the same holds for the rest of the cyclic type homology (i.e. Hochschild, negative cyclic and periodic cyclic homology).

The layers of $\mathcal{X}$ are

$$\bar{\mathcal{X}}^{2n+1}: \quad \frac{I^{n+1} + [I^n, R]}{I^{n+1}} \underset{0}{\leftrightarrows} \frac{\natural(I^n dR)}{\natural(I^{n+1}dR + I^n dI)}$$

$$\bar{\mathcal{X}}^{2n}: \quad \frac{I^n}{I^{n+1} + [I^n, R]} \leftrightarrows \frac{\natural(I^n dR + I^{n-1}dI)}{\natural(I^n dR)}$$

In $\bar{\mathcal{X}}^{2n+1}$ the differential $\bar{b}$ from odd degree to even is clearly surjective. We now examine $\bar{\mathcal{X}}^{2n}$, starting with the subspace of even degree.

Lemma 6.1. *One has canonical isomorphisms*

$$(I/I^2)\otimes_R \overset{n}{\cdots} \otimes_R (I/I^2) \xrightarrow{\sim} I^n/I^{n+1},$$

$$[(I/I^2)\otimes_R]^{(n)} \xrightarrow{\sim} I^n\big/I^{n+1} + [I^n, R],$$

induced by $x_1 \otimes \cdots \otimes x_n \mapsto x_1 \cdots x_n$ *for* $x_i \in I$.

Proof. The first isomorphism holds whenever I is flat as left or right R-module, which is the case for R quasi-free, cf. [2, Section 5]. By taking commutator quotient spaces we obtain the second isomorphism, where the left side is the nth circular tensor product of the R-bimodule I/I^2. □

Let us denote the second isomorphism of the lemma by

$$u : W \xrightarrow{\sim} \frac{I^n}{I^{n+1} + [I^n, R]} = \bar{\mathcal{X}}^{2n}_+,$$

where W is the circular tensor product. The cyclic group $\mathbb{Z}/n$ acts on W with the generator acting as the forward shift cyclic permutation σ, and W splits into the direct sum of the subspaces W^σ and $(1-\sigma)W$. We consider the two subspaces of I^n containing $I^{n+1} + [I^n, R]$, which correspond to these subspaces under the isomorphism u. The subspace corresponding to $(1-\sigma)W$ is clearly $I^{n+1} + [I^{n-1}, I]$; let $I^{n,\sigma}$ denote the subspace corresponding to W^σ. We then have isomorphisms

$$u : (1-\sigma)W \xrightarrow{\sim} \frac{I^{n+1} + [I^{n-1}, I]}{I^{n+1} + [I^n, R]},$$

$$u : W^\sigma \xrightarrow{\sim} \frac{I^{n,\sigma}}{I^{n+1} + [I^n, R]},$$

and $\bar{\mathcal{X}}^{2n}_+$ splits into the direct sum of the subspaces on the right.

In order to extend this splitting to all of $\bar{\mathcal{X}}^{2n}$ we introduce the map

$$v : I^{\otimes n} \to \Omega^1 R_\natural, \qquad v(x_1 \otimes \cdots \otimes x_n) = \natural(x_1 \cdots x_{n-1} dx_n).$$

Proposition 6.2. *The supercomplex $\bar{\mathcal{X}}^{2n}$ splits into the direct sum of the subcomplexes*

$$\frac{I^{n,\sigma}}{I^{n+1} + [I^n, R]} \underset{\twoheadrightarrow}{\overset{0}{\longleftarrow}} \frac{\natural(I^n dR + dI^n)}{\natural(I^n dR)},$$

$$\frac{I^{n+1} + [I^{n-1}, I]}{I^{n+1} + [I^n, R]} \underset{0}{\overset{\sim}{\rightleftarrows}} \frac{\natural(I^n dR + v(1-\sigma)(I^{\otimes n}))}{\natural(I^n dR)},$$

where in the former the differential is surjective from even to odd, and in the latter the differential is an isomorphism from odd to even.

Proof. One checks easily that v descends to the circular tensor product W to give a well-defined surjection

$$v : W \twoheadrightarrow \frac{\natural(I^n dR + I^{n-1} dI)}{\natural(I^n dR)} = \bar{\mathcal{X}}^{2n}_-$$

and that the following relations with the differential in $\bar{\mathcal{X}}^{2n}$ hold:

$$\bar{b}v = u(1-\sigma), \qquad (\natural d)u = vN_\sigma.$$

We thus have

$$v((1-\sigma)W) = \frac{\natural(I^n dR + v(1-\sigma)(I^{\otimes n}))}{\natural(I^n dR)},$$

$$v(W^\sigma) = v(N_\sigma W) = (\natural d)\bar{\mathcal{X}}^{2n}_+ = \frac{\natural(I^n dR + d(I^n))}{\natural(I^n dR)}.$$

Now $u((1-\sigma)W) \oplus v((1-\sigma)W)$ is a subcomplex of $\bar{\mathcal{X}}^{2n}$ such that the differential from odd to even is an isomorphism. This follows from $\bar{b}v = u(1-\sigma)$, the fact that

$(1-\sigma)$ is bijective on $(1-\sigma)W$, and the fact that u is an isomorphism. Moreover $u(W^\sigma)\oplus v(W^\sigma)$ is a subcomplex of $\bar{\mathcal{X}}^{2n}$ such that the differential is surjective from even to odd. These two subcomplexes intersect trivially as the differential $\bar{b}$ is injective on $v((1-\sigma)W)$ and zero on $v(W^\sigma)$, and their sum is $\bar{\mathcal{X}}^{2n}$ as v is surjective, which proves the proposition. □

We can describe the layers in the tower $\mathcal{X}$ schematically as follows:

$$\begin{aligned}
\bar{\mathcal{X}}^{2n+1}&: \qquad (\twoheadleftarrow)\\
\bar{\mathcal{X}}^{2n}&: (\overset{\sim}{\longleftarrow}) \oplus (\twoheadrightarrow)\\
\bar{\mathcal{X}}^{2n-1}&: \qquad (\twoheadleftarrow)
\end{aligned}$$

The tower $\mathcal{X}$ is normally not isomorphic to αP for some divisible S-module P because of the even layers. However, we can modify $\mathcal{X}$ so as to shift the contractible part $(\overset{\sim}{\longleftarrow})$ of each even layer to either the preceding or following layer. This gives various divisible S-modules P such that $\mathcal{X}$ and αP are quasi-isomorphic, hence homotopy equivalent by Corollary 3.6.

If all contractible parts are shifted upward we obtain the tower

$$\begin{aligned}
X^{2n+1}&: \quad R/I^{n+1} \quad \rightleftarrows \quad \Omega^1 R_\natural/\natural(I^{n+1}dR + I^n dI)\\
X^{2n}&: \quad R/I^{n+1}+[I^{n-1},I] \rightleftarrows \Omega^1 R_\natural/\natural(I^n dR + v(1-\sigma)(I^{\otimes n}))\\
X^{2n-1}&: \quad R/I^n \quad \rightleftarrows \quad \Omega^1 R_\natural/\natural(I^n dR + I^{n-1}dI)
\end{aligned}$$

which corresponds to the divisible S-module

$$\longrightarrow R/I^{n+1} \longrightarrow \Omega^1 R_\natural/\natural(I^n dR + v(1-\sigma)(I^{\otimes n})) \longrightarrow R/I^n \longrightarrow, \tag{6.1}$$

where R/I^{n+1} is in degree $2n$.

If all contractible parts are shifted downward we obtain the tower

$$\begin{aligned}
Y^{2n+1}&: \quad R/I^{n+1,\sigma} \quad \rightleftarrows \quad \Omega^1 R_\natural/\natural(I^{n+1}dR + d(I^{n+1}))\\
Y^{2n}&: \quad R/I^{n+1}+[I^n,R] \rightleftarrows \Omega^1 R_\natural/\natural(I^n dR)\\
Y^{2n-1}&: \quad R/I^{n,\sigma} \quad \rightleftarrows \quad \Omega^1 R_\natural/\natural(I^n dR + d(I^n))
\end{aligned}$$

which corresponds to the divisible S-module

$$\longrightarrow R/I^{n+1,\sigma} \longrightarrow \Omega^1 R_\natural/\natural(I^n dR) \longrightarrow R/I^{n,\sigma} \longrightarrow, \tag{6.2}$$

There are surjective quasi-isomorphisms $Y \to \mathcal{X} \to X$ of towers and a surjective quasi-isomorphism from (6.2) to (6.1). The homology of both complexes is the cyclic homology of $A = R/I$.

The complex (6.1) is closely related to the spectral sequence [13, I.5.5], and one can derive from it without much difficulty the exact sequence

$$0 \to HC_{2n}A \to HC_0(R/I^{n+1}) \to H_1(R, R/I^n)_\sigma \to HC_{2n-1}A \to 0$$

of [13, I.5.13]. Moreover, (6.2) is the quotient of the periodic complex

$$\longrightarrow R \longrightarrow \Omega^1 R_\natural \longrightarrow R \longrightarrow$$

by the complex

$$\longrightarrow I^{n+1,\sigma} \longrightarrow \natural(I^n dR) \longrightarrow I^{n,\sigma} \longrightarrow . \tag{6.3}$$

Assuming R is free, the periodic complex has homology $\mathbb{C}$ in even degrees and zero in odd degrees, and the homology of (6.3) is the reduced cyclic homology of A. The complex (6.3) is closely related to the spectral sequence [13, I.5.5], and one can obtain from it the exact sequence

$$0 \to \bar{HC}_{2n+1}A \to I^{n+1}/[I^n, I] \to H_1(R, I^n) \to \bar{HC}_{2n}A \to 0$$

of [13, I.5.11].

Finally, supposing only R quasi-free, we apply the preceding discussion to determine the homology of the supercomplex

$$X(R/I^n) : R/I^n \leftrightarrows \Omega^1 R_\natural \big/ \natural(I^n dR + d(I^n)),$$

cf. [4, Section 4]. We observe that there are surjections

$$Y^{2n-1} \to X(R/I^n) \to \mathcal{X}^{2n-1} = X^{2n-1}$$

whose composition is a quasi-isomorphism, and that the kernel of the former is

$$I^n/I^{n,\sigma} \simeq W/W^\sigma \simeq (1-\sigma)W$$

in even degree and zero in odd degree. Thus $H_*(X(R/I^n))$ is the direct sum of $H_*(\mathcal{X}^{2n-1})$ and the homology of this kernel, so we find

$$\begin{aligned} HD_0(R/I^n) &= H_+\left(X(R/I^n)\right) = HD_{2n-2}A, \\ HC_1(R/I^n) &= H_-\left(X(R/I^n)\right) = HC_{2n-1}A \oplus (1-\sigma)W. \end{aligned} \tag{6.4}$$

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