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The finite volume method based on stabilized finite element for the stationary Navier–Stokes problem ☆

Guoliang He[a], Yinnian He[b,*]

[a] *Faculty of Science, Xi'an Jiaotong University, Xi'an 710049, PR China*

[b] *Faculty of Science, State Key Laboratory of Multiphase Flow in Power Engineering, Xi'an Jiaotong University, Xi'an 710049, PR China*

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Abstract

A finite volume method based on stabilized finite element for the two-dimensional stationary Navier–Stokes equations is investigated in this work. A macroelement condition is introduced for constructing the local stabilized formulation for the problem. We obtain the well-posedness of the FVM based on stabilized finite element for the stationary Navier–Stokes equations. Moreover, for quadrilateral and triangular partition, the optimal H^1 error estimate of the finite volume solution u_h and L^2 error estimate for p_h are introduced. Finally, we provide a numerical example to confirm the efficiency of the FVM.

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Keywords: Stationary Navier–Stokes equations; Finite volume method; Stabilized finite element; Error estimation

1. Introduction

Finite difference and finite element methods have been widely used in computational fluid dynamics. On the one hand, the finite difference methods are easy to set up and implement, and conserve mass locally; on the other hand, they are not flexible in the treatment of complicated geometry and general boundary conditions. The finite element methods have the intrinsic grid flexibility but do not conserve mass locally (i.e., at the element level). Recently, finite volume methods (FVMs) have been employed to enforce such a local conservation property [10]. Generally speaking, the FVMs can be treated as an efficient middle ground between the finite difference and finite element methods. They were developed as an attempt to use the finite element idea in the finite difference setting. Based on volumes or control volumes, their basic idea is to approximate discrete fluxes of a partial differential equation using a finite element procedure. Flux-oriented approximation methods are collectively known as box methods [2,24], generalized difference methods [32,25,26,18], FVMs [6,4,14,20], control volume methods [9], balance methods [5], MAC (marker and cell) methods [12,27,16], covolume methods [11], and cell-centered approximation methods [7].

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* Corresponding author.

E-mail addresses: hegl@mailst.xjtu.edu.cn (G. He), heyn@mail.xjtu.edu.cn (Y. He).

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Regardless of their physical interpretations, FVMs can be mathematically treated as Petrov–Galerkin methods with trial function spaces associated with certain finite element spaces and test spaces related to finite volumes. In this paper we relate these methods for the Navier–Stokes equations to standard Galerkin or mixed finite element methods through introducing an interpolation (or lumping) operator that maps the trial spaces into the test spaces. The interpolation operator itself does not play a large role in the implementation of the FVMs.

FVMs have been widely used, but their analysis is far behind that of finite element methods, not to mention their analysis for complex problems in computational fluid dynamics. The development of efficient discretization methods is key for numerically solving the transient Navier–Stokes equations. On the one hand, their numerical solution requires the development of accurate, locally conservative discretization methods. On the other hand, these discretization methods must satisfy the discrete inf–sup (stability) condition to ensure the compatibility of approximations for velocity and pressure. It is also well known that the simplest conforming low-order elements like the $P_1 - P_0$ (linear velocity, constant pressure) triangular element and $Q_1 - P_0$ (bilinear velocity, constant pressure) quadrilateral element are not stable.

During the last two decades there has been a rapid development in practical stabilization techniques for the $P_1 - P_0$ element and the $Q_1 - P_0$ element for solving the Stokes problem. For this purpose a local "macroelement condition" and some energy methods have been used. The use of such a macroelement condition as a means of verifying the (Babuška–Brezzi) inf–sup condition is a standard technique (see, for example [15]); the basic idea was first introduced by Boland and Nicolaides [3], and independently by Stenberg [31]. Recently, Kechkar and Silvester [22,30], Kay and Silvester [21], Norburn and Silvester [28] pursued work which laid the foundations of the mathematical analysis and numerics of locally stabilized mixed finite element methods for the Stokes problem. For stationary Navier–Stokes, the results in He [17] are also valuable.

The aim of this paper is to construct the FVM based on the stabilized finite element method for solving the stationary Navier–Stokes problem. A macroelement condition is introduced for constructing the local stabilized formulation of the problem. By satisfying this condition the stability of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element are established. Moreover, we obtain the well-posedness and the optimal error estimate of the stabilized FVM for the stationary Navier–Stokes problem.

An outline of the paper is as follows. In the next section we introduce the mathematical setting of the stationary Navier–Stokes problem. In Section 3, we recall the notion of local stabilization of the $Q_1 - P_0$ quadrilateral element and the $P_1 - P_0$ triangular element based on the macroelement condition. We, in Section 4, prove the well-posedness of the stabilized FVM and the optimal H^1 error estimate of u_h and L^2 error estimate of p_h introduced in Section 3. Finally, we provide a numerical example to confirm the efficiency of the FVM. Below c (with or without a subscript) is a generic positive constant depending only on Ω.

2. Functional setting of the Navier–Stokes problem

Let Ω be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$ and to satisfy a further condition stated in (A1) below. We consider the stationary Navier–Stokes equations

$$\begin{cases} -\nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \operatorname{div} u = 0, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{2.1}$$

where $u = (u_1(x), u_2(x))$ represents the velocity vector, $p = p(x)$ the pressure, $f = f(x)$ the prescribed body force, and $\nu > 0$ the viscosity.

For the mathematical setting of problem (2.1), we introduce the following Hilbert spaces

$$X = (H_0^1(\Omega))^2, \quad Y = (L^2(\Omega))^2, \quad M = \left\{ q \in L^2(\Omega) : \int_\Omega q \, \mathrm{d}x = 0 \right\},$$

$$H = \{ v \in L^2(\Omega)^2;\ \operatorname{div} v = 0 \text{ in } \Omega, \text{ and } v \cdot \mathbf{n}|_{\partial\Omega} = 0 \}.$$

The spaces $(L^2(\Omega))^m$ $(m = 1, 2, 4)$ are endowed with the usual L^2-scalar product $(\cdot, \cdot)$ and norm $\|\cdot\|_0$, as appropriate. The space X is equipped with the scalar product $(\nabla u, \nabla v)$ and norm $\|\nabla u\|_0$.

Define $Au = -\Delta u$, which is the operator associated with the Navier–Stokes equations. It's positive self-adjoint operator from $D(A) = (H^2(\Omega))^2 \cap X$ onto Y, so, for $\alpha \in \mathbb{R}$, the powers A^α of A is well defined. In particular, $D(A^{1/2}) = X$, $D(A^0) = Y$, and

$$(A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \quad \|A^{1/2}u\|_0 = (\nabla u, \nabla u)^{1/2}$$

for all $u, v \in X$.

As mentioned above, we need a further assumption on Ω:

(A1) Assume that Ω is regular so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\Delta v + \nabla q = g, \quad \operatorname{div} v = 0 \text{ in } \Omega, \quad v|_{\partial\Omega} = 0$$

for a prescribed $g \in Y$ exists and satisfies

$$\|v\|_2 + \|q\|_1 \leqslant c\|g\|_0,$$

where $\|\cdot\|_i$ denotes the usual norm of the Sobolev space $H^i(\Omega)$ or $H^i(\Omega)^2$ for $i = 1, 2$.

We also introduce the following bilinear operator

$$B(u, v) = (u \cdot \nabla)v + \tfrac{1}{2}(\operatorname{div} u)v \quad \forall u, v \in X.$$

Moreover, we define the continuous bilinear forms $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = \nu((u, v)) \quad \forall u, v \in X, \quad d(v, q) = -(v, \nabla q) = (q, \operatorname{div} v) \quad \forall v \in X, \ q \in M,$$

and a generalized bilinear form on $(X, M) \times (X, M)$ by

$$\mathscr{B}((u, p); (v, q)) = a(u, v) - d(v, p) + d(u, q),$$

and a trilinear form on $X \times X \times X$ by

$$\begin{aligned} b(u, v, w) &= \langle B(u, v), w\rangle_{X' \times X} = ((u \cdot \nabla)v, w) + \tfrac{1}{2}((\operatorname{div} u)v, w) \\ &= \tfrac{1}{2}((u \cdot \nabla)v, w) - \tfrac{1}{2}((u \cdot \nabla)w, v) \quad \forall u, v, w \in X. \end{aligned}$$

We remark that the validity of assumption (A1) is known (see [19,23]) if $\partial\Omega$ is of C^2, or if Ω is a two-dimensional convex polygon. From assumption (A1), it is easily shown [19] that

$$\|v\|_0 \leqslant \gamma_0 \|A^{1/2}v\|_0 \quad \forall v \in X, \quad \|v\|_2 \leqslant \gamma_1 \|\widetilde{\Delta} v\|_0 \quad \forall v \in H^2(\Omega)^2 \cap X, \tag{2.2}$$

where $\widetilde{\Delta} = P\Delta$, and P is the L^2-orthonormal projection of $L^2(\Omega)^2$ onto the space H, and $\gamma_0, \gamma_1, \ldots$ are positive constants depending only on Ω.

It is easy to verify that $\mathscr{B}$ and b satisfy the following important properties (see [15,19,21,22]): there hold

$$\begin{cases} \nu\|A^{1/2}u\|_0^2 = \mathscr{B}((u, p); (u, p)), \\ |\mathscr{B}((u, p); (v, q))| \leqslant \gamma_2(\|A^{1/2}u\|_0 + \|p\|_0)(\|A^{1/2}v\|_0 + \|q\|_0), \\ \alpha_0(\|A^{1/2}u\|_0 + \|p\|_0) \leqslant \sup\limits_{(v,q)\in(X,M)} \dfrac{\mathscr{B}((u, p); (v, q))}{\|A^{1/2}v\|_0 + \|q\|_0} \end{cases} \tag{2.3}$$

for all $(u, p), (v, q) \in (X, M)$ and constants $\gamma_2 > 0$ and $\alpha_0 > 0$,

$$b(u, v, w) = -b(u, w, v), \tag{2.4}$$

$$|b(u, v, w)| \leqslant \tfrac{1}{2}c_0\|u\|_0^{1/2}\|A^{1/2}u\|_0^{1/2}(\|A^{1/2}v\|_0\|w\|_0^{1/2}\|A^{1/2}w\|_0^{1/2} + \|v\|_0^{1/2}\|A^{1/2}v\|_0^{1/2}\|A^{1/2}w\|_0), \tag{2.5}$$

$$|b(u, v, w)| \leqslant c\|u\|_0^{1/2}\|A^{1/2}u\|_0^{1/2}\|A^{1/2}v\|_0\|w\|_0^{1/2}\|A^{1/2}w\|_0^{1/2} \tag{2.6}$$

for all $u, v, w \in X$ and

$$|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)| \leqslant c\|A^{1/2}u\|_0\|Av\|_0\|w\|_0 \tag{2.7}$$

for all $u \in X$, $v \in D(A)$, $w \in Y$.

Under the above notations, the variational formulation of the problem (2.1) reads as: find $(u, p) \in (X, M)$ such that for all $(v, q) \in (X, M)$:

$$\mathscr{B}((u, p); (v, q)) + b(u, u, v) = (f, v). \tag{2.8}$$

The following existence and uniqueness results are classical (see [33]).

Theorem 2.1. *Assume that* ν *and* $f \in Y$ *satisfy the following uniqueness condition*:

$$1 - \frac{N_1}{\nu^2}\|f\|_{-1} > 0,$$

where

$$N_1 = \sup_{u,v,w \in H_0^1(\Omega)} \frac{b(u, v, w)}{\|A^{1/2}u\|_0 \|A^{1/2}v\|_0 \|A^{1/2}w\|_0}.$$

Then the problem (2.8) *admits a unique solution* $(u, p) \in (D(A) \cap X, H^1(\Omega) \cap M)$ *such that*

$$\|A^{1/2}u\|_0 \leqslant \nu^{-1}\|f\|_{-1}, \quad \|u\|_2 + \|p\|_1 \leqslant c\|f\|_0. \tag{2.9}$$

3. FVM based on stabilized finite element approximation

Based on stabilized finite element method developed for the incompressible Navier–Stokes equations [17], we, in this section, consider the FVM for two-dimensional stationary incompressible Navier–Stokes equations (2.1). Let $h > 0$ be a real positive parameter. The finite element subspace (X_h, M_h) of (X, M) is characterized by $T_h = T_h(\Omega)$, a partitioning of $\bar{\Omega}$ into triangles or quadrilaterals, assumed to be regular in the usual sense (see [13,15,21,22]), i.e., for some σ and ω with $\sigma > 1$ and $0 < \omega < 1$,

$$h_K \leqslant \sigma \rho_K \quad \forall K \in \tau_h, \tag{3.1}$$

$$|\cos \theta_{iK}| \leqslant \omega, \quad i = 1, 2, 3, 4 \;\; \forall K \in \tau_h, \tag{3.2}$$

where h_K is the diameter of element K, ρ_K is the diameter of the inscribed circle of element K, and θ_{iK} are the angles of K in the case of a quadrilateral partitioning. The mesh parameter h is given by $h = \max\{h_K\}$, and the set of all interelement boundaries will be denoted by Γ_h.

We also construct a dual partition T_h^* of T_h. The dual partition, consisting of dual elements of T_h usually called boxes or control volumes, is arbitrary, but the choice of appropriate dual elements is crucial for the analysis of the underlying discretization method. In general, the choices of dual partition based on circumcenter, orthocenter, incenter, or centroid of $K \in T_h$ are adaptable with some restricts, more or less, of the partition T_h (see [2,14,7]).

We call the dual partition T_h^* *regular* or *quasi-uniform* if there exist a positive constant $c > 0$ such that

$$c^{-1}h^2 \leqslant \text{meas}(K_i^*) \leqslant ch^2 \quad \forall K_i^* \in T_h^*;$$

here h is the maximum diameter of all element $K \in T_h$.

In this paper, we use a popular configuration in which the interior point p_i is chosen to the barycenter of element $K_i \in T_h$, and the midpoint m_{ij} on side of $\overline{v_i v_j}$ in Fig. 1. This type of dual partition for triangular and quadrilateral leads to relatively simple calculations for both two- and three-dimensional problems. In addition, if T_h is locally regular, then this dual partition is also regular.

Finite element subspaces of interest in this paper are defined by setting

$$R_1(K) = \begin{cases} P_1(K) & \text{if } K \text{ is triangular,} \\ Q_1(K) & \text{if } K \text{ is quadrilateral,} \end{cases} \tag{3.3}$$

giving the continuous piecewise (bi)linear velocity subspace

$$X_h = \{v \in X : v_h|_K \in (R_1(K))^2 \;\forall K \in T_h\},$$

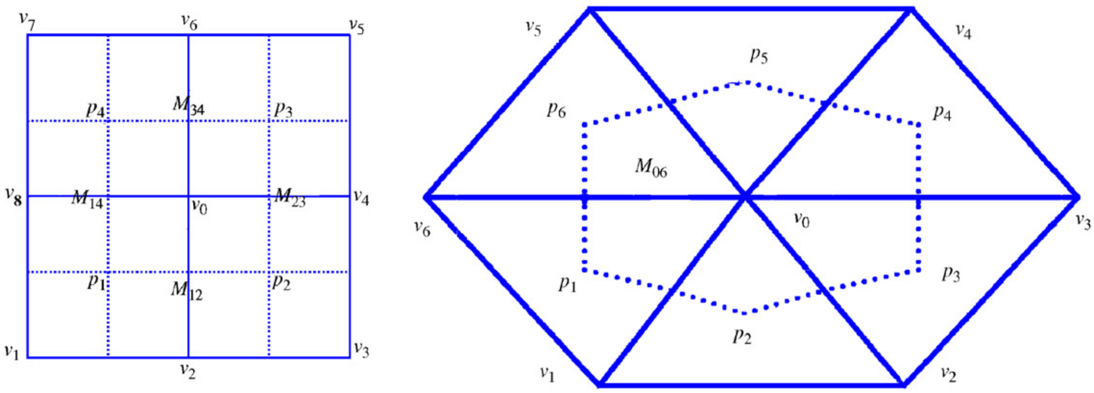

Fig. 1. The partition and dual partition of a rectangular and triangular.

the piecewise constant pressure subspace

$$M_h = \{q \in M : q|_K \in P_0(K)\ \forall K \in T_h\},$$

and the dual space of velocity subspace X_h^*,

$$X_h^* = \{v \in (L^2(\Omega))^2 : v|_{K^*} \in (P_0(K^*))^2\ \forall K^* \in T_h^*\}.$$

Actually, this choice of X_h^* is the span of the characteristic functions of the volume K^*.

Note that neither of these finite element methods are stable in standard Babuška–Brezzi sense; $P_1 - P_0$ triangle "locks" on regular grids (since there are more discrete incompressibility constraints than velocity degrees of freedom), and the $Q_1 - P_0$ quadrilateral is the most infamous example of an unstable mixed method, as elucidated by Sani et al. [29].

Furthermore, we need to introduce the discrete analogue $A_h : X_h \to X_h$ of the operator $A = -\Delta$, through the condition:

$$(A_h v_h, \phi_h) = (A^{1/2} v_h, A^{1/2}\phi_h) \quad \forall v_h, \phi_h \in X_h,$$
$$\|A_h^{1/2} u_h\|_0^2 = (A^{1/2} u_h, A^{1/2} u_h) \quad \forall u_h \in X_h.$$

We have

$$\|A_h^{1/2} u_h\|_0 = \|A^{1/2} u_h\|_0 \quad \forall u_h \in X_h.$$

Let interpolation operator $I_h^* : X_h \to X_h^*$,

$$I_h^* u_h = \sum_{x_i \in N_h} u_h(x_i)\chi_i(x),$$

where $N_h = \{P_i :$ vertices of quadrilateral or triangles in $T_h\}$.

We define the continuous bilinear forms $\widetilde{a}(\cdot,\cdot)$ and $\widetilde{d}(\cdot,\cdot)$ on $X_h \times X_h$ and $X_h \times M_h$, respectively, by

$$\widetilde{a}(u_h, I_h^* v_h) = \nu((u_h, I_h^* v_h)) = -\nu \sum_{K_i^* \in T_h^*} \int_{\partial K_i^*} v_h(x_i) \frac{\partial u_h}{\partial n}\, \mathrm{d}s \quad \forall u_h, v_h \in X_h,$$

$$\widetilde{d}(I_h^* v_h, p_h) = (I_h^* v_h, \nabla p_h) = \sum_{K_i^* \in T_h^*} \int_{\partial K_i^*} v_h(x_i) p_h \cdot n\, \mathrm{d}s \quad \forall u_h \in X_h, \quad p_h \in M_h,$$

where n is the outnormal vector. We also define the trilinear forms $\widetilde{\widetilde{b}}(\cdot,\cdot,\cdot)$, $\widetilde{b}(\cdot,\cdot,\cdot)$, and $\overline{b}(\cdot,\cdot,\cdot)$ on $X_h \times X_h \times X_h$ by

$$\widetilde{\widetilde{b}}(u_h, v_h, I_h^* w_h) = ((u_h \cdot \nabla)v_h, I_h^* w_h),$$

$$\widetilde{b}(u_h, v_h, I_h^* w_h) = ((u_h \cdot \nabla)v_h, I_h^* w_h) + \tfrac{1}{2}((\operatorname{div} u_h)v_h, I_h^* w_h),$$

$$\overline{b}(u_h, v_h, w_h - I_h^* w_h) = ((u_h \cdot \nabla)v_h, w_h - I_h^* w_h) + \tfrac{1}{2}((\operatorname{div} u_h)v_h, w_h - I_h^* w_h),$$

for all $u_h, v_h, w_h \in X_h$, the right side of term

$$(f, I_h^* v_h) = \sum_{K_i^* \in T_h^*} \int_{K_i^*} v_h(x_i) f \,\mathrm{d}x \quad \forall v_h \in X_h,$$

and a generalized bilinear form on

$$\widetilde{\mathscr{B}}((u_h, p_h); (I_h^* v_h, q_h)) = \widetilde{a}(u_h, I_h^* v_h) - \widetilde{d}(I_h^* v_h, p_h) + d(u_h, q_h).$$

We define the norms and semi-norms

$$\|u_h\|_{0,h} = \left(\sum_{K \in T_h} \|u_h\|_{0,h,K}^2\right)^{1/2},$$

$$\|\widetilde{A}_h^{1/2} u_h\|_0 = \left(\sum_{K \in T_h} \|\widetilde{A}_h^{1/2} u_h\|_{0,h,K}^2\right)^{1/2},$$

$$\|u_h\|_{1,h} = (\|u_h\|_{0,h}^2 + \|\widetilde{A}_h^{1/2} u_h\|_0^2)^{1/2},$$

where

$$\|u_h\|_{0,h,K} = \begin{cases} \left[\dfrac{S_v}{3}(u_{P_i}^2 + u_{P_j}^2 + u_{P_k}^2)\right]^{1/2}, & \text{if } K \text{ is triangular}, \\ \left[\dfrac{S_p}{4}(u_{P_1}^2 + u_{P_2}^2 + u_{P_3}^2 + u_{P_4}^2)\right]^{1/2}, & \text{if } K \text{ is rectangular}, \end{cases}$$

$$\|\widetilde{A}_h^{1/2} u_h\|_{0,h,K} = \begin{cases} \left\{\left[\left(\dfrac{\partial u_h(p)}{\partial x}\right)^2 + \left(\dfrac{\partial u_h(p)}{\partial y}\right)^2\right] S_v\right\}^{1/2}, & \text{if } K \text{ is triangular}, \\ \left\{\left[\left(\dfrac{\partial u_h(M_{12})}{\partial x}\right)^2 + \left(\dfrac{\partial u_h(M_{34})}{\partial x}\right)^2 \right.\right. & \\ \quad \left.\left. + \left(\dfrac{\partial u_h(M_{41})}{\partial y}\right)^2 + \left(\dfrac{\partial u_h(M_{23})}{\partial y}\right)^2\right] S_p\right\}^{1/2}, & \text{if } K \text{ is rectangular}, \end{cases}$$

with S_v, S_p the area of $\triangle v_i v_j v_k$ and $\square p_1 p_2 p_3 p_4$, respectively (see Fig. 1).

In order to define a locally stabilized formulation of the stationary Navier–Stokes problem, we introduce *a macroelement partitioning* Λ_h as follows: given any subdivision T_h, a macroelement partitioning Λ_h may be defined such that each macroelement $\mathscr{K}$ is a connected set of adjoining elements from T_h. Every element K must lie in exactly one macroelement, which implies that macroelements do not overlap. For each $\mathscr{K}$, the set of interelement edges which are strictly in the interior of $\mathscr{K}$ will be denoted by $\Gamma_{\mathscr{K}}$, and the length of an edge $e \in \Gamma_{\mathscr{K}}$ is denoted by h_e. For a macroelement $\mathscr{K}$ the restricted pressure space is given by

$$M_{0,h} = \{q \in L_0^2(\mathscr{K}) : q|_K \in P_0(K)\ \forall K \in \mathscr{K}\}.$$

With the above choices of the velocity–pressure finite element spaces X_h, X_h^*, M_h and these additional definitions, a locally stabilized formulation of the Navier–Stokes problem (2.8) can be stated as follows.

Definition 3.1 (*Locally stabilized FVM formulation*). Find $(u_h, p_h) \in (X_h, M_h)$, such that for all $(v, q) \in (X_h, M_h)$,

$$\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v, q)) + \widetilde{\widetilde{b}}(u_h, u_h, I_h^* v_h) = (f, I_h^* v), \tag{3.4}$$

where

$$\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v, q)) = \widetilde{\mathscr{B}}((u_h, p_h); (I_h^* v, q)) + \beta \mathscr{C}_h(p_h, q) \quad \forall (u_h, p_h), (v, q) \in (X_h, M_h),$$

$$\mathscr{C}_h(p, q) = \sum_{\mathscr{K} \in \Lambda_h} \sum_{e \in \Gamma_{\mathscr{K}}} h_e \int_e [p]_e [q]_e \, \mathrm{d}s,$$

for all p, q in the algebraic sum $H^1(\Omega) + M_h$, and $[\cdot]_e$ is the jump operator across $e \in \Gamma_{\mathscr{K}}$ and $\beta > 0$ is the local stabilization parameter.

A general framework for analyzing the locally stabilized formulation (3.4) can be developed using the notion of equivalence class of macroelements. As in Stenberg [31], each equivalence class, denoted by $\mathscr{E}_{\hat{\mathscr{K}}}$, contains macroelements which are topologically equivalent to a reference macroelement $\hat{\mathscr{K}}$. To illustrate the idea, two practical examples of locally stabilized mixed approximations are given below.

Example 3.1. A locally stabilized formulation (3.4) can be constructed in this case, if T_h is such that the elements K can be grouped into 2×2 macroelements $\mathscr{K} = \{K_1, K_2, K_3, K_4\}$, with the reference macroelement

$$\hat{\mathscr{K}} = \{\hat{K}_1, \hat{K}_2, \hat{K}_3, \hat{K}_4\}.$$

An obvious way of constructing such a partitioning in practice is to form the grid T_h by uniformly refining a coarse grid Λ_h, for example, by joining the mid-edge points.

Example 3.2. The triangular $P_1 - P_0$ approximation pair can similarly be established if the partitioning T_h is constructed such that the elements can be grouped into disjoint macroelements, all consisting of four elements.

For the above finite element spaces X_h and M_h, it is well known that the following approximation properties and inverse inequality

$$\begin{aligned}
&\|A_h^{1/2} v_h\|_0 \leqslant c h^{-1} \|v_h\|_0 \quad \forall v_h \in X_h,\\
&\|v - I_h v\|_0 + h\|A_h^{1/2}(v - I_h v)\|_0 \leqslant c h^2 \|A_h v\|_0 \quad \forall v \in D(A),\\
&\|v - I_h^* v\|_0 \leqslant c h \|A_h^{1/2} v\|_0 \quad \forall v \in X_h,\\
&\|q - J_h q\|_0 \leqslant c h \|q\|_1 \quad \forall q \in H^1(\Omega) \cap M,
\end{aligned} \tag{3.5}$$

hold (see [1,13]), where $I_h : D(A) \to X_h$ is the interpolation operator and $J_h : H^1(\Omega) \cap M \to M_h$ is the L^2-orthogonal projection.

With all the above notation, we have the following [14,25,26].

Lemma 3.2. *There exist constants* $c_1, c_1', c_2, c_2' > 0$, *independent of* h, *such that*

$$\begin{aligned}
&c_1' \|u_h\|_{0,h} \leqslant \|u_h\|_0 \leqslant c_1 \|u_h\|_{0,h} \quad \forall u_h \in X_h,\\
&c_2' \|\widetilde{A}_h^{1/2} u_h\|_0 \leqslant \|A_h^{1/2} u_h\|_0 \leqslant c_2 \|\widetilde{A}_h^{1/2} u_h\|_0 \quad \forall u_h \in X_h.
\end{aligned} \tag{3.6}$$

Following [31], we define Q_h^* to the L^2 projection from M_h onto the subspace

$$Q_h = \{\mu \in L_0^2(\Omega) : \ \mu|_{\mathscr{K}} \in P_0(\mathscr{K}) \ \forall \mathscr{K} \in \Lambda_h\}.$$

Set

$$\gamma_{\mathscr{K}} = \inf_{p\in M_{0,K},\|p\|_{0,M}=1} \mathscr{C}_h(p, p),$$

we have the following "global" inequality [22]:

$$\mathscr{C}_h(q, q) \geqslant \alpha_1 \|(I - Q_h^*)q\|_0^2 \quad \forall q \in M_h, \tag{3.7}$$

where $\alpha_1 = \min\{\gamma_{\widehat{M}_i}, i = 1, \ldots, n\}$ and is independent of h.

The following stability results of these mixed methods for the macroelement partitioning defined above were formally established by Kay and Silvester [21] and Kechkar and Silvester [22]. Throughout the article we shall assume that $\beta \geqslant \beta_0$.

Theorem 3.3. *Given a stabilization parameter $\beta \geqslant \beta_0 > 0$, suppose that every macroelement $\mathscr{K} \in \Lambda_h$ belongs to one of the equivalence classes $\mathscr{E}_{\hat{\mathscr{K}}}$, and that the following macroelement connectivity condition is valid: for any two neighboring macroelements $\mathscr{K}_1$ and $\mathscr{K}_2$ with $\int_{\mathscr{K}_1\cap\mathscr{K}_2} \mathrm{d}s \neq 0$ there exists $v \in X_h$ such that*

$$\operatorname{supp} v \subset \mathscr{K}_1 \cup \mathscr{K}_2 \quad \textit{and} \quad \int_{\mathscr{K}_1\cap\mathscr{K}_2} v \cdot n \, \mathrm{d}s \neq 0. \tag{3.8}$$

Then,

$$|\mathscr{C}_h(p, q)| \leqslant c \sum_{K\in T_h} \left(\int_K (|p|^2 + h^2|\nabla p|^2)\,\mathrm{d}x\right)^{1/2} \left(\int_K (|q|^2 + h^2|\nabla q|^2)\,\mathrm{d}x\right)^{1/2} \tag{3.9}$$

for all $p, q \in H^1(\Omega) + M_h$, and

$$\mathscr{C}_h(p, q_h) = 0, \quad \mathscr{C}_h(p_h, q) = 0, \quad \mathscr{C}_h(p, q) = 0 \quad \forall p, q \in H^1(\Omega), \quad p_h, q_h \in M_h, \tag{3.10}$$

where $c > 0$ is a constant independent of h and β, and β_0 is some fixed positive constant.

4. Error estimates

Our view of the FVE method as a Petrov–Galerkin finite element method suggests that we treat the FVE method as a perturbation of the Galerkin finite element method so that we can derive optimal-order H^1 error estimate for u_h and L^2 for p_h with a reasonable assumption.

The following two lemmas can be found in [14,12,25,26,8].

Lemma 4.1. *For any $u_h, v_h \in X_h$, we have*

$$a(u_h, v_h) = \widetilde{a}(u_h, I_h^* v_h) + E(u_h, v_h), \tag{4.1}$$

$$d(v_h, p_h) = \widetilde{d}(I_h^* v_h, p_h), \tag{4.2}$$

with

$$E(u_h, v_h) = \begin{cases} 0 & \textit{if } K \textit{ is triangular partition}, \\ \frac{1}{24}\sum_K [h_x^s (h_y^s)^3 + (h_x^s)^3 h_y^s](u_{h\ xy} \cdot v_{h\ xy}) & \textit{if } K \textit{ is rectangular partition}. \end{cases}$$

Lemma 4.2. *For any $u_h, v_h \in X_h$, $q_h \in M_h$, there exist constants $c, \alpha_2, c_3, c_3' > 0$, independent of h, such that*

$$\widetilde{a}(u_h, I_h^* v_h) \leqslant c\|\widetilde{A}_h^{1/2} u_h\|_0 \|A_h^{1/2} v_h\|_0, \quad \widetilde{a}(u_h, I_h^* u_h) \geqslant \alpha_2 \|\widetilde{A}_h^{1/2} u_h\|_0^2, \tag{4.3}$$

$$c_3\|\widetilde{A}_h^{1/2} u_h\|_0^2 \leqslant E(u_h, u_h) \leqslant c_3' \|\widetilde{A}_h^{1/2} u_h\|_0^2, \tag{4.4}$$

$$\widetilde{d}(u_h, p_h) \leqslant c\|\widetilde{A}_h^{1/2} u_h\|_0 \|p_h\|_0. \tag{4.5}$$

In order to derive error estimates of (u_h, p_h) in the FVM, we need to prove the existence of the variational problem (3.4).

Lemma 4.3. *Under the assumptions of Theorem* 3.3, *there exist constants* γ' *and* $\alpha > 0$ *such that*

$$\nu\|\widetilde{A}_h^{1/2}u_h\|_0^2 + \beta\mathscr{C}_h(p_h, p_h) = \widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* u_h, p_h)), \tag{4.6}$$

$$|\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v_h, q_h))| \leqslant \gamma_2'(\|\widetilde{A}_h^{1/2}u_h\|_0 + \|p_h\|_0)(\|\widetilde{A}_h^{1/2}v_h\|_0 + \|q_h\|_0), \tag{4.7}$$

$$\alpha(\|\widetilde{A}_h^{1/2}u_h\|_0 + \|p_h\|_0) \leqslant \sup_{(v_h,q_h)\in(X_h,M_h)} \frac{\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v_h, q_h))}{\|\widetilde{A}_h^{1/2}v_h\|_0 + \|q_h\|_0}, \tag{4.8}$$

hold for all (u_h, p_h) *and* $(v_h, q_h) \in (X_h, M_h)$.

Proof. Eqs. (4.6) and (4.7) are the results of (2.3) and Lemmas 3.2 and 4.1.

For quadrilateral partition, let $(u_h, p_h) \in (X_h, M_h)$. Because of the condition (3.8), there exist a positive constant α_3 [22], independent of *h*, and a $g_h \in X_h$ satisfy

$$(Q_h^* p_h, \operatorname{div} g_h) = \|Q_h^* p_h\|_0^2 \quad \text{and} \quad \|\widetilde{A}_h^{1/2} g_h\|_0 \leqslant \alpha_3 \|Q_h^* p_h\|_0. \tag{4.9}$$

If $v_h \in X_h$ and $q_h \in M_h$ are now chosen such that $v_h = u_h - \delta g_h$ and $q_h = -p_h$, where

$$\delta = \frac{1}{\alpha_3^2}\left(1 + c_3' + \frac{1}{\alpha_1\beta_0}\right)^{-1}. \tag{4.10}$$

With (3.7), Lemma (4.1), (4.9), (4.10), it follows that

$$\begin{aligned}
\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v_h, q_h)) &= \widetilde{a}(u_h, I_h^* u_h) + E(u_h, u_h) - \delta\widetilde{a}(u_h, I_h^* g_h) \\
&\quad - \delta E(u_h, g_h) - \delta d(p_h, \operatorname{div} g_h) + \mathscr{C}_h(p_h, p_h) \\
&\geqslant (1 + c_3)\|\widetilde{A}_h^{1/2}u_h\|_0^2 - (1 + c_3')\delta\|\widetilde{A}_h^{1/2}u_h\|_0\|\widetilde{A}_h^{1/2}g_h\|_0 \\
&\quad + \delta d(Q_h^* p_h, \operatorname{div} g_h) - \delta d((Q_h^* - I)p_h, \operatorname{div} g_h) + \alpha_1\beta\|(I - Q_h^*)p_h\|_0^2 \\
&\geqslant (1 + c_3)\|\widetilde{A}_h^{1/2}u_h\|_0^2 - \alpha_3\delta(1 + c_3')\|\widetilde{A}_h^{1/2}u_h\|_0\|Q_h^* p_h\|_0 + \delta\|Q_h^* p_h\|_0^2 \\
&\quad - \delta\alpha_3\|(I - Q_h^*)p_h\|_0\|Q_h^* p_h\|_0 + \alpha_1\beta_0\|(I - Q_h^*)p_h\|_0^2 \\
&\geqslant \left(\frac{1}{2} + c_3\right)\|\widetilde{A}_h^{1/2}u_h\|_0^2 - (1 + c_3')\frac{\delta^2\alpha_3^2}{2}\|Q_h^* p_h\|_0^2 + \delta\|Q_h^* p_h\|_0^2 \\
&\quad - \frac{\delta^2\alpha_3^2}{2\alpha_1\beta_0}\|Q_h^* p_h\|_0^2 + \frac{\alpha_1\beta_0}{2}\|(I - Q_h^*)p_h\|_0^2,
\end{aligned}$$

that is,

$$\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v_h, q_h)) \geqslant \left(\frac{1}{2} + c_3\right)\|\widetilde{A}_h^{1/2}u_h\|_0^2 + \frac{\delta}{2}\|Q_h^* p_h\|_0^2 + \frac{\alpha_1\beta_0}{2}\|(I - Q_h^*)p_h\|_0^2,$$

i.e., there is a positive constant κ_1, independent of β, such that

$$\widetilde{\mathscr{B}}_h((u_h, p_h); (I_h^* v_h, q_h)) \geqslant \kappa_1(\|\widetilde{A}_h^{1/2}u_h\|_0 + \|p_h\|_0)^2. \tag{4.11}$$

On the other hand,

$$\|\widetilde{A}_h^{1/2}v_h\|_0 + \|q_h\|_0 \leqslant \kappa_2(\|\widetilde{A}_h^{1/2}u_h\|_0 + \|p_h\|_0), \tag{4.12}$$

for some positive constant κ_2.

Finally, combining (4.11) and (4.12) establishes inequality (4.8) with $\alpha = \kappa_1/\kappa_2$, independent of β. □

For the trilinear term $\tilde{\tilde{b}}(u_h, v_h, I_h^* w_h)$, the following properties are useful. Set

$$N_2 = \sup_{u,v,w \in H_0^1(\Omega)} \frac{\tilde{\tilde{b}}(u, v, I_h^* w)}{\|\tilde{A}_h^{1/2} u\|_0 \|\tilde{A}_h^{1/2} v\|_0 \|\tilde{A}_h^{1/2} w\|_0},$$

$$N = \max\{c_2^2 N_1, N_2\}. \tag{4.13}$$

Lemma 4.4. *The trilinear forms* $\tilde{b}$ *and* $\tilde{\tilde{b}}$ *satisfy*

$$|\tilde{b}(u_h, v_h, I_h^* w_h)| \leqslant N \|\tilde{A}_h^{1/2} u_h\|_0 \|\tilde{A}_h^{1/2} v_h\|_0 \|\tilde{A}_h^{1/2} w_h\|_0,$$

$$|\bar{b}(u_h, v_h, w_h - I_h^* w_h)| \leqslant N \|\tilde{A}_h^{1/2} u_h\|_0 \|\tilde{A}_h^{1/2} v_h\|_0 \|\tilde{A}_h^{1/2} w_h\|_0,$$

$$|\tilde{\tilde{b}}(u_h, v_h, I_h^* w_h)| \leqslant c \|u_h\|_0^{1/2} \|\tilde{A}_h^{1/2} u_h\|_0^{1/2} \|\tilde{A}_h^{1/2} v_h\|_0 \|\tilde{A}_h w_h\|_0^{1/2} \|\tilde{A}_h^{1/2} w_h\|_0^{1/2}, \tag{4.14}$$

for any $u_h, v_h, w_h \in X_h$.

Proof. For any $u_h, v_h, w_h \in X_h$, the first inequality is the result of the definition of $\tilde{b}(u_h, v_h, I_h^* w_h)$. Since $u_h, v_h, w_h \in X_h$ are all (bi)linear function, direct computation gives the second one. To prove the third, in addition to the approximation property and the inverse inequality in (3.5), we also need the discrete analogue of the Sobolev inequality [33,19]

$$\|\phi_h\|_{L^4} \leqslant c \|\phi_h\|_0^{1/2} \|A_h^{1/2} \phi_h\|_0^{1/2} \quad \forall \phi_h \in X_h. \tag{4.15}$$

Moreover, note that, for any $u_h, v_h, w_h \in X_h$,

$$|\tilde{\tilde{b}}(u_h, v_h, I_h^* w_h)| \leqslant c \|u_h\|_{L^4} \|A_h^{1/2} v_h\|_0 \|w_h\|_{L^4},$$

which, together with (4.15) and Lemma 3.2, implies the second inequality of (4.14). □

Theorem 4.5. *Suppose the assumptions of Theorems* 2.1 *and* 3.3 *holds, and the body force f satisfies the following uniqueness condition*

$$1 - \frac{4N}{v^2} \|f\|_{-1} > 0. \tag{4.16}$$

Then there exists a unique solution (u_h, p_h) *of problem* (3.4) *satisfying the following estimate*:

$$v \|\tilde{A}_h^{1/2} u_h\|_0^2 + \|p_h\|_0^2 \leqslant \kappa. \tag{4.17}$$

Proof. Let the Hilbert space $H_h = (X_h, M_h)$ be supplied with the scalar product and norm:

$$((v, q); (I_h^* w, r))_{H_h} = ((v, I_h^* w)) + (q, r), \quad \|(v, q)\|_{H_h}^2 = \|\tilde{A}_h^{1/2} v\|_0^2 + \|q\|_0^2,$$

and K_h be a non-void, convex and compact subset of H_h defined by

$$K_h = \left\{ (v, q) \in H_h : \|\tilde{A}_h^{1/2} v\|_0 \leqslant \frac{2}{v} \|f\|_{-1}, \|q\|_0 \leqslant \frac{\gamma_2'}{\alpha} \|f\|_{-1} + \frac{4\gamma_2' N}{\alpha v^2} \|f\|_{-1}^2 \right\}.$$

We now define a continuous mapping from K_h into H_h as follows: Given $(\bar{v}, \bar{q}) \in K_h$ find $(u, p) = \Psi(\bar{v}, \bar{q})$ such that for all $(v, q) \in H_h$

$$\tilde{\mathscr{B}}_h((u, p); (I_h^* v, q)) + \tilde{\tilde{b}}(\bar{v}, u, I_h^* v) = (f, I_h^* v). \tag{4.18}$$

Taking $(v, q) = (u, p)$ in (4.18) and using (4.13), we obtain

$$v \|\tilde{A}_h^{1/2} u\|_0^2 \leqslant \|f\|_{-1} \|\tilde{A}_h^{1/2} u\|_0 + N \|\tilde{A}_h^{1/2} \bar{v}\|_0 \|\tilde{A}_h^{1/2} u\|_0 \|\tilde{A}_h^{1/2} u\|_0. \tag{4.19}$$

Since $\bar{v} \in K_h$ and f satisfies the uniqueness condition (4.16), we see that

$$\|\widetilde{A}_h^{1/2} u\|_0 \leqslant \frac{2}{v}\|f\|_{-1},$$

$$\alpha(\|\widetilde{A}_h^{1/2} u\|_0 + \|p\|_0) \leqslant \gamma_2'(\|f\|_{-1} + N\|\widetilde{A}_h^{1/2}\bar{v}\|_0\|\widetilde{A}_h^{1/2}u\|_0) \leqslant \gamma_2'\|f\|_{-1} + \frac{4\gamma_2' N}{v^2}\|f\|_{-1}^2,$$

which implies $\Psi(\bar{v}, \bar{q}) = (u, p) \in K_h$. By the fixed point theorem (see [15]), the mapping $\Psi(\bar{v}, \bar{q})$ has at least a fixed point $(u_h, p_h) \in K_h$, namely, $(u_h, p_h) \in K_h$ is a FVM solution of problem (3.4).

Next, we shall prove that problem (3.4) has only one solution (u_h, p_h). In fact, if (u_1, p_1) and (u_2, p_2) all satisfy formulation (3.4), then for all $(v, q) \in (X_h, M_h)$,

$$\widetilde{\mathscr{B}}_h((u_1 - u_2, p_1 - p_2); (I_h^* v, q)) + \widetilde{\widetilde{b}}(u_1 - u_2, u_1, I_h^* v) + \widetilde{\widetilde{b}}(u_2, u_1 - u_2, I_h^* v) = 0. \tag{4.20}$$

Taking $v = u_1 - u_2, q = p_1 - p_2$ in Eq. (4.20), we see that

$$v\|\widetilde{A}_h^{1/2}(u_1 - u_2)\|_0^2 \leqslant |\widetilde{\widetilde{b}}(u_1 - u_2, u_1, I_h^* v)| + |\widetilde{\widetilde{b}}(u_2, u_1 - u_2, I_h^* v)|. \tag{4.21}$$

Combining (4.21) with (4.13) yields

$$v\|\widetilde{A}_h^{1/2}(u_1 - u_2)\|_0^2 \leqslant N(\|\widetilde{A}_h^{1/2}u_1\|_0 + \|\widetilde{A}_h^{1/2}u_2\|_0)\|\widetilde{A}_h^{1/2}(u_1 - u_2)\|_0^2,$$

which implies

$$v\left(1 - \frac{4N}{v^2}\|f\|_{-1}\right)\|\widetilde{A}_h^{1/2}(u_1 - u_2)\|_0^2 \leqslant 0.$$

Hence by uniqueness condition (4.16), we have proven that the solution, u_h, of problem (3.4) is unique. Using again (4.8), (4.13) and (4.20), we also obtain the uniqueness of p_h.

The estimate (4.17) is simple. □

For the error estimate of (u_h, p_h), we need the following Galerkin projection $(\widetilde{R}_h, \widetilde{Q}_h) : (X, M) \to (X_h, M_h)$ defined by

$$\widetilde{\mathscr{B}}_h((\widetilde{R}_h(v, q), \widetilde{Q}_h(v, q)); (I_h^* v_h, q_h)) = \widetilde{\mathscr{B}}((v, q); (I_h^* v_h, q_h)) \quad \forall (v_h, q_h) \in (X_h, M_h), \tag{4.22}$$

for each $(v, q) \in (X, M)$.

Note that, due to Lemma 4.3, $(\widetilde{R}_h, \widetilde{Q}_h)$ is well defined. By using a similar argument to ones used by He in [17], the following approximate properties can be obtained.

Lemma 4.6. *Under the assumptions of Lemma* 4.3, *the projection* $(\widetilde{R}_h, \widetilde{Q}_h)$ *satisfies*

$$\|\widetilde{A}_h^{1/2}(v - \widetilde{R}_h(v, q))\|_0 + \|q - \widetilde{Q}_h(v, q)\|_0 \leqslant c(\|A^{1/2}v\|_0 + \|q\|_0), \tag{4.23}$$

for all $(v, q) \in (X, M)$ *and*

$$\|\widetilde{A}_h^{1/2}(v - \widetilde{R}_h(v, q))\|_0 + \|q - \widetilde{Q}_h(v, q)\|_0 \leqslant ch(\|Av\|_0 + \|A^{1/2}q\|_0), \tag{4.24}$$

for all $(v, q) \in (D(A), H^1(\Omega) \cap M)$.

Proof. The stability of the projection follows simply by Lemma 4.3, namely

$$\begin{aligned} \|\widetilde{A}_h^{1/2}\widetilde{R}_h(v, q)\|_0 + \|\widetilde{Q}_h(v, q)\|_0 &\leqslant \alpha^{-1} \sup_{(w_h, r_h) \in (X_h, M_h)} \frac{\widetilde{\mathscr{B}}_h((\widetilde{R}_h(v, q), \widetilde{Q}_h(v, q)); (I_h^* w_h, r_h))}{\|\widetilde{A}_h^{1/2}w_h\|_0 + \|r_h\|_0} \\ &\leqslant \alpha^{-1} \sup_{(w_h, r_h) \in (X_h, M_h)} \frac{\mathscr{B}((v, q); (I_h^* w_h, r_h))}{\|\widetilde{A}_h^{1/2}w_h\|_0 + \|r_h\|_0} \\ &\leqslant c(\|A^{1/2}v\|_0 + \|q\|_0) \quad \forall (v, q) \in (X, M). \end{aligned} \tag{4.25}$$

Now the triangle inequality gives

$$\|\widetilde{A}_h^{1/2}(v-\widetilde{R}_h(v,q))\|_0+\|q-\widetilde{Q}_h(v,q)\|_0\leqslant c(\|A^{1/2}v\|_0+\|q\|_0), \tag{4.26}$$

for all $(v,q)\in(X,M)$, which is (4.23).

Next, let $(v,q)\in(D(A),H^1(\Omega)\cap M)$, using the standard interpolation $(I_hv,J_hp)\in(X_h,M_h)$ and Lemma 4.3, we have

$$\begin{aligned}&\|\widetilde{A}_h^{1/2}(I_hv-\widetilde{R}_h(v,q))\|_0+\|J_hq-\widetilde{Q}_h(v,q)\|_0\\&\quad\leqslant\alpha^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\widetilde{\mathscr{B}}_h((I_hv-\widetilde{R}_h(v,q),J_hq-\widetilde{Q}_h(v,q));(I_h^*v_h,q_h))}{\|\widetilde{A}_h^{1/2}v_h\|_0+\|q_h\|_0}\\&\quad\leqslant\alpha^{-1}\sup_{(v_h,q_h)\in(X_h,M_h)}\frac{\widetilde{\mathscr{B}}_h((I_hv-v,J_hq-q);(I_h^*v_h,q_h))}{\|\widetilde{A}_h^{1/2}v_h\|_0+\|q_h\|_0}\\&\quad\leqslant c(\|\widetilde{A}_h^{1/2}(I_hv-v)\|_0+\|J_hq-q\|_0).\end{aligned} \tag{4.27}$$

Thus the triangles inequality and approximate properties (3.5) give

$$\|\widetilde{A}_h^{1/2}(v-\widetilde{R}_h(v,q))\|_0+\|q-\widetilde{Q}_h(v,q)\|_0\leqslant ch(\|Av\|_0+\|A^{1/2}q\|_0).\qquad\Box \tag{4.28}$$

Now, we derive the following optimal H^1 error estimate of u_h and L^2 error estimate of p_h defined in (3.4).

Theorem 4.7. *Assume that the assumptions of Theorem* 4.5 *hold. Then there exists a positive constant* κ *such that the stabilized finite volume solution* (u_h,p_h) *satisfies the error estimate*:

$$\|\widetilde{A}_h^{1/2}(u-u_h)\|_0+\|p-p_h\|_0\leqslant\kappa h. \tag{4.29}$$

Proof. By subtracting (3.4) from (2.8) and using the properties of the Galerkin projection $(\widetilde{R}_h,\widetilde{Q}_h)$, we obtain

$$\begin{aligned}&\widetilde{\mathscr{B}}_h((e_h,\mu_h);(I_h^*v,q))+E(u,v)+b(u,u-u_h,v)+b(u-u_h,u,v)\\&\quad-b(u-u_h,u-u_h,v)-\overline{b}(u_h,u_h,v-I_h^*v)=(f,v-I_h^*v),\end{aligned} \tag{4.30}$$

for all $(v,q)\in(X_h,M_h)$, where

$$e_h=\widetilde{R}_h(u,p)-u_h,\quad \mu_h=\widetilde{Q}_h(u,p)-p_h,\quad u-u_h=w_h+e_h.$$

Then, setting $(v,q)=(e_h,\mu_h)$ in (4.30) and using (2.4), it follows that

$$\begin{aligned}&\nu\|\widetilde{A}_h^{1/2}e_h\|^2+\beta\mathscr{C}_h(\mu_h,\mu_h)+b(e_h+w_h,u,e_h)+b(u,w_h,e_h)-b(e_h+w_h,w_h,e_h)\\&\quad-\overline{b}(e_h+w_h,u,e_h-I_h^*e_h)-\overline{b}(u_h,e_h+w_h,e_h-I_h^*e_h)\\&\quad+\overline{b}(u,u,e_h-I_h^*e_h)+\overline{b}(u-u_h,u-u_h,e_h-I_h^*e_h)+E(u,e_h)\\&\qquad=(f,e_h-I_h^*e_h),\end{aligned} \tag{4.31}$$

where $w_h = u - \widetilde{R}_h(u,p)$. We rewrite (4.31) as

$$\begin{aligned} \nu\|u-u_h\|^2 + \beta\mathscr{C}_h(\mu_h,\mu_h) = &-b(u,u-u_h,w_h) + b(u-u_h,u,u-u_h) \\ &- b(u-u_h,u_h,w_h) - \bar{b}(u-u_h,u,(u-u_h)-I_h^*(u-u_h)) \\ &- \bar{b}(u_h,u-u_h,(u-u_h)-I_h^*(u-u_h)) \\ &+ \bar{b}(u,u,(u-u_h)-I_h^*(u-u_h)) - \bar{b}(u-u_h,u,w_h-I_h^*w_h) \\ &- \bar{b}(u_h,u-u_h,w_h-I_h^*w_h) + \bar{b}(u,u,w_h-I_h^*w_h) - \widetilde{a}(w_h,w_h) \\ &+ 2\widetilde{a}(w_h,u-u_h) - E(u,(u-u_h)) + E(u,w_h) \\ &+ (f,(u-u_h)-I_h^*(u-u_h)) + (f,w_h-I_h^*w_h). \end{aligned} \tag{4.32}$$

Note that

$$E(u,v_h) \leqslant ch\|Au\|_0\|v_h\|_0 \quad \forall u \in D(A)\cap X, \quad v_h \in X_h.$$

We shall make the following estimates for some nonlinear terms on the right-hand side in (4.32):

$$\begin{aligned} |b(u,u-u_h,w_h)| &\leqslant c\|\widetilde{A}_h^{1/2}w_h\|_0\|A_h^{1/2}u\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0 \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa\|\widetilde{A}_h^{1/2}w_h\|_0^2\|A_h^{1/2}u\|_0^2, \end{aligned}$$

$$\begin{aligned} |b(u-u_h,u_h,w_h)| &\leqslant c\|\widetilde{A}_h^{1/2}w_h\|_0\|\widetilde{A}_h^{1/2}u\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0 \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa\|\widetilde{A}_h^{1/2}w_h\|_0^2\|\widetilde{A}_h^{1/2}u\|_0^2, \end{aligned}$$

$$\begin{aligned} |b(u-u_h,u,u-u_h)| &\leqslant N_1\|A_h^{1/2}u\|_0\|A_h^{1/2}(u-u_h)\|_0^2 \\ &\leqslant N\|A_h^{1/2}u\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2, \end{aligned}$$

$$|\bar{b}(u_h,u-u_h,(u-u_h)-I_h^*(u-u_h))| \leqslant N\|\widetilde{A}_h^{1/2}u_h\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2,$$

$$\begin{aligned} |\bar{b}(u-u_h,u,(u-u_h)-I_h^*(u-u_h))| &\leqslant ch\|Au\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa h^2\|Au\|_0^2\|A^{1/2}u\|_0^2, \end{aligned}$$

$$\begin{aligned} |\bar{b}(u,u,(u-u_h)-I_h^*(u-u_h))| &\leqslant ch\|Au\|_0\|A^{1/2}u\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\|_0 \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa h^2\|Au\|_0^2\|A^{1/2}u\|_0^2, \end{aligned}$$

$$\begin{aligned} |\bar{b}(u-u_h,u,w_h-I_h^*w_h)| &\leqslant c\|\widetilde{A}_h^{1/2}w_h\|_0\|A^{1/2}u\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\| \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa\|\widetilde{A}_h^{1/2}w_h\|_0^2\|A^{1/2}u\|_0^2, \end{aligned}$$

$$\begin{aligned} |\bar{b}(u_h,u-u_h,w_h-I_h^*w_h)| &\leqslant c\|\widetilde{A}_h^{1/2}w_h\|_0\|\widetilde{A}_h^{1/2}u_h\|_0\|\widetilde{A}_h^{1/2}(u-u_h)\| \\ &\leqslant \frac{\nu}{32}\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \kappa\|\widetilde{A}_h^{1/2}w_h\|_0^2\|\widetilde{A}_h^{1/2}u_h\|_0^2, \end{aligned}$$

$$|\bar{b}(u,u,w_h-I_h^*w_h)| \leqslant c\|\widetilde{A}_h^{1/2}w_h\|_0\|A^{1/2}u\|_0^2.$$

Table 1
Numerical results of the FVM and FEM

h	Method	$\frac{\\|\widetilde{A}_h^{1/2}(u-u_h)\\|_0}{\\|\widetilde{A}_h^{1/2}u\\|_0}$	$\frac{\\|u-u_h\\|_0}{\\|u\\|_0}$	$\frac{\\|p-p_h\\|_0}{\\|p\\|_0}$
$\frac{1}{16}$	FEM	0.13493775	0.05282056	0.08907621
$\frac{1}{16}$	FVM	0.13754968	0.05310531	0.09503956
$\frac{1}{32}$	FEM	0.06749820	0.04425722	0.04428041
$\frac{1}{32}$	FVM	0.07662815	0.04921347	0.05417745
$\frac{1}{64}$	FEM	0.03653298	0.02869463	0.01700283
$\frac{1}{64}$	FVM	0.04927651	0.03424916	0.02715132

Combining the above estimates with (4.32) and using Theorems 2.1, 4.5 yields

$$\left(\frac{3v}{4} - N(\|A_h^{1/2}u\|_0 + \|\widetilde{A}_h^{1/2}u_h\|_0)\right)\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 + \beta\mathscr{C}_h(\mu_h,\mu_h)$$
$$\leqslant \kappa\|\widetilde{A}_h^{1/2}w_h\|^2 + \kappa h^2(\|Au\|_0^2\|A^{1/2}u\|_0^2 + \|Au\|_0^2 + \|A^{1/2}u\|_0^2 + \|w_h\|_0^2 + \|f\|_0^2).$$

Using (4.16), Theorems 2.1, 4.5 and Lemma 4.6, one finds

$$\|\widetilde{A}_h^{1/2}(u-u_h)\|_0^2 \leqslant \kappa h^2. \tag{4.33}$$

Since $C_h(p,q_h)=0, \forall p \in (H^1(\Omega)\cap M), \forall q_h \in M_h$, using again (4.8), (4.30), (2.9) and (4.17), we have

$$\|\mu_h\|_0 \leqslant \alpha^{-1}c(\|A^{1/2}u\|_0 + \|\widetilde{A}_h^{1/2}u_h\|_0)\|\widetilde{A}_h^{1/2}(u-u_h)\|_0 \leqslant \kappa\|\widetilde{A}_h^{1/2}(u-u_h)\|_0. \tag{4.34}$$

Finally, one finds

$$\begin{aligned}\|p-p_h\|_0 &= \|p-\widetilde{Q}_h(u,p)+\widetilde{Q}_h(u,p)-p_h\|_0\\ &\leqslant \|p-\widetilde{Q}_h(u,p)\|_0 + \|\widetilde{Q}_h(u,p)-p_h\|_0 \leqslant \kappa h.\end{aligned} \tag{4.35}$$

Hence, (4.29) follows. □

5. Numerical example

For the easy comparison with locally stabilized finite element method (FEM) [17], we also set that the exact solution is given by

$$u(x,y) = (u_1(x,y), u_2(x,y)), \quad p(x,y) = 10(2x-1)(2y-1),$$
$$u_1(x,y) = 10x^2(x-1)^2y(y-1)(2y-1), \quad u_2(x,y) = -10x(x-1)(2x-1)y^2(y-1)^2,$$

with $v = 0.005$ and f is determined by (2.1). Next, we provide the convergence accuracy of the stabilized FVM and FEM with $h = \frac{1}{16}, \frac{1}{32}$ and $\frac{1}{64}$ when the parameter value $\beta = 9.18$ is used to solve the flow problem on a uniformly refined sequence of grids in Table 1.

The results show the anticipated first-order convergence rate.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[2] R.E. Bank, D.J. Rose, Some error estimate for the box method, SIAM J. Numer. Anal. 24 (1987) 351–375.
[3] J. Boland, R.A. Nicolaides, Stability of finite elements under divergence constraints, SIAM J. Numer. Anal. 20 (1983) 722–731.
[4] Z. Cai, On the finite volume method, Numer. Math. 58 (1991) 713–735.
[5] Z. Cai, J.E. Jones, S.F. McCormick, T.F. Russell, Control-volume mixed finite element methods, Comput. Geosci. 1 (1997) 289–315.

[6] Z. Cai, J. Mandel, S. Mccormick, The finite volume element method for diffusion equations on general triangulations, SIAM J. Numer. Anal. 28 (1991) 392–402.
[7] P. Chatzipantelidies, A finite volume method based on the Crouziex–Raviart element for elliptic PDE's in two dimension, Numer. Math. 82 (1999) 409–432.
[8] Z. Chen, Finite Element Methods and Their Applications, Springer, Heidelberg, New York, 2005.
[9] Z. Chen, The control volume finite element methods and their applications to multiphase flow, Networks Heterogeneous Media, to appear.
[10] Z. Chen, G. Huan, Y. Ma, Computational Methods for Multiphase Flows in Porous Media, Computational Science and Engineering Series, vol. 2, SIAM, Philadelphia, PA, 2006.
[11] S.H. Chou, Analysis and convergence of a covolume method for the generalized Stokes problem, Math. Comp. 66 (1997) 85–104.
[12] S.H. Chou, D.Y. Kwak, Analysis and convergence of the MAC scheme for the generalized Stokes problem, Numer. Methods Partial Differential Equations 13 (1997) 147–162.
[13] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.
[14] R.E. Ewing, T. Lin, Y. Lin, On the accuracy of the finite volume element method based on piecewise linear polynomials, SIAM J. Numer. Anal. 39 (2002) 1865–1888.
[15] V. Girault, P.A. Raviart, Finite Element Method for Navier–Stokes Equations: Theory and Algorithms, Springer, Berlin, Heidelberg, 1987.
[16] F.H. Harlow, F.E. Welch, Numerical calculations of time dependent viscous incompressible flow of fluid with a free surface, Phys. Fluids 8 (1995) 2181–2197.
[17] Y. He, A. Wang, L. Mei, Stabilized finite element method for the stationary Navier–Stokes equations, J. Eng. Math. 51 (2005) 367–380.
[18] B. Heinrich, Finite Difference Method on Irregular Networks, ISNM 82, Birkhauser Verlag, Basel, 1987.
[19] J.G. Heywood, R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem I: regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal. 19 (1982) 275–311.
[20] J. Huang, On the finite volume element method for general self-adjoint elliptic problem, SIAM J. Numer. Anal. 35 (1998) 1762–1774.
[21] D. Kay, D. Silvester, A posteriori error estimation for stabilized mixed approximations of the Stokes equations, SIAM J. Sci. Comput. 21 (2000) 1321–1337.
[22] N. Kechkar, D. Silvester, Analysis of locally stabilized mixed finite element methods for the Stokes problem, Math. Comp. 58 (1992) 1–10.
[23] R.B. Kellogg, J.E. Osborn, A regularity result for the Stokes problem in a convex polygon, J. Funct. Anal. 21 (1976) 397–431.
[24] T. Kerkhoven, Piecewise linear Petrov–Galerkin error estimates for the box method, SIAM J. Numer. Anal. 33 (1996) 1864–1884.
[25] R. Li, Generalized difference methods for a nonlinear Dirichlet problem, SIAM J. Numer. Anal. 24 (1987) 77–88.
[26] R. Li, Z. Chen, W. Wu, Generalized Difference Methods for Differential Equations, Marcel Dekker, New York, 2000.
[27] R.A. Nicolaides, Analysis and convergence of the MAC scheme I the linear problem, SIAM J. Numer. Anal. 29 (1992) 1579–1591.
[28] S. Norburn, D. Silvester, Stabilised vs stable mixed methods for incompressible flow, Comput. Methods Appl. Mech. Eng. 166 (1998) 1–10.
[29] R.L. Sani, P.M. Gresho, R.L. Lee, D.F. Griffiths, The cause and cure(?) of the spuious pressures generated by certain finite element method solutions of the incompressible Navier–Stokes equations, Parts 1 and 2, Internat. J. Numer. Methods Fluids 1 (1981) 17–43, 171–206.
[30] D.J. Silvester, N. Kechkar, Stabilised bilinear-constant velocity-pressure finite elements for the conjugate gradient solution of the Stokes problem, Comput. Methods Appl. Mech. Eng. 79 (1990) 71–86.
[31] R. Stenberg, Analysis of mixed finite elements for the Stokes problem: a unified approach, Math. Comp. 42 (1984) 9–23.
[32] E. Suli, Convergence of finite volume schemes for Poissons equation on nonuniform meshes, SIAM J. Numer. Anal. 28 (1991) 1419–1430.
[33] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, third ed., North-Holland, Amsterdam, 1984.