On the Hurewicz image of the Steinberg summand $M(n)$

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ARTICLE INFO

Article history:
Received 25 August 2009
Received in revised form 8 July 2010
Accepted 10 July 2010

Keywords:
Steinberg idempotent
Hurewicz map
Brown–Peterson homology
Adams spectral sequence

ABSTRACT

Given a prime $p$, we define a spectrum $M(n)$ by the stable summand of the classifying space $B(\mathbb{Z}/p)^n$ induced by the Steinberg idempotent. We show that the mod $p$ Hurewicz map $\pi_*(M(n)) \to H_*(M(n); \mathbb{F}_p)$ has trivial image in most dimensions.

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1. Introduction

Let $p$ be a prime, and the cohomology $H^*(-)$ and the homology $H_*(-)$ with the coefficient $\mathbb{F}_p$. There are many researches on the classifying space $B(\mathbb{Z}/p)^n$. The following results about the Hurewicz image of $B(\mathbb{Z}/p)^n$ are known. Hansen [7] has determined the image of the stable Hurewicz map $\pi^*_2(B\mathbb{Z}/p) \to H_2(B\mathbb{Z}/p; \mathbb{Z})$ completely. Minami [18] has shown that the mod $p$ stable Hurewicz map $\pi^*_n(B(\mathbb{Z}/p)^n) \to H_*(B(\mathbb{Z}/p)^n)$ has trivial image in some dimensions, but it is far from the complete determination. Many works (e.g., [2, 4–6, 13, 29]) have been done on the $\mathbb{A}$-generating set for $H_*(B(\mathbb{Z}/p)^n)$, or equivalently the indecomposable quotient module $QH_*(B(\mathbb{Z}/p)^n) = B_p \otimes_{\mathbb{F}_p} H_*(B(\mathbb{Z}/p)^n)$, which lifts to a minimal $\mathbb{A}$-generating set for $H_*(B(\mathbb{Z}/p)^n)$. Here $\mathbb{A}$ and $\mathbb{A}_n$ are the Steenrod algebra and its dual, respectively. One of the reasons is the existence of the Singer’s transfer [26]

$$\varphi_n : \mathbb{F}_p \otimes \text{GL}_n(B(\mathbb{Z}/p)^n) \longrightarrow \text{Ext}^{n+n}_{\mathbb{A}}(\mathbb{F}_p, \mathbb{F}_p),$$

where $\text{Ext}^{n+n}_{\mathbb{A}}(\mathbb{F}_p, \mathbb{F}_p)$ is the $E_2$-term of the Adams spectral sequence converging to $\pi^*_n(S^0)$.

As explained in [21, 22], for each idempotent $e$ in the group ring $\mathbb{F}_p[\text{GL}_n(\mathbb{Z}/p)]$, we obtain a stable summand $e B(\mathbb{Z}/p)^n$ of $B(\mathbb{Z}/p)^n$ with $H^*(e B(\mathbb{Z}/p)^n) \cong H^*(B(\mathbb{Z}/p)^n)e$. Each idempotent corresponds to a modular representation of $\text{GL}_n(\mathbb{Z}/p)$ over $\mathbb{F}_p$. Mitchell [20] has shown that the transfer $t_{n} : B(\mathbb{Z}/p)^n \to S^0$ factors through the summand corresponding to the trivial representation. The Singer’s transfer $\varphi_n$ is the algebraic analog of $t_{n}$.

We define a spectrum $M(n)$ by $e_n B(\mathbb{Z}/p)^n$, where $e_n$ is the Steinberg idempotent corresponding to the Steinberg representation. See [21, 22, 27] for details of the Steinberg idempotent. In [14–16, 21, 22], Kuhn, Mitchell and Priddy have shown that $M(n)$ has various good properties. Though it is difficult to determine $H^*(e B(\mathbb{Z}/p)^n) \cong H^*(B(\mathbb{Z}/p)^n)e$ in terms of modular representation theory, Mitchell and Priddy [21, 22] have described $H^*(M(n))$ in terms of the Steenrod algebra. Moreover $B(\mathbb{Z}/p)^n$ has $p^{\langle i \rangle}$ stable summands equivalent to $M(n)$. In this paper, we study the mod $p$ Hurewicz image of $M(n)$. This describes a certain part of the mod $p$ stable Hurewicz image of $B(\mathbb{Z}/p)^n$.

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In [8,9], we have determined a basis of $QH^*(M(n))$. The results imply $\dim QH^k(M(n)) = \dim PH_k(M(n)) \leq 1$. For a spectrum $X$, let $h(X)_a : \pi_*(X) \to H_*(X)$ be the mod $p$ Hurewicz map. Since the image of $h(X)_a$ is included in $PH_*(X)$, the image of $h(M(n))_a$ is trivial if $\dim QH^k(M(n)) = 0$. If $\dim QH^k(M(n)) = 1$, it is isomorphic to 0 or $\mathbb{F}_p$.

Our main result is that the image of $h(M(n))_a$ is trivial for most dimensions. Since we have a stable splitting $M(n) \cong L(n) \vee L(n-1)$ [21,22], we actually study $L(n)$. We give a partial computation of the BP-primitive elements $PB_{\ast}(L(n))$ by using the BP-Adams operation $\psi^{p-1}$. By refinement of [8,9], we describe a basis of $PH_*(L(n))$. The Hurewicz map $h(L(n)) : \pi_*(L(n)) \to PH_*(L(n))$ factors through the usual reduction $PB_{\ast}(L(n)) \to PH_*(L(n))$. These induce the triviality of $h(L(n))_a$ for certain dimensions.

This paper is organized as follows. In Section 2, we recall the Brown–Peterson homology $BP_*(B(\mathbb{Z}/p)^n)$ from [10,11]. In Section 3, we recall spectra $M(n)$ and $L(n)$ from [21,22]. In Section 4, we describe the elements of $BP_*(L(n))$, and show that they have large order. In Section 5, we study the action of prime ideals $I$ on $BP_*(L(n))$ by the usage of the universal coefficient spectral sequence based on the chromatic tower, which is used in [18]. The results induce that any $BP_*$-primitive element in certain dimensions has relatively small order. In Section 6, we refine the results of [8,9], and describe a basis of $PH_*(L(n))$ exactly. In Section 7, we apply the results of Sections 4, 5 and 6 to proving Theorem 7.2. This leads to main theorems. In Section 8, we state main theorems and prove them. In Appendix A, we conjecture on the image of $h(L(n))_a$ in the dimensions which we cannot determine in Theorem B.

2. The Brown–Peterson homology of $B(\mathbb{Z}/p)^n$

Let $p$ be a prime, $X$ a spectrum, and $E$ the mod $p$ Eilenberg–MacLane spectrum $H = H\mathbb{Z}/p$ or the Brown–Peterson spectrum $BP$. We denote the primitive elements of an $E_\ast E$-comodule $E_\ast(X)$ by $PE_\ast(X)$, and the $E$-Hurewicz map of $X$ by $h(E)(X)_a : \pi_*(X) \to E_\ast(X)$.

We recall $BP_*(B(\mathbb{Z}/p)^n)$ from [10,11]. We have the coefficient ring $BP_\ast \cong \mathbb{Z}_p[v_1, v_2, \ldots, v_n]$. Let $I_0 = (0)$, $I_n = (p, v_1, v_2, \ldots, v_{n-1})$. The $BP$-formal group law gives the $[p]$-series:

$$[p](x) = \sum_{i \geq 0} a_i x^{q_i}, \quad a_i \in BP_{2i}, \ x \in BP^2(CP^\infty).$$

Here $a_0 = p$, and $a_{p-1} = v_n$ modulo $I_n$. We denote the standard generator in $BP_{2i-1}(B\mathbb{Z}/p)$ by $[i]$.

There is a unique expression $\sum c_{i,i} v^i[I]$, where $c_{i,i} \in [0, 1, \ldots, p-1]$, $v^i = v_n^j \cdots v_1^j$ for finite sequences $L = (i_0, \ldots, i_m)$ of non-negative integers, and $I$ ranges over all sequences of $n$ positive integers. Especially $T$ is free over $BP_*/I_n$.

Theorem 2.1. (10, Theorem 3.2, Remark 3.4) Any element of $T$ has a unique expression $\sum c_{i,i} v^i[I]$, where $c_{i,i} \in [0, 1, \ldots, p-1]$, $v^i = v_n^j \cdots v_1^j$ for finite sequences $L = (i_0, \ldots, i_m)$ of non-negative integers, and $I$ ranges over all sequences of $n$ positive integers. Especially $T$ is free over $BP_*/I_n$.

Corollary 2.2. (10, Corollary 3.3) The iterated Künneth homomorphism

$$\chi : T \to BP_*(\bigwedge^n B\mathbb{Z}/p)$$

is injective.

Therefore we identify $[i_1, \ldots, i_n]$ with the element in $BP_*(\bigwedge^n B\mathbb{Z}/p)$.

For $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$, we set $I - J = (i_1 - j_1, \ldots, i_n - j_n)$.

Corollary 2.3. (11, Corollary 2.7) Let $\Gamma = (p^{n-1}, \ldots, p^2, p, 1)$, and $[I] = [i_1, \ldots, i_n]$. Modulo terms of lexicographic order lower than that of $[I - s(p - 1)\Gamma]$, we have

$$p^s[I] = (-1)^{ns}(a_{p^{n-1}})^s[I - s(p - 1)\Gamma].$$

Lemma 2.4. (11, Lemma 2.4) All elements of $T$ are $v_j$-torsion for $0 \leq j < n$. 

3. The spectra $M(n)$ and $L(n)$

We recall $M(n)$ and $L(n)$ from [21,22]. The general linear group $GL_n(\mathbb{Z}/p)$ acts on $B(\mathbb{Z}/p)^n$ and has the resulting right action

$$P_n = H^*(B(\mathbb{Z}/p)^n) = \Lambda(x_1, \ldots, x_n) \otimes \mathbb{F}_p[y_1, \ldots, y_n].$$

where $\Lambda$ is an exterior algebra, and $x_i \in H^1(B(\mathbb{Z}/p)^n)$, $y_i \in H^2(B(\mathbb{Z}/p)^n)$ are the standard generators. Since $P_n = \mathbb{F}_p[x_1, \ldots, x_n]$ for $p = 2$, we set $x_i^2 = y_i$ as usual. The group ring $\mathbb{F}_p[GL_n(\mathbb{Z}/p)]$ acts on $P_n$. For each idempotent $e \in \mathbb{F}_p[GL_n(\mathbb{Z}/p)]$, we obtain a stable summand $eB(\mathbb{Z}/p)^n$ of $B(\mathbb{Z}/p)^n$ with $H^*(eB(\mathbb{Z}/p)^n) \cong P_{ne}$. For the Steenrod idempotent $e_{2^n} \in \mathbb{F}_p[GL_n(\mathbb{Z}/p)]$, we define $M(n)$ by $e_{2^n}B(\mathbb{Z}/p)^n$. We follow the notation of [21]. Note that the notation $M(n)$ in [19] is used for a different spectrum. The spectrum $M(n)$ in [19] is the same as $D(n)$ in [21].

Let $A$ be the mod $p$ Steenrod algebra, $\phi^i$ the ith reduced power operation, and $\beta$ the Bockstein operation. For $p = 2$, we identify $\phi^i$ and $\beta$ with $S^2$ and $S^1$, respectively. The Steenrod algebra has $\text{deg} \phi^i = 2(p - 1)i$ and $\text{deg} \beta = 1$. Let $I$ be a finite sequence $(\epsilon_0; a_1, \epsilon_1; a_2, \ldots)$ with integers $a_i \geq 0$, $\epsilon_i \in \{0, 1\}$. Then we denote the Steenrod operation $\beta \phi^{a_1} \phi^{\epsilon_1} \phi^{a_2} \ldots \phi^{\epsilon_n}$ by $\phi^I$. If $\epsilon_i = 0$ for all $i$, we may write $(a_1, a_2, \ldots, a_n)$ for $I = (a_0; 0, a_1; 0, a_2, \ldots)$ and $\phi^I$ for $\phi^i$. If $\epsilon_i \geq p a_{i + 1} + \epsilon_i$ for all $i$, we call $I$ admissible. The length of $I$ is defined by the smallest number $n$ which satisfies $a_i = 0$ for $i > n$ and $\epsilon_i = 0$ for $i \geq n$, and it is denoted by $l(I)$. Let $G_n$ be the vector subspace of $A$ spanned by $[\phi^I | I: \text{admissible}, \ l(I) > n]$. Then $G_n$ is a left $A$-ideal, and we have an $A$-module isomorphism $H^*(M(n)) \cong \Sigma^\infty G_n/G_{n+1}$. We regard $[\phi^I \in A | I: \text{admissible}, \ l(I) = n]$ as a basis of $H^*(M(n))$, and we write $\phi^I$ for the element corresponding to $\phi^I$. Exactly $H^*(\Sigma^\infty M(n))$ is a subquotient module of $A$, and $\text{deg} \phi^I = \text{deg} \phi^I - n$.

Let $L(n) = \Sigma^{-n} \Sigma^n (S^1) = \Sigma^{n} (S^1)$, where $\Sigma^{n} (S^1)$ is the nth symmetric product of $S^1$. Since $M(n) \cong L(n) \cup L(n - 1)$ [21, Proposition 5.15], we have an $A$-module isomorphism

$$H^*(M(n)) \cong H^*(L(n)) \otimes H^*(L(n - 1)).$$

Via this isomorphism, the set $[\phi^I | I: \text{admissible}, \ l(I) = n, \ \phi^I \not\in A\beta]$ is a basis of $H^*(L(n))$, and the set $[\phi^I | I: \text{admissible}, \ l(I) = n - 1, \ \phi^I \not\in A\beta]$ is a basis of $H^*(L(n - 1))$.

From now on, we actually study $L(n)$. The spectrum $L(n)$ is a stable summand of $\bigwedge^n B\mathbb{Z}/p$. Note that $\bigwedge^n B\mathbb{Z}/p$ is a stable summand of $B(\mathbb{Z}/p)^n$.

Let $Q_i$ be the ith Milnor element in $A$. Namely $Q_0 = \beta$, $Q_{i+1} = [\phi^i, Q_i]$.

Lemma 3.1. ([9, Lemma 5.1]) $H^*(L(n))$ is a free $A(Q_0, \ldots, Q_{n - 1})$-module on the set $[\phi^I | I: \text{admissible}, \ l(I) = n]$.

From now on, any admissible sequence of length $k$ has the form

$$(a_1, a_2, \ldots, a_k) = (0; a_1, 0; a_2, \ldots, 0; a_k),$$

and if we write an admissible sequence $(a_1, \ldots, a_k)$, it indicates that the sequence is of length $k$.

Let

$$l_n : B(\mathbb{Z}/p)^n \to e_{2^n}B(\mathbb{Z}/p)^n = M(n) \cong L(n) \cup L(n - 1) \to L(n)$$

be the projection. Recall $l_n^* : H^*(L(n)) \to H^*(B(\mathbb{Z}/p)^n)$ from [22]. Let $S_n$ denote the localization of $P_n$ obtained by inverting all non-zero linear forms in $y_1, \ldots, y_n$, i.e. all elements $\sum a_i y_i \neq 0$ ($a_i \in \mathbb{F}_p$). Then $S_n$ has a unique $A$-module structure extending that of $P_n$ by [28], and for any integer $k$, we have

$$\phi^j(y^k) = \left(\frac{k}{j}\right) y^{k+j(p-1)}, \quad \beta(y^k) = 0, \quad \phi^j(x_i) = 0, \quad \beta(x_i) = y_i. \quad (1)$$

Since $\left(\frac{1}{j}\right) = (-1)^{j}$, we see $\phi^1(y^{j}) = (-1)^{j} y^{j(p-1)-1}$. Let $M_n$ be the $A$-submodule of $S_n$ generated by $X_1 Y_n^{-1} - X_1 \cdot \cdots \cdot X_n Y_1^{-1} \cdots Y_n^{-1}$. We define $F_n : H^*(L(n)) \to M_n$ by $F_n(\phi^I) = \phi^I(X_1 Y_n^{-1})$. Then $F_n$ is an $A$-module homomorphism, and we have $F_n = l_n^* : H^*(L(n)) \to M_n$. Furthermore $F_n$ is injective, and $\text{Im}(F_n) = \text{Im}(l_n^*) \subset P_n \cap M_n$.

For $I = (i_1, \ldots, i_n)$, we set $X_n Y_i^j = x_1 \cdots x_n y_1^{-1} \cdots y_n^{-1} \in P_n$. The order on $I$ induces that on $X_n Y_i^j$. Let $e_{2f-1} \in H_{2f-1}(B(\mathbb{Z}/p)$ be the dual element of $xy^{f-1} \in H^{2f-1}(B\mathbb{Z}/p)$. The order on $X_n Y_i^j$ induces that on $[e_{2f-1} \wedge \cdots \wedge e_{2k-1}] \in H_{2f-1}(\bigwedge^n B\mathbb{Z}/p)$.

Let $J = (j_1(p - 1), \ldots, j_n(p - 1))$ for $J = (j_1, \ldots, j_n)$. The following results are similar to [11, Section 3].

Lemma 3.2. For an admissible sequence $I$ of $l(I) = n$, we have

$$\phi^J(X_n Y_i^j) = X_n Y_i^j w + w \in P_n \cap M_n,$$

where $w$ is a linear combination of $[X_n Y_i^j | I < J]$. 

Proof. The lemma follows from the action (1) and the Cartan formula. □

Corollary 3.3. For an admissible sequence \( I \) of \( l(I) = n \), we have \( I_n^*(\widetilde{\rho}) = X_n^{Y_n^I} + w \), where \( w \) is a linear combination of \( \{X_n^{Y_n^I} \mid I < J \} \).

We consider the following commutative diagram:

\[
\begin{array}{c}
T = \bigotimes_{BP} BP_*(Z/p) \xrightarrow{X} \bigotimes_{BP} BP_*(\Lambda^n BZ/p) \xrightarrow{\ln \rho} BP_*(L(n)) \\
\downarrow{\rho} \quad \downarrow{\rho} \quad \downarrow{\rho} \\
\bigotimes_{BP} H_*((BZ/p) \xrightarrow{\sim} \bigotimes_{BP} H_*((\Lambda^n BZ/p) \xrightarrow{\ln \rho} H_*((L(n))),
\end{array}
\]

where \( \rho : BP_*(\rightarrow) \rightarrow H_*(-) \) is the usual reduction. We see that \( \rho(\{j_1, \ldots, j_n\}) = c_{2j_1-1} \wedge \cdots \wedge c_{2j_n-1} \), and that \( \rho \) preserves the order.

Definition 3.4. For an admissible sequence \( I = (i_1, \ldots, i_n) \), we define \( (I) \in BP_*(L(n)) \) by \( l_{n*}([I]) \).

Note that \( (I) \) is of dimension \( 2(i_1 + \cdots + i_n)(p - 1) - n \).

Proposition 3.5. Let \( K \) be an admissible sequence of \( l(K) = n \), and \( [M] = [m_1, \ldots, m_n] \) an element of \( T \) with \( M \leq K \). Under the duality \( H^*(L(n)) \otimes H_*(L(n)) \xrightarrow{(\cdot, \cdot)} \bigotimes_{Z/p} \), we have

\[
\langle \rho^K, \rho l_{n*}([M]) \rangle = \begin{cases} 1 & \text{if } M = K, \\ 0 & \text{if } M \neq K. \end{cases}
\]

Proof. By the diagram (2) and Corollary 3.3, we have

\[
\langle \rho^K, \rho l_{n*}([M]) \rangle = \langle \rho^K, l_{n*}(\rho([M])) \rangle = \langle l_n^K(\rho^K), \rho([M]) \rangle = \langle X_n^{Y_n^K} + w, c_{2m_1-1} \otimes \cdots \otimes c_{2m_n-1} \rangle
\]

where \( w \) is a linear combination of \( \{X_n^{Y_n^K} \mid K < J \} \). We obtain \( (w, c_{2m_1-1} \otimes \cdots \otimes c_{2m_n-1}) = 0 \), and \( (X_n^{Y_n^K}, c_{2m_1-1} \otimes \cdots \otimes c_{2m_n-1}) = 1 \) if and only if \( M = K \). □

4. The Brown–Peterson homology of \( L(n) \)

We calculate \( BP_*(L(n)) = \pi_* (BP \wedge L(n)) \) by the Adams spectral sequence. The methods in this section is similar to [19].

Let \( A_* \) be the dual Steenrod algebra. We recall

\[
A_* = \left\{ \begin{array}{ll}
A(\tau_0, \tau_1, \ldots) \otimes F_p [\xi_1, \xi_2, \ldots] & \text{deg} \tau_i = 2p^i - 1, \text{deg} \xi_i = 2(p^i - 1) & \text{for } p > 2, \\
A(\tau_0, \tau_1, \ldots) \otimes F_p [\xi_1, \xi_2, \ldots] & \text{deg} \xi_i = 2(p^i - 1) & \text{for } p = 2.
\end{array} \right.
\]

Set \( P = F_p[\xi_1, \xi_2, \ldots] \) for \( p > 2 \), and \( P = F_p[\xi_1^2, \xi_2^2, \ldots] \) for \( p = 2 \). We define \( E = A_* \otimes P F_p \). By abuse of notation, we write \( E = A(\tau_0, \tau_1, \ldots) \) in the case \( p = 2 \).

To compute \( BP_*(X) = \pi_* (BP \wedge X) \), we can use the Adams spectral sequence

\[
\text{Ext}_{A_*}^*(F_p, H_*(BP \wedge X)) \Rightarrow \pi_*(BP \wedge X).
\]

By [24, Lemma 3.18], we have

\[
\text{Ext}_{A_*}^*(F_p, H_*(BP \wedge X)) \cong \text{Ext}_{A_*}^*(F_p, H_*(X)).
\]

For \( X = S^p \), we have

\[
E_2 = \text{Ext}_{A_*}^*(F_p, H_*(BP)) = F_p[v_0, v_1, \ldots] \Rightarrow BP_*,
\]

and this collapses. Here \( v_1 \) in \( E_2^p,2p^i-1 \) corresponds to \( v_i \) in \( BP_*. \)

For \( X = L(n) \), we have

\[
E_2' = \text{Ext}_{A_*}^*(F_p, H_*(BP \wedge L(n))) \cong \text{Ext}_{E_*}^*(F_p, H_*(L(n))) \Rightarrow BP_*(L(n)).
\]

Let \( I_n' = \Lambda(\tau_0, \tau_{n+1}, \ldots) \) and \( I_n = \Lambda(\tau_0, \ldots, \tau_{n-1}) \). Then we have a short exact sequence

\[
0 \rightarrow I_n' \rightarrow E \rightarrow I_n' \rightarrow 0.
\]
We define $\gamma_1 \in H_*(L(n))$ by the dual element of $\gamma \in H^*(L(n))$ with respect to the basis of Lemma 3.1. By Lemma 3.1, we obtain a $\gamma_n$-comodule isomorphism

$$H_*(L(n)) \cong \Gamma_n \otimes N,$$

where $N$ is an $E$-comodule with a basis $\{\gamma_i\}$ as a graded vector space. Since $\Sigma^n N$ is concentrated in even dimensions, the $E$-comodule structure of $\Sigma^n N$ is trivial, and so is $N$.

**Proposition 4.1.** We have

$$E_2' \cong \text{Ext}^{*,*}_{\Lambda} (\mathbb{F}_p, H_*(L(n))) \cong \text{Ext}^{*,*}_{\Gamma_n} (\mathbb{F}_p, N).$$

**Proof.** By Corollary 2.3, we have $E_1 \cong \text{Ext}^{*,*}_{\Gamma_n} (\mathbb{F}_p, H_*(L(n))) \cong \text{Ext}^{*,*}_{\Gamma_n} (\mathbb{F}_p, N)$. By Corollary 2.1 and Lemma 2.4, the right-hand side of the relation is not equal to 0, since $\Sigma^n N$ is trivial. Especially BP $\mathbb{E}$-comodule structure of $N$ is trivial, and so is $N$.

**Lemma 4.2.** The Adams spectral sequence

$$E_2 = \text{Ext}^{*,*}_{\Lambda} (\mathbb{F}_p, H_*(BP \wedge L(n))) \cong \mathbb{F}_p[v_n, v_{n+1}, \ldots] \otimes N \implies BP_*(L(n))$$

collapses.

**Proof.** To avoid case analysis, we consider $\Sigma^n L(n)$ instead of $L(n)$. We have

$$E_2' = \text{Ext}^{*,*}_{\Lambda} (\mathbb{F}_p, H_*(BP \wedge \Sigma^n L(n))) \cong \mathbb{F}_p[v_n, v_{n+1}, \ldots] \otimes \Sigma^n N \implies BP_*(\Sigma^n L(n)).$$

Since $E_2'$ is concentrated in even dimensions, this collapses. Therefore $E_2$ also collapses. □

**Theorem 4.3.** Any element of $BP_*(L(n))$ has a unique representation as a sum

$$\sum_{L,J} c_{L,J} v^L(J),$$

where $c_{L,J} \in \{0, 1, \ldots, p-1\}$, $v^L = v_1^{l_1} \cdots v_m^{l_m}$ for finite sequences $L = (l_1, \ldots, l_m)$ of non-negative integers, and $J = (j_1, \ldots, j_n)$ ranges over all admissible sequences. Especially $BP_*(L(n))$ is $v_n$-local.

**Proof.** The set $\{\gamma_i\}$ is a basis of $N$, and so is $\{J\}$ by Proposition 3.5. Therefore $\text{Ext}^{*,*}_{\Lambda} (\mathbb{F}_p, H_*(BP \wedge L(n)))$ is a free $\mathbb{F}_p[v_n, v_{n+1}, \ldots]$-module on $\{J\}$. The theorem follows from Lemma 4.2. □

**Proposition 4.4.** $BP_*(L(n))$ is $v_j$-torsion for $0 \leq j < n$.

**Proof.** The $BP_*$-module homomorphism $l_n \circ \xi : T = \otimes_{BP} BP_*(B\mathbb{F}_p) \to BP_*(L(n))$ is surjective by Theorem 2.1, Corollary 2.2 and Theorem 4.3. By Theorem 2.1 and Lemma 2.4, $T$ is $v_j$-torsion for $0 \leq j < n$. Therefore $BP_*(L(n))$ is $v_j$-torsion. □

**Proposition 4.5.** For an admissible sequence $J = (j_1, \ldots, j_n)$, we have $p^{j_n}(J) = 0$ and $p^{j_n-1}(J) \neq 0$ in $BP_*(L(n))$.

**Proof.** By Corollary 2.3, we have

$$p^{j_n-1} [J] \equiv (-1)^{0(j_n-1)}(a_{p^{j_n-1}})_{j_n-1}[(j_1 - j_n + 1)(p - 1), (j_2 - j_n + 1)(p - 1), \ldots, (p - 1)],$$

modulo lower order terms. Since $(j_1 - j_n + 1, j_2 - j_n + 1, \ldots, 1)$ is an admissible sequence, we see

$$p^{j_n-1} [J] \equiv (-1)^{0(j_n-1)}(a_{p^{j_n-1}})_{j_n-1}[(j_1 - j_n + 1, j_2 - j_n + 1, \ldots, 1)]$$

modulo lower order terms. The right-hand side of the relation is not equal to 0, since $a_{p^{j_n-1}} \equiv v_n$ modulo $l_n$. By Corollary 2.3, we obtain

$$p^{j_n} [J] \equiv (-1)^{j_n}(a_{p^{j_n}})_{j_n}[(j_1 - j_n)(p - 1), \ldots, 0] + w,$$

where $w$ is a sum of lower order terms. Since $[(j_1 - j_n)(p - 1), \ldots, 0] = 0$, we have $w = 0$. Hence $p^{j_n}(J) = 0$. □
5. **BP-Adams operations on** \(BP_*(L(n))\)

We recall BP-Adams operations \(\psi^k\) from [1,3,23]. For each \(k\) prime to \(p\), the Adams operations \(\psi^k : BP \to BP\) are stable. These satisfy the following properties:

(i) \(\psi^k\) is multiplicative;
(ii) \(\psi^k \psi^l = \psi^{kl}\);
(iii) \(\psi^k(x) = k^l x\) for \(x \in BP_{2^{2^l}}\); and
(iv) \(\psi^{p+1}(y) = (p + 1)^n y\) for \(y \in BP_{2^{p+2}}(B\mathbb{Z}/p)\) [7, Proposition 3.3].

We recall the universal coefficient spectral sequence based on the chromatic tower from [18, Section 3]. Miller, Ravenel and Wilson [17] have introduced the chromatic resolution

\[
BP_* \longrightarrow M^0 \longrightarrow M^1 \longrightarrow M^2 \longrightarrow \ldots
\]

by splicing the short exact sequences

\[
0 \longrightarrow N^n \longrightarrow M^n \longrightarrow N^{n+1} \longrightarrow 0,
\]

where \(N^0 = BP_*\), \(M^n = v_n^{-1} N^n\). Let \(L_n\) be the Bousfield localization functor with respect to \(v_n^{-1} BP\). We define \(N_0 = S^0\), \(M_1 = L_n N_n\), and \(N_{n+1}\) by the cofiber of the localization map \(N_n \longrightarrow L_n N_n = M_n\). Ravenel [25] has shown \(BP_*(N_n) = N^n\), \(BP_*(M_n) = M^n\), and that the cofibrations \(N_n \to M_n \to N_{n+1}\) realize the above short exact sequences. Thus we have the geometric chromatic resolution of sphere:

\[
\begin{align*}
S^0 & \leftarrow p_1 \Sigma^{-1} N_1 & & \leftarrow p_2 \Sigma^{-2} N_2 & & \leftarrow p_3 \ldots \\
L_0 S^0 = M_0 & \downarrow q_0 & \Sigma^{-1} N_1 = \Sigma^{-1} M_1 & \downarrow q_1 & \Sigma^{-2} N_2 = \Sigma^{-2} M_2 & \downarrow q_2 \\
L_1 S^0 & \downarrow q_1 & \Sigma^{-1} N_1 & \downarrow q_1 & \Sigma^{-2} N_2 & \downarrow q_2
\end{align*}
\]

Here \(p_n\) is the fiber of the localization map \(q_{n-1}\).

**Theorem 5.1.** ([18, Theorem 3.1(iii)]) Suppose \(Y = \text{colim} \ Y_n\) is a countable union of finite spectra. Then there exists a spectral sequence converging to \(BP_*(Y)\) such that \(E_1^{s,t} = \text{Hom}_{\text{cont-BP}_*}(BP^*(Y), M^t)\), where the topology of \(BP^*(Y)\) is the inverse limit topology \(BP^*(Y) \to \text{lim} BP^*(Y_n)\). In other words,

\[
E_1^{s,t} = \text{lim} \text{Hom}_{\text{BP}_*}(BP^*(Y_n), M^t).
\]

Hence the \(E_2\)-term becomes the homology of this continuous cochain complex, which we denote by \(\text{Ext}_{\text{cont-BP}_*}^{s,t}(BP^*(Y), M^t)\) in analogy with the continuous cohomology. Thus we may write

\[
E_2^{s,t} = \text{Ext}_{\text{cont-BP}_*}^{s,t}(BP^*(Y), M^t) = \text{lim} \text{Ext}_{BP_*}^{s,t}(BP^*(Y_n), M^t).
\]

**Proof.** We introduce a decreasing filtration \(BP_*(Y) = F^0 \supset F^1 \supset \cdots\) such that \([g] \in F^s\) if and only if \(g\) factorizes as

\[
S^* \to BP \times \Sigma^{-s} M \times Y \to BP \times S^0 \times Y = BP \times Y.
\]

We can show that

\[
E_1^{s,t} = \left\{ S^0, BP \times M \times Y \right\}^* \cong \text{lim} \left\{ S^0, BP \times M \times Y_n \right\}^* \\
\cong \text{lim} \text{Hom}_{BP_*}^{s,t}(BP^*(Y_n), M^t).
\]

\(\square\)

**Proposition 5.2.**

(i) For \(Y = L(n)\), the filtration associated with the spectral sequence of Theorem 5.1 is as follows:

\[BP_d(L(n)) = F^0 \supset F^1 \supset F^2 = \cdots = F^n \supset F^{n+1} \supset 0.\]

(ii) The above filtration is preserved by the action of \(\psi^{p+1}\). Furthermore, when \(2|d + n\), then \(\psi^{p+1}\) acts on \(E_\infty^{n,0+d} = F^n/F^{n+1} = BP_d(L(n))\) as multiplication by \((p + 1)^{d/n}\).
Proof. (i) A decreasing filtration $BP_*(Y) = F^0 \supset F^1 \supset \cdots$ of Theorem 5.1 is defined by $[g] \in F^i$ if and only if $g$ factorizes as

$$S^* \longrightarrow BP \wedge \Sigma^{-s} N_1 \wedge Y \xrightarrow{\beta \wedge (\rho \circ \sigma_p) \wedge Y} BP \wedge S^0 \wedge Y = BP \wedge Y.$$ 

Since $BP_*(L(n))$ is $v_j$-torsion for $0 \leq j < n$, and $v_n$-local by Theorem 4.3 and Proposition 4.4, we see $F^0 = F^1 = \cdots = F^n$ and $F^{n+1} = 0$.

(ii) Since $L(n)$ is a stable summand of $B(\mathbb{Z}/p)^n$, this part follows from [18, Proposition 3.2(ii)]. □

Corollary 5.3. Let $d$ be an integer such that $d + n$ is even, and $v_p(k)$ the power of $p$ in $k$. The $p$-exponent of any element in $PB_{P_d}(L(n))$ is at most

$$\begin{cases} 
\alpha + 1 & \text{if } p \text{ is odd}, \\
1 & \text{if } p = 2 \text{ and } \alpha = 0, \\
\alpha + 2 & \text{if } p = 2 \text{ and } \alpha \geq 1,
\end{cases}$$

where $\alpha = v_p((d+n)/2)$.

Proof. Since $\psi^{p+1}$ is a stable multiplicative operation, we have $PB_{P_d}(L(n)) \subset \text{Ker}(\psi^{p+1} - 1)$. It is enough to know the $p$-exponent of any element in $\text{Ker}(\psi^{p+1} - 1)|_{BP_{P_d}(L(n))}$. We recall the usual result in elementary number theory:

$$v_p((p+1)^n - 1) = v_p(n) + \epsilon_p(n), \quad \text{where } \epsilon_p(n) = \begin{cases} 
2 & \text{if } p = 2 \text{ and } v_2(n) \geq 1, \\
1 & \text{otherwise}.
\end{cases}$$

For $x \in BP_{P_d}(L(n))$, we have $\psi^{p+1}(x) = (p+1)^{p^\alpha}x$ by Proposition 5.2(ii). Here $v_p(\beta) = 0$. If $x \in PB_{P_d}(L(n))$, then

$$\psi^{p+1}(x) - x = \{(p+1)^{p^\alpha} - 1\}x = p^{\alpha + \epsilon_p(p^\alpha)k}kx = 0,$$

where $k \neq 0$ and $v_p(k) = 0$. □

6. A basis of $PH_*(L(n))$

In [8,9], we have determined a basis of the indecomposable quotient $QH^*(L(n)) = F_p \otimes_A H^*(L(n))$, which lifts to a minimal $A$-generating set for $H^*(L(n))$.

Theorem 6.1. ([8,9]) A basis of $QH^*(L(n))$ is

$$\begin{cases} 
\Sigma^{p^k_1} \cdots \Sigma^{p^k_n} & (k_1 > \cdots > k_n \geq 0) \quad \text{for } p > 2, \\
\Sigma^{q^k_1} \cdots \Sigma^{q^k_n} & (k_1 > k_2 > \cdots > k_n \geq 1) \quad \text{for } p = 2.
\end{cases}$$

We can easily see this by the following theorem and Lemma 3.1. For $l = (i_1, \ldots, i_n)$, we denote $Sq^{i_1} \cdots Sq^{i_n}$ by $Sq^l$.

Theorem 6.2. ([8, Theorem 4.2], [9, Theorem 4.2]) For an admissible sequence $l = (a_1, a_2, \ldots, a_n)$, we have

$$\begin{cases} 
Sq^l = \lambda Sq^{k_1} \cdots Sq^{k_n} + \sum_j \lambda_j Sq^{a_j} \cdot Sq^l & \text{for } p = 2, \\
\phi^l = \lambda \phi^{k_1} \cdots \phi^{k_n} + \sum_j \lambda_j \phi^{a_j} \cdot \phi^l & \text{for } p > 2,
\end{cases}$$

where $\lambda, \lambda_j \in \mathbb{F}_p$, $a_j > 0$, $k_1 > \cdots > k_n \geq 0$, and $j$ ranges over all admissible sequences of $l(j) = n$.

We study the action of $A$ on $H^*(BT^n) = F_p[y_1, \ldots, y_n]$, where $y_j \in H^2(BT^n)$ are the standard generators. Through this section, we discuss only the cases $p > 2$. We can proceed in a similar way for $p = 2$.

Lemma 6.3. For $k \geq 0$, we have

$$\phi^l(y_j^p) = \begin{cases} 
y_j^p & (i = 0), \\
y_j^{p+1} & (i = p^k), \\
0 & (i \neq 0, p^k).
\end{cases}$$

Proof. The lemma follows immediately from the Cartan formula and $\phi^l(y_j) = y_j^p$. □
Lemma 6.4. For $a > 0$, we have
\[ g^a(y_1^{p_1} \cdots y_n^{p_n}) = \sum_{\alpha_1, \ldots, \alpha_n \in \{0, 1\}} c(\alpha_1, \ldots, \alpha_n) y_1^{p_1+\alpha_1} \cdots y_n^{a \sum_{j=1}^n \alpha_j}, \]
where $\sum_{j=1}^n \alpha_j \geq 1$, and $c(\alpha_1, \ldots, \alpha_n) \in \{0, 1\}$. If $c(\alpha_1, \ldots, \alpha_n) \neq 0$ and there exists $i$ which satisfies $a > p^{k_i}$ and $\alpha_i = 1$, then $\sum_{j=1}^n \alpha_j > 1$.

Proof. The lemma follows from the Cartan formula and Lemma 6.3.

Lemma 6.5. For $k_1 > \cdots > k_n \geq 0$, we have
\[ g \circ \cdots \circ g^{a_i} (y_1^{p_1} \cdots y_n^{p_n}) = \sum_{\alpha_1, \ldots, \alpha_n \geq 0} c(\alpha_1, \ldots, \alpha_n) y_1^{p_1+\alpha_1} \cdots y_n^{p_n+\alpha_n}, \]
where $(\alpha_1, \ldots, \alpha_n) \neq (1, \ldots, 1)$, $\sum_{j=1}^n \alpha_j > n$, and $c(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p$. Especially $\alpha_i \geq 2$ for some $i$.

Proof. We apply Lemma 6.3 and Lemma 6.4 repeatedly.

Proposition 6.6. Under the condition and the notation of Theorem 6.2, if $l = (a_1, \ldots, a_0) \neq (p^{k_1}, \ldots, p^{k_n})$, then $\lambda = 0$.

Proof. We investigate the coefficients of $y_1^{p_1+\alpha_i} \cdots y_n^{p_n+\alpha_i}$ in the polynomials $g \circ \cdots \circ g^{a_i} (y_1^{p_1} \cdots y_n^{p_n})$ for $i \geq 1$. By Lemma 6.5, we have
\[ g_i (y_1^{p_1} \cdots y_n^{p_n}) = g_1 \cdots g_{a_i-1} (y_1^{p_1} \cdots y_n^{p_n}), \]
where $a_i \geq 0$, $\sum_{j=1}^n \alpha_j = n - i + 1$, $(\alpha_1, \ldots, \alpha_n) \neq (1, \ldots, 1)$, and $c(\alpha_1, \ldots, \alpha_n) \in \mathbb{F}_p$. Since $\alpha_i \geq 2$ for some $i \leq s \leq n$, it is enough to consider the polynomial $g_i (y_1^{p_1} \cdots y_n^{p_n})$. We discuss two cases: (a) $a_i < p^{k_{i-1}}$; and (b) $a_i > p^{k_{i-1}}$.

(a) If $a_i < p^{k_{i-1}}$, we have
\[ g_1 \cdots g_{a_i-1} (y_1^{p_1} \cdots y_n^{p_n}) = g_1 \cdots g_{a_i-2} (y_1^{p_1} \cdots y_n^{p_n+\alpha_i+1}). \]
By Lemma 6.4, we have
\[ g_{a_i-1} (y_1^{p_1+\alpha_i} \cdots y_n^{p_n+\alpha_i}) = \sum_{\alpha_1', \ldots, \alpha_n' \geq 1} c(\alpha_1', \ldots, \alpha_n') y_1^{p_1+\alpha_1'} \cdots y_n^{p_n+\alpha_n'}, \]
where $c(\alpha_1', \ldots, \alpha_n') \in \{0, 1\}$ and $\sum_{j=1}^n \alpha_j' = n - i + 2$. Since $\alpha_i' \geq 2$ for some $i \leq t \leq n$, the coefficient of $y_1^{p_1+\alpha_1'} \cdots y_n^{p_n+\alpha_n'}$ is equal to 0.

(b) If $a_i > p^{k_{i-1}}$, we have $a_j > p^{k_{j-1}}$ for $j < i - 1$. By using Lemma 6.4 repeatedly, we obtain
\[ g_1 \cdots g_{a_i-1} (y_1^{p_1} \cdots y_n^{p_n+\alpha_i}) = \sum_{\alpha_1'', \ldots, \alpha_n'' \geq 1} c(\alpha_1'', \ldots, \alpha_n'') y_1^{p_1+\alpha_1''} \cdots y_n^{p_n+\alpha_n''}. \]
where $c(\alpha_1'', \ldots, \alpha_n'') \in \mathbb{F}_p$ and $\sum_{j=1}^n \alpha_j'' = n + 1$. Therefore the coefficient of $y_1^{p_1+\alpha_1''} \cdots y_n^{p_n+\alpha_n''}$ is equal to 0.
(iii) By using Lemma 6.4 repeatedly,
\[ g^\alpha \cdot g^\beta (y_1^{p_{k_1}} \cdots y_n^{p_{k_n}}) = \sum_{\alpha_1, \ldots, \alpha_n \geq 0} c_{(\alpha_1, \ldots, \alpha_n)} y_1^{p_{k_1} + \alpha_1} \cdots y_n^{p_{k_n} + \alpha_n}, \]
where \( c_{(\alpha_1, \ldots, \alpha_n)} \in \mathbb{F}_p \), \( \sum_{i=1}^n \alpha_i \geq n + 1 \). Hence the coefficient of \( y_1^{p_{k_1} + 1} \cdots y_n^{p_{k_n} + 1} \) is equal to 0.
The three cases concludes \( \lambda = 0 \). \( \square \)

**Corollary 6.7.** The set \( \{\varphi_I | I = (p^{k_1}, p^{k_2}, \ldots, p^{k_n}) \} \) is not in \( \mathbb{F}_p \), \( k_1 > k_2 > \cdots > k_n \geq 0 \) is a basis of \( PH_s(L(n)) \).

**Proof.** By Lemma 7.1, the element (3) is equal to \( (\gamma \varphi)^{p_{k_1} + 1} \cdots (\gamma \varphi)^{p_{k_n} + 1} = 0 \in QH^s(L(n)) \).
By Lemma 3.1, we see that \( Q_i \cdots Q_n \varphi^j = 0 \in QH^*(L(n)) \) for \( s > 0 \). The corollary follows from the dual statement of these and Theorem 6.2. \( \square \)

**7. \( PBP_s(L(n)) \) and \( PH_s(L(n)) \)**

Let \( I = (i_1, \ldots, i_n) \) be an admissible sequence. For \( (I) \in BP_s(L(n)) \), we define \( ||I|| = \rho((I)) \in H_s(L(n)) \), where \( \rho : BP_s(L(n)) \rightarrow H_s(L(n)) \) is the usual reduction.

**Lemma 7.1.** We have
\[ \varphi_I = ||I|| + \sum_J \varepsilon_J ||J||, \]
where \( J \) ranges over all admissible sequences of length \( n \), and \( \varepsilon_J \in \mathbb{F}_p \).

**Proof.** By Proposition 3.5, we have \( ||I|| = \varphi_I + \sum' \varepsilon'_J \varphi_J \), where \( J \) ranges over admissible sequences with \( l(I') = n \) and \( l(I') > l(I) \), and \( \varepsilon'_J \in \mathbb{F}_p \). Applying the above equality to \( \varphi_I \) repeatedly, we have the lemma. \( \square \)

By Theorem 4.3, any element of \( BP_s(L(n)) \) is expressed as \( \sum_{l,j} c_{L,j} v^L(J) \). If \( L \neq (0, \ldots, 0) \), then \( \rho(v^L(J)) = 0 \). By abuse of notation, we write \( \varphi_I \) for the element \( \sum_J c_{(0),J} v^J(J) \) of \( BP_s(L(n)) \) such that \( \rho(\sum_J c_{(0),J} v^J(J)) = \varphi_I \). By Corollary 6.7, any element of \( \rho^{-1}(PH_s(L(n)) - \{0\}) \) is written as
\[ \gamma \varphi(p^{k_1}, p^{k_2}, \ldots, p^{k_n}) + \sum_{L \neq (0), J'} c_{L,J'} v^L(J'), \]
where \( \gamma \neq 0 \), and \( J' \) ranges over all admissible sequences of \( l(J') = n \).

**Theorem 7.2.** The reduction of the primitive part \( \rho|_{BP_s(L(n))} : BP_s(L(n)) \rightarrow PH_s(L(n)) \) is trivial, when
\[ \begin{align*}
&= 2(p - 1)(p^{k_1} + \cdots + p^{k_n}) - n \quad (k_1 > \cdots > k_n \geq 1) \quad \text{if } p > 2, \\
&= 2(2^{k_1} + \cdots + 2^{k_n}) - n \quad (k_1 > \cdots > k_n \geq 3) \quad \text{if } p = 2.
\end{align*} \]

**Proof.** By Lemma 7.1, the element (3) is equal to
\[ \gamma \varphi(p^{k_1}, p^{k_2}, \ldots, p^{k_n}) + \sum_{l' \neq (0)} c_{(0),l'} v^{l'}(J') + \sum_{L \neq (0), J'} c_{L,J'} v^L(J'), \]
where \( l', J' \) range over all admissible sequences of length \( n \), and \( c_{(0),l'} c_{L,J'} \in \{0, \ldots, p - 1\} \). Since \( \gamma \neq 0 \), the \( p \)-exponent of the element (4) is at least \( p^{k_n} \) by Proposition 4.5. If \( p > 2 \) and \( k_n \geq 1 \), then \( p^{k_n} > k_n + 1 \). By Corollary 5.3, the element (4) is not in \( PBP_s(L(n)) \). The theorem for \( p = 2 \) follows in a similar way. \( \square \)

**8. Main theorems**

If \( \dim QH^k(L(n)) = \dim PH_k(L(n)) = 0 \), then \( h_k(L(n))_s \) has trivial image. Theorem 6.1 implies the following corollary.

**Corollary 8.1.** The image of \( h_k(L(n))_k : \pi_k(L(n)) \rightarrow H_k(L(n)) \) is trivial
\[ \begin{align*}
&\text{if } \begin{cases} k \neq 2(p - 1)(p^{k_1} + p^{k_2} + \cdots + p^{k_n}) - n \quad (k_1 > \cdots > k_n \geq 0) & \text{for } p > 2, \\
&k \neq 2(2^{k_1} + 2^{k_2} + \cdots + 2^{k_n}) - n \quad (k_1 > k_2 > \cdots > k_n \geq 0) & \text{for } p = 2.
\end{cases}
\end{align*} \]
Since \( \dim QH_k(L(n)) = \dim PH_k(L(n)) \leq 1 \) by Theorem 6.1, it is enough to consider the case \( \dim QH_k(L(n)) = \dim PH_k(L(n)) = 1 \). Then the image of \( h_H(L(n))_* \) is equal to 0 or \( F_p \). We have the following main theorems.

**Theorem A.** The image of \( h_H(L(n))_k \) is trivial

\[
\begin{aligned}
&k = 2(p - 1)(p^{k_1} + \cdots + p^{k_{n-2}}) - n \quad (k_1 > \cdots > k_{n-2} \geq 1) \\
&k = 2(2^{k_1} + \cdots + 2^{k_{n-2}}) - n \quad (k_1 > \cdots > k_{n-2} \geq 3)
\end{aligned}
\]

for \( p > 2 \).

**Proof.** We have the following commutative diagram:

\[
\begin{array}{ccc}
\pi_*(L(n)) & \xrightarrow{h_{BP}(L(n))_*} & PB_{BP}(L(n)) \subset BP_{BP}(L(n)) \\
& \downarrow{h_H(L(n))_*} & \downarrow{\rho} \\
& PH_{BP}(L(n)) \subset H_{BP}(L(n)). & \\
\end{array}
\]

The image of \( h_H(L(n))_* \) is included in \( PH_{BP}(L(n)) \) via \( PB_{BP}(L(n)) \). The theorem follows from Theorem 7.2. \( \square \)

Adding Corollary 8.1 to Theorem A, we have the following theorem.

**Theorem B.** The image of \( h_H(L(n))_k \) is trivial

\[
\begin{aligned}
&k \neq 2(p - 1)(p^{k_1} + \cdots + p^{k_{n-1}}) - n \quad (k_1 > \cdots > k_{n-1} \geq 1) \\
&k = 2(2^{k_1} + \cdots + 2^{k_{n-2}}) - n \quad (k_1 > \cdots > k_{n-2} \in \{0, 1, 2\})
\end{aligned}
\]

Since \( M(n) \supseteq L(n) \lor L(n - 1) \), we can obtain the result for \( M(n) \).

**Theorem C.** The image of \( h_H(M(n))_k : \pi_k(M(n)) \to H_k(M(n)) \) is trivial

\[
\begin{aligned}
&k \neq 2(p - 1)(p^{k_1} + \cdots + p^{k_{n-1}}) - n \quad (k_1 > \cdots > k_{n-1} \geq 1), \\
&2(p - 1)(p^{k_1} + \cdots + p^{k_{n-1}}) - n + 1 \quad (k_1' > \cdots > k_{n-2}' \geq 1) \\
&k \neq 2(2^{k_1} + \cdots + 2^{k_{n-2}}) - n \quad (k_1 > \cdots > k_{n-2} \in \{0, 1, 2\}), \\
&2(2^{k_1} + \cdots + 2^{k_{n-2}}) - n + 1 \quad (k_1' > \cdots > k_{n-2}' \in \{0, 1, 2\})
\end{aligned}
\]

for \( p > 2 \).

**Acknowledgements**

The author would like to thank Professor Norihiko Minami for suggesting this problem, and Professor Hirofumi Nakai for English correction and helpful discussions.

This work is partially supported by the Grant-in-Aid for Young Scientists (B) No. 19740037 from The Ministry of Education, Culture, Sports, Science and Technology, Japan.

**Appendix A**

Since \( L(n) \) is connective and \( H_1(L(n)) = 0 \) for \( t < 2(p - 1)(p^{n-1} + \cdots + 1) - n \), the image of \( h(L(n))_{2(p - 1)(p^{n-1} + \cdots + 1) - n} \) is equal to \( F_p \) by the Hurewicz theorem. However we do not determine the image of \( h(L(n))_* \) completely. We can obtain the result for only the case \( n = 1 \) by [7].

**Proposition 9.1.** ([7, Proposition 3.12]) For \( p > 2 \), the stable Hurewicz image of \( B\mathbb{Z}/p \) is

\[
\text{Im}(\pi_{2n+1}(B\mathbb{Z}/p) \to H_{2n+1}(B\mathbb{Z}/p; \mathbb{Z})) = \begin{cases} F_p & \text{for } 0 \leq n \leq p - 1, \\ 0 & \text{otherwise.} \end{cases}
\]

For \( p = 2 \), the Hurewicz image of \( B\mathbb{Z}/2 \) is

\[
\text{Im}(\pi_{2n+1}(B\mathbb{Z}/2) \to H_{2n+1}(B\mathbb{Z}/2; \mathbb{Z})) = \begin{cases} F_2 & \text{for } n = 0, 1, 3, \\ 0 & \text{otherwise.} \end{cases}
\]

This induces the following corollary.

**Corollary 9.2.** For \( p > 2 \), the Hurewicz image of \( L(1) \) is

\[
\text{Im}(\pi_{2(p - 1)k - 1}(L(1)) \to H_{2(p - 1)k - 1}(L(1))) = \begin{cases} F_p & \text{for } k = 1, \\ 0 & \text{otherwise.} \end{cases}
\]
For $p = 2$, the Hurewicz image of $L(1)$ is

$$\text{Im}((\pi_{2k-1}(L(1)) \to H_{2k-1}(L(1)))) = \begin{cases} \mathbb{F}_2 & \text{for } k = 1, 2, 4, \\ 0 & \text{otherwise}. \end{cases}$$

We see that the image of $h(L(1))_k$ is equal to $\mathbb{F}_p$ except the condition in Theorem B. We conjecture that the image of $h(L(n))_k$ is equal to $\mathbb{F}_p$ except the condition in Theorem B.

References