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Efficiency Conditions and Duality Models for Multiobjective Fractional Subset Programming Problems with Generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-Convex Functions

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Abstract—In this paper, we first introduce a new class of generalized convex n-set functions, called $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-*convex functions*, and then present numerous sets of parametric and semiparametric sufficient efficiency conditions under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity assumptions for a multiobjective fractional subset programming problem. Moreover, we construct three parametric and three semiparametric duality models and prove appropriate duality theorems. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Multiobjective fractional subset programming, Generalized n-set convex functions, Efficiency criteria, Duality theorems.

1. INTRODUCTION

In this paper, we shall present a fairly large number of global parametric and semiparametric sufficient efficiency conditions and duality results under various generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity hypotheses for the following multiobjective fractional subset programming problem:

$$\begin{aligned} &\text{Minimize:} && \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \dots, \frac{F_p(S)}{G_p(S)}\right) \\ &\text{subject to:} && H_j(S) \leqq 0, \qquad j \in \underline{q}, \quad S \in \mathbb{A}^n, \end{aligned} \tag{P}$$

where $\mathbb{A}^n$ is the n-fold product of the σ-algebra $\mathbb{A}$ of subsets of a given set X, F_i, G_i, $i \in \underline{p} \equiv \{1, 2, \dots, p\}$, and H_j, $j \in \underline{q}$, are real-valued functions defined on $\mathbb{A}^n$, and for each $i \in \underline{p}$, $G_i(S) > 0$ for all $S \in \mathbb{A}^n$ such that $H_j(S) \leqq 0$, $j \in \underline{q}$.

The point-function counterparts of (P) are known in the area of mathematical programming as *multiobjective fractional programming problems*. These problems have been the focus of intense interest in the past few years, which has resulted in numerous publications. A fairly extensive list of references concerning various aspects of these problems is given in [1]. For more information

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about general multiobjective problems with point-functions, the reader may consult the recently published monograph by Miettinen [2].

In the area of subset programming, multiobjective problems have been investigated in [3–17], and multiobjective *fractional* problems in [18–24]. We next give a brief overview of the available results pertaining to the latter class of problems.

A parametric dual problem for (P) is constructed in [18] and a number of weak and strong duality theorems involving generalized ρ-convexity assumptions are proved. In [20], two parametric dual problems, which are slightly different from the one considered in [18], are formulated and some weak, strong, and strict converse duality results are established using generalized ρ-convexity hypotheses. Some of these results are further extended in [22] by using generalized $\mathcal{F}$-convex n-set functions. A multiobjective fractional problem like (P) in which the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are assumed to be convex is considered in [19] where parametric, semiparametric, and Lagrangian-type dual problems are formulated and weak, strong, and strict converse duality theorems are proved; in addition, a set of sufficient conditions characterizing properly efficient solutions of the problem under consideration is given. A problem similar to the one studied in [19], but with one additional restriction is discussed in [23]. In this paper, it is assumed that the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are convex and that the denominators of the objective functions are equal. With these assumptions, the authors establish necessary and sufficient proper efficiency results, formulate a dual problem that has the same objective function as the primal problem, and prove weak and strong duality theorems. In [21], Preda defines a (ρ, b)-vex n-set function, discusses some of its properties, and then establishes weak, strong, and converse duality results for a parametric dual problem for (P) under appropriate (ρ, b)-vexity conditions. b-vex n-set functions are utilized in [14] for obtaining sufficient proper efficiency criteria and some duality relations for a nonfractional multiobjective subset programming problem. The relevance and applicability of these results to a problem like (P) in which the functions $F_i, -G_i$, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are convex, and for each $i \in \underline{p}$, $F_i(S) \geqq 0$ and $G_i(S) > 0$ for all $S \in \mathbb{A}^n$, are also discussed. Recently, saddle-point-type proper efficiency conditions and Lagrangian-type duality results were obtained in [24] under cone-convexity assumptions for a cone-constrained multiobjective subset programming problem.

For brief surveys and additional references dealing with different aspects of subset programming problems, including areas of applications, optimality conditions, and duality models, the reader is referred to [14,17,25–28].

The rest of this paper is organized as follows. In Section 2, we recall the definitions of differentiability, convexity, and certain types of generalized convexity for n-set functions, which will be used frequently throughout the sequel. We begin our discussion of parametric sufficient efficiency criteria for (P) in Section 3 where we state and prove a number of sufficiency results under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity assumptions. These include two general sets of sufficiency conditions that are formulated with the help of two partitioning schemes. The first of these schemes was originally used in [29] for constructing generalized dual problems for nonlinear programs with point-functions, whereas the second, which is a variant of the first, was recently utilized in [30] for constructing a general dual problem for a multiobjective programming problem with point-functions. Semiparametric versions of these sufficiency results are discussed in Section 4. In Section 5, we turn to an investigation of the notion of parametric duality for (P). Here, we consider a simple dual problem and prove weak, strong, and strict converse duality theorems. In Sections 6 and 7, we formulate two general parametric duality models which are, in fact, two families of dual problems whose members can readily be identified by appropriate choices of certain sets and functions. In each case, we prove appropriate weak, strong, and strict converse duality theorems under generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity hypotheses. In Sections 8–10, we discuss the semiparametric counterparts of the duality models presented in Sections 5–7.

Evidently, all these efficiency and duality results are also applicable, when appropriately specialized, to the following three classes of problems with multiple, fractional, and conventional objective functions, which are particular cases of (P):

$$\text{Minimize}_{S\in\mathbb{F}} \quad (F_1(S), F_2(S), \ldots, F_p(S)); \tag{P1}$$

$$\text{Minimize}_{S\in\mathbb{F}} \quad \frac{F_1(S)}{G_1(S)}; \tag{P2}$$

$$\text{Minimize}_{S\in\mathbb{F}} \quad F_1(S), \tag{P3}$$

where $\mathbb{F}$ (assumed to be nonempty) is the feasible set of (P), that is,

$$\mathbb{F} = \left\{S \in \mathbb{A}^n : H_j(S) \leqq 0,\ j \in \underline{q}\right\}.$$

Since in most cases, the efficiency and duality results established for (P) can easily be modified and restated for each of the above problems, we shall not explicitly state these results.

2. PRELIMINARIES

In this section we gather, for convenience of reference, a number of basic definitions which will be used often throughout the sequel, and recall some auxiliary results.

Let $(X, \mathbb{A}, \mu)$ be a finite atomless measure space with $L_1(X, \mathbb{A}, \mu)$ separable, and let d be the pseudometric on $\mathbb{A}^n$ defined by

$$d(R, S) = \left[\sum_{i=1}^{n} \mu^2(R_i \triangle S_i)\right]^{1/2}, \qquad R = (R_1, \ldots, R_n), \quad S = (S_1, \ldots, S_n) \in \mathbb{A}^n,$$

where $\triangle$ denotes symmetric difference; thus, $(\mathbb{A}^n, d)$ is a pseudometric space. For $h \in L_1(X, \mathbb{A}, \mu)$ and $T \in \mathbb{A}$ with characteristic function $\chi_T \in L_\infty(X, \mathbb{A}, \mu)$, the integral $\int_T h\, d\mu$ will be denoted by $\langle h, \chi_T \rangle$.

We next define the notions of differentiability and convexity for n-set functions. They were originally introduced by Morris [25] for set functions, and subsequently extended by Corley [26] for n-set functions.

DEFINITION 2.1. *A function $F : \mathbb{A} \to \mathbb{R}$ is said to be differentiable at S^* if there exists $DF(S^*) \in L_1(X, \mathbb{A}, \mu)$, called the derivative of F at S^*, such that for each $S \in \mathbb{A}$,*

$$F(S) = F(S^*) + \langle DF(S^*), \chi_S - \chi_{S^*} \rangle + V_F(S, S^*),$$

where $V_F(S, S^*)$ is $o(d(S, S^*))$, that is, $\lim_{d(S,S^*)\to 0} V_F(S, S^*)/d(S, S^*) = 0$.

DEFINITION 2.2. *A function $G : \mathbb{A}^n \to \mathbb{R}$ is said to have a partial derivative at $S^* = (S_1^*, \ldots, S_n^*) \in \mathbb{A}^n$ with respect to its i^{th} argument if the function $F(S_i) = G(S_i^*, \ldots, S_{i-1}^*, S_i, S_{i+1}^*, \ldots, S_n^*)$ has derivative $DF(S_i^*), i \in \underline{n}$; in that case, the i^{th} partial derivative of G at S^* is defined to be $D_iG(S^*) = DF(S_i^*)$, $i \in \underline{n}$.*

DEFINITION 2.3. *A function $G : \mathbb{A}^n \to \mathbb{R}$ is said to be differentiable at S^* if all the partial derivatives $DG_i(S^*), i \in \underline{n}$, exist and*

$$G(S) = G(S^*) + \sum_{i=1}^{n} \langle DG_i(S^*), \chi_{S_i} - \chi_{S_i^*} \rangle + W_G(S, S^*),$$

where $W_G(S, S^*)$ is $o(d(S, S^*))$ for all $S \in \mathbb{A}^n$.

It was shown by Morris [25] that for any triple $(S, T, \lambda) \in \mathbb{A} \times \mathbb{A} \times [0, 1]$, there exist sequences $\{S_k\}$ and $\{T_k\}$ in $\mathbb{A}$ such that

$$\chi_{S_k} \xrightarrow{w^*} \lambda\chi_{S\setminus T} \qquad \text{and} \qquad \chi_{T_k} \xrightarrow{w^*} (1-\lambda)\chi_{T\setminus S} \tag{2.1}$$

imply

$$\chi_{S_k \cup T_k \cup (S\cap T)} \xrightarrow{w^*} \lambda\chi_S + (1-\lambda)\chi_T, \tag{2.2}$$

where $\xrightarrow{w^*}$ denotes weak* convergence of elements in $L_\infty(X, \mathbb{A}, \mu)$, and $S\setminus T$ is the complement of T relative to S. The sequence $\{V_k(\lambda)\} = \{S_k \cup T_k \cup (S \cap T)\}$ satisfying (2.1) and (2.2) is called the *Morris sequence* associated with (S, T, λ).

DEFINITION 2.4. *A function $F : \mathbb{A}^n \to \mathbb{R}$ is said to be (strictly) convex if for every $(S, T, \lambda) \in \mathbb{A}^n \times \mathbb{A}^n \times [0, 1]$, there exists a Morris sequence $\{V_k(\lambda)\}$ in $\mathbb{A}^n$ such that*

$$\limsup_{k\to\infty} F(V_k(\lambda)) \; (<) \leqq \lambda F(S) + (1 - \lambda)F(T).$$

It was shown in [25,26] that if a differentiable function $F : \mathbb{S} \to \mathbb{R}$ is (strictly) convex, then

$$F(S) \; (>) \geqq F(T) + \sum_{i=1}^{n} \langle D_i F(T), \chi_{S_i} - \chi_{T_i} \rangle$$

for all $S, T \in \mathbb{A}^n$.

For the purpose of formulating and proving various collections of sufficiency criteria and duality results for (P), in this study we shall use a new class of generalized convex n-set functions, called $(\mathcal{F}, \alpha, \rho, \theta)$-*V-convex functions*, which will be defined later in this section. This class of functions may be viewed as an n-set version of a combination of three classes of point-functions: $\mathcal{F}$-convex functions, ρ-invex functions, and V-invex functions, which were introduced in [31–33], respectively.

Prior to giving the definitions of the new classes of n-set functions, it will be useful for purposes of reference and comparison to recall the definitions of the point-function analogues of the principal components of these functions mentioned above. We shall keep this review to a bare minimum because our primary objective is only to put a number of interrelated generalized convexity concepts in proper perspective. For this reason, we shall only reproduce the essential forms of the definitions without elaborating on their refinements, variants, special cases, and other manifestations. For full discussions of the consequences and applications of the underlying ideas, the reader may consult the original sources. We begin by defining an invex function, which occupies a central position in a vast array of generalized convex functions some of which are specified in the following definitions.

DEFINITION 2.5. *(See [34].) Let f be a real-valued differentiable function defined on an open set $\mathbb{S}$ of $\mathbb{R}^n$. Then f is said to be η-invex (invex with respect to η) at x^* if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for each $x \in \mathbb{S}$,*

$$f(x) - f(x^*) \geqq \nabla f(x^*)^{\top} \eta(x, x^*),$$

where $\nabla f(x^*)$ is the gradient of f at x^*, and the superscript $\top$ denotes transposition; f is said to be η-invex (invex with respect to η) on $\mathbb{S}$ if there exists a function $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ such that for all $x, y \in \mathbb{S}$,

$$f(x) - f(y) \geqq \nabla f(y)^{\top} \eta(x, y).$$

From the above definition it is clear that every real-valued differentiable convex function is invex with respect to $\eta(x, y) = x - y$. This generalization of the concept of convexity was originally proposed by Hanson [34] who showed that for a nonlinear programming problem of the form

$$\begin{aligned} &\text{Minimize:} && f(x) \\ &\text{subject to:} && g_i(x) \leqq 0, \qquad i \in \underline{m}, \quad x \in \mathbb{R}^n, \end{aligned} \tag{P$_0$}$$

where the differentiable functions $f, g_i : \mathbb{R}^n \to \mathbb{R}$ are invex with respect to the same function η, the Karush-Kuhn-Tucker necessary optimality conditions are also sufficient. The term *invex* (for *in*variant con*vex*) was coined by Craven [35] to signify the fact that the invexity property, unlike convexity, remains invariant under bijective coordinate transformations.

In a similar manner, one can readily define η-pseudoinvex and η-quasi-invex functions as generalizations of differentiable pseudoconvex and quasiconvex functions.

The notion of invexity has been extended in several directions. Some recent surveys and synthesis of results pertaining to various generalizations of invex functions and their applications along with extensive lists of relevant references are available in [36–41]. One of the earliest extensions of invexity, called $\mathcal{F}$-*convexity*, was proposed by Hanson and Mond [31]. An $\mathcal{F}$-convex function is defined in terms of a sublinear function, that is, a function that is subadditive and positively homogeneous.

DEFINITION 2.6. *A function $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}$ is said to be sublinear if $\mathcal{F}(x + y) \leqq \mathcal{F}(x) + \mathcal{F}(y)$ for all $x, y \in \mathbb{R}^n$, and $\mathcal{F}(ax) = a\mathcal{F}(x)$ for all $x \in \mathbb{R}^n$ and $a \in \mathbb{R}_+ \equiv [0, \infty)$.*

Now combining the definitions of $\mathcal{F}$-convex and (ρ, η)-invex functions given in [31,32], respectively, we can define $(\mathcal{F}, \rho)$-convex, $(\mathcal{F}, \rho)$-pseudoconvex, and $(\mathcal{F}, \rho)$-quasiconvex functions.

Let h be a real-valued differentiable function defined on the open subset $\mathbb{S}$ of $\mathbb{R}^n$, and assume that for each $x, y \in \mathbb{S}$, the function $\mathcal{F}(x, y; \cdot) : \mathbb{R}^n \to \mathbb{R}$ is sublinear.

DEFINITION 2.7. *The function h is said to be $(\mathcal{F}, \rho)$-convex at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,*

$$h(x) - h(y) \geqq \mathcal{F}(x, y; \nabla h(y)) + \rho \|x - y\|^2,$$

where $\|z\|$ is the Euclidean norm of $z \in \mathbb{R}^n$.

DEFINITION 2.8. *The function h is said to be $(\mathcal{F}, \rho)$-pseudoconvex at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,*

$$\mathcal{F}(x, y; \nabla h(y)) \geqq -\rho \|x - y\|^2 \Rightarrow h(x) \geqq h(y).$$

DEFINITION 2.9. *The function h is said to be $(\mathcal{F}, \rho)$-quasiconvex at y if there exists a real number ρ such that for each $x \in \mathbb{S}$,*

$$h(x) \leqq h(y) \Rightarrow \mathcal{F}(x, y; \nabla h(y)) \leqq -\rho \|x - y\|^2.$$

Evidently, if in Definitions 2.7–2.9 we choose $\mathcal{F}(x, y; \nabla h(y)) = \nabla h(y)^\top \eta(x, y)$, where $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ is a given function, and set $\rho = 0$, then we see that they reduce to the definitions of η-invexity, η-pseudoinvexity, and η-quasi-invexity for the function h.

One of the most recent generalizations of invexity for vector-valued functions, named *V-invexity*, is due to Jeyakumar and Mond [33]. Below, we recall the definitions of V-invex, V-pseudoinvex, and V-quasi-invex functions.

Let $f = (f_1, f_2, \dots, f_p)$ be a differentiable function defined on an open subset $\mathbb{S}$ of $\mathbb{R}^n$.

DEFINITION 2.10. *The function f is said to be V-invex if there exist functions $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ and $\alpha_i : \mathbb{S} \times \mathbb{S} \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, such that for each $x, y \in \mathbb{S}$ and $i \in \underline{p}$,*

$$f_i(x) - f_i(y) \geqq \alpha_i(x, y) \nabla f_i(y)^\top \eta(x, y).$$

DEFINITION 2.11. *The function f is said to be V-pseudoinvex if there exist functions $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ and $\beta_i : \mathbb{S} \times \mathbb{S} \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, such that for each $x, y \in \mathbb{S}$,*

$$\sum_{i=1}^{p} \nabla f_i(y)^\top \eta(x, y) \geqq 0 \Rightarrow \sum_{i=1}^{p} \beta_i(x, y) f_i(x) \geqq \sum_{i=1}^{p} \beta_i(x, y) f_i(y).$$

DEFINITION 2.12. *The function f is said to be V-quasi-invex if there exist functions $\eta : \mathbb{S} \times \mathbb{S} \to \mathbb{R}^n$ and $\gamma_i : \mathbb{S} \times \mathbb{S} \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, such that for each $x, y \in \mathbb{S}$,*

$$\sum_{i=1}^{p} \gamma_i(x, y) f_i(x) \leqq \sum_{i=1}^{p} \gamma_i(x, y) f_i(y) \Rightarrow \sum_{i=1}^{p} \nabla f_i(y)^\top \eta(x, y) \leqq 0.$$

From Definitions 2.10–2.12 it is clear that every V-invex functions is both V-pseudoinvex (with $\beta_i = 1/\alpha_i$, $i \in \underline{p}$) and V-quasi-invex (with $\gamma_i = 1/\alpha_i$, $i \in \underline{p}$). Moreover, we observe that if we set $p = 1$, $\alpha_i(x,y) = \beta_i(x,y) = \gamma_i(x,y) = 1$, and $\eta(x,y) = x - y$, the Definitions 2.10–2.12 reduce to those of convexity, pseudoconvexity, and quasiconvexity of f, respectively.

Apparently, the motivation for introducing V-invex functions was to relax the rather stringent requirement that in an invex programming problem like (P_0) the invexity property be satisfied for both the objective function and the constraints for the same kernel function η. It was demonstrated in [33] that this improvement enables one to investigate the optimality and duality aspects of a number of mathematical programming problems, including pseudolinear multiobjective problems and certain types of multiobjective fractional programming problems, in a unified framework.

Most of the above classes of generalized convex functions have been utilized for establishing numerous sets of sufficient optimality conditions and a variety of duality results for several categories of static and dynamic optimization problems. For a wealth of information, as well as long lists of references concerning these results, the reader is referred to [36–41].

Finally, we are in a position to give our definitions of generalized $(\mathcal{F},\alpha,\rho,\theta)$-$V$-convex n-set functions. They are formulated by combining the n-set versions of Definitions 2.5–2.12.

Let $S, S^* \in \mathbb{A}^n$, let the function $F : \mathbb{A}^n \to \mathbb{R}^p$, with components F_i, $i \in \underline{p}$, be differentiable at S^*, let $\mathcal{F}(S,S^*;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$ be a sublinear function, and let $\theta : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n \times \mathbb{A}^n$ be a function such that $S \neq S^* \Rightarrow \theta(S,S^*) \neq (0,0)$.

DEFINITION 2.13. *The function F is said to be (strictly) $(\mathcal{F},\alpha,\rho,\theta)$-$V$-convex at S^* if there exist functions $\alpha_i : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, and $\rho \in \mathbb{R}$ such that for each $S \in \mathbb{A}^n$ and $i \in \underline{p}$,*

$$F_i(S) - F_i(S^*)\,(>) \geqq \mathcal{F}\left(S,S^*;\alpha_i(S,S^*)\,DF_i(S^*)\right) + \rho d^2\left(\theta(S,S^*)\right).$$

DEFINITION 2.14. *The function F is said to be (strictly) $(\mathcal{F},\bar{\alpha},\bar{\rho},\theta)$-$V$-pseudoconvex at S^* if there exist functions $\bar{\alpha}_i : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, and $\bar{\rho} \in \mathbb{R}$ such that for each $S \in \mathbb{A}^n$ $(S \neq S^*)$,*

$$\mathcal{F}\left(S,S^*;\sum_{i=1}^{p} DF_i(S^*)\right) \geqq -\bar{\rho} d^2\left(\theta(S,S^*)\right) \Rightarrow \sum_{i=1}^{p} \bar{\alpha}_i(S,S^*)\,F_i(S)(>) \geqq \sum_{i=1}^{p} \bar{\alpha}_i(S,S^*)\,F_i(S^*).$$

DEFINITION 2.15. *The function F is said to be $(\mathcal{F},\tilde{\alpha},\tilde{\rho},\theta)$-$V$-quasiconvex at S^* if there exist functions $\tilde{\alpha}_i : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, and $\tilde{\rho} \in \mathbb{R}$ such that for each $S \in \mathbb{A}^n$,*

$$\sum_{i=1}^{p} \tilde{\alpha}_i(S,S^*)\,F_i(S) \leqq \sum_{i=1}^{p} \tilde{\alpha}_i(S,S^*)\,F_i(S^*) \Rightarrow \mathcal{F}\left(S,S^*;\sum_{i=1}^{p} DF_i(S^*)\right) \leqq -\tilde{\rho} d^2\left(\theta(S,S^*)\right).$$

DEFINITION 2.16. *The function F is said to be prestrictly $(\mathcal{F},\hat{\alpha},\hat{\rho},\theta)$-$V$-quasiconvex at S^* if there exist functions $\hat{\alpha}_i : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{R}_+ \setminus \{0\}$, $i \in \underline{p}$, and $\hat{\rho} \in \mathbb{R}$ such that for each $S \in \mathbb{A}^n$,*

$$\sum_{i=1}^{p} \hat{\alpha}_i(S,S^*)\,F_i(S) < \sum_{i=1}^{p} \hat{\alpha}_i(S,S^*)\,F_i(S^*) \Rightarrow \mathcal{F}\left(S,S^*;\sum_{i=1}^{p} DF_i(S^*)\right) \leqq -\hat{\rho} d^2\left(\theta(S,S^*)\right).$$

Prestrict quasiconvexity for point-functions was first considered in [42].

Using the sublinearity property of $\mathcal{F}(S,S^*;\cdot)$, one can easily see from the above definitions that an $(\mathcal{F},\alpha,\rho,\theta)$-$V$-convex function is both $(\mathcal{F},\alpha,\rho,\theta)$-$V$-pseudoconvex (with $\bar{\alpha}_i = 1/\alpha_i$, $i \in \underline{p}$ and $\bar{\rho} = \sum_{i=1}^{p}[1/\alpha_i(S,S^*)]\rho$) and $(\mathcal{F},\alpha,\rho,\theta)$-$V$-quasiconvex (with $\tilde{\alpha}_i = 1/\alpha_i$, $i \in \underline{p}$, and $\tilde{\rho} = \sum_{i=1}^{p}[1/\alpha_i(S,S^*)]\rho$). Moreover, if a function is $(\mathcal{F},\alpha,\rho,\theta)$-$V$-quasiconvex at S^*, then it is prestrictly $(\mathcal{F},\alpha,\rho,\theta)$-$V$-quasiconvex at S^*, and if function is strictly $(\mathcal{F},\alpha,\rho,\theta)$-$V$-pseudoconvex at S^*, then it is $(\mathcal{F},\alpha,\rho,\theta)$-$V$-quasiconvex at S^*. Obviously, the converses of these assertions are not necessarily true.

In the proofs of the duality theorems, sometimes it may be more convenient to use certain alternative but equivalent forms of the above definitions. These are obtained by considering the contrapositive statements. For example, $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-quasiconvexity can be defined in the following equivalent way.

F is said to be $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-quasiconvex at S^* if for each $S \in \mathbb{A}^n$,

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} DF_i(S^*)\right) > -\rho d^2(\theta(S, S^*)) \Rightarrow \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S) > \sum_{i=1}^{p} \alpha_i(S, S^*) F_i(S^*).$$

Following the introduction of the notion of convexity for set functions by Morris [25] and its extension for n-set functions by Corley [26], various generalizations of convexity for set and n-set functions were proposed in [14,21,27,43–45]. More specifically, quasiconvexity and pseudoconvexity for set functions were defined in [43], and for n-set functions in [45]; generalized ρ-convexity for n-set functions was defined in [27], $(\mathcal{F}, \rho)$-convexity in [44], b-vexity in [14], and (ρ, b)-vexity in [21]. For predecessors and point-function counterparts of these convexity concepts, the reader is referred to the original papers where the extensions to set and n-set functions are discussed. A survey of recent advances in the area of generalized convex functions and their role in developing optimality conditions and duality relations for optimization problems is given in [40].

In the sequel, we shall also need a consistent notation for vector inequalities. For all $a, b \in \mathbb{R}^m$, the following order notation will be used:

$a \geqq b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$;
$a \geqslant b$ if and only if $a_i \geqq b_i$ for all $i \in \underline{m}$, but $a \neq b$;
$a > b$ if and only if $a_i > b_i$ for all $i \in \underline{m}$;
$a \ngeqslant b$ is the negation of $a \geqslant b$.

Throughout this paper, we shall deal exclusively with *efficient* solutions of (P). We recall that an $S^* \in \mathbb{F}$ is said to be an *efficient* solution of (P) if there is no $S \in \mathbb{F}$ such that

$$(F_1(S)/G_1(S), \ldots, F_p(S)/G_p(S)) \leqslant (F_1(S^*)/G_1(S^*), \ldots, F_p(S^*)/G_p(S^*)).$$

In order to derive a set of necessary efficiency conditions for (P), we employ a Dinkelbach-type [46] indirect approach via the following auxiliary parametric problem:

$$\text{Minimize}_{S \in \mathbb{F}} \quad (F_1(S) - \lambda_1 G_1(S), \ldots, F_p(S) - \lambda_p G_p(S)), \tag{Pλ}$$

where λ_i, $i \in \underline{p}$, are parameters. This problem is equivalent to (P) in the sense that for particular choices of λ_i, $i \in \underline{p}$, the two problems have the same set of efficient solutions. This equivalence is stated more precisely in the following lemma whose proof is straightforward, and hence, omitted.

LEMMA 2.1. *An $S^* \in \mathbb{F}$ is an efficient solution of (P) if and only if it is an efficient solution of (Pλ^*) with $\lambda_i^* = F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$.*

Now applying Theorem 3.23 of [9] to (Pλ) and using Lemma 2.1, we obtain the following necessary efficiency result for (P).

THEOREM 2.1. *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exists $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j(S^*) + \sum_{k=1}^{n} \left\langle D_k H_j(S^*), \chi_{\hat{S}_k^i} - \chi_{S_k^*} \right\rangle < 0, \qquad j \in \underline{q},$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \left\langle D_k F_\ell(S^*) - \lambda_\ell^* D_k G_\ell(S^*), \chi_{\hat{S}_k^\ell} - \chi_{S_k^*} \right\rangle < 0.$$

If S^* is an efficient solution of (P) and $\lambda_i^* = F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$, then there exist $u^* \in U = \{u \in \mathbb{R}^q : u > 0,\ \sum_{i=1}^p u_i = 1\}$ and $v^* \in \mathbb{R}_+^q$ such that

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^* \left[D_k F_i\left(S^*\right) - \lambda_i^* D_k G_i\left(S^*\right)\right] + \sum_{j=1}^{q} v_j^* D_k H_j\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0,$$
$$\text{for all } S \in \mathbb{A}^n, \quad v_j^* H_j\left(S^*\right) = 0, \quad j \in \underline{q}.$$

The above theorem contains two sets of parameters u_i^* and λ_i^*, $i \in \underline{p}$. It is possible to eliminate one of these two sets of parameters, and thus, obtain a semiparametric version of Theorem 2.1. Indeed, this can be accomplished by simply replacing λ_i^* by $F_i(S^*)/G_i(S^*)$, $i \in \underline{p}$, and redefining u^* and v^*. For future reference, we state this in the next theorem.

THEOREM 2.2. *Assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at $S^* \in \mathbb{A}^n$, and that for each $i \in \underline{p}$, there exists $\hat{S}^i \in \mathbb{A}^n$ such that*

$$H_j\left(S^*\right) + \sum_{k=1}^{n} \left\langle D_k H_j\left(S^*\right), \chi_{\hat{S}_k^i} - \chi_{S_k^*} \right\rangle < 0, \qquad j \in \underline{q}, \tag{2.3}$$

and for each $\ell \in \underline{p} \setminus \{i\}$,

$$\sum_{k=1}^{n} \left\langle G_\ell\left(S^*\right) D_k F_\ell\left(S^*\right) - F_\ell\left(S^*\right) D_k G_\ell\left(S^*\right), \chi_{\hat{S}_k^\ell} - \chi_{S_k^*} \right\rangle < 0.$$

If S^* is an efficient solution of (P), then there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{p} u_i^* \left[G_i\left(S^*\right) D_k F_i\left(S^*\right) - F_i\left(S^*\right) D_k G_i\left(S^*\right)\right] + \sum_{j=1}^{q} v_j^* D_k H_j\left(S^*\right), \chi_{S_k} - \chi_{S_k^*} \right\rangle \geqq 0,$$
$$\text{for all } S \in \mathbb{A}^n, \quad v_j^* H_j\left(S^*\right) = 0, \quad j \in \underline{q}.$$

For simplicity, we shall henceforth refer to an efficient solution S^* of (P) satisfying (2.3) and (2.4) for some $\hat{S}^i$, $i \in \underline{p}$, as a *normal* efficient solution.

The form and contents of the necessary efficiency conditions given in Theorem 2.2 provide clear guidelines for devising numerous sets of semiparametric sufficient efficiency criteria as well as for constructing various types of semiparametric duality models for (P).

3. PARAMETRIC SUFFICIENT EFFICIENCY CONDITIONS

In this section, we present several sets of parametric sufficient efficiency conditions for (P) under various generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity assumptions. To simplify the statements and proofs of these sufficiency results, we shall introduce along the way some additional notation. For stating our first sufficiency theorem, we use the real-valued functions $\mathcal{A}_i(\cdot, \lambda, u)$ and $\mathcal{B}_j(\cdot, v)$ defined, for fixed λ, u, and v, on $\mathbb{A}^n$ by

$$\mathcal{A}_i(S, \lambda, u) = u_i[F_i(S) - \lambda_i G_i(S)], \qquad i \in \underline{p},$$

and

$$\mathcal{B}_j(S, v) = v_j H_j(S), \qquad j \in \underline{q}.$$

THEOREM 3.1. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^* and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* DG_i(S^*)\right] + \sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \geqq 0, \qquad \text{for all } S \in \mathbb{A}^n, \tag{3.1}$$

$$F_i(S^*) - \lambda_i^* G_i(S^*) = 0, \qquad i \in \underline{p}, \tag{3.2}$$

$$v_j^* H_j(S^*) = 0, \qquad j \in \underline{q}, \tag{3.3}$$

where $\mathcal{F}(S, S^*; \cdot) : L_1^n(X, \mathbb{A}, \mu) \to \mathbb{R}$ is a sublinear function. Assume furthermore that any one of the following four sets of hypotheses is satisfied:

(a) (i) $(F_1, \dots, F_p)$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*convex at* S^*;
  (ii) $(-G_1, \dots, -G_p)$ *is* $(\mathcal{F}, \beta, \hat{\rho}, \theta)$-$V$-*convex at* S^*;
  (iii) $(H_1, \dots, H_q)$ *is* $(\mathcal{F}, \gamma, \tilde{\rho}, \theta)$-$V$-*convex at* S^*;
  (iv) $\alpha_1 = \alpha_2 = \dots = \alpha_p = \beta_1 = \beta_2 = \dots = \beta_p = \gamma_1 = \gamma_2 = \dots = \gamma_q = \delta$;
  (v) $\sum_{i=1}^{p} u_i^*(\bar{\rho}_i + \lambda_i^* \hat{\rho}_i) + \sum_{j=1}^{q} v_j^* \tilde{\rho}_j \geqq 0$;
(b) (i) $(\mathcal{A}_1(\cdot, \lambda^*, u^*), \dots, \mathcal{A}_p(\cdot, \lambda^*, u^*))$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*pseudoconvex at* S^*;
  (ii) $(\mathcal{B}_1(\cdot, v^*), \dots, \mathcal{B}_q(\cdot, v^*))$ *is* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-*quasiconvex at* S^*;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;
(c) (i) (b)(ii) *holds;*
  (ii) $(\mathcal{A}_1(\cdot, \lambda^*, u^*), \dots, \mathcal{A}_p(\cdot, \lambda^*, u^*))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*quasiconvex at* S^*;
  (iii) $\bar{\rho} + \tilde{\rho} > 0$;
(d) (i) (c)(ii) *holds;*
  (ii) $(\mathcal{B}_1(\cdot, v^*), \dots, \mathcal{B}_q(\cdot, v^*))$ *is strictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*pseudoconvex at* S^*;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then S^* is an efficient solution of (P).

PROOF. Let S be an arbitrary feasible solution of (P).

(a) From the sublinearity of $\mathcal{F}(S, S^*; \cdot)$ and (3.1) it follows that:

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* DG_i(S^*)\right]\right) + \mathcal{F}\left(S, S^*; \sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \geqq 0. \tag{3.4}$$

Keeping in mind that $u^* > 0$, $v^* \geqq 0$, and $\lambda^* \geqq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^* \left[F_i(S) - \lambda_i^* G_i(S)\right] \\
&\quad = \sum_{i=1}^{p} u_i^* \left\{F_i(S) - F_i(S^*) - \lambda_i^* \left[G_i(S) - G_i(S^*)\right]\right\} \quad (\text{by } (3.2)) \\
&\quad \geqq \sum_{i=1}^{p} u_i^* \left\{\mathcal{F}(S, S^*; \delta(S, S^*)\, DF_i(S^*)) + \bar{\rho}_i d^2(\theta(S, S^*)) + \lambda_i^* \left[\mathcal{F}(S, S^*; -\delta(S, S^*)\, DG_i(S^*))\right.\right. \\
&\qquad \left.\left. + \hat{\rho}_i d^2(\theta(S, S^*))\right]\right\} \quad (\text{by (i), (ii), and (iv)}) \\
&\quad \geqq \mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \delta(S, S^*) \left[DF_i(S^*) - \lambda_i^* DG_i(S^*)\right]\right) + \sum_{i=1}^{p} u_i^* (\bar{\rho}_i + \lambda_i^* \hat{\rho}_i)\, d^2(\theta(S, S^*)) \\
&\qquad (\text{by the sublinearity of } \mathcal{F}(S, S^*; \cdot)) \\
&\quad \geqq -\mathcal{F}\left(S, S^*; \sum_{j=1}^{q} v_j^* \delta(S, S^*)\, DH_j(S^*)\right) + \sum_{i=1}^{p} u_i^* (\bar{\rho}_i + \lambda_i^* \hat{\rho}_i)\, d^2(\theta(S, S^*)) \quad (\text{by } (3.4))
\end{aligned}$$

$$\geqq \sum_{j=1}^{q} v_j^*[H_j(S^*) - H_j(S)] + \left(\sum_{i=1}^{p} u_i^*(\bar{\rho}_i + \lambda_i^* \hat{\rho}_i) + \sum_{j=1}^{q} v_j^* \tilde{\rho}_j\right) d^2(\theta(S,S^*))$$

(by (iii) and (iv))

$\geqq 0$ (by (3.3), feasibility of S, and (v)).

Since $u^* > 0$, the above inequality implies that $(F_1(S) - \lambda_1^* G_1(S), \dots, F_p(S) - \lambda_p^* G_p(S)) \nleq (0, \dots, 0)$, which in turn implies that

$$\phi(S) \equiv \left(\frac{F_1(S)}{G_1(S)}, \frac{F_2(S)}{G_2(S)}, \dots, \frac{F_p(S)}{G_p(S)}\right) \nleq \lambda^*.$$

Because $S \in \mathbb{F}$ was arbitrary and (3.2) holds, that is, $\lambda^* = \phi(S^*)$, we conclude that S^* is an efficient solution of (P).

(b) Since $v^* \geqq 0$, $S \in \mathbb{F}$, and (3.3) holds, it is clear that $v_j^* H_j(S) \leqq v_j^* H_j(S^*)$ for each $j \in \underline{q}$, and hence,

$$\sum_{j=1}^{q} \beta_j(S,S^*) v_j^* H_j(S) \leqq \sum_{j=1}^{q} \beta_j(S,S^*) v_j^* H_j(T),$$

which by virtue of (ii) implies that

$$\mathcal{F}\left(S, S^*; \sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \leqq -\tilde{\rho}\, d^2(\theta(S,S^*)). \qquad (3.5)$$

From (3.4) and (3.5), we see that

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^*[DF_i(S^*) - \lambda_i^* DG_i(S^*)]\right) \geqq \bar{\rho}\, d^2(\theta(S,S^*)) \geqq -\tilde{\rho}\, d^2(\theta(S,S^*)),$$

where the second inequality follows from (iii). By (i), this inequality implies that

$$\sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geqq \sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*[F_i(S^*) - \lambda_i^* G_i(S^*)]$$

which in view of (3.2) becomes

$$\sum_{i=1}^{p} \alpha_i(S,S^*) u_i^*[F_i(S) - \lambda_i^* G_i(S)] \geqq 0. \qquad (3.6)$$

Since $\alpha_i(S,S^*)u_i^* > 0$ for each $i \in \underline{p}$, (3.6) implies that $(F_1(S) - \lambda_1^* G_1(S), \dots, F_1(S) - \lambda_p^* G_p(S)) \nleq (0, \dots, 0)$, which in turn implies that $\phi(S) \nleq \lambda^*$. Because $\lambda^* = \phi(S^*)$ and $S \in \mathbb{F}$ was arbitrary, we conclude that S^* is an efficient solution of (P).

(c),(d) The proofs are similar to that of Part (b). ∎

Next, we discuss several families of sufficient efficiency conditions under various generalized $(\mathcal{F}, \alpha, \rho, \theta)$-$V$-convexity hypotheses imposed on certain combinations of the problem functions. For this, we need to introduce some additional notation.

Let $\{J_0, J_1, \dots, J_m\}$ be a partition of the index set $\underline{q}$; thus, $J_r \subset \underline{q}$ for each $r \in \{0, 1, \dots, m\}$, $J_r \cap J_s = \emptyset$ for each $r, s \in \{0, 1, \dots, m\}$ with $r \neq s$, and $\bigcup_{r=0}^{m} J_r = \underline{q}$.

In addition, in this section we use the real-valued functions $\Gamma_i(\cdot, \lambda, u, v)$ and $\Delta_t(\cdot, v)$ defined, for fixed λ, u, and v, on $\mathbb{A}^n$ as follows:

$$\Gamma_i(S, \lambda, u, v) = u_i\left[F_i(S) - \lambda_i G_i(S) + \sum_{j \in J_0} v_j H_j(S)\right], \qquad i \in \underline{p},$$

$$\Delta_t(S, v) = \sum_{j \in J_t} v_j H_j(S), \qquad t \in \underline{m}.$$

Making use of this notation, we next state some generalized sufficiency criteria for (P).

THEOREM 3.2. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^* and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (3.1)–(3.3) hold. Assume furthermore, that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Gamma_1(\cdot, \lambda^*, u^*, v^*), \ldots, \Gamma_p(\cdot, \lambda^*, u^*, v^*))$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-pseudoconvex at* S^*;
 (ii) $(\Delta_1(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$*-quasiconvex at* S^*;
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(b) (i) (a)(ii) *holds;*
 (ii) $(\Gamma_1(\cdot, \lambda^*, u^*, v^*), \ldots, \Gamma_p(\cdot, \lambda^*, u^*, v^*))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{b}, \bar{\rho}, \theta)$-$V$*-quasiconvex at* S^*;
 (iii) $\bar{\rho} + \tilde{\rho} > 0$;

(c) (i) (b)(ii) *holds;*
 (ii) $(\Delta_1(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is strictly* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$*-pseudoconvex at* S^*;
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then S^* is an efficient solution of (P).

PROOF.

(a) Let S be an arbitrary feasible solution of (P). Since $v^* \geqq 0$, it follows from (3.3) that for each $t \in \underline{m}$:

$$\Delta_t(S, v^*) = \sum_{j \in J_t} v_j^* H_j(S) \leqq 0 = \sum_{j \in J_t} v_j^* H_j(S^*) = \Delta_t(S^*, v^*),$$

and so

$$\sum_{t=1}^{m} \beta_t(S, S^*) \Delta_t(S, v^*) \leqq \sum_{t=1}^{m} \beta_t(S, S^*) \Delta_t(S^*, v^*),$$

which because of (ii) implies that

$$\mathcal{F}\left(S, S^*; \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* \, DH_j(S^*)\right) \leqq -\tilde{\rho} \, d^2(\theta(S, S^*)). \tag{3.7}$$

From the sublinearity of $\mathcal{F}(S, S^*; \cdot)$ and (3.1) it is easily seen that

$$\begin{aligned}\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* \, DG_i(S^*)\right] + \sum_{j \in J_0} v_j^* \, DH_j(S^*)\right) \\ + \mathcal{F}\left(S, S^*; \sum_{t=1}^{m} \sum_{j \in J_t} v_j^* \, DH_j(S^*)\right) \geqq 0,\end{aligned} \tag{3.8}$$

which in view of (3.7) reduces to

$$\begin{aligned}\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* \, DG_i(S^*)\right] + \sum_{j \in J_0} v_j^* \, DH_j(S^*)\right) \\ \geqq \tilde{\rho} \, d^2(\theta(S, S^*)) \geqq -\bar{\rho} \, d^2(\theta(S, S^*)),\end{aligned}$$

where the second inequality follows from (iii). Since $\sum_{i=1}^{p} u_i^* = 1$, this inequality can be expressed as

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* \, DG_i(S^*) + \sum_{j \in J_0} v_j^* \, DH_j(S^*)\right]\right) \geqq -\bar{\rho} \, d^2(\theta(S, S^*)),$$

which by virtue of (i) implies that

$$\sum_{i=1}^{p} \alpha_i(S,S^*)\Gamma_i(S,\lambda^*,u^*,v^*) \geqq \sum_{i=1}^{p} \alpha_i(S,S^*)\Gamma_i(S^*,\lambda^*,u^*,v^*) = 0,$$

where the equality follows from (3.2) and (3.3). Since $v_j^* H_j(S) \leqq 0$ for each $j \in \underline{q}$, and $\alpha_i(S,S^*) > 0$ for each $i \in \underline{p}$, we deduce from this inequality that

$$\sum_{i=1}^{p} \alpha_i(S,S^*) u_i^* [F_i(S) - \lambda_i^* G_i(S)] \geqq 0,$$

which is precisely (3.6). Therefore, the rest of the proof is identical to that of Part (a) of Theorem 3.1.

(b),(c) The proofs are similar to that of Part (a). ∎

Evidently, Theorem 3.2 contains a number of special cases that can easily be identified by appropriate choices of the partitioning sets $J_0, J_1, \dots, J_m$, and $\mathcal{F}$, α, β, $\bar{\rho}$, $\tilde{\rho}$, and θ. By way of illustration, below we state explicitly some important special cases of Part (a) of Theorem 3.2 resulting from the different choices of the sets J_t, $t = 0, 1, \dots, m$.

COROLLARY 3.1. *Let $S \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^* and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (3.1)–(3.3) hold. Assume furthermore that any one of the following four sets of hypotheses is satisfied:*

(a) (i) $(u_1^*(F_1 - \lambda_1^* G_1), \dots, u_p^*(F_p - \lambda_p^* G_p))$ *is* $(\mathcal{F},\alpha,\bar{\rho},\theta)$*-V-pseudoconvex at* S^**;*
 (ii) $(\sum_{j\in J_1} v_j^* H_j, \dots, \sum_{j\in J_m} v_j^* H_j)$ *is* $(\mathcal{F},\beta,\tilde{\rho},\theta)$*-V-quasiconvex at* S^**;*
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$*;*

(b) (i) $(u_1^*(F_1 - \lambda_1^* G_1 + \sum_{j=1}^q v_j^* H_j), \dots, u_p^*(F_p - \lambda_p^* G_p + \sum_{j=1}^q v_j^* H_j))$ *is* $(\mathcal{F},\alpha,\bar{\rho},\theta)$*-V-pseudoconvex at* S^**;*
 (iii) $\bar{\rho} \geqq 0$*;*

(c) (i) (a)(i) *holds;*
 (ii) $(v_1^* H_1, \dots, v_q^* H_q)$ *is* $(\mathcal{F},\beta,\tilde{\rho},\theta)$*-V-quasiconvex at* S^**;*
 (iii) $\bar{\rho} \geqq 0$*;*

(d) (i) $(u_1^*(F_1 - \lambda_1^* G_1 + \sum_{j=r+1}^q v_j^* H_j), \dots, u_p^*(F_p - \lambda_p^* G_p + \sum_{j=r+1}^q v_j^* H_j))$ *is* $(\mathcal{F},\alpha,\bar{\rho},\theta)$*-V-pseudoconvex at* S^**;*
 (ii) $(v_1^* H_1, \dots, v_r^* H_r)$ *is* $(\mathcal{F},\tilde{\rho},\theta)$*-V-quasiconvex at* S^**;*
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then S^* is an efficient solution of (P).

PROOF. In Part (a) of Theorem 3.2, let

(a) $J_0 = \emptyset$,
(b) $J_0 = \underline{q}$,
(c) $J_t = \{t\}, t = 1, 2, \dots, q$,
(d) $J_t = \{t\}, t = 1, 2, \dots, r$, $J_0 = \{r+1, \dots, q\}$, $r < q$. ∎

In a similar manner, various special cases of the other two sets of sufficient efficiency conditions given in Theorem 3.2 can readily be identified.

In the remainder of this section, we present some additional sets of general parametric sufficient efficiency conditions using a variant of the partitioning scheme employed in Theorem 3.2. In these results, appropriate generalized $(\mathcal{F},\alpha,\rho,\theta)$-$V$-convexity assumptions are imposed on certain combinations of the functions $F_i - \lambda_i^* G_i$, $i \in \underline{p}$, and $v_j^* H_j$, $j \in \underline{q}$.

Let $\{I_0, I_1, \dots, I_k\}$ be a partition of $\underline{p}$ such that $K = \{0, 1, \dots, k\} \subset M = \{0, 1, \dots, m\}$, $k < m$, and let the function $\Theta_t(\cdot, \lambda^*, u^*, v^*) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed λ^*, u^*, and v^*, by

$$\Theta_t(S,\lambda^*,u^*,v^*) = \sum_{i\in I_t} u_i^* [F_i(S) - \lambda_i^* G_i(S)] + \sum_{j\in J_t} v_j^* H_j(S), \qquad t \in K.$$

THEOREM 3.3. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (3.1)–(3.3) hold. Assume furthermore that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Theta_0(\cdot, \lambda^*, u^*, v^*), \ldots, \Theta_k(\cdot, \lambda^*, u^*, v^*))$ *is strictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-pseudoconvex at* S^*;
  (ii) $(\Delta_{k+1}(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$*-quasiconvex at* S^*;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(b) (i) $(\Theta_0(\cdot, \lambda^*, u^*, v^*), \ldots, \Theta_k(\cdot, \lambda^*, u^*, v^*))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-quasiconvex at* S^*;
  (ii) $(\Delta_{k+1}(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is strictly* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$*-pseudoconvex at* S^*;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(c) (i) (a)(ii) *and* (b)(i) *hold;*
  (iii) $\bar{\rho} + \tilde{\rho} > 0$.

Then S^* is an efficient solution of (P).

PROOF.

(a) Suppose to the contrary that S^* is not an efficient solution of (P). Then there is $\bar{S} \in \mathbb{F}$ such that $(F_1(\bar{S})/G_1(\bar{S}), \ldots, F_p(\bar{S})/G_p(\bar{S})) \leqslant (F_1(S^*)/G_1(S^*), \ldots, F_p(S^*)/G_p(S^*))$, which in view of (3.2) implies that $F_i(\bar{S}) - \lambda_i^* G_i(\bar{S}) \leqq 0$, $i \in \underline{p}$, with strict inequality holding for at least one index $\ell \in \underline{p}$. Since $u^* > 0$, these inequalities yield

$$\sum_{i \in I_t} u_i^* \left[F_i\left(\bar{S}\right) - \lambda_i^* G_i\left(\bar{S}\right)\right] \leqq 0, \qquad t \in K. \tag{3.9}$$

Inasmuch as $v^* \geqq 0$ and $\bar{S}, S^* \in \mathbb{F}$, it follows from (3.2), (3.3), and (3.9) that for each $t \in K$,

$$\begin{aligned}
\Theta_t\left(\bar{S}, \lambda^*, u^*, v^*\right) &= \sum_{i \in I_t} u_i^* \left[F_i\left(\bar{S}\right) - \lambda_i^* G_i\left(\bar{S}\right)\right] + \sum_{j \in J_t} v_j^* H_j\left(\bar{S}\right) \\
&\leqq \sum_{i \in I_t} u_i^* \left[F_i\left(\bar{S}\right) - \lambda_i^* G_i\left(\bar{S}\right)\right] \\
&\leqq 0 \\
&= \sum_{i \in I_t} u_i^* \left[F_i\left(S^*\right) - \lambda_i^* G_i\left(S^*\right)\right] + \sum_{j \in J_t} v_j^* H_j\left(S^*\right) = \Theta_t\left(S^*, \lambda^*, u^*, v^*\right),
\end{aligned}$$

and so

$$\sum_{t \in K} \alpha_t\left(\bar{S}, S^*\right) \Theta_t\left(\bar{S}, \lambda^*, u^*, v^*\right) < \sum_{t \in K} \alpha_t\left(\bar{S}, S^*\right) \Theta_t\left(S^*, \lambda^*, u^*, v^*\right),$$

which in view of (i) implies that

$$\mathcal{F}\left(\bar{S}, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i\left(S^*\right) - \lambda_i^* \, DG_i\left(S^*\right)\right] + \sum_{t \in K} \sum_{j \in J_t} v_j^* \, DH_j\left(S^*\right)\right) < -\bar{\rho} d^2\left(\theta\left(\bar{S}, S^*\right)\right). \tag{3.10}$$

As for each $t \in M \backslash K$, $\Delta_t(\bar{S}, v^*) \leqq 0 = \Delta_t(S^*, v^*)$, and hence,

$$\sum_{t \in M \backslash K} \beta_t\left(\bar{S}, S^*\right) \Delta_t\left(\bar{S}, v^*\right) \leqq \sum_{t \in M \backslash K} \beta_t\left(\bar{S}, S^*\right) \Delta_t\left(S^*, v^*\right),$$

(ii) implies that

$$\mathcal{F}\left(\bar{S}, S^*; \sum_{t \in M \backslash K} \sum_{j \in J_t} v_j^* \, DH_j\left(S^*\right)\right) \leqq -\tilde{\rho} \, d^2\left(\theta\left(\bar{S}, S^*\right)\right). \tag{3.11}$$

Now adding (3.10) and (3.11) and using the sublinearity of $\mathcal{F}(\bar{S}, S^*; \cdot)$, and (iii), we see that

$$\mathcal{F}\left(\bar{S}, S^*; \sum_{i=1}^{p} u_i^* \left[DF_i(S^*) - \lambda_i^* \, DG_i(S^*)\right] + \sum_{j=1}^{q} v_j^* \, DH_j(S^*)\right) < -(\bar{\rho} + \tilde{\rho})\, d^2\left(\theta\left(\bar{S}, S^*\right)\right),$$

which contradicts (3.1). Hence, S^* is an efficient solution of (P).

(b),(c) The proofs are similar to that of Part (a). ∎

Following the pattern employed in generating Corollary 3.1, one can easily identify numerous special cases of the three families of sufficient efficiency conditions formulated in Theorem 3.3.

4. SEMIPARAMETRIC SUFFICIENT EFFICIENCY CONDITIONS

In this section, we present the semiparametric versions of the general parametric sufficiency results discussed in the preceding section. These sufficiency criteria are motivated by the form and features of Theorem 2.2. As we shall see in Sections 8–10, these results lead to the formulation of a number of semiparametric duality models for (P).

In the statements and proofs of our sufficiency theorems, we use the functions $\mathcal{B}_j(\cdot, v^*)$ and $\Lambda_t(\cdot, v^*)$ defined in Section 3, and $\mathcal{C}_i(\cdot, S^*, u^*)$ and $\Lambda_i(\cdot, S^*, u^*, v^*)$ defined, for fixed S^*, u^*, and v^*, on $\mathbb{A}^n$ by

$$\mathcal{C}_i\left(S, S^*, u^*\right) = u_i^* \left[G_i\left(S^*\right) F_i(S) - F_i\left(S^*\right) G_i(S)\right]$$

and

$$\Lambda_i\left(S, S^*, u^*, v^*\right) = u_i^* \left[G_i\left(S^*\right) F_i(S) - F_i\left(S^*\right) G_i(S) + \sum_{j \in J_0} v_j^* H_j(S)\right], \qquad i \in \underline{p}.$$

THEOREM 4.1. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^* and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that*

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[G_i\left(S^*\right)\, DF_i\left(S^*\right) - F_i\left(S^*\right)\, DG_i\left(S^*\right)\right] + \sum_{j=1}^{q} v_j^* \, DH_j\left(S^*\right)\right) \geqq 0, \tag{4.1}$$

for all $S \in \mathbb{A}^n$,

$$v_j^* H_j\left(S^*\right) = 0, \qquad j \in \underline{q}, \tag{4.2}$$

where $\mathcal{F}(S, S^*; \cdot) : L_1^n(X, \mathbb{A}, \mu) \to \mathbb{R}$ is a sublinear function. Assume furthermore that any one of the following four sets of hypotheses is satisfied:

(a) (i) *for each $i \in \underline{p}$, $F_i(S^*) \geqq 0$ and $(F_1, \dots, F_p)$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-convex at S^*;*
 (ii) *$(-G_1, \dots, -G_p)$ is $(\mathcal{F}, \beta, \hat{\rho}, \theta)$-$V$-convex at S^*;*
 (iii) *$(H_1, \dots, H_q)$ is $(\mathcal{F}, \gamma, \tilde{\rho}, \theta)$-$V$-convex at S^*;*
 (iv) *$\alpha_1 = \alpha_2 = \cdots = \alpha_p = \beta_1 = \beta_2 = \cdots = \beta_p = \gamma_1 = \gamma_2 = \cdots = \gamma_q = \delta$;*
 (v) *$\sum_{i=1}^{p} u_i^* [G_i(S^*)\bar{\rho}_i + F_i(S^*)\hat{\rho}_i] + \sum_{j=1}^{q} v_j^* \tilde{\rho}_j \geqq 0$;*

(b) (i) *$(\mathcal{C}_1(\cdot, S^*, u^*), \dots, \mathcal{C}_p(\cdot, S^*, u^*))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at S^*;*
 (ii) *$(\mathcal{B}_1(\cdot, v^*), \dots, \mathcal{B}_q(\cdot, v^*))$ is $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-quasiconvex at S^*;*
 (iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$;*

(c) (i) *(b)(ii) holds;*
 (ii) *$(\mathcal{C}_1(\cdot, S^*, u^*), \dots, \mathcal{C}_p(\cdot, S^*, u^*))$ is prestrictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-quasiconvex at S^*;*
 (iii) *$\bar{\rho} + \tilde{\rho} > 0$;*

(d) (i) *(c)(ii) holds;*
 (ii) *$(\mathcal{B}_1(\cdot, v^*), \dots, \mathcal{B}_q(\cdot, v^*))$ is strictly $(\mathcal{F}, \alpha, \tilde{\rho}, \theta)$-$V$-pseudoconvex at S^*;*
 (iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$.*

Then S^* is an efficient solution of (P).

PROOF. Let S be an arbitrary feasible solution of (P).

(a) From the sublinearity of $\mathcal{F}(S, S^*; \cdot)$ and (4.1) it follows that:

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[G_i(S^*) DF_i(S^*) - F_i(S^*) DG_i(S^*)\right]\right) + \mathcal{F}\left(S, S^*; \sum_{j=1}^{q} v_j^* DH_j(S^*)\right) \geqq 0. \quad (4.3)$$

Keeping in mind that $u^* > 0$, $v^* \geqq 0$, $F_i(S^*) \geqq 0$, and $G_i(S^*) > 0$, $i \in \underline{p}$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^* \left[G_i(S^*) F_i(S) - F_i(S^*) G_i(S)\right] \\
&= \sum_{i=1}^{p} u_i^* \left\{G_i(S^*) \left[F_i(S) - F_i(S^*)\right] - F_i(S^*) \left[G_i(S) - G_i(S^*)\right]\right\} \\
&\geqq \sum_{i=1}^{p} u_i^* G_i(S^*) \left[\mathcal{F}(S, S^*; \delta(S, S^*)\, DF_i(S^*)) + \bar{\rho}_i d^2(\theta(S, S^*))\right] \\
&\quad + \sum_{i=1}^{p} u_i^* F_i(S^*) \left[\mathcal{F}(S, S^*; -\delta(S, S^*)\, DG_i(S^*)) + \hat{\rho}_i d^2(\theta(S, S^*))\right] \\
&\quad \text{(by (i), (ii), and (iv))} \\
&\geqq \mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \delta(S, S^*) \left[G_i(S^*)\, DF_i(S^*) - F_i(S^*)\, DG_i(S^*)\right]\right) \\
&\quad + \sum_{i=1}^{p} u_i^* \left[G_i(S^*) \bar{\rho}_i + F_i(S^*) \hat{\rho}_i\right] d^2(\theta(S, S^*)) \qquad \text{(by the sublinearity of } \mathcal{F}(S, S^*; \cdot)) \\
&\geqq -\mathcal{F}\left(S, S^*; \sum_{j=1}^{q} v_j^* \delta(S, S^*) DH_j(S^*)\right) + \sum_{i=1}^{p} u_i^* \left[G_i(S^*) \bar{\rho}_i + F_i(S^*) \hat{\rho}_i\right] d^2(\theta(S, S^*)) \\
&\quad \text{(by (4.3))} \\
&\geqq \sum_{j=1}^{q} v_j^* \left[H_j(S^*) - H_j(S)\right] + \left(\sum_{i=1}^{p} u_i^* \left[G_i(S^*) \bar{\rho}_i + F_i(S^*) \hat{\rho}_i\right] + \sum_{j=1}^{q} v_j^* \tilde{\rho}_j\right) d^2(\theta(S, S^*)) \\
&\quad \text{(by (iii) and (iv))} \\
&\geqq 0 \qquad \text{(by (4.2), feasibility of } S\text{, and (v))}.
\end{aligned}$$

Since $u^* > 0$, the above inequality implies that $(G_1(S^*)F_1(S) - F_1(S^*)G_1(S), \ldots, G_p(S^*) F_p(S) - F_p(S^*)G_p(S)) \nleq (0, \ldots, 0)$, which in turn implies that $\varphi(S) \nleq \varphi(S^*)$. Since $S \in \mathbb{F}$ was arbitrary, we conclude that S^* is an efficient solution of (P).

(b) Combining (4.3) with (3.5), which is valid for the present case because of our assumption specified in (ii), and using (iii), we obtain

$$\mathcal{F}\left(S, S^*; \sum_{i=1}^{p} u_i^* \left[G_i(S^*)\, DF_i(S^*) - F_i(S^*)\, DG_i(S^*)\right]\right) \geqq -\bar{\rho}\, d^2(\theta(S, S^*)),$$

which in view of (i) implies that

$$\begin{aligned}
&\sum_{i=1}^{p} u_i^* \alpha_i(S, S^*) \left[G_i(S^*) F_i(S) - F_i(S^*) G_i(S)\right] \\
&\geqq \sum_{i=1}^{p} u_i^* \alpha_i(S, S^*) \left[G_i(S^*) F_i(S^*) - F_i(S^*) G_i(S^*)\right] = 0.
\end{aligned} \quad (4.4)$$

Since $u_i^*\alpha_i(S,S^* > 0$ for each $i \in \underline{p}$, (4.4) implies that $(G_1(S^*)F_1(S) - F_1(S^*)G_1(S), \dots, G_p(S^*)F_p(S) - F_1(S^*)G_1(S) \nleq (0,\dots,0)$, which in turn implies that $\varphi(S) \nleq \varphi(S^*)$. Hence, we conclude that S^* is an efficient solution of (P).

(c),(d) The proofs are similar to that of Part (b). ∎

THEOREM 4.2. *Let $S \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^* and that there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (4.1) and (4.2) hold. Assume furthermore that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Lambda_1(\cdot,S^*,u^*,v^*),\dots,\Lambda_p(\cdot,S^*,u^*,v^*))$ *is* $(\mathcal{F},\alpha,\bar{\rho},\theta)$-*V*-*pseudoconvex at* S^**;*
(ii) $(\Delta_1(\cdot,v^*),\dots,\Delta_m(\cdot,v^*))$ *is* $(\mathcal{F},\beta,\tilde{\rho},\theta)$-*V*-*quasiconvex at* S^**;*
(iii) $\bar{\rho}+\tilde{\rho} \geqq 0$*;*

(b) (i) (a)(ii) *holds;*
(ii) $(\Lambda_1(\cdot,S^*,u^*,v^*),\dots,\Lambda_p(\cdot,S^*,u^*,v^*))$ *is prestrictly* $(\mathcal{F},\alpha,\bar{\rho},\theta)$-*V*-*quasiconvex at* S^**;*
(iii) $\bar{\rho}+\tilde{\rho} > 0$*;*

(c) (i) (b)(ii) *holds;*
(ii) $(\Delta_1(\cdot,v^*),\dots,\Delta_m(\cdot,v^*))$ *is strictly* $(\mathcal{F},\beta,\tilde{\rho},\theta)$-*V*-*pseudoconvex at* S^**;*
(iii) $\bar{\rho}+\tilde{\rho} \geqq 0$.

Then S^* is an efficient solution of (P).

PROOF.

(a) Let S be an arbitrary feasible solution of (P). Then it is clear that (3.7) holds because of the assumption specified in (ii). Since $\mathcal{F}(S,S^*;\cdot)$ is sublinear, it follows from (4.1) that:

$$\mathcal{F}\left(S,S^*;\sum_{i=1}^{p} u_i^*\left[G_i(S^*)\,DF_i(S^*) - F_i(S^*)\,DG_i(S^*)\right] + \sum_{j\in J_0} v_j^*\,DH_j(S^*)\right) + \mathcal{F}\left(S,S^*;\sum_{t=1}^{m}\sum_{j\in J_t} v_j^*\,DH_j(S^*)\right) \geqq 0. \tag{4.5}$$

From (3.7), (4.5), and (iii) we deduce that

$$\mathcal{F}\left(S,S^*;\sum_{i=1}^{p} u_i^*\left[G_i(S^*)\,DF_i(S^*) - F_i(S^*)\,DG_i(S^*)\right] + \sum_{j\in J_0} v_j^* DH_j(S^*)\right) \geqq -\bar{\rho}d^2(\theta(S,S^*)),$$

which in view of (i) implies that

$$\sum_{i=1}^{p}\alpha_i(S,S^*)\,\Lambda_i(S,S^*,u^*,v^*) \geqq \sum_{i=1}^{p}\alpha_i(S,S^*)\,\Lambda_i(S^*,S^*,u^*,v^*).$$

Since $\Lambda_i(S^*,S^*,u^*,v^*) = 0$ for each $i \in \underline{p}$, $v_j^*H_j(S) \leqq 0$ for each $j \in \underline{q}$, and $\alpha_i(S,S^*) > 0$ for each $i \in \underline{p}$, the above inequality reduces to

$$\sum_{i=1}^{p} u_i^*\alpha_i(S,S^*)\left[G_i(S^*)\,F_i(S) - F_i(S^*)\,G_i(S)\right] \geqq 0,$$

which leads, as seen in the proof of Theorem 4.1, to the desired conclusion that S^* is an efficient solution of (P).

(b),(c) The proofs are similar to that of Part (a). ∎

Let the function $\Pi_t(\cdot,S^*,u^*,v^*) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed S^*, u^*, and v^*, by

$$\Pi_t(S,S^*,u^*,v^*) = \sum_{i\in I_t} u_i^*\left[G_i(S^*)\,F_i(S) - F_i(S^*)\,G_i(S)\right] + \sum_{j\in J_t} v_j^* H_j(S), \qquad t \in K.$$

THEOREM 4.3. *Let $S^* \in \mathbb{F}$ and assume that F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable at S^*, and that there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (4.1) and (4.2) hold. Assume furthermore that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Pi_0(\cdot, S^*, u^*, v^*), \ldots, \Pi_k(\cdot, S^*, u^*, v^*))$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*pseudoconvex at* S^**;*
(ii) $(\Delta_{k+1}(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is* $(\mathcal{F}, \tilde{\rho}, \theta)$-$V$-*quasiconvex at* S^**;*
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$*;*

(b) (i) $(\Pi_0(\cdot, S^*, u^*, v^*), \ldots, \Pi_k(\cdot, S^*, u^*, v^*))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*quasiconvex at* S^**;*
(ii) $(\Delta_{k+1}(\cdot, v^*), \ldots, \Delta_m(\cdot, v^*))$ *is strictly* $(\mathcal{F}, \tilde{\rho}, \theta)$-$V$-*pseudoconvex at* S^**;*
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$*;*

(c) (i) (a)(ii) *and* (b)(i) *hold;*
(iii) $\bar{\rho} + \tilde{\rho} > 0$.

Then S^* is an efficient solution of (P).

PROOF.

(a) Suppose to the contrary that S^* is not an efficient solution of (P). As seen in the proof of Theorem 3.3, this supposition leads to the inequalities $G_i(S^*)F_i(\bar{S}) - F_i(S^*)G_i(\bar{S}) \leqq 0$, $i \in \underline{p}$, with strict inequality holding for at least one index $\ell \in \underline{p}$, for some $\bar{S} \in \mathbb{F}$. Since $u^* > 0$, these inequalities yield

$$\sum_{i \in I_t} u_i^* \left[G_i(S^*) F_i(\bar{S}) - F_i(S^*) G_i(\bar{S})\right] \leqq 0, \qquad t \in K. \tag{4.6}$$

Inasmuch as $v^* \geqq 0$ and $\bar{S}, S^* \in \mathbb{F}$, it follows from (4.2) and (4.6) that for each $t \in K$,

$$\begin{aligned}
\Pi_t\left(\bar{S}, S^*, u^*, v^*\right) &= \sum_{i \in I_t} u_i^* \left[G_i(S^*) F_i(\bar{S}) - F_i(S^*) G_i(\bar{S})\right] + \sum_{j \in J_t} v_j^* H_j(\bar{S}) \\
&\leqq \sum_{i \in I_t} u_i^* \left[G_i(S^*) F_i(\bar{S}) - F_i(S^*) G_i(\bar{S})\right] \\
&\leqq 0 \\
&= \sum_{i \in I_t} u_i^* \left[G_i(S^*) F_i(S^*) - F_i(S^*) G_i(S^*)\right] \\
&\quad + \sum_{j \in J_t} v_j^* H_j(S^*) = \Pi_t\left(S^*, S^*, u^*, v^*\right),
\end{aligned}$$

and so

$$\sum_{t \in K} \alpha_t\left(\bar{S}, S^*\right) \Pi_t\left(\bar{S}, S^*, u^*, v^*\right) < \sum_{t \in K} \alpha_t\left(\bar{S}, S^*\right) \Pi_t\left(S^*, S^*, u^*, v^*\right),$$

which in view of (i) implies that

$$\begin{aligned}
\mathcal{F}\left(\bar{S}, S^*; \sum_{i=1}^{p} u_i^* \left[G_i(S^*)\, DF_i(S^*) - F_i(S^*)\, DG_i(S^*)\right] + \sum_{t \in K} \sum_{j \in J_t} v_j^*\, DH_j(S^*)\right) \\
< -\bar{\rho}\, d^2\left(\theta\left(\bar{S}, S^*\right)\right).
\end{aligned} \tag{4.7}$$

As for each $t \in M \backslash K$, $\Delta_t(\bar{S}, v^*) \leqq 0 = \Delta_t(S^*, v^*)$, and hence,

$$\sum_{t \in M \backslash K} \beta_t\left(\bar{S}, S^*\right) \Delta_t\left(\bar{S}, v^*\right) \leqq \sum_{t \in M \backslash K} \beta_t\left(\bar{S}, S^*\right) \Delta_t\left(S^*, v^*\right),$$

it follows from (ii) that:

$$\mathcal{F}\left(\bar{S}, S^*; \sum_{t \in M \backslash K} \sum_{j \in J_t} v_j^*\, DH_j(S^*)\right) \leqq -\tilde{\rho}\, d^2\left(\theta\left(\bar{S}, S^*\right)\right). \tag{4.8}$$

Now adding (4.7) and (4.8) and using the sublinearity of $\mathcal{F}(\bar{S}, S^*; \cdot)$, and (iii), we see that

$$\mathcal{F}\left(\bar{S}, S^*; \sum_{i=1}^{p} u_i^* \left[G_i(S^*)\, DF_i(S^*) - F_i(S^*)\, DG_i(S^*)\right] + \sum_{j=1}^{q} v_j^*\, DH_j(S^*)\right) < -(\bar{\rho} + \tilde{\rho})\, d^2\left(\theta\left(\bar{S}, S^*\right)\right),$$

which contradicts (4.1). Hence, S^* is an efficient solution of (P).

(b),(c) The proofs are similar to that of Part (a). ∎

As in the case of Theorems 3.2 and 3.3, the six sets of sufficient efficiency conditions given in Theorems 4.2 and 4.3 contain a fairly large number of interesting and important special cases that can be identified, as in Corollary 3.1, in a straightforward manner by different choices of the partitioning sets $J_0, J_1, \dots, J_m$, and $\mathcal{F}(S, S^*; \cdot), \alpha, \beta, \bar{\rho}, \tilde{\rho}$, and θ.

5. DUALITY MODEL I

In the remainder of this paper, we present six duality models for (P). Three parametric models whose forms and properties are based on Theorems 2.1, 3.1, 3.2, and 3.3; and three semiparametric models whose structure and contents are motivated by Theorems 2.2, 4.1, 4.2, and 4.3. In each case, we state and prove appropriate weak, strong, and strict converse duality theorems. We begin our discussion of parametric duality in the present section by considering the following dual problem:

$$\text{Maximize} \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \qquad \text{subject to} \tag{DI}$$

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i \left[DF_i(T) - \lambda_i\, DG_i(T)\right] + \sum_{j=1}^{q} v_j\, DH_j(T)\right) \geqq 0, \qquad \text{for all } S \in \mathbb{A}^n, \tag{5.1}$$

$$u_i[F_i(T) - \lambda_i G_i(T)] \geqq 0, \qquad i \in \underline{p}, \tag{5.2}$$

$$v_j H_j(T) \geqq 0, \qquad j \in \underline{q}, \tag{5.3}$$

$$T \in \mathbb{A}^n, \quad \lambda \in \mathbb{R}_+, \quad u \in U, \quad v \in \mathbb{R}_+^q,$$

where $\mathcal{F}(S, T; \cdot) : L_1^n(X, A, \mu) \to \mathbb{R}$ is a sublinear function. Throughout our discussion of duality for (P), we assume that the functions F_i, G_i, $i \in \underline{p}$, and H_j, $j \in \underline{q}$, are differentiable on $\mathbb{A}^n$.

In this section, we use the functions $\mathcal{A}_i(\cdot, \lambda, u)$ and $\mathcal{B}_j(\cdot, v)$ introduced in Section 3.

The next two theorems show that (DI) is a dual problem for P.

THEOREM 5.1. WEAK DUALITY. *Let S and (T, λ, u, v) be arbitrary feasible solutions of (P) and (DI), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) *for each $i \in \underline{p}$, $F_i(T) \geqq 0$ and $(F_1, \dots, F_p)$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-convex at T;*
  (ii) *$(-G_1, \dots, -G_p)$ is $(\mathcal{F}, \beta, \hat{\rho}, \theta)$-$V$-convex at T;*
  (iii) *$(H_1, \dots, H_q)$ is $(\mathcal{F}, \gamma, \tilde{\rho}, \theta)$-$V$-convex at T;*
  (iv) *$\alpha_1 = \alpha_2 = \cdots = \alpha_p = \beta_1 = \beta_2 = \cdots = \beta_p = \gamma_1 = \gamma_2 = \cdots = \gamma_q = \delta$;*
  (v) *$\sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda_i \hat{\rho}_i) + \sum_{j=1}^{q} v_j \tilde{\rho}_j \geqq 0$;*

(b) (i) *$(\mathcal{A}_1(\cdot, \lambda, u), \dots, \mathcal{A}_p(\cdot, \lambda, u))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at T;*
  (ii) *$(\mathcal{B}_1(\cdot, v), \dots, \mathcal{B}_q(\cdot, v))$ is $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-quasiconvex at T;*
  (iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$;*

(c) (i) *(b)(ii) holds;*
  (ii) *$(\mathcal{A}_1(\cdot, \lambda, u), \dots, \mathcal{A}_p(\cdot, \lambda, u))$ is prestrictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-quasiconvex at T;*
  (iii) *$\bar{\rho} + \tilde{\rho} > 0$;*

(d) (i) (c)(ii) *holds;*
(ii) $(\mathcal{B}_1(\cdot, v), \ldots, \mathcal{B}_q(\cdot, v))$ *is strictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*pseudoconvex at* T;
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\varphi(S) \nleq \lambda$.

PROOF.

(a) From the sublinearity of $\mathcal{F}(S, T; \cdot)$ and (5.1) it follows that:

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[DF_i(T) - \lambda_i\, DG_i(T)]\right) + \mathcal{F}\left(S, T; \sum_{j=1}^{q} v_j\, DH_j(T)\right) \geqq 0. \tag{5.4}$$

Keeping in mind that $u > 0$, $v \geqq 0$, and $\lambda \geqq 0$, we have

$$\begin{aligned}
&\sum_{i=1}^{p} u_i[F_i(S) - \lambda_i G_i(S)] \\
&\quad \geqq \sum_{i=1}^{p} u_i\{F_i(S) - F_i(T) - \lambda_i[G_i(S) - G_i(T)]\} \quad \text{(by (5.2))} \\
&\quad \geqq \sum_{i=1}^{p} u_i \left\{\mathcal{F}(S, T; \delta(S,T)\, DF_i(T)) + \bar{\rho}_i\, d^2(\theta(S,T)) + \lambda_i \left[\mathcal{F}(S, T; -\delta(S,T) DG_i(T)\right.\right. \\
&\qquad \left.\left. +\ \hat{\rho}_i\, d^2(\theta(S,T))\right]\right\} \quad \text{(by (i), (ii), and (iv))} \\
&\quad \geqq \mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i \delta(S,T)[DF_i(T) - \lambda_i\, DG_i(T)]\right) + \sum_{i=1}^{p} u_i \left(\bar{\rho}_i + \lambda_i \hat{\rho}_i\right) d^2(\theta(S,T)) \\
&\qquad \text{(by the sublinearity of } \mathcal{F}(S, T; \cdot)) \\
&\quad \geqq -\mathcal{F}\left(S, T; \sum_{j=1}^{q} v_j \delta(S,T)\, DH_j(T)\right) + \sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda_i \hat{\rho}_i)\, d^2(\theta(S,T)) \quad \text{(by (5.4))} \\
&\quad \geqq \sum_{j=1}^{q} v_j[H_j(T) - H_j(S)] + \left(\sum_{i=1}^{p} u_i(\bar{\rho}_i + \lambda_i \hat{\rho}_i) + \sum_{j=1}^{q} v_j \tilde{\rho}_j\right) d^2(\theta(S,T)) \quad \text{(by (iii) and (iv))} \\
&\quad \geqq 0 \quad \text{(by (5.3), primal feasibility of } S\text{, and (v)).}
\end{aligned}$$

Since $u > 0$, the above inequality implies that $(F_1(S) - \lambda_1 G_1(S), \ldots, F_p(S) - \lambda_p G_p(S)) \nleq (0, \ldots, 0)$, which in turn implies that $\varphi(S) \nleq \lambda$.

(b) Since for each $j \in \underline{q}$, $v_j H_j(S) \leqq 0$, it follows from (5.3) that $v_j H_j(S) \leqq 0 \leqq v_j H_j(T)$, and hence, we have that:

$$\sum_{j=1}^{q} \beta_j(S,T) v_j H_j(S) \leqq \sum_{j=1}^{q} \beta_j(S,T) v_j H_j(T),$$

which by virtue of (ii) implies that

$$\mathcal{F}\left(S, T; \sum_{j=1}^{q} v_j\, DH_j(T)\right) \leqq -\tilde{\rho}\, d^2\left(\theta(S,T)\right). \tag{5.5}$$

Combining (5.4), (5.5), and (iii), we find that

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[DF_i(T) - \lambda_i\, DG_i(T)]\right) \geqq -\bar{\rho}\, d^2(\theta(S,T)).$$

which by (i) implies that

$$\sum_{i=1}^{p} u_i\alpha_i(S,T)[F_i(S) - \lambda_i G_i(S)] \geqq \sum_{i=1}^{p} u_i\alpha_i(S,T)[F_i(T) - \lambda_i G_i(T)].$$

By (5.2) this inequality reduces to

$$\sum_{i=1}^{p} u_i\alpha_i(S,T)[F_i(S) - \lambda_i G_i(S)] \geqq 0. \tag{5.6}$$

Since $u_i\alpha_i(S,T) > 0$ for each $i \in \underline{p}$, (5.6) implies that $(F_1 - \lambda_1 G_1(S), \ldots, F_1 - \lambda_p G_p(S)) \nleq (0, \ldots, 0)$, which in turn implies that $\varphi(S) \nleq \lambda$.

(c),(d) The proofs are similar to that of Part (b). ∎

THEOREM 5.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, and assume that any one of the four sets of hypotheses specified in Theorem 5.1 holds for all feasible solutions of (DI). Then there exist $\lambda^* \in \mathbb{R}_+^p$, $u^* \in U$, and $v^* \in \mathbb{R}_+^q$ such that $(S^*, \lambda^*, u^*, v^*)$ is an efficient solution of (DI) and $\varphi(S^*) = \lambda^*$.*

PROOF. By Theorem 2.1, there exist $\lambda^*(= \varphi(S^*))$, u^*, and v^*, as specified above, such that $(S^*, \lambda^*, u^*, v^*)$ is a feasible solution of (DI). If it were not efficient, then there would exist a feasible solution $(\bar{S}, \bar{\lambda}, \bar{u}, \bar{v})$ of (DI) such that $\varphi(\bar{S}) \geqslant \lambda^*$. Since $\lambda^* = \varphi(S^*)$, this inequality implies that $\varphi(\bar{S}) \geqslant \varphi(S^*)$, which contradicts the assertion of Theorem 5.1. Therefore, $(S^*, \lambda^*, u^*, v^*)$ is an efficient solution of (DI). ∎

THEOREM 5.3. STRICT CONVERSE DUALITY. *Let S^* be a feasible solution of (P), let $\mathcal{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \langle D_k F(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, let $(\tilde{S}, \tilde{\lambda}, \tilde{u}, \tilde{v})$ be a feasible solution of (DI) such that*

$$\sum_{i=1}^{p} \tilde{u}_i\alpha_i\left(S^*, \tilde{S}\right)\left[F_i\left(S^*\right) - \tilde{\lambda}_i G_i\left(S^*\right)\right] \leqq 0. \tag{5.7}$$

Assume furthermore that any one of the following four sets of hypotheses is satisfied:

(a) *the assumptions specified in Part (a) of Theorem 5.1 are satisfied for all feasible solutions of (DI), and $(F_1, \ldots, F_p)$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-V-convex at $\tilde{S}$, or $(-G_1, \ldots, -G_p)$ is strictly $(\mathcal{F}, \beta, \bar{\rho}, \theta)$-V-convex at $\tilde{S}$, or $(H_1, \ldots, H_q)$ is strictly $(\mathcal{F}, \gamma, \tilde{\rho}, \theta)$-V-convex at $\tilde{S}$, with $\tilde{v} \neq 0$, or $\sum_{i=1}^{p} \tilde{u}_i[\bar{\rho}_i + \tilde{\lambda}_i\hat{\rho}_i] + \sum_{j=1}^{q} \tilde{v}_j\tilde{\rho}_j > 0$;*

(b) *the assumptions specified in Part (b) of Theorem 5.1 are satisfied for all feasible solutions of (DI), and $(\mathcal{A}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{A}_p(\cdot, \tilde{\lambda}, \tilde{u}))$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-V-pseudoconvex at $\tilde{S}$;*

(c) *the assumptions specified in Part (c) of Theorem 5.1 are satisfied for all feasible solutions of (DI), and $(\mathcal{A}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{A}_p(\cdot, \tilde{\lambda}, \tilde{u}))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-V-quasiconvex at $\tilde{S}$;*

(d) *the assumptions specified in Part (d) of Theorem 5.1 are satisfied for all feasible solutions of (DI), and $(\mathcal{A}_1(\cdot, \tilde{\lambda}, \tilde{u}), \ldots, \mathcal{A}_p(\cdot, \tilde{\lambda}, \tilde{u}))$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-V-pseudoconvex at $\tilde{S}$.*

Then $\tilde{S} = S^*$.

PROOF.

(a) Suppose, on the contrary, that $\tilde{S} \neq S^*$. Proceeding as in the proof of Theorem 5.1 (with S replaced by S^* and (T, λ, u, v) by $(\tilde{S}, \tilde{\lambda}, \tilde{u}, \tilde{v})$), we arrive at the strict inequality

$$\sum_{i=1}^{p} \tilde{u}_i\alpha_i\left(S^*, \tilde{S}\right)\left[F_i\left(S^*\right) - \tilde{\lambda}_i G_i\left(S^*\right)\right] > 0,$$

which contradicts (5.7). Hence, $\tilde{S} = S^*$.

(b)–(d) The proofs are similar to that of Part (a). ∎

6. DUALITY MODEL II

In this section, we formulate a relatively more general parametric duality model by making use of the partitioning scheme introduced in Section 3. This duality model has the form

$$\begin{aligned} &\text{Maximize} \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \\ &\text{subject to} \end{aligned} \tag{DII}$$

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[DF_i(T) - \lambda_i\, DG_i(T)] + \sum_{j=1}^{q} v_j\, DH_j(T)\right) \geqq 0, \qquad \text{for all } S \in \mathbb{A}^n, \tag{6.1}$$

$$u_i \left[F_i(T) - \lambda_i G_i(T) + \sum_{j \in J_0} v_j H_j(T)\right] \geqq 0, \qquad i \in \underline{p}, \tag{6.2}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \qquad t \in \underline{m}, \tag{6.3}$$

$$T \in \mathbb{A}^n, \quad \lambda \in \mathbb{R}^p_+, \quad u \in U, \quad v \in \mathbb{R}^q_+,$$

where $\mathcal{F}(S, T; \cdot) : L_1^n(X, A, \mu) \to \mathbb{R}$ is a sublinear function.

Next, we show that (DII) is a dual problem for (P) by establishing weak and strong duality theorems. Here we use the functions $\Gamma_i(\cdot, \lambda, u, v)$ and $\Delta_t(\cdot, v)$ defined in Section 3.

THEOREM 6.1. WEAK DUALITY. *Let S and (T, λ, u, v) be arbitrary feasible solutions of (P) and (DII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Gamma_1(\cdot, \lambda, u, v), \dots, \Gamma_p(\cdot, \lambda, u, v))$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*pseudoconvex at* T;
  (ii) $(\Delta_1(\cdot, v), \dots, \Delta_m(\cdot, v))$ *is* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-*quasiconvex at* T;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(b) (i) (a)(ii) *holds;*
  (ii) $(\Gamma_1(\cdot, \lambda, u, v), \dots, \Gamma_p(\cdot, \lambda, u, v))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-*quasiconvex at* T;
  (iii) $\bar{\rho} + \tilde{\rho} > 0$;

(c) (i) (b)(ii) *holds;*
  (ii) $(\Delta_1(\cdot, v), \dots, \Delta_m(\cdot, v))$ *is strictly* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-*pseudoconvex at* T;
  (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\varphi(S) \nleq \lambda$.

PROOF.

(a) From the sublinearity of $\mathcal{F}(S, T; \cdot)$ and (6.1) it is easily seen that

$$\begin{aligned} &\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[DF_i(T) - \lambda_i\, DG_i(T)] + \sum_{j \in J_0} v_j\, DH_j(T)\right) \\ &\quad + \mathcal{F}\left(S, T; \sum_{t=1}^{m} \sum_{j \in J_t} v_j\, DH_j(T)\right) \geqq 0. \end{aligned} \tag{6.4}$$

Since $S \in \mathbb{F}$ and $v \geqq 0$, it follows from (6.3) that for each $t \in \underline{m}$:

$$\Delta_t(S, v) = \sum_{j \in J_t} v_j H_j(S) \leqq 0 \leqq \sum_{j \in J_t} v_j H_j(T) = \Delta_t(T, v),$$

and so

$$\sum_{t=1}^{m} \beta_t(S, T) \Delta_t(S, v) \leqq \sum_{t=1}^{m} \beta_t(S, T) \Delta_t(T, v),$$

which in view of (ii) implies that

$$\mathcal{F}\left(S,T;\sum_{t=1}^{m}\sum_{j\in J_t} v_j DH_j(T)\right) \leqq -\tilde{\rho}\, d^2(\theta(S,T)). \tag{6.5}$$

Because of (6.5), (6.4) reduces to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p} u_i[DF_i(T)-\lambda_i\, DG_i(T)]+\sum_{j\in J_0} v_j\, DH_j(T)\right) \geqq \tilde{\rho}\, d^2(\theta(S,T)) \geqq -\tilde{\rho}\, d^2(\theta(S,T)),$$

where the second inequality follows from (iii). By virtue of (i) this implies that

$$\sum_{i=1}^{p}\alpha_i(S,T)\Gamma_i(S,\lambda,u,v) \geqq \sum_{i=1}^{p}\alpha_i(S,T)\Gamma_i(T,\lambda,u,v).$$

Inasmuch as $\alpha_i(S,T) > 0$, $u_i \geqq 0$, $i \in \underline{p}$, and (6.2) holds, the right-hand side of the above inequality is greater than or equal to zero, and so it reduces to

$$\sum_{i=1}^{p}\alpha_i(S,T)\Gamma_i(S,\lambda,u,v) \geqq 0,$$

which simplifies to

$$\sum_{i=1}^{p} u_i\alpha_i(S,T)[F_i(S)-\lambda_i G_i(S)] \geqq 0$$

because $\alpha_i(S,T) > 0$, $i \in \underline{p}, v \geqq 0$, and $S \in \mathbb{F}$. As demonstrated in the proof of Theorem 5.1, this inequality leads to the desired conclusion that $\varphi(S) \nleq \lambda$.

(b),(c) The proofs are similar to that of Part (a). ∎

THEOREM 6.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, and assume that any one of the three sets of hypotheses specified in Theorem 6.1 holds for all feasible solutions of (DII). Then there exist $\lambda^*\in\mathbb{R}_+^p$, $u^*\in U$, and $v^*\in\mathbb{R}_+^q$ such that (S^*,λ^*,u^*,v^*) is an efficient solution of (DII) and $\varphi(S^*)=\lambda^*$.*

PROOF. By Theorem 2.1, there exist $\lambda^*(=\varphi(S^*))$, u^*, and v^*, as specified above, such that (S^*,λ^*,u^*,v^*) is a feasible solution of (DII). That it is an efficient solution follows from Theorem 6.1.

THEOREM 6.3. STRICT CONVERSE DUALITY. *Let S^* be a feasible solution of (P), let $\mathcal{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, let $(\tilde{S},\tilde{\lambda},\tilde{u},\tilde{v})$ be a feasible solution of (DII) such that*

$$\sum_{i=1}^{p}\tilde{u}_i\alpha_i\left(S^*,\tilde{S}\right)\left[F_i\left(S^*\right)-\tilde{\lambda}_i G_i\left(S^*\right)\right] \leqq 0.$$

Assume furthermore that any one of the following three sets of conditions is satisfied:

(a) *the assumptions of Part (a) of Theorem 6.1 hold for all feasible solutions of (DII), and $(\Gamma_1(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}),\ldots,\Gamma_p(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}))$ is strictly $(\mathcal{F},\alpha,\bar{\rho},\theta)$-$V$-pseudoconvex at $\tilde{S}$;*

(b) *the assumptions of Part (b) of Theorem 6.1 hold for all feasible solutions of (DII), and $(\Gamma_1(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}),\ldots,\Gamma_p(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}))$ is $(\mathcal{F},\alpha,\bar{\rho},\theta)$-$V$-quasiconvex at $\tilde{S}$;*

(c) *the assumptions of Part (c) of Theorem 6.1 hold for all feasible solutions of (DII), and $(\Gamma_1(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}),\ldots,\Gamma_p(\cdot,\tilde{\lambda},\tilde{u},\tilde{v}))$ is strictly $(\mathcal{F},\alpha,\bar{\rho},\theta)$-$V$-pseudoconvex at $\tilde{S}$.*

Then $\tilde{S}=S^*$.

PROOF. The proof is similar to that of Theorem 5.3. ∎

Obviously, (DII) contains numerous special cases that can easily be identified by appropriate choices of $F_i, G_i, H_j, \mathcal{F}, J_0, J_1, \ldots, J_m, m$, and p.

7. DUALITY MODEL III

In this section, we present another general parametric duality model for (P). It is based on the partitioning scheme employed earlier in the definition of the function $\Theta_t(\cdot, \lambda^*, u^*, v^*)$ and in the formulation of Theorem 3.3, and can be stated as follows:

$$\text{Maximize} \quad \lambda = (\lambda_1, \lambda_2, \dots, \lambda_p) \qquad \text{(DIII)}$$
subject to

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[DF_i(T) - \lambda_i\, DG_i(T)] + \sum_{j=1}^{q} v_j\, DH_j(T)\right) \geqq 0, \qquad \text{for all } S \in \mathbb{A}^n, \tag{7.1}$$

$$F_i(T) - \lambda_i G_i(T) \geqq 0, \qquad i \in \underline{p}, \tag{7.2}$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \qquad t \in \underline{m} \cup \{0\}, \tag{7.3}$$

$$T \in \mathbb{A}^n, \quad \lambda \in \mathbb{R}_+^p, \quad u \in U, \quad v \in \mathbb{R}_+^q,$$

where $\mathcal{F}(S, T; \cdot) : L_1^n(X, A, \mu) \to \mathbb{R}$ is a sublinear function.

We next show that (DIII) is a dual problem for (P) by proving weak and strong duality theorems.

THEOREM 7.1. WEAK DUALITY. *Let S and (T, λ, u, v) be arbitrary feasible solutions of (P) and (DIII), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Theta_0(\cdot, \lambda, u, v), \dots, \Theta_k(\cdot, \lambda, u, v))$ *is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at T;*
 (ii) $(\Delta_{k+1}(\cdot, v), \dots, \Delta_m(\cdot, v))$ *is $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-quasiconvex at T;*
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(b) (i) $(\Theta_0(\cdot, \lambda, u, v), \dots, \Theta_k(\cdot, \lambda, u, v))$ *is prestrictly $(\mathcal{F}, \beta, \bar{\rho}, \theta)$-$V$-quasiconvex at T;*
 (ii) $(\Delta_{k+1}(\cdot, v), \dots, \Delta_m(\cdot, v))$ *is strictly $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$-pseudoconvex at T;*
 (iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(c) (i) (a)(ii) *and* (b)(i) *hold;*
 (ii) $\bar{\rho} + \tilde{\rho} > 0$.

Then $\varphi(S) \nleq \lambda$.

PROOF.

(a) Suppose to the contrary that $\varphi(S) \leqslant \lambda$. This implies that $F_i(S) - \lambda_i G_i(S) \leqq 0$ for each $i \in \underline{p}$, with strict inequality holding for at least one index $\ell \in \underline{p}$. From these inequalities, nonnegativity of v, primal feasibility of S, and (7.2) it is easily seen that for each $t \in K$,

$$\begin{aligned}
\Theta_t(S, \lambda, u, v) &= \sum_{i \in I_t} u_i[F_i(S) - \lambda_i G_i(S)] + \sum_{j \in J_t} v_j H_j(S) \\
&\leqq \sum_{i \in I_t} u_i[F_i(S) - \lambda_i G_i(S)] \\
&\leqq 0 \\
&\leqq \sum_{i \in I_t} u_i[F_i(T) - \lambda_i G_i(T)] + \sum_{j \in J_t} v_j H_j(T) = \Theta_t(T, \lambda, u, v),
\end{aligned}$$

and so

$$\sum_{t \in K} \alpha_t(S, T)\Theta_t(S, \lambda, u, v) < \sum_{t \in K} \alpha_t(S, T)\Theta_t(T, \lambda, u, v),$$

which in view of (i) implies that

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i[DF_i(T)-\lambda_i\,DG_i(T)]+\sum_{t\in K}\sum_{j\in J_t}v_j\,DH_j(T)\right)<-\bar{\rho}\,d^2(\theta(S,T)). \tag{7.4}$$

As for each $t\in M\backslash K$,

$$\Delta_t(S,v)=\sum_{j\in J_t}v_jH_j(S)\leqq 0=\sum_{j\in J_t}v_jH_j(T)=\Delta_t(T,v),$$

and hence,

$$\sum_{t\in M\backslash K}\beta_t(S,T)\Delta_t(S,v)\leqq\sum_{t\in M\backslash K}\beta_t(S,T)\Delta_t(T,v),$$

(ii) implies that

$$\mathcal{F}\left(S,T;\sum_{t\in M\backslash K}\sum_{j\in J_t}v_j\,DH_j(T)\right)\leqq-\tilde{\rho}\,d^2(\theta(S,T)). \tag{7.5}$$

Now adding (7.4) and (7.5) and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$, and (iii), we see that

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i[DF_i(T)-\lambda_i\,DG_i(T)]+\sum_{j=1}^{q}v_j\,DH_j(T)\right)<-\left(\bar{\rho}+\tilde{\rho}\right)\,d^2(\theta(S,T)),$$

which contradicts (7.1). Hence, $\varphi(S)\nleq\lambda$.

(b),(c) The proofs are similar to that of Part (a).

THEOREM 7.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S,S^*;DF(S^*))=\sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, and assume that any one of the three sets of hypotheses specified in Theorem 7.1 holds for all feasible solutions of (DIII). Then there exist $\lambda^*\in\mathbb{R}_+^p$, $u^*\in U$, and $v^*\in\mathbb{R}_+^q$ such that (S^*,λ^*,u^*,v^*) is an efficient solution of (DIII) and $\varphi(S^*)=\lambda^*$.*

PROOF. By Theorem 2.1, there exist $\lambda^*(=\varphi(S^*))$, u^*, and v^*, as specified above, such that (S^*,λ^*,u^*,v^*) is a feasible solution of (DIII). That it is an efficient solution follows from Theorem 7.1.

8. DUALITY MODEL IV

In this section, we investigate the following duality model for (P), which may be viewed as the semiparametric counterpart of (DI):

$$\text{Maximize}\quad\left(\frac{F_1(T)}{G_1(T)},\dots,\frac{F_p(T)}{G_p(T)}\right) \tag{DIV}$$

subject to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i[G_i(T)\,DF_i(T)-F_i(T)\,DG_i(T)]+\sum_{j=1}^{q}v_jDH_j(T)\right)\geqq 0,\qquad\text{for all } S\in\mathbb{A}^n, \tag{8.1}$$

$$v_jH_j(T)\geqq 0,\qquad j\in\underline{q}, \tag{8.2}$$

$$T\in\mathbb{A}^n,\quad u\in U,\quad v\in\mathbb{R}_+^q,$$

where $\mathcal{F}(S,T;\cdot):L_1^n(X,\mathbb{A},\mu)\to\mathbb{R}$ is a sublinear function. In the remainder of this paper, we assume that $G_i(T)>0$ and $F_i(T)\geqq 0$, $i\in\underline{p}$, for all T and u such that (T,u,v) is a feasible solution of the dual problem under consideration.

We next state and prove weak, strong, and strict converse duality theorems for (P)–(DIV).

THEOREM 8.1. WEAK DUALITY. *Let S and (T, u, v) be arbitrary feasible solutions of (P) and (DIV), respectively, and assume that any one of the following four sets of hypotheses is satisfied:*

(a) (i) $(F_1, \dots, F_p)$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-convex at* T;
(ii) $(-G_1, \dots, -G_p)$ *is* $(\mathcal{F}, \beta, \hat{\rho}, \theta)$-$V$*-convex at* T;
(iii) $(H_1, \dots, H_q)$ *is* $(\mathcal{F}, \gamma, \tilde{\rho}, \theta)$-$V$*-convex at* T;
(iv) $\alpha_1 = \alpha_2 = \cdots = \alpha_p = \beta_1 = \beta_2 = \cdots = \beta_p = \gamma_1 = \gamma_2 = \cdots = \gamma_q = \delta$;
(v) $\sum_{i=1}^p u_i[G_i(T)\bar{\rho}_i + F_i(T)\hat{\rho}_i] + \sum_{j=1}^q v_j\tilde{\rho}_j \geqq 0$;

(b) (i) $(\mathcal{C}_1(\cdot, T, u), \dots, \mathcal{C}_p(\cdot, T, u))$ *is* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-pseudoconvex at* T;
(ii) $(\mathcal{B}_1(\cdot, v), \dots, \mathcal{B}_q(\cdot, v))$ *is* $(\mathcal{F}, \beta, \tilde{\rho}, \theta)$-$V$*-quasiconvex at* T;
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$;

(c) (i) (a)(ii) *holds;*
(ii) $(\mathcal{C}_1(\cdot, T, u), \dots, \mathcal{C}_p(\cdot, T, u))$ *is prestrictly* $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$*-quasiconvex at* T;
(iii) $\bar{\rho} + \tilde{\rho} > 0$;

(d) (i) (b)(ii) *holds;*
(ii) $(\mathcal{B}_1(\cdot, v), \dots, \mathcal{B}_q(\cdot, v))$ *is strictly* $(\mathcal{F}, \alpha, \tilde{\rho}, \theta)$-$V$*-pseudoconvex at* T;
(iii) $\bar{\rho} + \tilde{\rho} \geqq 0$.

Then $\varphi(S) \nleq \psi(T, u, v)$*, where* $\psi = (\psi_1, \psi_2, \dots, \psi_p)$ *is the objective function of (DIV).*

PROOF.

(a) From the sublinearity of $\mathcal{F}(S, T; \cdot)$ and (8.1) it follows that:

$$\mathcal{F}\left(S, T; \sum_{i=1}^p u_i[G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)]\right) + \mathcal{F}\left(S, T; \sum_{j=1}^q v_j\, DH_j(T)\right) \geqq 0. \tag{8.3}$$

Keeping in mind that $u > 0$, $v \geqq 0$, $F_i(T) \geqq 0$, and $G_i(T) > 0$, $i \in \underline{p}$, we have

$$\begin{aligned}
&\sum_{i=1}^p u_i[G_i(T)F_i(S) - F_i(T)G_i(S) \\
&= \sum_{i=1}^p u_i\{G_i(T)[F_i(S) - F_i(T)] - F_i(T)[G_i(S) - G_i(T)]\} \\
&\geqq \sum_{i=1}^p u_i G_i(T)\left[\mathcal{F}(S, T; \delta(S, T)\, DF_i(T) + \bar{\rho}_i\, d^2(\theta(S, T))\right] \\
&\quad + \sum_{i=1}^p u_i F_i(T)\left[\mathcal{F}(S, T; -\delta(S, T)\, DG_i(T) + \hat{\rho}_i\, d^2(\theta(S, T))\right] \quad \text{(by (i), (ii), and (iv))} \\
&\geqq \mathcal{F}\left(S, T; \sum_{i=1}^p u_i \delta(S, T)[G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)]\right) \\
&\quad + \sum_{i=1}^p u_i\left[G_i(T)\bar{\rho}_i + F_i(T)\hat{\rho}_i\right] d^2(\theta(S, T)) \quad \text{(by the sublinearity of } \mathcal{F}(S, T; \cdot)) \\
&\geqq -\mathcal{F}\left(S, T; \sum_{j=1}^q v_j \delta(S, T)\, DH_j(T)\right) + \sum_{i=1}^p u_i\left[G_i(T)\bar{\rho}_i + F_i(T)\hat{\rho}_i\right] d^2(\theta(S, T)) \quad \text{(by (8.3))} \\
&\geqq \sum_{j=1}^q v_j\left[H_j(T) - H_j(S)\right] + \left(\sum_{i=1}^p u_i\left[G_i(T)\bar{\rho}_i + F_i(T)\hat{\rho}_i\right] + \sum_{j=1}^q v_j \tilde{\rho}_j\right) d^2(\theta(S, T)) \\
&\quad \text{(by (iii) and (iv))} \\
&\geqq 0 \quad \text{(by (8.2), feasibility of } S\text{, and (v)).}
\end{aligned}$$

Since $u > 0$, the above inequality implies that $(G_1(T)F_1(S) - F_1(T)G_1(S), \ldots, G_p(T)F_p(S) - F_p(T)G_p(S)) \nleq (0, \ldots, 0)$, which in turn implies that $\varphi(S) \nleq \psi(T, u, v)$.

(b) As shown in the proof of Theorem 5.1, our assumption in (ii) leads to (5.5), which when combined with (8.3) yields

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i[G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)]\right) \geqq \tilde{\rho}\, d^2\left(\theta(S,T)\right) \geqq -\bar{\rho}\, d^2(\theta(S,T)),$$

where the second inequality follows from (iii). By (i), this inequality implies that

$$\sum_{i=1}^{p} u_i\alpha_i(S,T)[G_i(T)F_i(S) - F_i(T)G_i(S)] \geqq \sum_{i=1}^{p} u_i\alpha_i(S,T)[G_i(T)F_i(T) - F_i(T)G_i(T)] = 0.$$

Since $u_i\alpha_i(S,T) > 0$ for each $i \in \underline{p}$, the above inequality implies that $(G_1(T)F_1(S) - F_1(T)G_1(S), \ldots, G_p(T)F_p(S) - F_p(T)G_p(S)) \nleq (0, \ldots, 0)$, which in turn implies that $\varphi(S) \nleq \psi(T, u, v)$.

(c),(d) The proofs are similar to that of Part (b). ∎

THEOREM 8.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \langle D_kF(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, and assume that any one of the four sets of hypotheses specified in Theorem 8.1 holds for all feasible solutions of (DIV). Then there exist $u^* \in U$ and $v^* \in \mathbb{R}_+^q$ such that (S^*, u^*, v^*) is an efficient solution of (DIV) and $\varphi(S^*) = \psi(S^*, u^*, v^*)$.*

PROOF. By Theorem 2.2, there exist u^* and v^*, as specified above, such that (S^*, u^*, v^*) is a feasible solution of (DIV) and $\varphi(S^*) = \psi(S^*, u^*, v^*)$. Efficiency of (S^*, u^*, v^*) for (DIV) follows from Theorem 8.1. ∎

The proof of the next theorem is similar to that of Theorem 5.3, and hence, omitted.

THEOREM 8.3. STRICT CONVERSE DUALITY. *Let S^* be a feasible solution of (P), let $\mathcal{F}(S, S^*; DF(S^*)) = \sum_{k=1}^{n} \langle D_kF(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, let $(\tilde{S}, \tilde{u}, \tilde{v})$ be a feasible solution of (DIV) such that*

$$\sum_{i=1}^{p} \tilde{u}_i\alpha_i\left(S^*, \tilde{S}\right)\left[G_i\left(\tilde{S}\right)F_i\left(S^*\right) - F_i\left(\tilde{S}\right)G_i\left(S^*\right)\right] \leqq 0.$$

Moreover, assume that any one of the following four sets of hypotheses is satisfied:

(a) *the assumptions specified in Part (a) of Theorem 8.1 are satisfied for all feasible solutions of (DIV), and $(F_1, \ldots, F_p)$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-convex at $\tilde{S}$, or $(-G_1, \ldots, -G_p)$ is strictly $(\mathcal{F}, \beta, \bar{\rho}, \theta)$-$V$-convex at $\tilde{S}$, or $(H_1, \ldots, H_q)$ is strictly $(\mathcal{F}, \gamma, \bar{\rho}, \theta)$-$V$-convex at $\tilde{S}$, with $\tilde{v} \neq 0$, or $\sum_{i=1}^{p} \tilde{u}_i[G_i(\tilde{S})\bar{\rho}_i + F_i(\tilde{S})\hat{\rho}_i] + \sum_{j=1}^{q} \tilde{v}_j\tilde{\rho}_j > 0$;*
(b) *the assumptions specified in Part (b) of Theorem 8.1 are satisfied for all feasible solutions of (DIV), and $(\mathcal{C}_1(\cdot, \tilde{S}, \tilde{u}), \ldots, \mathcal{C}_p(\cdot, \tilde{S}, \tilde{u}))$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at $\tilde{S}$;*
(c) *the assumptions specified in Part (c) of Theorem 8.1 are satisfied for all feasible solutions of (DIV), and $(\mathcal{C}_1(\cdot, \tilde{S}, \tilde{u}), \ldots, \mathcal{C}_p(\cdot, \tilde{S}, \tilde{u}))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-quasiconvex at $\tilde{S}$;*
(d) *the assumptions specified in Part (d) of Theorem 8.1 are satisfied for all feasible solutions of (DIV), and $(\mathcal{C}_1(\cdot, \tilde{S}, \tilde{u}), \ldots, \mathcal{C}_p(\cdot, \tilde{S}, \tilde{u}))$ is strictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at $\tilde{S}$.*

Then $\tilde{S} = S^*$.

9. DUALITY MODEL V

In this section, we present a more general semiparametric duality model for (P) whose structure and features are based on Theorems 2.2 and 4.2. It has the following form:

$$\text{Maximize}\quad \omega(T,u,v) = \left(\frac{F_1(T) + \sum_{j\in J_0} v_j H_j(T)}{G_1(T)}, \dots, \frac{F_p(T) + \sum_{j\in J_0} v_j H_j(T)}{G_p(T)}\right) \tag{DV}$$

subject to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p} u_i\left\{\{G_i(T)\left[DF_i(T) + \sum_{j\in J_0} v_j\, DH_j(T)\right] - [F_i(T) + \Delta_0(T,v)]\, DG_i(T)\right\} + \sum_{j\in \underline{q}\setminus J_0} v_j DH_j(T)\right) \geqq 0, \qquad \text{for all } S \in \mathbb{A}^n, \tag{9.1}$$

$$\sum_{j\in J_t} v_j H_j(T) \geqq 0, \qquad t \in \underline{m} \cup \{0\}, \tag{9.2}$$

$$T \in \mathbb{A}^n, \quad u \in U, \quad v \in \mathbb{R}^q_+,$$

where $\mathcal{F}(S,T;\cdot) : L_1^n(X,\mathbb{A},\mu) \to \mathbb{R}$ is a sublinear function.

For each $i \in \underline{p}$, let the function $\Phi_i(\cdot,T,u,v) : \mathbb{A}^n \to \mathbb{R}$ be defined, for fixed T, u, and v, by

$$\Phi_i(R,T,u,v) = u_i\left\{G_i(T)\left[F_i(R) + \sum_{j\in J_0} v_j\, DH_j(R)\right] - [F_i(T) + \Delta_0(T,v)]G_i(R)\right\}.$$

THEOREM 9.1. WEAK DUALITY. *Let S and (T,u,v) be arbitrary feasible solutions of (P) and (DV), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) $(\Phi_1(\cdot,T,u,v),\dots,\Phi_p(\cdot,T,u,v))$ *is $(\mathcal{F},\alpha,\bar{\rho},\theta)$-$V$-pseudoconvex at T;*
  (ii) $(\Delta_1(\cdot,v),\dots,\Delta_m(\cdot,v))$ *is $(\mathcal{F},\beta,\tilde{\rho},\theta)$-$V$-quasiconvex at T;*
  (iii) $\bar{\rho}+\tilde{\rho} \geqq 0$;

(b) (i) (a)(ii) *holds;*
  (ii) $(\Phi_1(\cdot,T,u,v),\dots,\Phi_p(\cdot,T,u,v))$ *is prestrictly $(\mathcal{F},\alpha,\bar{\rho},\theta)$-$V$-quasiconvex at T;*
  (iii) $\bar{\rho}+\tilde{\rho} > 0$;

(c) (i) (b)(ii) *holds;*
  (ii) $(\Delta_1(\cdot,v),\dots,\Delta_m(\cdot,v))$ *is strictly $(\mathcal{F},\beta,\tilde{\rho},\theta)$-$V$-pseudoconvex at T;*
  (iii) $\bar{\rho}+\tilde{\rho} \geqq 0$.

Then $\varphi(S) \nleq \omega(T,u,v)$, where $\omega = (\omega_1,\omega_2,\dots,\omega_p)$ is the objective function of (DV).

PROOF.

(a) From the sublinearity of $\mathcal{F}(S,T;\cdot)$ and (9.1) we deduce that

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p} u_i\left\{G_i(T)\left[DF_i(T) + \sum_{j\in J_0} v_j\, DH_j(T)\right] - [F_i(T) + \Delta_0(T,v)]DG_i(T)\right\}\right) + \mathcal{F}\left(S,T;\sum_{t=1}^{m}\sum_{j\in J_t} v_j\, DH_j(T)\right) \geqq 0. \tag{9.3}$$

Because of (5.5), which is valid for the present case due to our assumption specified in (ii), and (iii), the above inequality reduces to

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p}u_i\left\{G_i(T)\left[DF_i(T)+\sum_{j\in J_0}v_j\,DH_j(T)\right]-[F_i(T)+\Delta_0(T,v)]\,DG_i(T)\right\}\right) \geqq -\bar{\rho}\,d^2(\theta(S,T)),$$

which in view of (i) implies that

$$\sum_{i=1}^{p}\alpha_i(S,T)\Phi_i(S,T,u,v) \geqq \sum_{i=1}^{p}\alpha_i(S,T)\Phi_i(T,T,u,v).$$

Since $\Phi_i(T,T,u,v) = 0$ for each $i \in \underline{p}$ and $v_jH_j(S) \leqq 0$ for each $j \in \underline{q}$, this inequality takes the form

$$\sum_{i=1}^{p}u_i\alpha_i(S,T)\{G_i(T)F_i(S)-[F_i(T)+\Delta_0(T,v)]G_i(S)\} \geqq 0.$$

Because $u_i\alpha_i(S,T) > 0$ for each $i \in \underline{p}$, the above inequality implies that $(G_1(T)F_1(S) - [F_1(T)+\Delta_0(T,v)]G_1(S),\ldots,G_p(T)F_p(S)-[F_p(T)+\Delta_0(T,v)]G_p(S)) \nleq (0,\ldots,0)$, which in turn implies that $\varphi(S) \nleq \omega(T,u,v)$.

(b),(c) The proofs are similar to that of Part (a).

THEOREM 9.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*),\chi_{S_k}-\chi_{S_k^*}\rangle$ for any differentiable function $F:\mathbb{A}^n\to\mathbb{R}$ and $S\in\mathbb{A}^n$, and assume that any one of the three sets of hypotheses specified in Theorem 9.1 holds for all feasible solutions of (DV). Then there exist $u^*\in U$ and $v^*\in\mathbb{R}^q_+$ such that (S^*,u^*,v^*) is an efficient solution of (DV) and $\varphi(S^*)=\omega(S^*,u^*,v^*)$.*

PROOF. By Theorem 2.2, there exist $u^*\in U$ and $\bar{v}\in\mathbb{R}^q_+$ such that

$$\left\langle\sum_{i=1}^{p}u_i^*\left[G_i(S^*)\,D_kF_i(S^*)-F_i(S^*)\,D_kG_i(S^*)\right] + \sum_{j=1}^{q}\bar{v}_jD_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle \geqq 0, \qquad \text{for all } S_k\in\mathbb{A},\quad k\in\underline{n}, \tag{9.4}$$

$$\bar{v}_jH_j(S^*)=0, \qquad j\in\underline{q}. \tag{9.5}$$

Now if we let $v_j^* = \bar{v}_j/G_i(S^*)$ for each $j\in J_0$, $v_j^*=\bar{v}_j$ for each $j\in\underline{q}\setminus J_0$, and observe that $\Delta_0(S^*,v^*)=0$, then (9.4) and (9.5) can be expressed as follows:

$$\left\langle\sum_{i=1}^{p}u_i^*\left\{G_i(S^*)\left[D_kF_i(S^*)+\sum_{j\in J_0}v_j^*\,D_kH_j(S^*)\right] - [F_i(S^*)+\Delta_0(S^*,v^*)]\,D_kG_i(S^*)\right\} + \sum_{j\in\underline{q}\setminus J_0}v_j^*\,D_kH_j(S^*),\chi_{S_k}-\chi_{S_k^*}\right\rangle \geqq 0, \qquad \text{for all } S_k\in\mathbb{A},\quad k\in\underline{n}, \tag{9.6}$$

$$\sum_{j\in J_t}v_j^*H_j(S^*)=0, \qquad j\in\underline{m}\cup\{0\}. \tag{9.7}$$

From (9.6) and (9.7), it is clear that (S^*,u^*,v^*) is a feasible solution of (DV), and $\varphi(S^*) = \omega(S^*,u^*,v^*)$. That it is an efficient solution follows from Theorem 9.1. ∎

The proof of the next theorem is similar to that of Theorem 5.3, and hence, omitted.

THEOREM 9.3. STRICT CONVERSE DUALITY. *Let $\mathcal{F}$ be as in Theorem 9.2, let S^* and $(\tilde{S}, \tilde{u}, \tilde{v})$ be feasible solutions of (P) and (DV), respectively, such that*

$$\sum_{i=1}^{p} \alpha_i \left(S^*, \tilde{S}\right) \Phi_i \left(S^*, \tilde{S}, \tilde{u}, \tilde{v}\right) \leqq 0,$$

and assume that any one of the following three sets of conditions is satisfied:

(a) *the assumptions of Part (a) of Theorem 9.1 hold for all feasible solutions of (DV), and $(\Phi_1(\cdot, \tilde{S}, \tilde{u}, \tilde{v}), \dots, \Phi_p(\cdot, \tilde{S}, \tilde{u}, \tilde{v}))$ is strictly $(\mathcal{F}, \bar{\rho}, \theta)$-$V$-pseudoconvex at $\tilde{S}$;*
(b) *the assumptions of Part (b) of Theorem 9.1 hold for all feasible solutions of (DV), and$(\Phi_1(\cdot, \tilde{S}, \tilde{u}, \tilde{v}), \dots, \Phi_p(\cdot, \tilde{S}, \tilde{u}, \tilde{v}))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-quasiconvex at $\tilde{S}$;*
(c) *the assumptions of Part (c) of Theorem 9.1 hold for all feasible solutions of (DV), and $(\Phi_1(\cdot, \tilde{S}, \tilde{u}, \tilde{v}), \dots, \Phi_p(\cdot, \tilde{S}, \tilde{u}, \tilde{v}))$ is strictly $(\mathcal{F}, \bar{\rho}, \theta)$-$V$-pseudoconvex at $\tilde{S}$.*

Then $\tilde{S} = S^*$.

10. DUALITY MODEL VI

In this section, we discuss another general duality model for (P) which may be viewed as the semiparametric version of (DIII). It can be stated as follows:

$$\text{Maximize} \quad \left(\frac{F_1(T)}{G_1(T)}, \dots, \frac{F_p(T)}{G_p(T)}\right) \tag{DVI}$$

subject to

$$\mathcal{F}\left(S, T; \sum_{i=1}^{p} u_i [G_i(T)\, DF_i(T) - F_i(T)\, DG_i(T)] + \sum_{j=1}^{q} v_j\, DH_j(T)\right) \geqq 0, \tag{10.1}$$

$$\text{for all } S \in \mathbb{A}^n,$$

$$\sum_{j \in J_t} v_j H_j(T) \geqq 0, \qquad t \in \underline{m} \cup \{0\}, \tag{10.2}$$

$$T \in \mathbb{A}^n, \quad u \in U, \quad v \in \mathbb{R}^q_+,$$

where $\mathcal{F}(S, T; \cdot) : L_1^n(X, \mathbb{A}, \mu) \to \mathbb{R}$ is a sublinear function.

We next show that (DVI) is a dual problem for (P) by proving weak and strong duality theorems.

THEOREM 10.1. WEAK DUALITY. *Let S and (T, u, v) be arbitrary feasible solutions of (P) and (DVI), respectively, and assume that any one of the following three sets of hypotheses is satisfied:*

(a) (i) *$(\Phi_0(\cdot, T, u, v), \dots, \Phi_k(\cdot, T, u, v))$ is $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-pseudoconvex at T;*
(ii) *$(\Delta_{k+1}(\cdot, v), \dots, \Delta_m(\cdot, v))$ is $(\mathcal{F}, \tilde{\rho}, \theta)$-$V$-quasiconvex at T;*
(iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$;*
(b) (i) (a)(ii) *holds;*
(ii) *$(\Phi_0(\cdot, T, u, v), \dots, \Phi_k(\cdot, T, u, v))$ is prestrictly $(\mathcal{F}, \alpha, \bar{\rho}, \theta)$-$V$-quasiconvex at T;*
(iii) *$\bar{\rho} + \tilde{\rho} > 0$;*
(c) (i) (b)(ii) *holds;*
(ii) *$(\Delta_{k+1}(\cdot, v), \dots, \Delta_m(\cdot, v))$ is strictly $(\mathcal{F}, \tilde{\rho}, \theta)$-$V$-pseudoconvex at T;*
(iii) *$\bar{\rho} + \tilde{\rho} \geqq 0$;*

Then $\varphi(S) \nleq \zeta(T, u, v)$, where $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_p)$ is the objective function of (DVI).

PROOF.

(a) Suppose to the contrary that $\varphi_i(S) \leqq F_i(T)/G_i(T)$, with strict inequality holding for some $\ell \in \underline{p}$. This implies that $G_i(T)F_i(S) - F_i(T)G_i(S) \leqq 0$ for each $i \in \underline{p}$, with strict inequality holding for some $\ell \in \underline{p}$. From these inequalities, nonnegativity of v, and primal feasibility of S, it is easily seen that for each $t \in K$,

$$\begin{aligned}\Phi_t(S,T,u,v) &= \sum_{i\in I_t} u_i[G_i(T)F_i(S) - F_i(T)G_i(S)] + \sum_{j\in J_t} v_j H_j(S)\\ &\leqq \sum_{i\in I_t} u_i[G_i(T)F_i(S) - F_i(T)G_i(S)]\\ &\leqq 0\\ &= \sum_{i\in I_t} u_i[G_i(T)F_i(T) - F_i(T)G_i(T)] + \sum_{j\in J_t} v_j H_j(T) = \Phi_t(T,T,u,v),\end{aligned}$$

and so

$$\sum_{t\in K} \alpha_t(S,T)\Phi_t(S,T,u,v) < \sum_{t\in K} \alpha_t(S,T)\Phi_t(T,T,u,v),$$

which in view of (i) implies that

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p} u_i[G_i(T)DF_i(T) - F_i(T)DG_i(T)] + \sum_{t\in K}\sum_{j\in J_t} v_j DH_j(T)\right) < -\bar{\rho}\, d^2(\theta(S,T)). \quad (10.3)$$

As for each $t \in M\backslash K$,

$$\Delta_t(S,v) = \sum_{j\in J_t} v_j H_j(S) \leqq 0 = \sum_{j\in J_t} v_j H_j(T) = \Delta_t(T,v),$$

and hence,

$$\sum_{t\in M\backslash K} \beta_t(S,T)\Delta_t(S,v) \leqq \sum_{t\in M\backslash K} \beta_t(S,T)\Delta_t(T,v),$$

(ii) implies that

$$\mathcal{F}\left(S,T;\sum_{t\in M\backslash K}\sum_{j\in J_t} v_j\, DH_j(T)\right) \leqq -\tilde{\rho}\, d^2(\theta(S,T)). \quad (10.4)$$

Now adding (10.3) and (10.4) and using the sublinearity of $\mathcal{F}(\bar{S},S^*;\cdot)$, and (iii), we see that

$$\mathcal{F}\left(S,T;\sum_{i=1}^{p} u_i[G_i(T)\,DF_i(T) - F_i(T)\,DG_i(T)] + \sum_{j=1}^{q} v_j\, DH_j(T)\right) < -(\bar{\rho}+\tilde{\rho})\; d^2(\theta(S,T)),$$

which contradicts (10.1). Hence, $\varphi(S) \nleq \zeta(T,u,v)$.

(b),(c) The proofs are similar to that of Part (a).

THEOREM 10.2. STRONG DUALITY. *Let S^* be a regular efficient solution of (P), let $\mathcal{F}(S,S^*;DF(S^*)) = \sum_{k=1}^{n}\langle D_kF(S^*), \chi_{S_k} - \chi_{S_k^*}\rangle$ for any differentiable function $F : \mathbb{A}^n \to \mathbb{R}$ and $S \in \mathbb{A}^n$, and assume that any one of the three sets of hypotheses specified in Theorem 10.1 holds for all feasible solutions of (DVI). Then there exist $u^* \in U$ and $v^* \in \mathbb{R}^q_+$ such that (S^*,u^*,v^*) is an efficient solution of (DVI) and $\varphi(S^*) = \zeta(S^*,u^*,v^*)$.*

PROOF. By Theorem 2.2, there exist u^* and v^*, as specified above, such that (S^*,u^*,v^*) is a feasible solution of (DVI). That it is an efficient solution follows from Theorem 10.1.

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