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A Characterization of Dual Spaces

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A Banach space is a dual space if and only if it is isometric to the space of uniform functions on a set with graded constraints.

1. STATEMENT OF THE THEOREM

For each set X let $V(X)$ denote the weak direct sum of X copies of the (real or complex) scalar field Λ so that $V(X)$ is the vector space of all finitely nonzero X-tuples (λ_x) of scalars. A *set with graded constraints* (henceforth sgc) is a pair (X, A) where $A = (A_n: n = 1, 2, 3, \ldots)$ is an arbitrary sequence of subsets of $V(X)$. Given such (X, A) and a Banach space $E, f: X \to E$ is *uniform* if f is bounded and if

$$\exists M > 0 \; \forall n \; \forall (\lambda_x) \in A_n \; \|\textstyle\sum \lambda_x f(x)\| \leqslant M/n,$$

in which case $\|f\|_A$ is defined to be the infimum of all such M. Let $\|f\|_\infty$ be the sup norm and let $\mathrm{Lip}(X, A; E)$ (just $\mathrm{Lip}(X, A)$ if $E = \Lambda$) denote the space of all uniform functions with norm

$$\|f\| = \mathrm{Max}\{\|f\|_\infty, \|f\|_A\}.$$

MAIN THEOREM. *If E is a dual space, so is* $\mathrm{Lip}(X, A; E)$ *for any* sgc (X, A). *A necessary and sufficient condition for E to be a dual space is that E be isometric to* $\mathrm{Lip}(X, A)$ *for some* sgc (X, A). *A necessary and sufficient condition that E be isometric to the dual of a separable space is that E be isometric to* $\mathrm{Lip}(X, A)$ *for some* sgc (X, A) *with X countable.*

EXAMPLE 1.1. Let d be a metric on X and let D be a dense subset of (X, d). Define $A_1 = \{(d(x, y))^{-1}(x - y): x \neq y, x, y \in D\}$ and $A_n = \varnothing$ for $n > 1$. Then $\mathrm{Lip}(X, A; E)$ is the usual space of bounded Lipschitz functions; in particular, Johnson's result ([1], Theor. 4.1) that this is a dual space if E is follows from Corollary 3.2 below.

EXAMPLE 1.2. *Let X be a subset of an abelian group and define* $A_n = \{x + y - z: x, y, z, x + y \in X$ and $x + y = z\}$ for all n. Then $\mathrm{Lip}(X, A; E)$ is the space of all additive functions. A similar construction proves that the space of all convex functionals on a convex set is a dual space.

2. Free Banach Spaces

Since X is a basis for $V(X)$, $g: X \to Y$ extends uniquely to a linear map $V(g): V(X) \to V(Y)$. An sgc *homomorphism* $g: (X, A) \to (Y, B)$ is a function $g: X \to Y$ such that $V(g)$ maps A_n into B_n for all n. Each Banach space E induces the sgc (X_E, A_E) where X_E is the unit ball of E and $A_{E,n} = \{(\lambda_x): \| \sum \lambda_x x \| \leqslant n^{-1}\}$. If $h: E \to E'$ is a contractive linear map (i.e., bounded of norm $\leqslant 1$), $h: (X_E, A_E) \to (X_{E'}, A_{E'})$ is an sgc homomorphism. Given the sgc (X, A) a *free Banach space over* (X, A) is a pair (F, η) where F is a Banach space and $\eta: (X, A) \to (X_F, A_F)$ is an sgc homomorphism subject to the following universal property:

$$
\begin{array}{ccccc}
(X, A) & \xrightarrow{\ \eta\ } & (X_F, A_F) & \qquad & F \\
 & \searrow^{g} & \downarrow{\scriptstyle g^{\#}} & & \downarrow{\scriptstyle g^{\#}} \\
 & & (X_E, A_E) & & E
\end{array}
$$

for all Banach spaces E and sgc homomorphisms $g: (X, A) \to (X_E, A_E)$ there exists a unique contractive linear map $g^{\#}: F \to E$ such that $g^{\#}\eta = g$. It is clear that (F, η), if it exists, is unique up to linear isometry.

THEOREM. *For every* sgc (X, A) *the free Banach space over* (X, A) *exists.*

Proof. This is an immediate consequence of the Freyd special adjoint functor theorem [2, V.8]. ∎

3. Proof of the Main Theorem

PROPOSITION 3.1. *Let (X, A) be an* sgc, *let (F, η) be the free Banach space over (X, A) and let E be a Banach space. Then* $\mathrm{Lip}(X, A; E)$ *is a Banach space and is isometric to the Banach space $\mathscr{L}(F, E)$ of bounded linear maps.*

Proof. The passage from $h: F \to E$ to $h\eta: X \to E$ is a linear map from $\mathscr{L}(F, E)$ to the vector space of all functions from X to E which must, by the universal property, map the unit ball of $\mathscr{L}(F, E)$ bijectively onto the set of sgc homomorphisms from (X, A) to (X_E, A_E). Because the uniform maps $(X, A) \to E$ are precisely the scalar multiples of sgc homomorphisms $(X, A) \to (X_E, A_E)$, the passage from h to $h\eta$ is a linear isomorphism from $\mathscr{L}(F, E)$ to the set of uniform maps $(X, A) \to E$ (which must therefore be a vector space); the unique norm on the latter rendering this passage an isometry must be given by the Minkowski functional

$$\| f \| = \mathrm{Inf}\{\lambda > 0 : \lambda^{-1} f \text{ is an sgc homomorphism } (X, A) \to (X_E, A_E)\}$$

which is indeed the norm $\mathrm{Max}\{\| f \|_\infty, \| f \|_A\}$. ∎

COROLLARY 3.2. *Let (X, A) be an* sgc *and let E be a dual space. Then* $\mathrm{Lip}(X, A; E)$ *is a dual space.*

Proof. Let E be isometric to G^*. Using a well-known property of the Schatten tensor product, $\mathrm{Lip}(X, A; E) \simeq \mathscr{L}(F, \mathscr{L}(G, \Lambda)) \simeq (F \otimes G)^*$. ∎

LEMMA 3.3. *Let (F, η) be the free Banach space over the* sgc (X, A). *Then F is the closed span of $\eta(X)$.*

Proof. Because of the universal property, any contractive linear functional h satisfying $h\eta = 0$ must be identically 0; now use the Hahn–Banach Theorem. ∎

PROPOSITION 3.4. *Let (X, A) be an* sgc *with X countable. Then* $\mathrm{Lip}(X, A)$ *is the dual of a separable space.*

Proof. Let (F, η) be free over (X, A). Then $\mathrm{Lip}(X, A)$ is isometric to F^* and, by 3.3, F is separable. ∎

Let E be a fixed Banach space and let C be any subset of the unit ball of E. (C, A) is an sgc where $A_n = \{(\lambda_c): \| \sum \lambda_c c \| \leqslant n^{-1}\}$. If inc: $C \to E$ denotes the inclusion function, inc is an sgc homomorphism $(C, A) \to (X_E, A_E)$. Thus, if (F, η) is the free Banach space over (C, A), the universal property induces the contractive linear map $\mathrm{inc}^{\#}\colon F \to E$ as shown below.

$$\begin{array}{ccccc} (C, A) & \xrightarrow{\ \eta\ } & (X_F, A_F) & \qquad & F \\ & \searrow^{\mathrm{inc}} & \downarrow{\scriptstyle \mathrm{inc}^{\#}} & & \downarrow{\scriptstyle \mathrm{inc}^{\#}} \\ & & (X_E, A_E) & & E \end{array}$$

LEMMA 3.5. $\ker(\mathrm{inc}^{\#}) = 0$.

Proof. Let $t \in \ker(\mathrm{inc}^{\#})$. By 3.3 and the continuity of $\mathrm{inc}^{\#}$, there exists a sequence t_n in the linear span of $\eta(C)$ such that t_n converges to t and $\| \mathrm{inc}^{\#}(t_n)\| \leqslant n^{-1}$. Write $t_n = \sum n_c \eta(c)$. Then $\| \sum n_c \,\mathrm{inc}(c)\| = \| \mathrm{inc}^{\#}(t_n)\| \leqslant n^{-1}$ so that, by the definition of (C, A), $(n_c) \in A_n$. Since η is an sgc homomorphism, $\| t_n \| = \| \sum n_c \eta(c)\| \leqslant n^{-1}$ and $t = 0$ as desired. ▮

LEMMA 3.6. *With respect to the metric on C induced by restricting* $\| x - y \|_E$, η *is an isometry into F.*

Proof. Let $x \neq y \in C$. As $\| x - y \|^{-1} (x - y) \in A_1$ and η is an sgc homomorphism, $\| x - y \|^{-1} (\eta x - \eta y) \in A_{F,1}$ (note: $\eta x \neq \eta y$ since inc is injective). Therefore, $\| \eta x - \eta y \|_F \leqslant \| x - y \|_E$. Conversely, because $\mathrm{inc}^{\#}$ has norm $\leqslant 1$, $\| x - y \|_E = \| \mathrm{inc}^{\#}(\eta x - \eta y)\|_E \leqslant \| \eta x - \eta y \|_F$. ▮

LEMMA 3.7. *If every unit vector in E is in the closure of C,* $\mathrm{inc}^{\#}\colon F \to E$ *is a linear isometry.*

Proof. By 3.6, $\mathrm{inc}^{\#}$ maps $\eta(C)$ isometrically onto C. Since "closure" coincides with "metric completion" for subsets of a Banach space, $\mathrm{inc}^{\#}$ maps $\mathrm{cls}(\eta(C))$ isometrically onto $\mathrm{cls}(C)$. It is now clear that $\mathrm{inc}^{\#}$ is bijective. Moreover, as $\eta(C)$ is a subset of the unit ball of F, $\mathrm{inc}^{\#}$ is an isometry. ▮

What remains to prove of the main theorem follows at once from 3.7 and 3.1, choosing C to be countable if E is separable.

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