INTRODUCTION

Let $G$ be a finite group and $\chi$ be a virtual character of $G$. Set $n = \chi(1)$ and $L = \{\chi(g); g \in G, g \neq 1\}$. Then it is known¹ (Blichfeldt [2], Cameron and Kiyota [4]) that

$$|G| \text{ divides } \prod_{l \in L} (n - l).$$

Note that the product $\prod_{l \in L} (n - l)$ is a rational integer. Cameron and Kiyota [4] called the pair $(G, \chi)$ $L$-sharp² (or sharp of type $L$) if $|G| = \prod_{l \in L} (n - l)$, and posed the problem of determining all the sharp pairs $(G, \chi)$ with $L = W$ for a given finite set $W$ of complex numbers. For several cases, those problems are settled or partially answered in Cameron and Kiyota [4] and Kataoka [9].

The purpose of this paper is to settle the case $W = \{-1, 1\}$. The result is as follows.

¹ The upper bound $\prod(n - l)$ is improved if $\chi$ is not rational valued. See Kataoka [7].
² The divisibility property $(\ast)$ holds in more general settings, and the concept of sharpness is generalised. See Cameron, Deza, and Kiyota [3].
Theorem. Let $G$ be a finite group and $\chi$ be a character of $G$ of degree $n$. Suppose that $(G, \chi)$ is sharp of type $\{-1, 1\}$ (i.e. $|G| = n^2 - 1$ and the set of values of $\chi$ at non-identity elements of $G$ is $\{-1, 1\}$). Then $G$ is isomorphic to one of the following 12 groups:

- $D_8$ and $Q_8$ ($n = 3$);
- $S_4$ and $SL(2, 3)$ ($n = 5$);
- $GL(2, 3)$ and the binary octahedral group ($n = 7$);
- $S_5$ and $SL(2, 5)$ ($n = 11$);
- $PSL(2, 7)$ ($n = 13$);
- $A_6$ ($n = 19$);
- the double cover $\hat{A}_7$ of $A_7$ ($n = 71$);
- $M_{11}$ ($n = 89$).

It is easily verified that the above twelve groups have sharp characters of type $\{-1, 1\}$ by inspection of character tables. More precise information on these characters is given in Section 3.

In Kataoka [8], a class of pairs $(G, \chi)$ with $L = \{-1, 1\}$ satisfying a weaker condition than sharpness is studied. Especially, for a pair $(G, \chi)$ in that class, the group structure of $G/O(G)$ is determined, where $O(G)$ is the largest normal subgroup of $G$ of odd order. The theorem is proved by picking the sharp pairs among pairs in the class.

1. Preliminaries

Let $\chi$ be a character of degree $n$ of a finite group $G$. Suppose that $(G, \chi)$ is sharp of type $\{-1, 1\}$. By definition,

$$|G| = n^2 - 1. \quad (1.1)$$

Since $(\chi, \chi)_G = 2$, $\chi$ is a sum of two irreducible characters of $G$,

$$\chi = \chi_1 + \chi_2,$$

where $\chi_i$ is irreducible.

Set $s = \chi_1(1)$ and $t = \chi_2(1)$.

Proposition 1. In the above notations, we have the following:

1. $s$ divides $t^2 - 1$ and $t$ divides $s^2 - 1$.
2. $(s, t) = 1$.
3. $\chi_1$ and $\chi_2$ are rational valued.
4. $n$ is odd.
Proof. (1) Since \( \chi_1 \) is irreducible, its degree \( s \) divides \( |G| = (s + t)^2 - 1 \). So \( s \) divides \( t^2 - 1 \).

(2) follows from (1).

(3) If \( s = t = 1 \), then we have a contradiction since a cyclic group of order three has no sharp characters of type \( \{-1, 1\} \). Hence \( \chi_1 \) and \( \chi_2 \) have different degrees by (2). As \( \chi = \chi_1 + \chi_2 \) is rational valued, (3) holds.

(4) Since groups of odd order have no real valued irreducible characters except the trivial one, \( G \) is of even order by (3) and so \( n \) is odd.

The next is a key result in the proof of the theorem.

PROPOSITION 2. Let assumptions and notations be as in Proposition 1. Let \( N \) be a normal subgroup of \( G \) contained in the kernel of \( \chi_1 \). Suppose that \( s = \chi_1(1) > 1 \) and \( N \) is non-trivial. Then we have the following two inequalities

\[
\frac{[G:N] + 2(s + 1)}{(s + 1)^2} \leq |N| \leq \frac{[G:N] - 2(s - 1)}{(s - 1)^2} \quad (1.2)
\]

and

\[
[G:N] \leq s(s + 1)(s + 3). \quad (1.3)
\]

Proof. Suppose that \( (\chi_2|_N, 1_N)|_N > 0 \). Since \( N \) is normal in \( G \), we have \( \chi_2|_N = t1_N \) by Clifford's theorem. As \( \chi_1|_N = s1_N \), we get \( \chi|_N = n1_N \), which contradicts the fact that \( \chi \) is faithful. Hence \( (\chi_2|_N, 1_N)|_N = 0 \) and so \( (\chi|_N, 1_N)|_N = s \) or equivalently

\[
n + \sum_{x \in N \setminus \{1\}} \chi(x) = s |N|.
\]

As \( \chi(x) = -1, 1 \) for \( x \neq 1 \), we have

\[
(s - 1) |N| + 1 \leq n \leq (s + 1) |N| - 1. \quad (1.4)
\]

It follows that

\[
((s - 1) |N| + 1)^2 - 1 \leq n^2 - 1 \leq ((s + 1) |N| - 1)^2 - 1,
\]

and so, by (1.1), we have

\[
(s - 1)^2 |N| + 2(s - 1) \leq [G:N] \leq (s + 1)^2 |N| - 2(s + 1).
\]

This immediately yields the inequality (1.2).

By Proposition 1 (1), \( t \leq s^2 - 1 \) and so \( n \leq s^2 + s - 1 \). From (1.4) we have

\[
|N| \leq s + 2. \quad (1.5)
\]

Now the second inequality (1.3) comes from (1.2) and (1.5).
The following proposition will be used to eliminate Case 1 in the proof of Lemma 2. This cannot be done by the above numerical conditions.

**Proposition 3.** Let assumptions and notations be as in Proposition 1. Let $p$ and $q$ be odd primes. Suppose that $p$ or $q$ divides $n - 1$ or $n + 1$ respectively.

1. $\chi(x) = 1$ for any $x$ of order $p$ in $G$.
2. $\chi(y) = -1$ for any $y$ of order $q$ in $G$.
3. $G$ has no element of order $pq$.

**Proof:** (1) Let $x$ be an element of $G$ of order $p$. Set $\chi(x) = \varepsilon$. Since $\chi$ is rational valued, $\chi(x^i) = \varepsilon$ for $i = 1, 2, \ldots, p - 1$. By considering the multiplicity of $1\chi\langle x \rangle$ in $\chi\langle x \rangle$, we have

$$n + (p - 1)\varepsilon \equiv 0 \pmod{p}.$$  

Then $\varepsilon \equiv n \equiv 1 \pmod{p}$, and so $\varepsilon = 1$ since $p$ is odd and $\varepsilon = -1, 1$.

(2) is proved in the same way as (1).

(3) Suppose that $G$ has an element $z$ of order $pq$. Then we have

$$\chi(z) \equiv \chi(z_q) \pmod{p},$$

where $z_q$ is the $q$-part of $z$. Note that $z_q$ is also the $p'$-part. As $z_q$ is of order $q$, (2) and the above congruence imply that $\chi(z) = -1$. By symmetry we have

$$\chi(z) = \chi(z_p) = 1.$$  

This contraction completes the proof.

2. **Proof of the Theorem**

In this section, we prove the theorem stated in the introduction. Results in Kataoka [8] needed for our proof are summarized as follows.

**Theorem A.** For a finite group $G$, the following conditions (a) and (b) are equivalent to each other:

(a) There exists a virtual character $\chi$ of $G$ satisfying

(i) $\chi(1)$ is odd,

(ii) the 2-part of $\chi(1)^2 - 1$ is equal to the 2-part of the order of $G$,
and

(iii) Values of $\chi$ at non-identity elements are roots of unity.

(b) A Sylow 2-subgroup $P$ of $G$ satisfies

(i) $P$ is dihedral of order divisible by 8, (generalized) quaternion, or semidihedral, and
(ii) $C_G(P)$ is contained in $P$.

**Theorem B.** Assume that a finite group $G$ satisfies (a) or (b) in Theorem A.

1. The number of irreducible characters of $G$ of odd degree is equal to 4. In particular, the index $[G:G']$ is at most 4.

2. $K$ denotes the intersection of kernels of irreducible representations of $G$ of odd degree. On the group structure of $G$, we have the following.

2.1. If the index $[G:G']$ is 4, then $G$ has a normal 2-complement.

2.2. If the index $[G:G']$ is at most 3, then $G/K$ is isomorphic to one of the following groups:

(a) $\text{PGL}(2, q)$, $q$ is a power of an odd prime number;
(b) $\text{PGL}^*(2, q)$, $q$ is an even power of an odd prime number;
(c) $\text{PSL}(2, q)$, $q$ is a power of an odd prime number;
(d) $\text{PSL}(3, 3)$;
(e) $\text{PSL}(3, 7)$;
(f) $\text{PSU}(3, 5)$;
(g) $A_7$;
(h) $M_{11}$.

Here $\text{PGL}^*(2, q)$ is the subgroup of $\text{PGL}(2, q)$ containing $\text{PSL}(2, q)$ with index 2 different from $\text{PGL}(2, q)$ and $C_2 \rtimes \text{PSL}(2, q)$.

Theorem A is an amalgam of (1.12) and (1.16) in [8]. Sources of Theorem B are (3.1), (3.6), (3.8), (3.9), and (3.10) in that paper. More precise results than (2.2) in Theorem B are obtained in [8].

Now we go back to the proof of the theorem. We assume that $(G, \chi)$ is sharp of type $\{-1, 1\}$ and use the notations in Section 1 namely

$\chi = \chi_1 + \chi_2$, \hspace{1em} $\chi_i$ is an irreducible character for $i = 1, 2$,

$n = \chi(1)$,

$s = \chi_1(1)$,
and

\[ t = \chi_2(1). \]

By Proposition 1 (4), we may (and will) assume that \( s \) is odd and \( t \) is even without loss of generality.

First we settle the case \( s = 1 \).

**Lemma 1.** If \( s = 1 \) holds, then \( G \) is isomorphic to \( D_8 \) or \( Q_8 \).

**Proof.** \( \chi_1 \) is a rational linear character, and so values of \( \chi_1 \) are \(-1\) or \(1\). Let \( \theta \) be \( \chi \chi_1 \) and let \( L \) denote the set \( \{ \theta(g); g \in G, g \neq 1 \} \). Then \( L = \{1\}, \{-1\}, \) or \(\{-1, 1\}\). If \( L \) consists of one element, then \( n^2 - 1 = |G| \) divides \( n - e, e \in L \). A contradiction. So \( L = \{-1, 1\} \), and \((G, \theta)\) is also sharp of type \(\{-1, 1\}\).

\((G, \theta - 1_G)\) is \(\{-2, 0\}\)-sharp and the multiplicity of \(1_G\) in \(\theta - 1_G\) is 0. Such pairs are determined in Cameron and Kiyota [4, Proposition 2.4]. The lemma follows from this result.

From now on we assume that \( s \) is larger than \( 1 \) and \( K \) denotes the normal subgroup of \( G \) defined in Theorem B (2). The character \( \chi \) trivially satisfies the conditions (i)-(iii) in Theorem A (a), and so \( G \) satisfies the assumption of Theorem B. Since the index \([G:G']\) is the number of linear characters of \( G \), \([G:G'] < 4\) follows from (1) in Theorem B. Therefore \( G/K \) is isomorphic to one of the groups listed in Theorem B (2.2).

Thus the proof of the theorem is reduced to the following three lemmas.

**Lemma 2.** Assume that \( G/K \) is isomorphic to a group listed in (a), (b) of Theorem B (2.2). Then \( G \) is isomorphic to \( S_4, \ GL(2, 3) \), the binary octahedral group or \( S_6 \).

**Proof.** \([G:K] = q(q^2 - 1)\) follows from the assumption. \([G:G'] = 2\) also follows. The set of irreducible characters of \( G \) of odd degree consists of two linear character and two irreducible characters of degree \( q \). Hence \( s = q \) holds.

From Proposition 1 (1), \( t^2 \equiv 1 \pmod{q} \) holds, and so

\[ t \equiv \varepsilon \pmod{q} \quad \text{where} \quad \varepsilon = -1 \text{ or } 1. \]

By the same lemma, we have a natural number \( u \) satisfying

\[ tu = q^2 - 1. \]

In particular \( u \equiv -\varepsilon \pmod{q} \) follows. Since \( t \) is an even natural number, \( t \) is not less than \( q + \varepsilon \). Hence we have that

\[ u = -\varepsilon \quad \text{or} \quad q - \varepsilon. \]
We divide into two cases according to the number $u$.

Case 1. $u = -\varepsilon$.

$\varepsilon = -1$, $u = 1$, $t = q^2 - 1$ and $n = q^2 + q - 1$ follow from the above discussion. The order of $K$ is $q + 2$ by (1.1).

Let $p$ be the prime number dividing $q$ and $x$ be an element of $G$ of order $p$. Since the order of $K$ divides $n - 1$ and the order of $x$ divides $n + 1$, we have

$$C_G(x) \cap K = \{1\}$$

because of Proposition 3 (3). So $C_G(x)$ is isomorphic to a subgroup of $C_{G/K}(xK)$. By the second orthogonality relations for $G$ and $G/K$, $\psi(x) = 0$ holds for any irreducible character $\psi$ of $G$ such that Ker $\psi$ does not contain $K$. Hence

$$\chi_2(x) = 0.$$

On the other hand, we have

$$\chi_1(x) = 0,$$

because $x$ is of order $p$ and the degree of $\chi_1$ is the highest power of $p$ dividing the order of $G$. Then we have

$$-1 = \chi(x) = \chi_1(x) + \chi_2(x) = 0,$$

a contradiction. Therefore Case 1 does not occur.

Case 2. $u = q - \varepsilon$.

t = q + $\varepsilon$ and $n = 2q + \varepsilon$ follow. By (1.1), the order of $K$ is equal to $4/(q - \varepsilon)$, and so is 1 or 2. Here we know that $G/K$ is not isomorphic to $PGL^*(2, q)$ because $q$ is at most 5.

If $K$ is trivial, then $(q, \varepsilon) = (3, -1)$ or $(5, 1)$ holds, and so $G$ is isomorphic to

$$PGL(2, 3) (= S_4) \quad \text{or} \quad PGL(2, 5) (= S_5).$$

If $K$ is of order 2, then $q = 3$ and $\varepsilon = 1$ hold. From condition (i) in Theorem A (b), $G$ is isomorphic to

$$GL(2, 3) \quad \text{or the binary octahedral group.}$$

Thus Case 2 is completed, and the lemma is proved.

**Lemma 3.** Assume that $G/K$ is isomorphic to a group listed in (c) of Theorem B (2.2). Then $G$ is isomorphic to $SL(2, 3)$, $SL(2, 5)$, $PSL(2, 7)$, or $A_6$. 
Proof. \([G:K]=q(q^2-1)/2\) follows. \([G':G'] = 1, 3\) holds on account of the assumption. The set of irreducible characters of \(G\) of odd degree consists of the principal character, two irreducible characters of degree \((q-\delta)/2\) and the irreducible character of degree \(q\), where \(\delta = -1, 1\). If \(\delta = 1\), then irreducible characters of degree \((q-\delta)/2\) are not real valued. So \(s = (q+1)/2\) or \(q\).

We divide into cases according to \(s = (q+1)/2\) or \(q\).

**Case 1.** \(s = (q+1)/2\).

Since \(\chi_1\) takes the value \((1+\sqrt{q})/2\), \(q\) is a square.

Suppose that \(K\) is non-trivial. By (1.3), we have an inequality

\[4q(q-1) \leq (q+3)(q+7).\]

\(q < 7\) follows. This contradicts the fact that \(q\) is an even power of an odd prime.

Suppose that \(K\) is trivial. An irreducible character of \(PSL(2, q)\) of even degree is of degree \(q-1\) or \(q+1\). So \(t = q - \varepsilon\) holds for \(\varepsilon = -1\) or 1. By Proposition 1 (1), \((q+1)/2\) divides \((q-\varepsilon)^2 - 1\) and so divides \(q - 2\varepsilon\). If \(\varepsilon = -1\), then \(q - 2\varepsilon\) is prime to \(q+1\) and so prime to \((q+1)/2\), a contradiction. If \(\varepsilon = 1\), then \(q - 2\varepsilon = (q+1)/2\) and so \(q = 5\), a contradiction.

Therefore Case 1 does not occur.

**Case 2.** \(s = q\).

By the same argument as in the beginning of Lemma 2, we obtain that \(t\) is equal to \(q^2 - 1\) or \(q + \varepsilon\), where \(\varepsilon = -1, 1\). From the inequality (1.2) we obtain that

the order of \(K\) is 1, \((q-1)/2\) or \((q+1)/2\).

The case \(t = q^2 - 1\) does not occur. In fact, if it holds, then \(n = q^2 + q - 1\) and so \(K\) is of order \(2(q+2)\) by (1.1), a contradiction. Hence we have

\[t = q + \varepsilon, \quad n = 2q + \varepsilon, \quad \text{and} \quad \#K = 8/(q-\varepsilon).\]

Therefore the order of \(K\) is 1 or 2.

If \(K\) is trivial, then \((q, \varepsilon) = (7, -1)\) or \((9, 1)\), and so \(G\) is isomorphic to

\[PSL(2, 7) \quad \text{or} \quad PSL(2, 9) (= A_6).\]

If \(K\) is of order 2, then \((q, \varepsilon) = (3, -1)\) or \((5, 1)\). From condition (i) in Theorem A (b), \(G\) is isomorphic to

\[SL(2, 3) \quad \text{or} \quad SL(2, 5).\]

Thus the proof of the lemma is completed.
LEMMA 4. Assume that $G/K$ is isomorphic to one of the five groups listed in (d)–(h) of Theorem B (2.2). Then $G$ is isomorphic to $A_7$ or $M_{11}$.

Proof. $G = G'$ follows from the assumption. In particular $G$ has exactly three irreducible characters of odd degree $> 1$. $[G:K]$ and the possible values of $s$ are given as follows.

(d) If $G/K \cong \text{PSL}(3,3)$, then $[G:K] = 5616$ and $s = 13, 27$ or $39$.
(e) If $G/K \cong \text{PSL}(3,7)$, then $[G:K] = 1,876,896$ and $s = 57, 343$, or $399$.
(f) If $G/K \cong \text{PSU}(3,5)$, then $[G:K] = 126,000$ and $s = 21, 105$, or $125$.
(g) If $G/K \cong A_7$, then $[G:K] = 2520$ and $s = 15, 21$, or $35$.
(h) If $G/K \cong M_{11}$, then $[G:K] = 7920$ and $s = 11, 45$, or $55$.

By the inequalities (1.2) and (1.3) in Proposition 2, we obtain the possible values of order $\# K$ of $K$:

(d) $\# K = 1$ if $s = 13$; $\# K = 1, 8$ if $s = 27$; and $\# K = 1$ if $s = 39$.
(e) $\# K = 1$ if $s = 57$; $\# K = 1, 16$ if $s = 343$; and $\# K = 1$ if $s = 399$.
(f) $\# K = 1$ if $s = 21$; $\# K = 1$ if $s = 105$; and $\# K = 1, 8$ if $s = 125$.
(g) $\# K = 1, 10, 11, 12$ if $s = 15$; $\# K = 1, 6$ if $s = 21$; and $\# K = 1, 2$ if $s = 35$.
(h) $\# K = 1$ if $s = 11$; $\# K = 1, 4$ if $s = 45$; and $\# K = 1$ if $s = 55$.

Among possible values of $\# K$, the equation (1.1) has an integral solution $n$ only if

(1) $G/K \cong A_7$ and $\# K = 2$ hold, or
(2) $G/K \cong M_{11}$ and $\# K = 1$ hold.

For the case (1), $G$ is isomorphic to $A_7$, because of (i) in Theorem A (b). Thus the lemma is proved.

From Lemma 2 to Lemma 4 we have exhausted all possibilities of $G/K$ and finished the proof of the theorem.

3. CONCLUDING REMARKS

In this final section, we make some comments on the theorem and the related facts.

(3.1) Let $(G, \chi)$ be a sharp pair of type $\{-1, 1\}$. Observing the theorem, the following facts are found. (Notations and assumptions are the same as in Section 1.)
(1) \( n \) is a prime number.

(2) \( \{ \chi_1(1), \chi_2(1) \} = \{(n-1)/2, (n+1)/2\} \).

(3) Both \( \chi_1 \) and \( \chi_2 \) belong to the principal 2-block of \( G \).

(4) \( O(G) = \{1\} \).

Note that (3) implies (4) by a well-known fact on modular representation theory.

(5) For another sharp character \( \chi' \) of \( G \) type \( \{-1, 1\} \), there exists a linear character \( \theta \) of order at most 2 with \( \chi' = \theta \chi \). In particular the number of sharp characters of \( G \) of type \( \{-1, 1\} \) is equal to the number of linear characters of \( G \) of order at most 2.

(3.2) \( \{-1, 1\} \)-sharp pairs \((G, \chi)\) are divided into the following three classes according to the rationalities of \( \chi' \):

(a) \( \chi + ml_G \) is a permutation character for some non-negative integer \( m \).

(b) \( \chi \) is a difference of two permutation characters, but an integer \( m \) with the condition (a) does not exist.

(c) \( \chi \) is not a \( Q \)-character.

Finite groups having \( \{-1, 1\} \)-sharp characters in the above classes are given as follows.

(a) \( D_8(2), S_4(2), GL(2, 3), PSL(2, 7) \).

(b) \( D_8(2), GL(2, 3), S_4(2), A_6, M_{11} \).

(c) \( Q_8(4), SL(2, 3), \) the binary octahedral group \((2), SL(2, 5), \hat{A}_7 \).

When a group has more than one sharp character in a class, the number of characters is given in the parentheses. See Section 3 of Cameron and Kiyota [4], Ito and Kiyota [6], and Section 2 of Kataoka [8].

(3.3) It is easily verified that, for a virtual character \( \chi \) of a finite group \( G \) such that \((G, \chi)\) is sharp of type \( \{-1, 1\} \), either \( \chi \) or \( -\chi \) is a character of \( G \). Therefore the theorem is still valid for virtual characters.

(3.4) The proof in [8] of Theorem B (2.2) depends on results of Alperin, Brauer, and Gorenstein [1] and Gorenstein and Walter [5]. So our proof of the theorem also depends on their results.

(3.5) For an application of the theorem, see Theorem 6.1 of Cameron and Kiyota [4].

(3.6) Among finite sets \( L \) of rational integers such that all the \( L \)-sharp pairs are determined, \( L = \{-1, 1\} \) is the first case that \((\chi, t)_G > 1\) holds for a \( L \)-sharp character \( \chi \).
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