On the existence and shape of least energy solutions for some elliptic systems

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Abstract

We establish an existence result for strongly indefinite semilinear elliptic systems with Neumann boundary condition, and we study the limiting behavior of the positive solutions of the singularly perturbed problem.

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1. Introduction

In the 70s, Keller and Segel [12] studied the chemotaxis of an amoeoba using a system of parabolic differential equations. Later, Gierer and Meinhardt [8] studied the activation–inhibition of two chemical components as a model of pattern formation again deducing a system of differential equations. Experimental evidence showed that the solutions must concentrate about a point, that is, a spike-layered behavior. Under some assumptions, the stationary solutions of these two biochemical systems are solutions of a nonlinear elliptic equation.

The first approach to solve these elliptic systems was given by Ni and Takagi. In a series of papers [13–17] they studied the shape of the positive solution of an equation

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of a prototype

\[-\varepsilon^2\Delta u + u = |u|^{p-1} u, \quad \text{in } \Omega, \tag{1}\]

\[\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{2}\]

where \(\Omega\) is a bounded domain in \(\mathbb{R}^N\) with smooth boundary \(\partial \Omega\), and \(\varepsilon\) is a parameter. Using the ground state solution \(w\) of the equation

\[-\Delta w + w = w^p \quad \text{in } \mathbb{R}^N, \tag{3}\]

and its exponential decay they obtained, for \(\varepsilon\) small, an upper estimate of the critical value given by the Mountain Pass Lemma. Moreover, they proved the existence of a nontrivial solution \(u_{\varepsilon}\) of (1) and (2). Using the uniqueness of the ground state, they proved that \(u_{\varepsilon}\) concentrates about a point \(P \in \partial \Omega\) where \(P\) maximizes the mean curvature of the boundary.

Our main goal is to study the same phenomena for the nonlinear elliptic system

\[-\varepsilon^2\Delta u + u = |u|^{q-1} v, \quad -\varepsilon^2\Delta v + v = |u|^{p-1} u \quad \text{in } \Omega, \tag{4}\]

\[\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \quad \text{on } \partial \Omega, \tag{5}\]

where the exponents \(p\) and \(q\) are below the critical parabola, that is,

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N}, \quad p, q > 1. \tag{6}\]

One of the main characteristic of this system is that the functional associated to (4) and (5) is strongly indefinite. Using linking theorems De Figueiredo and Felmer [4] and Hulshof and Van der Vorst [11] proved the existence of nontrivial solutions of Eqs. (4) for Dirichlet boundary conditions. However, the Neumann problem (4)–(5) allows constant solutions. A direct application of linking theorems cannot avoid this kind of solutions. To obtain nontrivial solutions, one estimates the critical value from above by \(\varepsilon^N\), and push it below the energy of constant solutions by taking \(\varepsilon > 0\) small. But the estimate of the critical value given by the linking approach is difficult to handle. We use a different framework, the dual variational formulation of the problem, which allow us to find a critical point of the related functional by the Mountain Pass Lemma. Indeed, we use the ground state solution of the problem

\[-\Delta u + u = v^q, \quad -\Delta v + v = u^p \quad \text{in } \mathbb{R}^N \tag{7}\]

to construct a test function. By a ground state solution of (7) we mean the least energy solution of (7). It was proved in [5,18] the existence of a ground state solution of (7) and the exponential decay at infinity of the solution and their derivatives. These facts
enable us to obtain an upper bound for the critical value described by the Mountain Pass Lemma. Then, we are able to show that (4) and (5) possesses at least one nontrivial solution. Furthermore, using a characterization of the critical value given by the Mountain Pass Lemma we show that the corresponding solutions are actually positive. Thus, we get our first result.

**Theorem 1.1.** There exists $\epsilon_0 > 0$ such that system (4)–(5) possesses at least a nontrivial positive solution $z_\epsilon = (u_\epsilon, v_\epsilon)$ provided that $0 < \epsilon < \epsilon_0$.

Next, we investigate the limit behavior of the positive solutions as $\epsilon \to 0$. Using a blow-up technique, we obtain a uniform $L^\infty$-bound in $\epsilon$. Then, we study the behavior of the maxima of $(u_\epsilon, v_\epsilon)$ obtained in Theorem 1.1. We obtain the following result.

**Theorem 1.2.** Let $P_\epsilon$ and $Q_\epsilon$ be a maximum point of $u_\epsilon$ and $v_\epsilon$, respectively and suppose that $\frac{N+2}{N-2} > p, q$. Then

(i) $P_\epsilon$ and $Q_\epsilon$ are on the boundary $\partial \Omega$ for $\epsilon$ small,

(ii) $P_\epsilon$ and $Q_\epsilon$ approach to a point $P$ on the boundary when $\epsilon \to 0$.

The restriction $p, q < \frac{N+2}{N-2} \$ is due to an application of Liouville Theorem for systems. The best result we know requires such a condition.

We also consider the asymptotic behavior of the critical value $c_\epsilon$ of the associated dual functional which is defined in (11). Let $(u, v)$ be a ground state solution of (7) and

$$c^\infty := \left( \frac{p}{p+1} - \frac{1}{2} \right) \int_{\mathbb{R}^N} u^{p+1}(x) \, dx + \left( \frac{q}{q+1} - \frac{1}{2} \right) \int_{\mathbb{R}^N} v^{q+1}(x) \, dx. \quad (8)$$

Denote by $z$ the maximum of the mean curvature of the boundary $\partial \Omega$.

Our result is as follows.

**Theorem 1.3.**

$$c_\epsilon = \epsilon^N \left( \frac{c^\infty}{2} - \epsilon \gamma + O(\epsilon^2) \right), \quad (9)$$

where $\gamma$ is a positive constant.

The conclusions of Theorem 1.3 will be completed by finding a lower estimate of the critical value. It is usually studied by using the uniqueness of the ground state of (3). However, up to our knowledge, there is no uniqueness result for (7) in the literature. To solve this problem and inspired by the work [7], we use a ground state solution of (7) to construct a test function which will give us the right lower bound.
Finally, we study the case \( \varepsilon > 0 \) large. We obtain

**Theorem 1.4.** There exists \( \varepsilon' > 0 \) such that if \( \varepsilon > \varepsilon' \), system (4)–(5) has only constant positive solutions. More precisely, if \( \varepsilon > \varepsilon' \), \( u = v = 1 \) is the only positive solution of system (4)–(5).

The work is divided as follows: in Section 2 we state the dual formulation of the system and prove the existence of a solution for (4)–(5). Then, in Section 3, we show that this solution is positive. In Section 4 we find an upper bound for the critical value associated to the solution. Also, it gives that the solution is nontrivial for \( \varepsilon > 0 \) small. This completes the proof of Theorem 1.1. In Section 5 we prove Theorem 1.2 and obtain the lower bound to conclude Theorem 1.3. Finally, we prove Theorem 1.4 in Section 6.

2. Dual functional and existence

The energy functional associated to (4) and (5) in \( H^1(\Omega) \times H^1(\Omega) \) is given by

\[
I_\varepsilon(u, v) = \int_\Omega (\varepsilon^2 \nabla u \cdot \nabla v + uv) \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx - \frac{1}{q+1} \int_\Omega |v|^{q+1} \, dx. \tag{10}
\]

Because of the indefinite sign nature of the quadratic part, one uses a linking theorem to find a critical point. See [4,5,10,11]. But the critical value described by the linking theorem is not easy to handle as the one described by the Mountain Pass Lemma. Thus, we need an alternative functional given by the dual variational principle to apply the Mountain Pass Lemma.

To define the dual functional, we consider the quadratic part and the nonlinear part separately. For the quadratic part we need to recall some analysis. We know that the inclusion

\[ i_\varepsilon : W^{1,r}(\Omega) \to L^{p+1}(\Omega) \]

is compact if \( 2 < p + 1 < \frac{N}{N-r} \) and \( N > r \). Also, for \( \varepsilon > 0 \) the operator \( A_\varepsilon := -\varepsilon^2 \Delta + id : \{ W^{1,r}(\Omega), \frac{\partial u}{\partial n} = 0 \} \to W^{-1,r}(\Omega) \) is an isomorphism. Hence, the operator

\[ T_\varepsilon := i_2 \circ (-\varepsilon^2 \Delta + id)^{-1} \circ i_2^*: L^{1+\frac{1}{q}}(\Omega) \to L^{p+1}(\Omega) \]

is linear, self-adjoint, and continuous. Thus, \( T_\varepsilon w = u \) if and only if \( A_\varepsilon u = w \).

For the nonlinear part, we denote by \( X = L^{p+1} \times L^{q+1}(\Omega) \) and by \( X^* \) its dual \( L^{1+\frac{1}{q}}(\Omega) \). We note that the Legendre–Fenchel transformation \( F^* \) of \( F(u) = \int_\Omega (u^2 + uv) \, dx \) is given by

\[ F^*(\lambda) = \inf_u \left( \frac{1}{2} \int_\Omega |u|^2 \, dx + \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx + \frac{1}{q+1} \int_\Omega |v|^{q+1} \, dx - \lambda \int_\Omega uv \, dx \right) \]

for \( \lambda \in X \).
\[ \frac{1}{p+1} |u|^{p+1} \] is

\[ F^*(s) = \sup_{t \in \mathbb{R}} \left\{ st - \frac{1}{p+1} |t|^{p+1} \right\} = \frac{p}{p+1} |s|^{\frac{1}{p}}. \]

With these two relations, we define on \( X^* \) the dual functional

\[ J_\varepsilon(w) = \int_\Omega \frac{p}{p+1} |w_1|^{\frac{1}{p+1}} dx + \int_\Omega \frac{q}{q+1} |w_2|^{\frac{1}{q+1}} dx - \frac{1}{2} \int_\Omega (w_1 T_\varepsilon w_2 + w_2 T_\varepsilon w_1) \, dx, \quad (11) \]

where \( w = (w_1, w_2) \in X^* \). The critical points of \( J_\varepsilon \) satisfy

\[ \int_\Omega (|w_1|^{\frac{1}{p}-1} w_1 \phi + |w_2|^{\frac{1}{q}-1} w_2 \varphi) \, dx = \frac{1}{2} \int_\Omega (\phi T_\varepsilon w_2 + w_2 T_\varepsilon \phi + \varphi T_\varepsilon w_1 + w_1 T_\varepsilon \varphi) \, dx \]

for \( (\phi, \varphi) \in X^* \). Because \( T_\varepsilon \) is self-adjoint, we have the following relation between the two components of \( w \):

\[ T_\varepsilon w_2 = |w_1|^{\frac{1}{p}-1} w_1, \quad T_\varepsilon w_1 = |w_2|^{\frac{1}{q}-1} w_2. \quad (12) \]

Let \( u = T_\varepsilon w_2, \ v = T_\varepsilon w_1 \). Then (12) is transformed to the original system

\[ -\varepsilon^2 \Delta u + u = |v|^{q-1} v, \quad -\varepsilon^2 \Delta v + v = |u|^{p-1} u, \]

with the boundary condition (5). Thus, \( (u, v) \) is a solution of system (4)–(5). On the other hand, the dual functional evaluated at the critical points but with respect to the \( u \) and \( v \) variables gives

\[ J_\varepsilon(w) = \frac{p}{p+1} \int_\Omega |u|^{p+1} dx + \frac{q}{q+1} \int_\Omega |v|^{q+1} dx - \frac{1}{2} \int_\Omega (|u|^{p+1} + |v|^{q+1}) \, dx \]

\[ = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega |u|^{p+1} dx + \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega |v|^{q+1} dx. \quad (13) \]

Replacing \( (u, v) \) in (10) we have that

\[ I_\varepsilon(u, v) = \left( \frac{1}{2} - \frac{1}{p+1} \right) \int_\Omega |u|^{p+1} dx + \left( \frac{1}{2} - \frac{1}{q+1} \right) \int_\Omega |v|^{q+1} dx. \]

Therefore, at a critical point both functionals coincide, that is,

\[ I_\varepsilon(u, v) = J_\varepsilon(w_1, w_2). \]

Hence, solutions of (4) and (5) can be obtained as critical points of \( J_\varepsilon \).
In the same way, we define the dual functional for functional (9) defined in the whole \( \mathbb{R}^N \) by

\[
J(w) = \int_{\mathbb{R}^N} \frac{p}{p+1} |w_1|^{1+\frac{1}{p}} dx + \int_{\mathbb{R}^N} \frac{q}{q+1} |w_2|^{1+\frac{1}{q}} dx - \frac{1}{2} \int_{\mathbb{R}^N} (w_1 T w_2 + w_2 T w_1) dx
\]

with \( w = (w_1, w_2) \).

As we mentioned, critical points of \( J_\varepsilon \) will be found by the Mountain Pass Lemma. Let us prove that \( J_\varepsilon \) satisfies the conditions of this lemma.

**Lemma 2.1.** There exist \( \rho > 0, \alpha > 0 \) such that

\[
J_\varepsilon(w) \geq \alpha, \quad \text{if } ||w||_{X^\varepsilon} = \rho.
\]

**Proof.** Let \( c_\delta = \min \{ \frac{p}{p+1} \delta^{\frac{1}{p}-1}, \frac{q}{q+1} \delta^{\frac{1}{q}-1} \} \). For \( 0 < \delta < 1 \) we have

\[
\frac{p}{p+1} |s|^{1+\frac{1}{p}} \geq c_\delta |s|^2 \quad \text{if } |s| \leq \delta,
\]

\[
= \frac{p}{p+1} |s|^{1+\frac{1}{p}} \quad \text{if } |s| \geq \delta.
\]

Hence, we have

\[
J_\varepsilon(w) \geq c_\delta \int_{|w_1| \leq \delta} |w_1|^2 dx + c_\delta \int_{|w_2| \leq \delta} |w_2|^2 dx
\]

\[
+ \frac{p}{p+1} \int_{|w_1| \geq \delta} |w_1|^{1+\frac{1}{p}} dx + \frac{q}{q+1} \int_{|w_2| \geq \delta} |w_2|^{1+\frac{1}{q}} dx
\]

\[
- \frac{1}{2} \left( |w_1|^{1+\frac{1}{p}} \| T_\varepsilon w_2 \|_{L^{p+1}} + |w_2|^{1+\frac{1}{q}} \| T_\varepsilon w_1 \|_{L^{q+1}} \right).
\]

Since \( T_\varepsilon \) is continuous, there is a \( \beta > 0 \) such that

\[
\| T_\varepsilon w_1 \|_{L^{p+1}} \leq \beta \| w_2 \|_{L^{p+1}}, \quad \| T_\varepsilon w_2 \|_{L^{q+1}} \leq \beta \| w_1 \|_{L^{q+1}}.
\]

By Hölder’s inequality, for any \( \omega \subset \Omega \)

\[
\int_\omega w_1^2 dx \geq |\omega|^{\frac{p-1}{p+1}} \| w_1 \|_{L^{p+1}}^2.
\]

Let \( \delta \) be small such that \( c_\delta \) satisfies

\[
|\omega|^{-\frac{p-1}{p+1}} c_\delta - \frac{E}{2} \geq c > 0,
\]
with \(c\) a positive constant. We have

\[
c\delta \int_{\{|w_1| \leq \delta\}} |w_1|^2 \, dx - \frac{\beta}{2} \left( \int_{\{|w_1| \leq \delta\}} |w_1|^{1+\frac{1}{p}} \right)^{\frac{2p}{p+1}} \geq c \left( \int_{\{|w_1| \leq \delta\}} |w_1|^{1+\frac{1}{p}} \right)^{\frac{2p}{p+1}} \, dx
\]

and for \(||w||_{X^*}\) small

\[
\frac{p}{p+1} \int_{\{|w_1| \geq \delta\}} |w_1|^{1+\frac{1}{p}} \, dx - \frac{\beta}{2} \left( \int_{\{|w_1| \geq \delta\}} |w_1|^{1+\frac{1}{p}} \right)^{\frac{2p}{p+1}} \geq c \left( \int_{\{|w_1| \geq \delta\}} |w_1|^{1+\frac{1}{p}} \right)^{\frac{2p}{p+1}} \, dx
\]

for some constant \(c > 0\). Thus,

\[
c \left( \int_{\{|w_1| \leq \delta\}} |w_1|^{1+\frac{1}{p}} \, dx \right)^{\frac{2p}{p+1}} + c \left( \int_{\{|w_1| \geq \delta\}} |w_1|^{1+\frac{1}{p}} \right)^{\frac{2p}{p+1}} \geq c ||w||_{L^{\frac{1}{p}}}^2.
\]

Similarly,

\[
c\delta \int_{\{|w_2| \leq \delta\}} |w_2|^2 \, dx - \frac{\beta}{2} \left( \int_{\{|w_2| \leq \delta\}} |w_2|^{1+\frac{1}{q}} \right)^{\frac{2q}{q+1}} \geq c \left( \int_{\{|w_2| \leq \delta\}} |w_2|^{1+\frac{1}{q}} \right)^{\frac{2q}{q+1}} \, dx
\]

and

\[
c \left( \int_{\{|w_2| \leq \delta\}} |w_2|^{1+\frac{1}{q}} \, dx \right)^{\frac{2q}{q+1}} + c \left( \int_{\{|w_2| \geq \delta\}} |w_2|^{1+\frac{1}{q}} \right)^{\frac{2q}{q+1}} \geq c ||w||_{L^{\frac{1}{q}}}^2.
\]

Consequently,

\[
J_e(w) \geq c_0 ||w||_{X^*}^2 = c_0 \rho^2.
\]

Lemma 2.2. There exist \(t > 0\) and \(\tilde{w} \in X^*\) such that \(J_e(t\tilde{w}) \leq 0\).

Proof. Let \(\tilde{w}\) be fixed and such that \(\int_{\Omega} \tilde{w}_1 T_e \tilde{w}_2 \, dx > 0\). Since

\[
J_e(t\tilde{w}) = \frac{p}{p+1} t^{1+\frac{1}{p}} \int_{\Omega} |\tilde{w}_1|^{1+\frac{1}{p}} \, dx + \frac{q}{q+1} t^{1+\frac{1}{q}} \int_{\Omega} |\tilde{w}_2|^{1+\frac{1}{q}} \, dx
- \frac{t^2}{2} \int_{\Omega} (\tilde{w}_1 T_e \tilde{w}_2 + \tilde{w}_2 T_e \tilde{w}_1) \, dx
\]

and \(1 + \frac{1}{p}, 1 + \frac{1}{q} < 2\), the conclusion follows for \(t > 0\) large enough.
Lemma 2.3. $J_\epsilon$ satisfies the (PS) condition.

Proof. Let $\{w_n\} \subset X^*$ be a (PS)-sequence of $J_\epsilon$, that is,
$$|J_\epsilon(w_n)| \leq C, \quad J'_\epsilon(w_n) \to 0.$$

Thus for $w_n = (w_n^1, w_n^2)$, we have,
$$\langle J'_\epsilon(w_n), w_n \rangle = \int_{\Omega} |w_n^1|^{1+\frac{1}{p}} \, dx + \int_{\Omega} |w_n^2|^{1+\frac{1}{q}} \, dx - \int_{\Omega} (w_n^1 T_\epsilon w_n^2 + w_n^2 T_\epsilon w_n^1) \, dx$$
$$= o(1) ||w_n||_{X^*}.$$

Because $\{J_\epsilon(w_n)\}$ is a bounded sequence, we obtain
$$\frac{p}{p+1} \int_{\Omega} |w_n^1|^{1+\frac{1}{p}} \, dx + \frac{q}{q+1} \int_{\Omega} |w_n^2|^{1+\frac{1}{q}} \, dx$$
$$\leq \frac{1}{2} \int_{\Omega} (w_n^1 T_\epsilon w_n^2 + w_n^2 T_\epsilon w_n^1) \, dx + C$$
$$= \frac{1}{2} \int_{\Omega} |w_n^1|^{1+\frac{1}{p}} \, dx + \frac{1}{2} \int_{\Omega} |w_n^2|^{1+\frac{1}{q}} \, dx + C + o(1) ||w_n||_{X^*}.$$

This gives
$$\frac{p-1}{2p+2} \int_{\Omega} |w_n^1|^{1+\frac{1}{p}} \, dx + \frac{q-1}{2q+2} \int_{\Omega} |w_n^2|^{1+\frac{1}{q}} \, dx \leq C + o(1) ||w_n||_{X^*}.$$

Therefore, $||w_n||_{X^*}$ is bounded. Let $z_n = (T_\epsilon w_n^2, T_\epsilon w_n^1)$. Since $T_\epsilon$ is bounded, it follows that
$$||z_n||_X \leq C,$$

and for $E = H^1(\Omega) \times H^1(\Omega)$ we obtain
$$||z_n||_E \leq C ||w_n||_E \leq C ||w_n||_{X^*} \leq C.$$

Solving the equations $A_\epsilon z_n^1 = w_n^2, A_\epsilon z_n^2 = w_n^1$ and using elliptic regularity theory [14] we obtain
$$z_n \in W^{2,1+\frac{1}{q}}(\Omega) \times W^{2,1+\frac{1}{p}}(\Omega).$$

Thus $z_n = (u_n, v_n)$ satisfies
$$||u_n||_{W^{2,1+\frac{1}{q}}(\Omega)} \leq C, \quad ||v_n||_{W^{2,1+\frac{1}{p}}(\Omega)} \leq C.$$
Hence we have a subsequence \( \{z_{k_n}\} \subset \{z_n\} \) such that \( z_{k_n} \to z \) in \( E \) and \( X \). Furthermore, \( z_{k_n} \to z \) in \( L^2(\Omega) \times L^\beta(\Omega) \) for all \( 2 < \alpha, \beta < \frac{2N}{N-2} \). Since \( \{w_n\} \) is bounded in \( X^* \), we have

\[
-\varepsilon^2 \Delta u_n + u_n - |v_n|^{q-1}v_n = \varepsilon_{1,n} \quad \text{in } L^{1+\frac{1}{q}},
\]

\[
-\varepsilon^2 \Delta v_n + v_n - |u_n|^{p-1}u_n = \varepsilon_{2,n} \quad \text{in } L^{1+\frac{1}{p}},
\]

where \( \varepsilon_{1,n} \) and \( \varepsilon_{2,n} \) go to 0 in the corresponding spaces. Therefore, for \( T_\varepsilon w = z = (u, v) \)

\[
||w_n - w||_{X^*} \leq \| |u_n|^{p-1}u_n - |u|^{p-1}u\|_{L^{1+\frac{1}{p}}} + \| |v_n|^{q-1}v_n - |v|^{q-1}v\|_{L^{1+\frac{1}{q}}} + \|\varepsilon_n\|_{X^*}.
\]

The right-hand side goes to zero as \( n \to 0 \) and the proof is completed. \( \square \)

Define

\[
c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{w \in \gamma} J_\varepsilon(w), \tag{15}
\]

where

\[
\Gamma = \{g \in C([0, 1], X^*): g(0) = 0, g(1) = \bar{w}, J_\varepsilon(\bar{w}) < 0\},
\]

for some fixed \( \bar{w} \in X^* \).

**Proposition 2.1.** The functional \( J_\varepsilon \) has a critical point \( w_\varepsilon \) such that \( J_\varepsilon(w_\varepsilon) = c_\varepsilon \).

**Proof.** Using Lemmas 2.1–2.3, and the Mountain Pass Lemma, we obtain a critical point \( w_\varepsilon \). \( \square \)

Obviously, system (4)–(5) has constant solutions \((0, 0)\), \((1, 1)\), and \((-1, -1)\). Although Proposition 2.1 implies that system (4)–(5) has a solution \( z_\varepsilon = T_\varepsilon w_\varepsilon \), with \( z_\varepsilon = (u_\varepsilon, v_\varepsilon) \), it does not confirm that the solution is nontrivial, that is, a nonconstant solution. To obtain a nontrivial solution of (4)–(5), in Section 4 we will find an upper estimate of \( c_\varepsilon \) such that \( c_\varepsilon \) is strictly less than the energy of the constant solutions for \( \varepsilon \) small. We first show in next section that the solution is positive.

### 3. Positive solutions

To obtain positive solutions we need to prove the following equivalent characterization of the critical value for systems. Let us define

\[
c_\varepsilon^* = \inf_{w \in X^*} \sup_{t \geq 0} J_\varepsilon(tw).
\]

**Lemma 3.1.** \( c_\varepsilon = c_\varepsilon^* \).
Proof. Clearly $c_e \leq c^*_e$. Let us prove that $c_e \geq c^*_e$. For $w \in X^*$ fixed,
\[
\frac{dJ_e(tw)}{dt} = \frac{1}{\nu^p} \int_{\Omega} |w_1|^p \, dx + \frac{1}{\eta^q} \int_{\Omega} |w_2|^q \, dx - t \int_{\Omega} (w_2 T_\epsilon w_1 + w_1 T_\epsilon w_2) \, dx.
\]
The nontrivial critical points of $J_e(tw)$ satisfy
\[
\frac{1}{\nu^p-1} \int_{\Omega} |w_1|^p \, dx + \frac{1}{\eta^q-1} \int_{\Omega} |w_2|^q \, dx = \int_{\Omega} (w_2 T_\epsilon w_1 + w_1 T_\epsilon w_2) \, dx.
\]
We can define $\tilde{t}$ as the unique solution of this equality for $w$ fixed. Then, we can write
\[
c^*_e = \inf_{w \in M} J_e(w)
\]
with
\[
M = \{ \tilde{w} = \tilde{t}w : w \in X^*, w \neq 0, \tilde{t} < \infty \}.
\]
Let $\gamma \in \Gamma$ be a path. If for all $\gamma \in \Gamma$, $\gamma \cap M \neq \emptyset$ then the inequality is proved. If there exists $\gamma = (\gamma_1, \gamma_2)$ such that $\gamma(t) \notin M$ for all $t \in [0, 1]$, then we have
\[
\int_{\Omega} |\gamma_1|^p \, dx + \int_{\Omega} |\gamma_2|^q \, dx > \int_{\Omega} (\gamma_1 T_\epsilon \gamma_2 + \gamma_2 T_\epsilon \gamma_1) \, dx
\]
and
\[
J_e(\gamma) = \frac{p}{p+1} \int_{\Omega} |\gamma_1|^p \, dx + \frac{q}{q+1} \int_{\Omega} |\gamma_2|^q \, dx - \frac{1}{2} \int_{\Omega} (\gamma_1 T_\epsilon \gamma_2 + \gamma_2 T_\epsilon \gamma_1) \, dx
\]
\[
> \left( \frac{p}{p+1} - \frac{1}{2} \right) \int_{\Omega} |\gamma_1|^p \, dx + \left( \frac{q}{q+1} - \frac{1}{2} \right) \int_{\Omega} |\gamma_2|^q \, dx \geq 0.
\]
This is a contradiction with the Mountain Pass characterization of $c_e$. Consequently,
\[
c_e = c^*_e.
\]

Let $(u_e, v_e)$ be the solution obtained by Proposition 2.1. We are ready to prove

**Proposition 3.1.** $u_e$ and $v_e$ are positive functions.

**Proof.** Let $w_e$ be a critical point of $J_e$ obtained by Proposition 2.1. We know that $w_e$ satisfies
\[
|w_e|^p - w_e^1 = T_\epsilon w_e^2 =: u_e,
\]
(16)
\[ w_e^2 \frac{1}{q-1} w_e^2 = T_e w_e^1 := v_e, \]

which is equivalent to (4) and (5).

By (16) and (17) we have

The critical points of \( h_+ (t, w) \) in \( t \), which we called \( t_e \), satisfy

\[
0 = \frac{\partial h_+ (t, w_e)}{\partial t} = \frac{1}{p+1} t \int_\Omega (w_e)_+^{p+1} dx + \frac{1}{q+1} \int_\Omega (w_e)_+^{q+1} dx
- t \int_\Omega ((w_e)_+ T_e w_e^1 + (w_e)_+ T_e w_e^2) dx.
\]

By (16) and (17) we have

\[
\frac{1}{r_e^p - t_e} \int_\Omega (w_e)_+^{p+1} dx + \frac{1}{r_e^q - t_e} \int_\Omega (w_e)_+^{q+1} dx = 0,
\]

which implies \( t_e = 0 \) or 1. Therefore,

\[
h_+ (1, w_e) = \max_{t \geq 0} h_+ (t, w_e).
\]

In the same way, we have

\[
h_- (1, w_e) = \max_{t \geq 0} h_- (t, w_e).
\]

Decomposing \( J_e (w_e) \) we obtain

\[
c_e = J_e (w_e) = h_+ (1, w_e) + h_- (1, w_e).
\]
Let
\[ f(t) := J_\varepsilon(t w^+_\varepsilon) = \frac{p}{p+1} t^{\frac{p+1}{p}} \int_{\Omega} (w^+_\varepsilon)^{\frac{p+1}{p}} \, dx + \frac{q}{q+1} t^{\frac{q+1}{q}} \int_{\Omega} (w^+_\varepsilon)^{\frac{q+1}{q}} \, dx - \frac{1}{2} t^2 \int_{\Omega} ((w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ \, dx. \tag{22}\]

The critical points of \( f(t) \) satisfy
\[
0 = f'(t) = \frac{1}{t^p - 1} \int_{\Omega} ((w^+_\varepsilon)_+^p \, dx + \frac{1}{t^q - 1} \int_{\Omega} (w^+_\varepsilon)_+^q \, dx - \int_{\Omega} ((w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ \, dx. \tag{23}\]

Again by (16) and (17) we obtain
\[
(w^+_\varepsilon)_+^{\frac{1+p}{p}} = (w^+_\varepsilon)_+ T_\varepsilon w^+_\varepsilon = (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ + (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_-,
\]
\[
(w^+_\varepsilon)_+^{\frac{1+q}{q}} = (w^+_\varepsilon)_+ T_\varepsilon w^+_\varepsilon = (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ + (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_-.
\]

Substituting them in (23) we obtain
\[
0 = (t^p - 1) \int_{\Omega} ((w^+_\varepsilon)_+^p \, dx + (t^q - 1) \int_{\Omega} (w^+_\varepsilon)_+^q \, dx - \int_{\Omega} ((w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+) \, dx. \tag{24}\]

By the maximum principle, one infers that
\[
\int_{\Omega} ((w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+) \, dx \leq 0. \tag{25}\]

It follows from (24) and (25) that the critical point \( t^+_\varepsilon \) of \( f(t) \) is uniformly bounded in \( \varepsilon \). Similarly, we may deduce that \( t^-_\varepsilon \) satisfying \( \frac{dJ_\varepsilon(w^+_\varepsilon)}{dt} |_{t=t^-_\varepsilon} = 0 \) is uniformly bounded in \( \varepsilon \). From (21), (25) and Lemma 3.1 we obtain
\[
c_\varepsilon = J_\varepsilon(w^+_\varepsilon) = h_+ (1, w^+_\varepsilon) + h_- (1, w^+_\varepsilon) \geq h_+ (t^+_\varepsilon, w^+_\varepsilon) + h_- (t^-_\varepsilon, w^+_\varepsilon)
\geq J(t^+_\varepsilon (w^+_\varepsilon)_+) + J(t^-_\varepsilon (w^+_\varepsilon)_-) - C \int_{\Omega} ((w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+ (w^+_\varepsilon)_+ T_\varepsilon(w^+_\varepsilon)_+)
\geq J(t^+_\varepsilon (w^+_\varepsilon)_+) + J(t^-_\varepsilon (w^+_\varepsilon)_-) \geq 2c_\varepsilon,
\]
a contradiction. \( \square \)
4. Upper bound for the critical value

Let $P \in \partial \Omega$ be a point at the boundary. We fix a coordinate system about $P$ such that $x = \Phi(y) \in C^1$ where $\Phi$ is related to the local parametrization of the boundary about $P$ and it is defined in [16] (2.8). Its domain is a ball $B_{3\kappa}$ for $\kappa > 0$ fixed, $P = \Phi(0)$, $\det D\Phi = I$, where $I$ denotes the $N$-dimensional identity matrix. Also, denote $\Phi^{-1}$ by $\Psi$.

Let us consider for $t > 0$ and $s > 0$ the value

$$J_\varepsilon(tw_1, sw_2) = \frac{p}{p+1} t^{\frac{1}{p}} \int_{\Omega} |w_1|^{\frac{1}{p} + \frac{1}{q}} dx + \frac{q}{q+1} s^{\frac{1}{q} + \frac{1}{q}} \int_{\Omega} |w_2|^{\frac{1}{q} + \frac{1}{q}} dx$$

$$- \frac{ts}{2} \int_{\Omega} (w_1 T \varepsilon w_2 + w_2 T \varepsilon w_1) dx. \quad (26)$$

Let $\zeta_\rho(t) = 1$ if $0 \leq t \leq \rho$, $\zeta_\rho(t) = 2 - \frac{t}{\rho}$ if $\rho \leq t \leq 2\rho$, and $\zeta_\rho(t) = 0$ if $t > 2\rho$, be a cut-off function, and $(u, v)$ be a positive ground state of

$$-\Delta u + u = \nu^\rho, \quad -\Delta v + v = \omega^\rho \quad \text{in} \quad \mathbb{R}^N. \quad (27)$$

We define $(u_\varepsilon(z), v_\varepsilon(z)) = \zeta_\varepsilon(|z|)(u(z), v(z))$. Denote by $D_j = \Phi(B_{\varepsilon j})$ for $j = 1, 2$.

Then $D_1 \subset D_2 \subset \Omega$. Now denote by

$$(\phi_\varepsilon(x), \psi_\varepsilon(x)) = \begin{cases} (u_\varepsilon(\frac{\Psi(x)}{\varepsilon}), v_\varepsilon(\frac{\Psi(x)}{\varepsilon})) & \text{if} \ x \in D_2, \\
0 & \text{otherwise.} \end{cases} \quad (28)$$

Let us denote $(w_1, w_2) = (\phi_\varepsilon^p, \psi_\varepsilon^q)$ in the whole section, we have from (26) that

$$J(tw_1, sw_2) = \frac{pt^{\frac{1}{p}}}{p+1} \int_{\Omega} \phi_\varepsilon^{p+1} dx + \frac{qs^{\frac{1}{q}}}{q+1} \int_{\Omega} \psi_\varepsilon^{q+1} dx - \frac{st}{2} \int_{\Omega} (\phi_\varepsilon^p T \varepsilon \phi_\varepsilon^p + \psi_\varepsilon^q T \varepsilon \phi_\varepsilon^q) dx. \quad (29)$$

We begin with the following proposition.

**Lemma 4.1.** We have the estimate

$$J(tw_1, sw_2) = \frac{pt^{\frac{1}{p}}}{p+1} \int_{\Omega} \phi_\varepsilon^{p+1} dx + \frac{qs^{\frac{1}{q}}}{q+1} \int_{\Omega} \psi_\varepsilon^{q+1} dx$$

$$- \frac{st}{2} \int_{\Omega} (\phi_\varepsilon^{p+1} + \psi_\varepsilon^{q+1}) dx + O(\varepsilon^{\frac{\mu k}{r}})$$

as $\varepsilon \to 0$.

**Proof.** Consider the system

$$T_\varepsilon \phi_\varepsilon^p = \psi_\varepsilon + \xi_\varepsilon, \quad T_\varepsilon \psi_\varepsilon^q = \phi_\varepsilon + \eta_\varepsilon \quad \text{in} \quad \Omega. \quad (30)$$
Thus, solving for $\xi_\varepsilon$ we have

$$(-\varepsilon^2 A + id)\xi_\varepsilon = \phi_\varepsilon^p - (-\varepsilon^2 A + id)\psi_\varepsilon. \quad (31)$$

Let $\tilde{\xi}_\varepsilon(x) = \xi_\varepsilon(\varepsilon x)$, $\tilde{\phi}_\varepsilon(x) = \phi_\varepsilon(\varepsilon x)$, and $\tilde{\psi}_\varepsilon(x) = \psi_\varepsilon(\varepsilon x)$. Then, system (31) changes to

$$(-\Delta + id)\tilde{\xi}_\varepsilon = \tilde{\phi}_\varepsilon^p - (-\Delta + id)\tilde{\psi}_\varepsilon,$$

in $\Omega_\varepsilon = \{x : \varepsilon x \in \Omega\}$. Since the right-hand side of (32) is a smooth function with compact support. Estimating by Newtonian potential and the interpolation theorem [9] we obtain as $\varepsilon \to 0$,

$$\|\tilde{\xi}_\varepsilon\|_{W^{2,2}(\Omega_\varepsilon)} \leq C\varepsilon^{-N}\|\phi_\varepsilon^p - (-\varepsilon^2 A + id)\psi_\varepsilon\|_{L^2(D_2)}$$

$$= C\varepsilon^{-N}\left(\int_{D_2} (\phi_\varepsilon^p - (-\varepsilon^2 A + id)\psi_\varepsilon)^2 \right)^{\frac{1}{2}} dx$$

$$= C\varepsilon^{-N}\left(\int_{D_2} \left(\left|u_\varepsilon\left(\frac{\psi(x)}{\varepsilon}\right)\right|^p - (-\varepsilon^2 A + id)v_\varepsilon\left(\frac{\psi(x)}{\varepsilon}\right)^2\right)^{\frac{1}{2}} dx$$

$$= C\left(\int_{B_{2\varepsilon}^+} |u_\varepsilon(y)|^p - (-\Delta + id)v_\varepsilon(y)^2(1 - \varepsilon y N + O(\varepsilon y)^2)^2 dy, \right)^{\frac{1}{2}}$$

where we used the estimate given in Lemma A.1 in [16]

$$\text{det } D\Phi = 1 - \varepsilon y N + O(\varepsilon y)^2 \quad (34)$$

for $\varepsilon > 0$ small, and $\varepsilon$ denotes the mean curvature at the point $P$. Denote $R = \frac{\xi}{\varepsilon}$. Using the decay of the ground state solutions $(u, v)$, we have on the half-ball $B_{2R}^+$ that

$$\left(\int_{B_{2\varepsilon}^+} |u_\varepsilon(y)|^p - (-\Delta + id)v_\varepsilon(y)^2(1 - \varepsilon y N + O(\varepsilon y)^2)^2 \right)^{\frac{1}{2}} dy = O(e^{-\frac{\mu k}{\varepsilon}}),$$

where $\mu > 0$ is a constant. By the Sobolev embeddings we get if $1 \leq p \leq \frac{2N}{N-4}$,

$$\varepsilon^{-N}\|\tilde{\xi}_\varepsilon\|_{L^p(\Omega)} = \|\tilde{\xi}_\varepsilon\|_{L^p(\Omega)} \leq c\|\tilde{\xi}_\varepsilon\|_{W^{2,2}(\Omega)},$$

This implies that $\|\tilde{\xi}_\varepsilon\|_{L^p(\Omega)} = O(e^{-\frac{\mu k}{\varepsilon}}).$ Similarly, we can obtain an estimate for $\eta_\varepsilon$. On the other hand, by (30),

$$\int_{\Omega} (w_1 T_\varepsilon w_2 + w_2 T_\varepsilon w_1) dx = \int_{\Omega} (\phi_\varepsilon^p + \phi_\varepsilon^p \eta_\varepsilon + \psi_\varepsilon^{p+1} + \psi_\varepsilon^{p+1} \xi_\varepsilon) dx.$$
Because \( ||\phi_\varepsilon||, ||\psi_\varepsilon|| \) are uniformly bounded,

\[
\left| \int_\Omega \phi_\varepsilon^{p+1} \eta_\varepsilon \, dx \right| \leq ||\phi_\varepsilon||_{p+1} ||\eta_\varepsilon||_p = O(e^{\frac{\mu_k}{\varepsilon}}), \quad \left| \int_\Omega \psi_\varepsilon^q \xi_\varepsilon \, dx \right| \leq O(e^{\frac{\mu_k}{\varepsilon}}).
\]

Thus, (29) follows. \( \square \)

Next, we have an asymptotic estimate for the integrals obtained in (29).

**Lemma 4.2.**

\[
\int_\Omega \phi_\varepsilon^{p+1} \, dx = \varepsilon^N \left\{ \int_{R_+^N} u^{p+1} \, dx - \varepsilon \int_{R_+^N} u^{p+1} y_N \, dx + O(\varepsilon^2) \right\}, \quad (35)
\]

\[
\int_\Omega \psi_\varepsilon^q \, dx = \varepsilon^N \left\{ \int_{R_+^N} v^{q+1} \, dx - \varepsilon \int_{R_+^N} v^{q+1} y_N \, dx + O(\varepsilon^2) \right\} \quad (36)
\]

as \( \varepsilon \to 0 \).

**Proof.** By definition of \( \phi_\varepsilon \), and changing variables \( \varepsilon y = \Psi(x) \), \( dx = \varepsilon^N \det D\Phi \, dy \), we get

\[
\int_\Omega \phi_\varepsilon^{p+1} \, dx = \varepsilon^N \int_{B_{2\kappa} / \varepsilon} u^{p+1}(y) \gamma_{p+1}(|y|) \det D\Phi \, dy.
\]

If we denote by \( R = \frac{\varepsilon}{\varepsilon} \), we can write \( B_{2R} = B_R \cup (B_{2R} \setminus B_R) \). Thus, the last integral can be split as

\[
\varepsilon^N \left( \int_{R_+^N} u^{p+1}(y) \det D\Phi \, dy - \int_{R_+^N \setminus B_R} u^{p+1}(y) \det D\Phi \, dy \right.
\]

\[
+ \int_{B_{2R} \setminus B_R} u^{p+1}(y) \gamma_{p+1}(|y|) \det D\Phi \, dy \Bigg). \]

On the other hand, it was proved in [5] the estimate \( u(x) \leq ce^{-\theta|x|} \), for \( 0 < \theta < 1 \) fixed and \( |x| \) large. Then, using again (34), the second integral can be estimated by

\[
\left| \int_{R_+^N \setminus B_R} u^{p+1}(y) \det D\Phi \, dy \right| \leq c \int_{\infty}^\infty e^{-pN-1}(1 + \varepsilon x + O((\varepsilon x)^2)) \, dr,
\]

where \( \gamma = \theta(p + 1) \) and \( c \) denotes a generic positive constant. This integral is bounded by \( cR^{N+2}e^{-\gamma R} \). Similarly, we get that
\[
\int_{B_{2R}\setminus B_R} u^{p+1}(y)\gamma^{p+1}(\|y\|) \det D\Phi \, dy \leq cR^{N+2}e^{-\gamma R}.
\]

But \( R = \frac{\xi}{\varepsilon} \) then for \( \varepsilon \) small
\[
\int_{\Omega} \phi^{p+1}_\varepsilon \, dx = \varepsilon^N \int_{R^N} u^{p+1}(y)(1 - \varepsilon x N + \varepsilon^2 O(\|y\|^2)) \, dy + O(e^{-\frac{\gamma R}{2}}).
\]  

(37)

We can compute a similar estimate for \( \int_{\Omega} \psi^{q+1}_\varepsilon \, dx. \)

Let \( \delta = \frac{q(p+1)}{p(q+1)} \). Solving for \( s \) in the derivatives of \( J_\varepsilon(tw_1, sw_2) \) with respect to \( t \) we get the relation \( t^{\frac{1}{1+p}} = s^{\frac{1}{1+\delta}}. \) Denote \( s = s(t) = t^{\delta}. \)

**Lemma 4.3.** Suppose that \( t_0 > 0 \) maximizes \( J(t w_1, s(t)w_2) \), then for \( \varepsilon > 0 \) small we have
\[
t_0(\varepsilon) = 1 + \beta \varepsilon + O(\varepsilon^2),
\]  

(38)

where \( \beta \) is a positive constant.

**Proof.** To prove this result, we will use the Implicit Function theorem. Choosing \( w \) as in Lemma 4.1 and using (29) we obtain
\[
J_\varepsilon(tw_1, sw_2) = \frac{p}{p+1} t^{\frac{1}{1+p}} \int_{\Omega} \phi^{p+1}_\varepsilon \, dx + \frac{q}{q+1} t^{\frac{1}{1+q}} \int_{\Omega} \psi^{q+1}_\varepsilon \, dx
\]
\[\quad - \frac{1}{2} t^{1+\delta} \left[ \int_{\Omega} (\phi^{p+1}_\varepsilon + \psi^{q+1}_\varepsilon) \, dx + O(e^{-\frac{\gamma R}{\varepsilon}}) \right].
\]

From Lemma 4.2, we can write
\[
\int_{\Omega} \phi^{p+1}_\varepsilon \, dx = \varepsilon^N (A_0 - \varepsilon A_1 + O(\varepsilon^2)), \quad \int_{\Omega} \psi^{q+1}_\varepsilon \, dx = \varepsilon^N (B_0 - \varepsilon B_1 + O(\varepsilon^2)),
\]  

(39)

where \( A_0, B_0, A_1, B_1 > 0. \) Taking derivative of \( J_\varepsilon(tw_1, tw_2) \) with respect to \( t \) and using (39) we have
\[
\sigma(\varepsilon, t) := \frac{1}{\varepsilon^N} \frac{d}{dt} J(tw_1, sw_2) = \left\{ \frac{1}{p} (A_0 - \varepsilon A_1 + O(\varepsilon^2)) \right\} + \frac{1}{q} (B_0 - \varepsilon B_1 + O(\varepsilon^2))
\]
\[\quad - \frac{1}{2} (1 + \delta) \delta^2 (A_0 + B_0 - \varepsilon (A_1 + B_1) + O(\varepsilon^2))
\]

which gives \( \sigma(0, 1) = 0 \) and
\[
\frac{d}{dt} \sigma(0, 1) = A_0 \left( \frac{1}{p} - \frac{\delta}{2}(1 + \delta) \right) + B_0 \left( \frac{\delta}{p} - \frac{\delta}{2}(1 + \delta) \right).
\]  

(40)
Since $\delta > \frac{1}{2}(1 + \frac{1}{p})$, suppose that the coefficient of $A_0$ is nonnegative, then
\[
\frac{2}{p} \geq \delta(1 + \delta) > \frac{1}{2} \left(1 + \frac{1}{p}\right) \left(1 + \frac{1}{2} \left(1 + \frac{1}{p}\right)\right),
\]
namely $3p^2 - 4p + 1 < 0$. The inequality has no solution if $p > 1$. We obtain a contradiction. If the coefficient of $B_0$ is nonnegative, we have $\frac{2}{p} \geq 1 + \delta > \frac{3}{2} + \frac{1}{2p}$, i.e., $1 > p$ which is again a contradiction. Thus, $\frac{de}{dt}(0, 1) < 0$. Using the Implicit Function Theorem we conclude the asymptotic formula. \qed

We are ready to prove the main result in this section.

**Proposition 4.1.**

\[
c_e \leq e^N \left(\frac{c^\infty}{2} - e \gamma + O(\varepsilon^2)\right),
\]

where $c^\infty$ is defined by (8), and
\[
\gamma = \frac{p - 1}{2(p + 1)} \int_{\Omega} u^{p+1} y_N \, dy + \frac{q - 1}{2(q + 1)} \int_{\Omega} v^{q+1} y_N \, dy.
\]

**Proof.** Let $w_e = (t_0 \phi_e^p, s(t_0)\psi_e^q)$ where $t_0$ is given in (38). Then,
\[
J_e(w_e) = e^N \left(\left(\frac{p}{p + 1} t_0^{1 + \frac{1}{p}} - t_0^{1 + \delta} \right) \int_{\Omega} \phi_e^{p+1} \, dx
\right.
\]
\[
+ \left. \left(\frac{q}{q + 1} t_0^{1 + \frac{1}{p}} - t_0^{1 + \delta} \right) \int_{\Omega} \psi_e^{q+1} \, dx - \frac{t_0^{1 + \delta}}{2} \int_{\Omega} (\phi_e^{p} \eta_e + \psi_e^{q} \xi_e) \, dx\right).
\]

By Lemma 4.2 and (39), we have
\[
J_e(w_e) = e^N \left(\left(\frac{p}{p + 1} t_0^{1 + \frac{1}{p}} - t_0^{1 + \delta} \right) (A_0 - \varepsilon A_1) + \left(\frac{q}{q + 1} t_0^{1 + \frac{1}{p}} - t_0^{1 + \delta} \right) (B_0 - \varepsilon B_1)
\right.
\]
\[
+ \left. O(\varepsilon^2) - \frac{t_0^{1 + \delta}}{2} O(\varepsilon^{\frac{\gamma}{\varepsilon}})\right).
\]

(42)

Lemma 4.3 implies
\[
\frac{p}{p + 1} t_0^{1 + \frac{1}{p}} - t_0^{1 + \delta} = \frac{p - 1}{2(p + 1)} + \beta \varepsilon \left(1 - \frac{1 + \delta}{2}\right) + O(\varepsilon^2),
\]
where $\beta$ is the constant given in (38). Similarly,

$$\frac{q}{q+1} \frac{t_0^{1+p}}{t_0^1} - \frac{t_0^{1+\delta}}{2} = \frac{q-1}{2(q+1)} + \beta \varepsilon \left( \delta - \frac{1+\delta}{2} \right) + O(\varepsilon^2).$$

Replacing in (42), we get

$$J_\varepsilon(w_\varepsilon) = \varepsilon^N \left( \frac{p-1}{2(p+1)} A_0 + \varepsilon \left( \beta \left( 1 - \frac{1+\delta}{2} \right) A_0 - \frac{p-1}{2(p+1)} A_1 \right) + \frac{q-1}{2(q+1)} B_0 + \varepsilon \left( \beta \left( \frac{1+\delta}{2} \right) B_0 - \frac{q-1}{2(q+1)} B_1 \right) + O(\varepsilon^2) \right).$$

Since $(u, v)$ is a ground state of (7) and $(u, v)$ is radially symmetric we have that $A_0 = B_0$. Therefore,

$$\beta \left( \frac{1}{2} - \frac{\delta}{2} \right) A_0 + \beta \left( \frac{\delta}{2} - \frac{1}{2} \right) B_0 = 0.$$

We conclude that

$$J_\varepsilon(w_\varepsilon) \leq \varepsilon^N \left( \frac{p-1}{2(p+1)} \int_{\mathbb{R}^N} u^{p+1} \, dx + \frac{q-1}{2(q+1)} \int_{\mathbb{R}^N} v^{q+1} \, dx - \varepsilon x_\gamma + O(\varepsilon^2) \right).$$

Now we complete the proof of Theorem 1.1 by Proposition 4.1.

**Proof of Theorem 1.1.** We remark that the only constant solutions of (4)–(5) are $(-1, -1)$, $(0, 0)$, and $(1, 1)$, and

$$c = \delta = \frac{p-1}{2(p+1)} |\Omega| + \frac{q-1}{2(q+1)} |\Omega| = \left( 1 - \frac{1}{p+1} - \frac{1}{q+1} \right) |\Omega| > 0.$$

By Proposition 4.1, it yields $J_\varepsilon(w_\varepsilon) = c < c$ for $\varepsilon > 0$ small. We conclude that $w_\varepsilon$ is nontrivial, and by results in Proposition 4.1 it is positive. □

5. **Lower bound for the critical value**

In this section we prove Theorems 1.2, 1.3, and we find a lower bound for the critical value. We assume that $(u_\varepsilon, v_\varepsilon)$ is the solution obtained in Theorem 1.1 and $1 < p, q < \frac{N+2}{N-2}$. We split the proofs in several lemmata.
Lemma 5.1. Let $x_1$ and $x_2$ be points where $u_e$ and $v_e$ attain their supremum respectively. Then

$$u_e(x_1) \geq 1, \quad v_e(x_2) \geq 1.$$ 

Proof. Let $x_0$ be a point where $u_e + v_e$ attains a local maximum. First, we prove that either $u_e(x_0) \geq 1$ or $v_e(x_0) \geq 1$. By contradiction, if $u_e(x_0) < 1$ and $v_e(x_0) < 1$, then there is a ball $B_R(x_0)$ such that $u_e(x), v_e(x) < 1$ for $x \in B_R(x_0) \cap \Omega$. If $x_0 \in \Omega$ we can choose a smaller ball $B_r(x_0) \subset \Omega$ and then

$$\varepsilon^2 \Delta (u_e + v_e) = u_e(1 - u_e^{p-1}) + v_e(1 - v_e^{q-1}) > 0, \quad \text{in } B_r(x_0).$$

However, $\varepsilon^2 \Delta (u_e + v_e)(x_0) \leq 0$. We obtain a contradiction. If $x_0 \in \partial \Omega$ we have that $u_e(x) + v_e(x) < u_e(x_0) + v_e(x_0)$ for $x$ in a neighborhood of $x_0$. By Hopf’s Lemma we have that $\frac{\partial (u_e + v_e)}{\partial n}(x_0) > 0$ contradicting the boundary condition $\frac{\partial u_e}{\partial n}(x_0) = 0$, and $\frac{\partial v_e}{\partial n}(x_0) = 0$. Thus, we conclude that at least one of the functions $u_e$ and $v_e$ is larger than 1 at $x_0$. Let us suppose that $u_e(x_0) \geq 1$.

By Lemma 2.2 in [6] we obtain, $1 \leq u_e(x_0) \leq v_e^{q+1}(x_2)$, where $x_2$ is the point where the supremum of $v_e$ is attained. Thus, we conclude that $v_e(x_2) \geq 1$. Obviously, we can conclude that $u_e(x_1) \geq u_e(x_0) \geq 1$ with $x_1$ is the maximum of $u_e$. \(\square\)

Lemma 5.2. There exists $\varepsilon_0 > 0$ such that if $0 < \varepsilon < \varepsilon_0$, then

$$\max \{ ||u_e||_{L^c}, ||v_e||_{L^c} \} \leq c,$$  \hspace{1cm} (43)

where $c > 0$ is a constant independent of $\varepsilon$.

Proof. We will use a blow-up argument as in [2]. We sketch the proof. Denote $u_n := u_{e_n}$ and $v_n := v_{e_n}$ and suppose there exists $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$, and such that

$$\max \{ ||u_n||_{L^c}, ||v_n||_{L^c} \} \rightarrow \infty.$$ 

We may assume for some constant $\lambda_n$ that

$$\frac{1}{\lambda_n^{\beta_1} ||u_n||_{L^c}} \geq \frac{1}{\lambda_n^{\beta_2} ||v_n||_{L^c}},$$

with $\beta_1, \beta_2 > 0$ to be determined later. Suppose $u_n(x_n) = ||u_n||_{L^c} \rightarrow \infty$, with $x_n \rightarrow x_0 \in \Omega$, and $\lambda_n ||u_n||_{L^c} = 1$. Then, $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Let $w_{1,n}(y) = \lambda_n^{\beta_1} u_n(\varepsilon_n \lambda_n y + x_n)$ and $w_{2,n}(y) = \lambda_n^{\beta_2} v_n(\varepsilon_n \lambda_n y + x_n)$. The Laplacian with respect to the $y$ variable is

$$-\Delta_y w_{1,n}(y) = \varepsilon_n^{2 \beta_1 + 2} A_y u_n(x).$$
which gives us the equation
\[-\Delta w_{1,n} + \lambda_n^2 w_{1,n} = \lambda_n^{\beta_1 + 2} w_{1,n}^q.\]

Similarly,
\[-\Delta w_{2,n} + \lambda_n^2 w_{2,n} = \lambda_n^{\beta_1 + 2} w_{2,n}^p.\]

Choosing $\beta_1, \beta_2$ such that
\[\beta_1 = \frac{2(1 + q)}{pq - 1}, \quad \beta_2 = \frac{2(1 + p)}{pq - 1},\]

and using $L^p$ and Schauder’s estimates we have
\[w_{1,n} \to w_1, \quad w_{2,n} \to w_2,\]
in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N) \cap C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N)$ for $0 < \alpha < 1$. Thus, $(w_1, w_2)$ satisfies the system
\[-\Delta w_1 = w_2^q, \quad -\Delta w_2 = w_1^p \quad \text{in} \quad \mathbb{R}^N,\]
with $w_1(0) = 1$. This is a contradiction with the Liouville-type results for systems [3].

In the case $x_n \to x_0 \in \partial \Omega$ we may argue in the same way by using Liouville theorem in the half-space. \hfill \Box

**Lemma 5.3.**
\[
\int_{\Omega} u_r^e \, dx \leq c_r e^N \quad \text{and} \quad \int_{\Omega} v_r^e \, dx \leq c_r e^N \quad \text{for} \quad r \geq 1
\]

provided that $0 < e < e_0$.

**Proof.** Since $(u_r, v_r)$ is a solution of (4) and (5),
\[
\int_{\Omega} (e^2 |\nabla u_r|^2 + u_r^2) \, dx = \int_{\Omega} v_r^q u_r \, dx, \quad \int_{\Omega} (e^2 |\nabla v_r|^2 + v_r^2) \, dx = \int_{\Omega} u_r^p v_r \, dx.
\]

We can assume that $p > q > 1$. Then by Lemma 5.2 and Hölder’s inequality we obtain
\[
\int_{\Omega} (e^2 |\nabla v_r|^2 + v_r^2) \, dx \leq \left( \int_{\Omega} u_r^{p+1} \, dx \right)^{\frac{p}{p+1}} \left( \int_{\Omega} v_r^{q+1} \, dx \right)^{\frac{1}{p+1}} \leq c \left( \int_{\Omega} u_r^{p+1} \, dx \right)^{\frac{p}{p+1}} \left( \int_{\Omega} v_r^{q+1} \, dx \right)^{\frac{1}{p+1}}.
\]
From Proposition 4.1 and the fact that \((u_\varepsilon, v_\varepsilon)\) is a critical point satisfying (13), we can get the following estimates:

\[
\int_{\Omega} u_\varepsilon^{p+1} \, dx = \int_{\Omega} v_\varepsilon^{q+1} \, dx = O(\varepsilon^N).
\]

Hence,

\[
\int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + v_\varepsilon^2) \, dx \leq C \varepsilon^{p+1} \varepsilon^{q+1} = C \varepsilon^N.
\]

On the other hand, using Young’s inequality we have

\[
\int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + v_\varepsilon^2) \, dx \leq \left( \int_{\Omega} u_\varepsilon^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} v_\varepsilon^{2q} \, dx \right)^{\frac{1}{2}}
\]

\[
\leq \delta \int_{\Omega} u_\varepsilon^2 \, dx + c_\delta \int_{\Omega} v_\varepsilon^{2q} \, dx.
\]

Thus,

\[
\int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + (1 - \delta)u_\varepsilon^2) \, dx \leq c_\delta \int_{\Omega} v_\varepsilon^{2q} \, dx.
\]

By (43)

\[
\int_{\Omega} v_\varepsilon^{2q} \, dx \leq C \int_{\Omega} v_\varepsilon^{q+1} \, dx.
\]

Choosing \(\delta = \frac{1}{2}\), we get

\[
\int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + \frac{1}{2} u_\varepsilon^2) \, dx \leq C \int_{\Omega} v_\varepsilon^{2q} \, dx \leq C \int_{\Omega} v_\varepsilon^{q+1} \, dx \leq C \varepsilon^N.
\]

In particular,

\[
\int_{\Omega} u_\varepsilon^2 \, dx \leq C \varepsilon^N.
\]

Thus we obtain

\[
\int_{\Omega} (\varepsilon^2 |\nabla u_\varepsilon|^2 + u_\varepsilon^2) \, dx \leq C \varepsilon^N.
\]

Using a bootstrap argument (see [13, Lemma 2.3]), we obtain the result. \(\square\)

**Lemma 5.4.** Let \(P_\varepsilon\) and \(Q_\varepsilon\) denote a local maximum of \(u_\varepsilon\) and \(v_\varepsilon\), respectively. Then for \(\varepsilon > 0\) small

\[
d(P_\varepsilon, \partial \Omega) \leq C \varepsilon \quad \text{and} \quad d(Q_\varepsilon, \partial \Omega) \leq C \varepsilon,
\]

where \(C > 0\) is independent of \(\varepsilon\).
Proof. Assume that one of the inequalities is false, for example, there exists a sequence $\varepsilon_j \to 0$ such that

$$\rho_j = \frac{d(P_{\varepsilon_j}, \partial \Omega)}{\varepsilon_j} \to \infty.$$ 

Let us define $\bar{u}_j = u_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)$ and $\bar{v}_j = v_{\varepsilon_j}(P_{\varepsilon_j} + \varepsilon_j z)$ for $z \in B_{\rho_j}$. These new functions satisfy the system

$$-\Delta \bar{u}_j + \bar{u}_j = \bar{v}_j^q,$$
$$-\Delta \bar{v}_j + \bar{v}_j = \bar{u}_j^p \quad \text{in} \ B_{\rho_j}. \ (44)$$

On the other hand, using Lemma 5.3 we see that there is a constant $C > 0$ such that

$$\int_{B_{\rho_j}} \bar{u}_j dx \leq C \quad \text{and} \quad \int_{B_{\rho_j}} \bar{v}_j dx \leq C \quad \text{for} \ r \geq 1.$$ 

By $L^p$ and Schauder estimates, we have $\bar{u}_j \to u$ and $\bar{v}_j \to v$ in $C^2_{\text{loc}}(\mathbb{R}^N)$. By Lemma 5.1, $u \geq 1$ and $v \geq 1$ at some points. Thus, $u$ and $v$ are nontrivial, positive, and satisfy Eq. (27).

From [5], we know that a solution $(u, v)$ of (24) has an exponential decay as $|x| \to \infty$. Thus, for $\rho_j \geq 2R$,

$$||\bar{u}_j - u||_{C^2(B_{2R})} \leq \delta_R := C e^{-\frac{R}{2}} \quad \text{and} \quad ||\bar{v}_j - v||_{C^2(B_{2R})} \leq \delta_R. \ (45)$$

Denoting $I_d$ the functional $I$, defined as in (10) in the whole $\mathbb{R}^N$, restricted to the set $A$. Noticing $J_{\varepsilon_j}(\bar{u}_j^q, \bar{v}_j^q) = I_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j})$ we get for $B(P_{\varepsilon_j}, \varepsilon R)$, a ball centered at $P_{\varepsilon}$ with radius $\varepsilon R$,

$$c_j = I_{\varepsilon_j}(u_{\varepsilon_j}, v_{\varepsilon_j}) \geq I_{\varepsilon_j}|_{B(P_{\varepsilon_j}, \varepsilon R)}(u_{\varepsilon_j}, v_{\varepsilon_j})$$
$$= \varepsilon_j^N(I_{B(0,R)}(u, v) + I_{B(0,R)}(\bar{u}_j, \bar{v}_j) - I_{B(0,R)}(u, v)). \ (46)$$

Using the exponential decay, we know that $||\bar{u}_j^{p+1} - u^{p+1}||_{C(B_{\rho_j})} \leq C \delta_R$ and $||\bar{v}_j^{q+1} - v^{q+1}||_{C(B_{\rho_j})} \leq C \delta_R$. Then, the difference of the last two integrals is small.

Since $u$ is positive and radial, it decays exponentially to zero at infinity. Therefore,

$$\int_{\mathbb{R}^N, B(0,R)} u^{p+1} dx \leq C \int_{\mathbb{R}^N} u^{p+1}(r)r^{N-1} dr$$
$$\leq C \int_{\mathbb{R}} e^{-\theta R(p+1)} r^{N-1} dr$$
$$= CR^N e^{-\theta R(p+1)} \leq Ce^{-\frac{\theta R(p+1)}{2}} \ (47)$$
with $\theta > 0$ constant. We get a similar estimate for $v$. This yields to

$$I_{R^N \setminus B(0, R)}(u, v) \leq Ce^{-\frac{\theta(p+1)R}{2}}. $$

Because

$$I_{B(0, R)}(u, v) = e_j^N (I(u, v) - I_{R^N \setminus B(0, R)}(u, v)),$$

we obtain the estimate

$$c_j \geq e_j^N \left( I(u, v) - Ce^{-\frac{\theta R(p+1)}{2}} \right).$$

Taking $\delta_R = e_j$ small and using (41), we obtain

$$e_j^N \frac{1}{2} I(u, v) \geq c_j \geq e_j^N (I(u, v) - O(e_j)),$$

which is the desired contradiction. $\square$

From Lemma 5.4 we know that $P_\varepsilon \to P \in \partial \Omega$ and $Q_\varepsilon \to Q \in \partial \Omega$ when $\varepsilon \to 0$. It is natural to ask if $P = Q$. We have the following lemma which completes the proof of Theorem 1.2.

**Lemma 5.5.** When $\varepsilon \to 0$, $P_\varepsilon$ and $Q_\varepsilon \to P$. Moreover, for $\varepsilon$ small both $P_\varepsilon$ and $Q_\varepsilon$ are on the boundary $\partial \Omega$.

**Proof.** First, we show that $P = Q$. Suppose by contradiction that there is a sequence $\varepsilon_j \to 0$ such that $P_j = P_{\varepsilon_j}$ satisfies $\rho_j = \frac{d(P_j, Q)}{\varepsilon_j} \to \infty$ as $j \to \infty$. Let $B_{2\kappa}$ be a ball centered at $Q$ with radius $\kappa > 0$. Let $\Phi$ be the parametrization of the boundary at $Q$, with $\Phi(0) = Q$. Define

$$\tilde{u}_j(z) = \begin{cases} u_j(\Phi(P_j + \varepsilon z)) & \text{if } z \in \tilde{B}_{\frac{2\kappa}{\varepsilon}}, \\ u_j(\Phi(p_j' + \varepsilon z', x_j \varepsilon)) & \text{if } z \in \tilde{B}_{\frac{2\kappa}{\varepsilon}}, \end{cases}$$

where $\tilde{B}_{\frac{2\kappa}{\varepsilon}} := \tilde{B}(P_j, \frac{2\kappa}{\varepsilon}) \cap \{ x : x_N \geq 0 \}$ and $\tilde{B}_{\frac{2\kappa}{\varepsilon}} := \tilde{B}(P_j, \frac{2\kappa}{\varepsilon}) \cap \{ x : x_N \}$. We define similarly $\tilde{v}_j$ and we denote $\tilde{z}_j = (\tilde{u}_j, \tilde{v}_j)$. Because $\{(u_j, v_j)\}$ is uniformly bounded in $\varepsilon$ in $H^1 \times H^1$, by Lemma 5.4 and $L^p$ and Schauder estimates $\{x_j\}$ is a bounded sequence. We can prove that $\tilde{z}_j \to z$ in $C^2_{\text{loc}}(\mathbb{R}^N)$, where $z$ a ground state of (7). Since $(u_j, v_j) := (u_{\varepsilon_j}, v_{\varepsilon_j})$ is a solution of (4) and (5) we obtain

$$c_j := c_{\varepsilon_j} = \frac{p-1}{2(p+1)} \int_{\Omega} |u_j|^{p+1} \, dx + \frac{q-1}{2(q+1)} \int_{\Omega} |v_j|^{q+1} \, dx.$$
We split $\Omega$ in two domains as $\Omega = \Phi(B_{R_e}) \cup (\Omega \setminus \Phi(B_{R_e}))$ with $\rho_j \geq 2R$. Using (47) and the estimate of $\det D \Phi$ given in (34) we get that

$$J_{\Phi(B_{R_e})}(w) \geq e^N \left( \int_{B_R^+} \left( \frac{p-1}{2(p+1)} |u|^{p+1} + \frac{q-1}{2(q+1)} |\tilde{v}|^{q+1} \right) dx - C e^{-\frac{\theta_R(p+1)}{2}} - C \epsilon \right).$$

Applying a Harnack-type inequality in a neighborhood of $P$, a ball $B(P', \epsilon)$ [13, Lemma 4.3], we get a constant $C^*$ independent of $\epsilon$ such that at least

$$J_{\Omega \setminus \Phi(B_{R_e})}(w) \geq e^N \int_{B(P', \epsilon)} \frac{p-1}{2(p+1)} |u|^{p+1} dx \geq C^* e^N.$$

By the estimates on $\Phi(B_{R_e})$ and $\Omega \setminus \Phi(B_{R_e})$, Proposition 3.1, and taking $\epsilon = e^{-\frac{R}{2}}$ we get

$$e^N \frac{1}{2} J(w) \geq c_j \geq e^N \left( \frac{1}{2} J(w) + C^* - C e^{\theta(p+1)} - C \epsilon \right),$$

which is again a contradiction. Hence $P = Q$.

Now we prove that $P_e \in \partial Q$ for $\epsilon > 0$ small. Suppose $P_e \notin \partial Q$ for all $\epsilon > 0$. Let $\Psi$ denote the parametrization of the boundary at $P$ such that $P = \Psi(0)$. Denote $p_e = \Psi(P_e) \notin \{z_n = 0\}$. We reflect the function $\tilde{u}_j$ about the hyperplane $z_n = 0$ then $\frac{\partial \tilde{u}_j}{\partial n} = 0$ on this hyperplane. Because $P_e$ is a local maximum, we also have that $\frac{\partial \tilde{u}_j}{\partial x_N}(P_e) = 0$.

Applying Rolle’s theorem to $\frac{\partial \tilde{u}_j}{\partial x_N}$ we see that there is a point $Z_e = \{z_e, z_N\}$ with $0 \leq z_N \leq p_e N$ such that

$$\frac{\partial^2 \tilde{u}_j}{\partial x_N^2}(Z_e) \geq 0.$$

But from Lemma 5.4, we know that $x_e = \frac{Z_e - p_e}{\epsilon} \to 0$ as $\epsilon \to 0$. Then, because 0 is a maximum point of $u$ we obtain that

$$0 \geq \frac{\partial^2 u}{\partial x_N^2}(0) = \lim_{\epsilon \to 0} \frac{\partial^2 \tilde{u}_j}{\partial x_N^2}(x_e) \geq 0,$$

which is a contradiction. Thus, $P_e \in \partial Q$. In the same way we may show that $Q_e \in \partial Q$.

**Remark 5.1.** Comparing with the single equation, we cannot prove that there is a unique maximum point. The main problem to prove this fact is the lack of uniqueness of the ground state. See [1], for example, where they proved the uniqueness of the maximum via uniqueness of the ground state.

Finally we prove the lower bound.
Proposition 5.1.

\[ J_e(w_e) \geq \varepsilon^N \left( \frac{e^{-\beta}}{2} - e\gamma + O(\varepsilon^2) \right). \]  

(49)

Proof. Let \( P_e \) be a maximum point of \( u_e \) on \( \tilde{\Omega} \). By Lemma 5.4, we know that \( P_e \in \partial \Omega \) and \( P_e \rightarrow P \in \partial \Omega \) as \( \varepsilon \rightarrow 0 \).

In the set \( \Omega_\varepsilon = \frac{1}{\varepsilon}(\Omega - P_e) \) let us define \( \tilde{u}_e(y) = u_e(P_e + \varepsilon y) \) and \( \tilde{v}_e(y) = v_e(P_e + \varepsilon y) \). Using an argument as in Lemma 5.2, we can prove that \( (\tilde{u}_e, \tilde{v}_e) \) converges as \( \varepsilon \rightarrow 0 \) in the \( C^2_{\text{loc}}(R^N_+) \) sense to \( (u, v) \), a solution of the system

\[ -\Delta u + u = \psi^0, \quad -\Delta v + v = \psi^0 \quad \text{in} \quad R^N_+ \]

with zero Neumann boundary condition. Extending \( (\tilde{u}, \tilde{v}) \) to \( R^N_+ \) as even functions we know by Theorem 2.2 in [5] that there are positive constants \( c \) and \( \theta \) such that

\[ \tilde{u}(y) \leq ce^{-\theta|y|}, \quad \tilde{v}(y) \leq ce^{-\theta|y|}. \]

Hence the \( C^2_{\text{loc}} \) convergence yields that \( \tilde{u}_e(y) \leq ce^{-\theta|y|}, \tilde{v}_e(y) \leq ce^{-\theta|y|} \) for \( y \in \Omega \).

Let \( V_e \) be a neighborhood of \( P_e \) and denote by \( \Psi_\varepsilon \) the local parametrization of the boundary at \( P \) such that \( \Psi_\varepsilon(0) = P_e, \quad \Psi_\varepsilon'(0) = 0 \) and \( \Omega = \{(x', x_N) : x_N > \Psi_\varepsilon(x') \} \). Because of the convergence of \( P_e \), we can consider that \( \Psi_\varepsilon \rightarrow \Psi \) in a \( C^2 \) sense, the local parametrization of the boundary at \( P \).

Denote by \( J_A \) the functional \( J_\varepsilon \) restricted to the set \( A \), then

\[ c^e = J_\varepsilon(w_e) \geq J_\varepsilon(tw_e) = \varepsilon^N J_{\Omega_e}(tw_e), \]  

(50)

where \( \tilde{w}_e = (\tilde{w}_e^i, \tilde{w}_e^j) \).

Now, let us extend \( (\tilde{u}_e, \tilde{v}_e) \) to \( R^N_+ \) by defining \( \tilde{u}_e(y) = \tilde{u}_e(y) \) if \( \Psi_e(ey') \leq y_N \) and \( \tilde{u}_e(y') = \tilde{u}_e(y', \Psi(ey')) \) if \( \Psi_e(ey') > y_N \), and \( \tilde{v}_e(y) \) in the same way. Denote \( \tilde{w}_e = (\tilde{w}_e^i, \tilde{w}_e^j) \). Then,

\[ J_{\Omega_e}(t\tilde{w}_e) \geq J_{\Omega_e \cap V_e}(t\tilde{w}_e) \]  

(51)

\[ = J_{R^N_+ \cap V_e}(tw_e) + J_{(\Omega_e \cap V_e) \setminus R^N_+}(tw_e) - J_{(R^N_+ \cap V_e) \setminus \Omega_e}(tw_e). \]  

(52)

Choosing \( t = t_e \) such that \( J_{R^N_+}(t_e\tilde{w}_e) \) maximizes at \( t_e \), we deduce

\[ J_{R^N_+ \cap V_e}(t_e\tilde{w}_e) = J_{R^N_+}(t_e\tilde{w}_e) - J_{(R^N_+ \cap V_e)(t_e\tilde{w}_e)} \geq \frac{1}{2} c^\infty - O(e^{-\beta}), \]  

(53)

with a positive constant \( \beta \).
Let us call $J_1$ and $J_2$ the second and third integrals in (52), using the exponential decay we get that

$$J_1 = \int_{B_{\varepsilon}} \int_0^1 \frac{\varphi_{\varepsilon}(y')}{\varepsilon} \left( \frac{p-1}{2(p+1)} |\tilde{u}_\varepsilon|^{p+1}(y', \varepsilon y') + \frac{q-1}{2(q+1)} |\tilde{v}_\varepsilon|^{q+1}(y', \varepsilon y') \right) dy_N + O(\varepsilon^{-\beta}),$$

and

$$J_2 = \int_{B_{\varepsilon}} \int_0^1 \frac{\varphi_{\varepsilon}(y')}{\varepsilon} \left( \frac{p-1}{2(p+1)} |\tilde{u}_\varepsilon|^{p+1}(y', \varphi_{\varepsilon}(y')) + \frac{q-1}{2(q+1)} |\tilde{v}_\varepsilon|^{q+1}(y', \varphi_{\varepsilon}(y')) \right) dy_N + O(\varepsilon^{-\beta}),$$

where $f_+ = \max\{f, 0\}$ and $f_- = \min\{f, 0\}$. To get the limit behavior of these integrals, we first notice that $\varphi_{\varepsilon} = \frac{\varepsilon}{2} \varphi_{\varepsilon}(0) + O(\varepsilon^2)$. Using the fact that the ground state solutions on $\mathbb{R}^N$ are radial, when $\varepsilon$ goes to zero we get

$$\frac{J_1 + J_2}{\varepsilon} = \frac{1}{2} \sum_{i,j}^{N-1} \int_{\mathbb{R}^N} \varphi_{i,j}(0) y_i y_j \left( \frac{p-1}{2(p+1)} |y|^{p+1} + \frac{q-1}{2(q+1)} |y|^{q+1} \right) y_j dy' = \frac{1}{N-1} \Delta \varphi(0) \int_{\mathbb{R}^N} \left( \frac{p-1}{2(p+1)} |y|^{p+1} + \frac{q-1}{2(q+1)} |y|^{q+1} \right) y_N dy = \alpha y,$$

(54)

where $\alpha$ is the same constant given in the upper bound, and $\alpha$ is the mean curvature at the point $P$. From (50), (53), and (54) we obtain the lower estimate. □

**Proof of Theorem 1.3.** Using the two bounds given in Propositions 4.1 and 5.1, we obtain the asymptotic formula for the critical value $c_\varepsilon$ of $J_\varepsilon$. □

### 6. Positive constant solutions

In this section we prove Theorem 1.4. We follow the arguments in [15]. First we have a priori estimates for the positive solutions of (4) and (5).

**Lemma 6.1.** Suppose $(u, v)$ is a nonnegative solution to (4) and (5). Then $(u, v)$ satisfies

$$\|(u, v)\|_{C^{0}(\bar{\Omega})} \leq C \max\{1, \varepsilon^{-\theta}\}$$

for some $\theta \in (0, 1)$, $\alpha > 1$ and $C > 0$ independent of $(u, v)$ and $\varepsilon$. 
Proof. Integrating (4) and (5) on $\Omega$ we find
\[
\int_{\Omega} u \, dx = \int_{\Omega} v^q \, dx, \quad \int_{\Omega} v \, dx = \int_{\Omega} u^p \, dx.
\]
Applying Hölder’s inequality to the right-hand side of
\[
\int_{\Omega} (u^p + v^q) \, dx = \int_{\Omega} (u + v) \, dx,
\]
we see that
\[
\int_{\Omega} (u^p + v^q) \, dx
\]
is uniformly bounded. Then by $L^p$ estimates (see [15]) we obtain
\[
\|u\|_{W^{1,p}(\Omega)} \leq C \max(1, \varepsilon^{-2}), \quad \|v\|_{W^{1,p}(\Omega)} \leq C \max(1, \varepsilon^{-2}).
\]
The assertion follows by bootstrap arguments as [15]. We omit the details. □

Proof of Theorem 1.4. We decompose $(u, v)$ as $u = u_0 + \phi$, $v = v_0 + \psi$, where
\[
u_0 = |\Omega|^{-1} \int_{\Omega} u \, dx, \quad v_0 = |\Omega|^{-1} \int_{\Omega} v \, dx
\]
and
\[
\int_{\Omega} \phi \, dx = 0, \quad \int_{\Omega} \psi \, dx = 0.
\]
Then
\[
\varepsilon^2 \Delta \phi - \phi + \left( q \int_0^1 (v_0 + t\psi)^{q-1} \, dt \right) \psi = u_0 - v_0^q.
\]
Multiply both sides by $\phi$ and integrate over $\Omega$, it yields
\[
\varepsilon^2 \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Omega} \phi^2 \, dx = \int_{\Omega} \left( \int_0^1 q(v_0 + t\psi)^{q-1} \, dt \right) \phi \psi \, dx.
\]
It follows from Lemma 6.1 that $(u, v)$ is uniformly bounded. Thus, an application of Young’s inequality and Poincaré inequality gives
\[
(1 + c_0 \varepsilon^2) \int_{\Omega} \phi^2 \, dx \leq C \int_{\Omega} (\phi^2 + \psi^2) \, dx
\]
for some \( c_0 > 0 \). Similarly, we obtain the same inequality for \( \psi \)

\[
(1 + c_0 \varepsilon^2) \int_{\Omega} \psi^2 \, dx \leq C \int_{\Omega} (\phi^2 + \psi^2) \, dx.
\]

Adding these two inequalities yields

\[
(1 + c_0 \varepsilon^2) \int_{\Omega} (\phi^2 + \psi^2) \, dx \leq C \int_{\Omega} (\phi^2 + \psi^2) \, dx.
\]

It implies \( \phi \equiv 0, \psi \equiv 0 \) if \( \varepsilon^2 > (C - 1)/c_0 \), i.e., \( (u, v) \) is a constant solution. \( \square \)

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