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Vanishing β-function for Grosse–Wulkenhaar model in a magnetic field ☆

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ABSTRACT

We prove that the β-function of the Grosse–Wulkenhaar model including a magnetic field vanishes at all order of perturbations. We compute the renormalization group flow of the relevant dynamic parameters and find a non-Gaussian infrared fixed point. Some consequences of these results are discussed.

1. Introduction

The Grosse–Wulkenhaar model (GWm) has been shown renormalizable at all orders of perturbations [1]. This noncommutative (NC) $\phi_4^{\star 4}$ scalar theory has been extensively studied in the very recent years [2–9]. Its most remarkable property, which holds both for the real and complex version, is asymptotic safeness. The β-function vanishes at all order of perturbations. This result has been proved in [9] by a careful combination of Dyson–Schwinger equations (DSe) and Ward identities (WI's). Aside from $\mathcal{N}=4$ supersymmetric non-Abelian gauge theory, it is the only yet known four-dimensional quantum field theory with this property.

It is natural to wonder whether there exist other NC renormalizable models with this striking property. The Gross–Neveu model [10] and the $O(N)$ and $U(N)$ invariant GWm's [11], when preserving Langmann–Szabo (LS) duality [12], have been also proved renormalizable. However the Gross–Neveu model is not asymptotically safe [13], and in the class of NC color models considered in [11], only the GWm itself (real or complex) is asymptotically safe.[1]

In this Letter, we consider the complex GWm with added magnetic field, namely the Langmann–Szabo–Zarembo (LSZ) model [14]. Langmann and co-workers proposed that small perturbations of this theory could produce solvable models with renormalizable interactions. Here, we study the RG flow of the coupling constant, still with $\Omega=1$ and a magnetic field satisfying $|B|<1$. We prove that this model is still asymptotically safe at all orders, and we calculate the RG flows of the two corresponding wave function constants q and p. Note that beyond its possible relevance for high energy physics, this LSZ model is a toy version of the quantum Hall effect, which can be considered a $2+1$ NC quantum field theory with non-relativistic propagator including Matsubara frequencies and Fermi surface [4].

The Letter is organized as follows. The next section introduces the GWm in a magnetic field. The main theorem on the vanishing β-function and its proof are developed in Section 3. Section 4 provides derivations of RG flows of the new parameters q and p. Further issues and conclusions are drawn in Section 5, while Appendix A summarizes some calculations.

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[1] Nevertheless, a large class of such models is UV asymptotic free hence susceptible of a full constructive analysis [15].

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2. Notations and considerations

The following action describes the complex NC $\phi_4^{\star 4}$ LSZ model in the Moyal–Euclidean space [14]

$$S = \int d^4x \Big\{ \partial_\mu \bar\phi \partial^\mu \phi + \mu^2 \bar\phi \phi + \Omega \bar\phi \tilde{x}_\mu \tilde{x}^\mu \phi + 2iB\bar\phi(\tilde{x}_\mu \partial^\mu)\phi + \frac{\lambda}{2} \bar\phi \star \phi \star \bar\phi \star \phi \Big\}, \tag{2.1}$$

where B is the magnetic field, $\tilde{x}_\nu = 2(\theta^{-1}_{\nu\mu})x^\mu$ and $\theta^{-1}_{\nu\mu}$ is the inverse of the antisymmetric matrix associated with the Moyal $\star$-product. The mass parameter is μ and $\Omega - B^2$ is an harmonic potential. The complex GWm is recovered at $B = 0$.

We use the matrix basis and the notations of Refs. [8,9], setting $\Omega = 1$.[2] As argued in [8,9], the flow of Ω goes rapidly to 1 in the UV limit. After rescaling the field and the coupling constant by two constants, the generating functional becomes

$$Z(\eta, \bar\eta) = \int d\phi\, d\bar\phi\, e^{-S(\bar\phi,\phi)+F(\bar\eta,\eta;\bar\phi,\phi)}, \tag{2.2}$$

$$S(\bar\phi,\phi) = \bar\phi X_R \phi + \phi X_L \bar\phi + A\bar\phi\phi + \frac{\lambda}{2}\phi\bar\phi\phi\bar\phi,$$

$$F(\bar\eta,\eta;\bar\phi,\phi) = \bar\phi\eta + \bar\eta\phi, \tag{2.3}$$

$$\phi = (\phi_{mn}), \qquad X_L = qm\delta_{mn}, \qquad X_R = pm\delta_{mn},$$

$$q = 1 + B, \quad p = 1 - B, \tag{2.4}$$

where traces are implicit, S is the action and F represents external sources. The quadratic part of the action is now expressed in term of the left and right matrix operators X_L and X_R with unequal weights q and p, respectively, and the new mass parameter is $A := 2 + \mu^2\theta/4$ [16].

The theory is stable for $|B| < \Omega \leqslant 1$. By convention, we consider B positive, hence $q > p$. Note that the model (2.3) can be seen as a (q, p)-deformed matrix theory with dual parameters $q > 1$ and $p < 1$. The GWm is recovered as $q \to 1$ and $p \to 1$ ($B \to 0$). Although this deformation is not in the ordinary sense of Chakrabarti and co-workers [17,18], the model renormalizability suggests that the deformed quantum algebras which are encountered in quantum group theory and quantum mechanics combined with nonlocal geometries may also be renormalizable. This is encouraging for the q-bosons studies and related theoretical models [18–20].

The bare propagator in the matrix base, at $\Omega = 1$, is

$$C_{mn;kl} = C_{mn}\delta_{ml}\delta_{nk}, \quad C_{mn} = \frac{1}{A + qm + pn}, \tag{2.5}$$

and we use notations

$$\delta_{ml} = \delta_{m_1 l_1}\delta_{m_2 l_2}, \qquad qm + pn = q(m_1 + m_2) + p(n_1 + n_2). \tag{2.6}$$

Feynman rules involve only orientable graphs with propagators oriented from $\bar\phi$ to ϕ. Arrows occur in alternating cyclic order at every vertex. For a field $\bar\phi_{mn}$, we call the index m a *left index* and n a *right index*. Consequently for the field ϕ_{kl}, k is a *right index* and l a *left index*.

In [4], the renormalizability of the model has been proved in the direct space at all order of perturbation. The renormalization of the four point function is essentially the same as the one of the real GWm. But the two point function renormalization is more subtle due to the left/right asymmetry of the model.

[2] The corresponding model is an independent non-identically distributed matrix model.

3. Coupling constant flow

We denote by $\Gamma^4(m,n,k,l)$ the amputated one particle irreducible (1PI) four point function with external indices m,n,k,l, and $\Sigma(m,n)$ the amputated 1PI two point function with external indices m,n (the self-energy). To define the wave function renormalization, we have to distinguish the left and right side of the ribbon and to attribute to each side its renormalization through the definitions

$$Z_L = 1 - \frac{1}{q}\partial_L \Sigma(0,0), \qquad Z_R = 1 - \frac{1}{p}\partial_R \Sigma(0,0) \tag{3.7}$$

which are the derivative of the self-energy with respect to left and right indices. The wave function renormalization is then $Z = \sqrt{Z_L Z_R}$ corresponding to a field rescaling $\phi \to Z^{1/2}\phi$. Therefore the effective coupling is defined as

$$\lambda^{\text{eff}} = -\frac{\Gamma^4}{Z^2} = -\frac{\Gamma^4}{Z_L Z_R}. \tag{3.8}$$

Theorem 3.1. *The equation*

$$\Gamma^4(0,0,0,0) = -\lambda\left(1 - \frac{1}{p}\partial_R \Sigma(0,0)\right)\left(1 - \frac{1}{q}\partial_L \Sigma(0,0)\right), \tag{3.9}$$

where λ is the bare constant, holds up to irrelevant terms to all orders of perturbation theory.

Irrelevant terms have to be understood with respect to power counting and include in particular all contributions of non-planar or planar graphs with more than one broken face. This theorem is proved in the remaining of this section following the method and ideas of [9] adapted to the left and right asymmetry.

3.1. (q, p)-Ward identities

The proof of Theorem 3.1 involve Ward identities (WI's) related to the $U(N)$ covariance of the theory. These WI's can be extended to a class of classical or quantum symmetry transformations (translations and dilatations) letting the action invariant up to a total derivative [21,22]. The following lemma holds.

Lemma 3.1. *The planar one broken external face correlation functions satisfy*

$$q(a-b)\big\langle[\bar\phi\phi]_{ab}\phi_{\nu a}\bar\phi_{b\nu}\big\rangle_c = \langle\phi_{\nu b}\bar\phi_{b\nu}\rangle_c - \langle\bar\phi_{a\nu}\phi_{\nu a}\rangle_c, \tag{3.10}$$

$$p(a-b)\big\langle[\phi\bar\phi]_{ab}\phi_{b\mu}\bar\phi_{\mu a}\big\rangle_c = \langle\bar\phi_{\mu b}\phi_{b\mu}\rangle_c - \langle\phi_{a\mu}\bar\phi_{\mu a}\rangle_c, \tag{3.11}$$

$$q(a-b)\big\langle\phi_{\alpha a}[\bar\phi\phi]_{ab}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\big\rangle_c = \langle\phi_{\alpha b}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c - \langle\phi_{\alpha a}\bar\phi_{a\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c, \tag{3.12}$$

$$p(a-b)\big\langle\bar\phi_{\alpha a}[\phi\bar\phi]_{ab}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\big\rangle_c = \langle\phi_{\alpha b}\bar\phi_{b\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c - \langle\phi_{\alpha a}\bar\phi_{a\nu}\phi_{\nu\delta}\bar\phi_{\delta\alpha}\rangle_c. \tag{3.13}$$

The rest of this subsection is devoted to the proof of this lemma.

Proof of Lemma 3.1.
$U(N)$ transformations. Let B be an infinitesimal hermitian matrix and consider the $U(N)$ group element $U = e^{\imath B}$ acting on the right and left on the matrix fields

$$\text{(right)} \quad \phi^U := \phi U, \qquad \bar\phi^U = U^\dagger\bar\phi, \tag{3.14}$$

$$\text{(left)} \quad \phi^U := U\phi, \qquad \bar\phi^U = \bar\phi U^\dagger. \tag{3.15}$$

The variation of the action under (3.14) and (3.15) is, at first order in B,

$$\delta_L S = \imath B(X_L\bar\phi\phi - \bar\phi\phi X_L), \qquad \delta_R S = \imath B(-X_R\phi\bar\phi + \phi\bar\phi X_R), \tag{3.16}$$

respectively. Similarly the variations of external sources are at first order

$$\delta_L F = \imath B(-\bar\phi\eta + \bar\eta\phi), \qquad \delta_R F = \imath B(-\eta\bar\phi + \phi\bar\eta). \tag{3.17}$$

As a consequence of the theory covariance, we have

$$\begin{aligned}\frac{\delta_L \ln Z}{\delta B_{ba}} = 0 &= \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi\left(-\frac{\delta_L S}{\delta B_{ba}} + \frac{\delta_L F}{\delta B_{ba}}\right)e^{-S+F}\\ &= \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi\, e^{-S+F}\big(-[X_L\bar\phi\phi - \bar\phi\phi X_L]_{ab}\\ &\quad + [-\bar\phi\eta + \bar\eta\phi]_{ab}\big),\end{aligned} \tag{3.18}$$

$$\begin{aligned}\frac{\delta_R \ln Z}{\delta B_{ba}} = 0 &= \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi\left(-\frac{\delta_R S}{\delta B_{ba}} + \frac{\delta_R F}{\delta B_{ba}}\right)e^{-S+F}\\ &= \frac{1}{Z(\bar\eta,\eta)}\int d\bar\phi\, d\phi\, e^{-S+F}\big(-[-X_R\phi\bar\phi + \phi\bar\phi X_R]_{ab}\\ &\quad + [-\eta\bar\phi + \phi\bar\eta]_{ab}\big).\end{aligned} \tag{3.19}$$

Two point function Ward identities. Applying the operator $\partial_\eta\partial_{\bar\eta}|_{\eta=\bar\eta=0}$ on the above expressions and analyzing the result in terms of connected components leads to

$$0 = \big\langle\partial_\eta\partial_{\bar\eta}\big(-[X_L\bar\phi\phi - \bar\phi\phi X_L]_{ab} + [-\bar\phi\eta + \bar\eta\phi]_{ab}\big)e^{F(\bar\eta,\eta)}\big|_0\big\rangle_c, \tag{3.20}$$

$$0 = \big\langle\partial_\eta\partial_{\bar\eta}\big([X_R\phi\bar\phi - \phi\bar\phi X_R]_{ab} + [-\eta\bar\phi + \phi\bar\eta]_{ab}\big)e^{F(\bar\eta,\eta)}\big|_0\big\rangle_c, \tag{3.21}$$

from which one deduces

$$\begin{aligned}&\left\langle\frac{\partial(\bar\eta\phi)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta} - \frac{\partial(\bar\phi\eta)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\right.\\ &\quad\left. - [X_L\bar\phi\phi - \bar\phi\phi X_L]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c = 0,\end{aligned} \tag{3.22}$$

$$\begin{aligned}&\left\langle\frac{\partial(\phi\bar\eta)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta} - \frac{\partial(\eta\bar\phi)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\right.\\ &\quad\left. + [X_R\phi\bar\phi - \phi\bar\phi X_R]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c = 0.\end{aligned} \tag{3.23}$$

By the definition of $X_{L,R}$, we get

$$\begin{aligned}&q(a-b)\left\langle[\bar\phi\phi]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c\\ &\quad = \left\langle\frac{\partial(\bar\eta\phi)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c - \left\langle\frac{\partial(\bar\phi\eta)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\right\rangle,\\ &-p(a-b)\left\langle[\phi\bar\phi]_{ab}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c\\ &\quad = \left\langle\frac{\partial(\phi\bar\eta)_{ab}}{\partial\bar\eta}\frac{\partial(\bar\phi\eta)}{\partial\eta}\right\rangle_c - \left\langle\frac{\partial(\eta\bar\phi)_{ab}}{\partial\eta}\frac{\partial(\bar\eta\phi)}{\partial\bar\eta}\right\rangle,\end{aligned}$$

and fixing $\bar\eta_{\beta\alpha}$ and $\eta_{\nu\mu}$, the previous relations become

$$q(a-b)\big\langle[\bar\phi\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\big\rangle_c = \langle\delta_{a\beta}\phi_{\alpha b}\bar\phi_{\mu\nu}\rangle_c - \langle\delta_{b\mu}\bar\phi_{a\nu}\phi_{\alpha\beta}\rangle_c, \tag{3.24}$$

$$-p(a-b)\big\langle[\phi\bar\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\big\rangle_c = \langle\delta_{b\alpha}\phi_{a\beta}\bar\phi_{\mu\nu}\rangle_c - \langle\delta_{a\nu}\bar\phi_{\mu b}\phi_{\alpha\beta}\rangle_c. \tag{3.25}$$

Restricting to planar with a single external face terms requires $[\alpha=\nu,\ a=\beta,\ b=\mu]$ and $[\mu=\beta,\ \nu=a,\ b=\alpha]$ for (3.24) and (3.25), respectively, and leads to (3.10)–(3.11).

Four point function Ward identities. Derivating further (3.20) and (3.21) yields

$$\begin{aligned}&q(a-b)\big\langle[\bar\phi\phi]_{ab}\partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\partial_{\bar\eta_2}(\bar\eta\phi)\partial_{\eta_2}(\bar\phi\eta)\big\rangle_c\\ &\quad = \big\langle\partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\big[\partial_{\bar\eta_2}(\bar\eta\phi)_{ab}\partial_{\eta_2}(\bar\phi\eta) - \partial_{\eta_2}(\bar\phi\eta)_{ab}\partial_{\bar\eta_2}(\bar\eta\phi)\big]\big\rangle_c + 1\\ &\quad \leftrightarrow 2,\end{aligned} \tag{3.26}$$

$$\begin{aligned}&-p(a-b)\big\langle[\phi\bar\phi]_{ab}\partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\partial_{\bar\eta_2}(\bar\eta\phi)\partial_{\eta_2}(\bar\phi\eta)\big\rangle_c\\ &\quad = \big\langle\partial_{\bar\eta_1}(\bar\eta\phi)\partial_{\eta_1}(\bar\phi\eta)\big[\partial_{\bar\eta_2}(\phi\bar\eta)_{ab}\partial_{\eta_2}(\bar\phi\eta) - \partial_{\eta_2}(\eta\bar\phi)_{ab}\partial_{\bar\eta_2}(\bar\eta\phi)\big]\big\rangle_c + 1\\ &\quad \leftrightarrow 2.\end{aligned} \tag{3.27}$$

A straightforward derivation at fixed $\bar\eta_{1,\beta\alpha}$, $\eta_{1,\nu\mu}$, $\bar\eta_{2,\delta\gamma}$ and $\eta_{2,\sigma\rho}$ gives

$$\begin{aligned}&q(a-b)\big\langle[\bar\phi\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\big\rangle_c\\ &\quad = \langle\phi_{\alpha\beta}\bar\phi_{\mu\nu}\delta_{a\delta}\phi_{\gamma b}\bar\phi_{\rho\sigma}\rangle_c - \langle\phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{a\sigma}\delta_{b\rho}\rangle_c\\ &\qquad + \langle\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\delta_{a\beta}\phi_{\alpha b}\bar\phi_{\mu\nu}\rangle_c - \langle\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\phi_{\alpha\beta}\bar\phi_{a\nu}\delta_{b\mu}\rangle_c,\end{aligned} \tag{3.28}$$

$$\begin{aligned}&-p(a-b)\big\langle[\phi\bar\phi]_{ab}\phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\big\rangle_c\\ &\quad = \langle\phi_{\alpha\beta}\bar\phi_{\mu\nu}\delta_{\gamma b}\phi_{a\delta}\bar\phi_{\rho\sigma}\rangle_c - \langle\phi_{\alpha\beta}\bar\phi_{\mu\nu}\phi_{\gamma\delta}\bar\phi_{\rho b}\delta_{a\sigma}\rangle_c\\ &\qquad + \langle\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\delta_{\alpha b}\phi_{a\beta}\bar\phi_{\mu\nu}\rangle_c - \langle\phi_{\gamma\delta}\bar\phi_{\rho\sigma}\phi_{\alpha\beta}\bar\phi_{\mu b}\delta_{a\nu}\rangle_c.\end{aligned} \tag{3.29}$$

Neglecting irrelevant graphs gives (3.12)–(3.13), completing the proof of the lemma. □

A simple induction proves that such identities hold for 2ℓ point functions with a left or right insertion, and for any integer ℓ, as depicted in Fig. 1.

3.2. Proof of the theorem

Besides the WI's, the proof of Theorem 3.1 uses left (right) DSe for four point functions with the two left (right) indices equal to m (see Fig. 2 for the left DSe), namely

$$\begin{aligned}G^4(0,m,0,m) &= G^4_{(1)}(0,m,0,m) + G^4_{(2)}(0,m,0,m)\\ &\quad + G^4_{(3)}(0,m,0,m),\end{aligned} \tag{3.30}$$

$$\begin{aligned}G^4(m,0,m,0) &= G^4_{(1)}(m,0,m,0) + G^4_{(2)}(0,m,0,m)\\ &\quad + G^4_{(3)}(m,0,m,0),\end{aligned} \tag{3.31}$$

where $G^4(m,n,k,l)$ is the connected planar single external face four point function. Eq. (3.30) is the left DSe whereas (3.31) is the right DSe. A clear comment of the meaning of these relations is provided in [9]. The term $G^4_{(2)}$ is zero by mass renormalization.

Denote $\partial_L F(m,n) := \partial_{m_i}F(m,n)$ and $\partial_R F(m,n) := \partial_{n_i}F(m,n)$, where m and n are respectively left and right indices. The following lemma holds.

Lemma 3.2. *Up to irrelevant terms we have*

$$G^4_{(1)}(0,m,0,m) = -\lambda\big(G^2(0,m)\big)^4\left(Z_R + \frac{A_r\partial_R\Sigma(0,0)}{p(pm+A_r)}\right)Z_L, \tag{3.32}$$

$$G^4_{(1)}(m,0,m,0) = -\lambda\big(G^2(m,0)\big)^4\left(Z_L + \frac{A_r\partial_L\Sigma(0,0)}{q(qm+A_r)}\right)Z_R, \tag{3.33}$$

$$G^4_{(3)}(0,m,0,m) = -G^4(0,m,0,m)\frac{A_r}{(pm+A_r)}\frac{\partial_R\Sigma^R(0,0)}{p-\partial_R\Sigma(0,0)}, \tag{3.34}$$

$$G^4_{(3)}(m,0,m,0) = -G^4(m,0,m,0)\frac{A_r}{(qm+A_r)}\frac{\partial_L\Sigma^L(0,0)}{q-\partial_L\Sigma(0,0)}, \tag{3.35}$$

where $\Sigma^{R,L}(0,0)$ is defined in (3.48) below and $G^2(m,n)$ is the connected planar one broken face two point function.

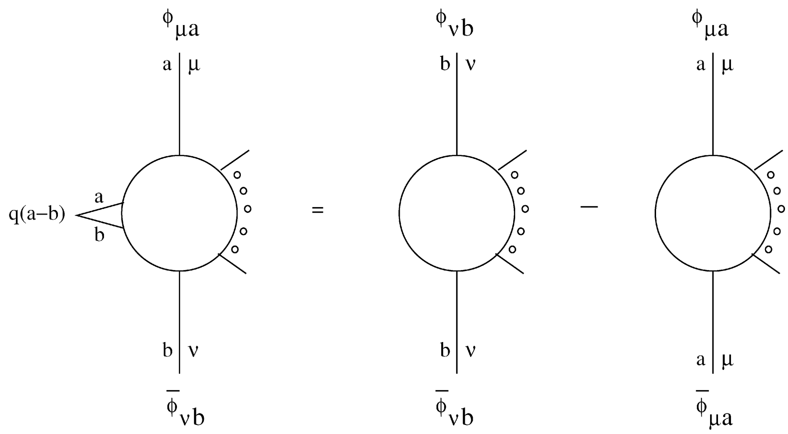

Fig. 1. q-Ward identity for a 2ℓ point function with insertion on the left face.

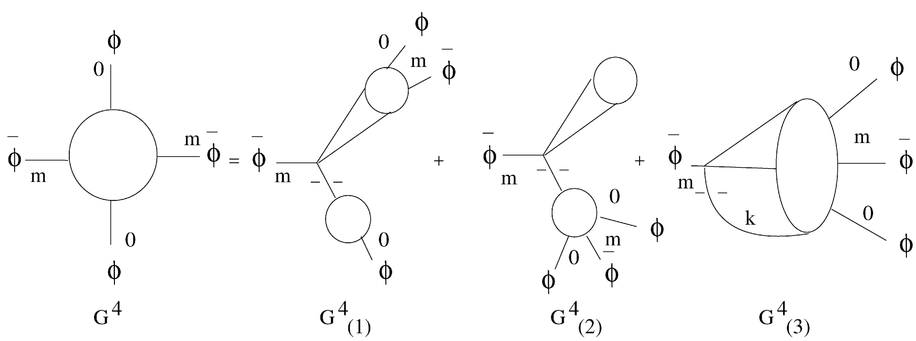

Fig. 2. The left Dyson equation.

Proof. Let us prove (3.32) and (3.34). The proof of the other expressions is analogous.

$G^2(m,n)$ is given by a well-known sum of a geometric series

$$G^2(m,n)=\frac{C_{mn}}{1-C_{mn}\Sigma(m,n)}=\frac{1}{C_{mn}^{-1}-\Sigma(m,n)}. \tag{3.36}$$

Let $G^L_{ins}(a,b;\ldots)$ be the planar one broken face connected function with one index jump on the left from a to b. Using (3.10), one writes

$$q(a-b)G^{2,L}_{ins}(a,b;\nu)=G^2(b,\nu)-G^2(a,\nu). \tag{3.37}$$

We note that WI's and the DSe have a meaning both in the bare (of mass $A=A_{\text{bare}}$) and in the mass renormalized theory ($A_r=A_{\text{bare}}-\Sigma(0,0)$). The latter case implies that every two point 1PI subgraph should be subtracted at 0 external indices. In the following, we use the mass-renormalized derivation.[3] The mass renormalized theory is free from quadratic divergences. Residual logarithmic divergences in the UV cutoff can be read off the effective series as argued in [8,15].

$G^4_{(1)}$ decomposes as

$$G^4_{(1)}(0,m,0,m)=-\lambda C_{0m}G^2(0,m)G^{2,L}_{ins}(0,0;m). \tag{3.38}$$

By the WI (3.37), we obtain

$$\begin{aligned}G^{2,L}_{ins}(0,0;m)&=\lim_{\zeta\to0}G^2_{ins}(\zeta,0;m)\\&=\frac{1}{q}\lim_{\zeta\to0}\frac{G^2(0,m)-G^2(\zeta,m)}{\zeta}\\&=-\frac{1}{q}\partial_L G^2(0,m).\end{aligned} \tag{3.39}$$

Using the form (3.36) of $G^2(0,m)$, we get

$$\begin{aligned}G^4_{(1)}(0,m,0,m)&=-\frac{\lambda}{q}C_{0m}\frac{C_{0m}C_{0m}^2[q-\partial_L\Sigma(0,m)]}{[1-C_{0m}\Sigma(0,m)](1-C_{0m}\Sigma(0,m))^2}\\&=-\frac{\lambda}{q}\big[G^2(0,m)\big]^4\frac{C_{0m}}{G^2(0,m)}\big[q-\partial_L\Sigma(0,m)\big].\end{aligned} \tag{3.40}$$

A Taylor expansion gives the self energy up to irrelevant terms [7],

$$\Sigma(m,n)=\Sigma(0,0)+m\partial_L\Sigma(0,0)+n\partial_R\Sigma(0,0). \tag{3.41}$$

Keeping in mind that $C_{0m}^{-1}=pm+A_r$, we have (again up to irrelevant terms)

$$G^2(0,m)=\frac{1}{pm+A_{\text{bare}}-\Sigma(0,m)}=\frac{1}{m[p-\partial_R\Sigma(0,0)]+A_r}, \tag{3.42}$$

and

$$\frac{C_{0m}}{G^2(0,m)}=\frac{1}{p}\big(p-\partial_R\Sigma(0,0)\big)+\frac{A_r}{p(pm+A_r)}\partial_R\Sigma(0,0). \tag{3.43}$$

Substituting (3.43) in (3.40) we get

$$\begin{aligned}&G^4_{(1)}(0,m,0,m)\\&\quad=-\lambda\big[G^2(0,m)\big]^4\left(\frac{1}{p}\big(p-\partial_R\Sigma(0,0)\big)+\frac{A_r}{p(pm+A_r)}\partial_R\Sigma(0,0)\right)\\&\qquad\times\left[\frac{1}{q}\big(q-\partial_L\Sigma(0,m)\big)\right].\end{aligned} \tag{3.44}$$

To evaluate $G^4_{(3)}(0,m,0,m)$, we need to “open” the face “on the right” in the k loop in the third term of Fig. 2. The left bare correlation functions are given by

[3] An equivalence with the bare theory could be deduced from [9].

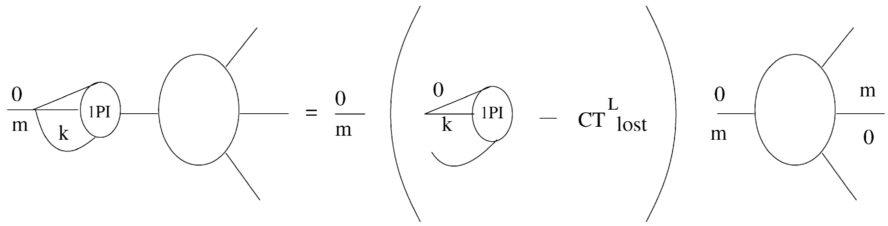

Fig. 3. Two point left insertion and opening of the loop with index k.

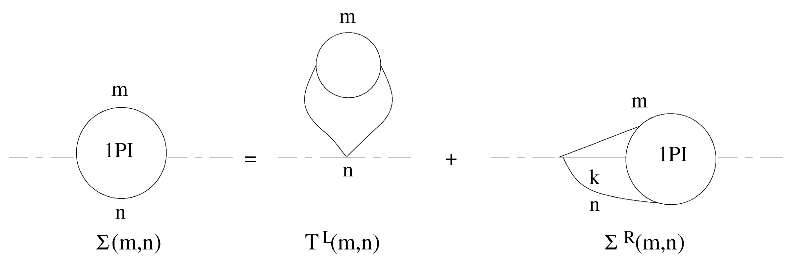

Fig. 4. The self energy.

$$G^{4,\text{bare}}_{(3)}(0,m,0,m) = -\lambda C_{0m} \sum_k G^{4,\text{bare},L}_{ins}(k,0;m,0,m). \tag{3.45}$$

The face indexed by k may belong to a 1PI two point insertion in $G^4_{(3)}$ (see Fig. 3). In that case, because we use the mass renormalized expansion, one has to introduce a counterterm in order to compensate the one lost during the "opening" process. In other terms, we have

$$\begin{aligned} G^4_{(3)}(0,m,0,m) &= -\lambda C_{0m} \sum_k G^{4,L}_{ins}(0,k;m,0,m) \\ &\quad - C_{0m}\big(CT^L_{lost}\big) G^4(0,m,0,m). \end{aligned} \tag{3.46}$$

It turns out that all two point function counterterms contribute to CT^L_{lost} except those of the generalized left tadpole. We write

$$\Sigma(m,n) = T^L(m,n) + \Sigma^R(m,n) \tag{3.47}$$

with T^L the generalized left tadpole contribution and Σ^R the rest. $T^L(m,n)$ is a left border insertion hence does not depend upon the right index n (see Fig. 4).

With these notations the missing mass counterterm is given by

$$CT^L_{lost} = \Sigma^R(0,0) = \Sigma(0,0) - T^L. \tag{3.48}$$

To compute $\Sigma^R(0,0)$, we open its face indexed by k and use (3.10) to get

$$\begin{aligned} \Sigma^R(0,0) &= -\frac{\lambda}{G^2(0,0)} \sum_k G^{2,L}_{ins}(0,k;0) \\ &= -\frac{\lambda}{q} \frac{1}{G^2(0,0)} \sum_k \frac{1}{k}\big[G^2(0,0) - G^2(k,0)\big] \\ &= -\frac{\lambda}{q} \sum_k \frac{1}{k}\left(1 - \frac{G^2(k,0)}{G^2(0,0)}\right). \end{aligned} \tag{3.49}$$

Then (3.46) and (3.49) imply that

$$\begin{aligned} &G^4_{(3)}(0,m,0,m) \\ &\quad = -\lambda C_{0m} \sum_k G^{4,L}_{ins}(0,k;m,0,m) \\ &\qquad - \frac{(-\lambda)}{q} C_{0m} G^4(0,m,0,m) \sum_k \frac{1}{k}\left(1 - \frac{G^2(k,0)}{G^2(0,0)}\right). \end{aligned} \tag{3.50}$$

Eq. (3.12) reexpresses the first term in (3.50)

$$\begin{aligned} &-\lambda C_{0m} \sum_k G^{4,L}_{ins}(0,k;m,0,m) \\ &\quad = -\frac{\lambda}{q} C_{0m} \sum_k \frac{1}{k}\big(G^4(0,m,0,m) - G^4(k,m,0,m)\big). \end{aligned} \tag{3.51}$$

The second term in (3.51) is at least cubic in k, hence irrelevant. The above sums over k for $G^4(k,m,0,m)$ are always convergent (see [9]). We inject (3.51) in (3.50) and obtain

$$G^4_{(3)}(0,m,0,m) = -\frac{\lambda}{q} C_{0m} \frac{G^4(0,m,0,m)}{G^2(0,0)} \sum_k \frac{G^2(k,0)}{k}. \tag{3.52}$$

From (3.42), we obtain

$$\begin{aligned} \sum_k \frac{G^2(k,0)}{k} &= \sum_k \frac{G^2(k,0)}{k}\left(\frac{1}{G^2(0,1)} - \frac{1}{G^2(0,0)}\right) \\ &\quad \times \frac{1}{(p - \partial_R \Sigma(0,0))}. \end{aligned} \tag{3.53}$$

Performing the same manipulations as in (3.49), we express

$$\begin{aligned} \Sigma^R(0,1) &= -\frac{\lambda}{q} \sum_k \frac{1}{k}\left(1 - \frac{G^2(k,1)}{G^2(0,1)}\right) \\ &= -\frac{\lambda}{q} \sum_k \frac{1}{k}\left(1 - \frac{G^2(k,0)}{G^2(0,1)}\right) \end{aligned} \tag{3.54}$$

up to an irrelevant term. Substituting (3.49) and (3.54) in (3.53),

$$\begin{aligned} -\lambda \sum_k \frac{G^2(k,0)}{k} &= \frac{q(\Sigma^R(0,0) - \Sigma^R(0,1))}{p - \partial_R \Sigma(0,0)} \\ &= -\frac{q \partial_R \Sigma^R(0,0)}{p - \partial_R \Sigma(0,0)}. \end{aligned} \tag{3.55}$$

Therefore,

$$\begin{aligned} G^4_{(3)}(0,m,0,m) &= -C_{0m} G^4(0,m,0,m) \frac{1}{G^2(0,0)} \frac{\partial_R \Sigma^R(0,0)}{(p - \partial_R \Sigma(0,0))} \\ &= -G^4(0,m,0,m) \frac{A_r \partial_R \Sigma^R(0,0)}{(pm + A_r)(p - \partial_R \Sigma(0,0))} \end{aligned} \tag{3.56}$$

which achieves the proof of Lemma 3.2. □

Proof of Theorem 3.1. Plugging (3.44) and (3.56) in (3.30), one has

$$G^4(0,m,0,m)\left(1+\frac{A_r\,\partial_R\Sigma^R(0,0)}{(pm+A_r)(p-\partial_R\Sigma(0,0))}\right)$$
$$=-\lambda\big(G^2(0,m)\big)^4\left(\frac{1}{p}\big(p-\partial_R\Sigma(0,0)\big)\right.$$
$$\left.+\frac{A_r}{p(pm+A_r)}\partial_R\Sigma(0,0)\right)\frac{1}{q}\big(q-\partial_L\Sigma(0,m)\big). \tag{3.57}$$

Multiplying (3.57) by $(p-\partial_R\Sigma(0,0))/p$, amputating four times and neglecting the irrelevant differences $\Gamma^4(0,m,0,m)-\Gamma^4(0,0,0,0)$ and $\partial_L\Sigma(0,m)-\partial_L\Sigma(0,0)$, we finally find

$$\Gamma^4(0,0,0,0)=-\lambda\left(1-\frac{1}{q}\partial_L\Sigma(0,0)\right)\left(1-\frac{1}{p}\partial_R\Sigma(0,0)\right) \tag{3.58}$$

which completes the proof of (3.9). □

4. One loop RG flow of (q, p) parameters

The left and right wave function renormalizations Z_L and Z_R, respectively, determine the RG flow of q and p. To compute RG flows, we need to introduce some slice decomposition [2]. After renormalization of the field $\phi\to\phi(Z_RZ_L)^{1/4}$, the discrete RG flow equation are

$$\lambda_{i-1}=\lambda_i,\qquad q_{i-1}=q_i\left(\frac{Z_L}{Z_R}\right)^{\frac12},\qquad p_{i-1}=p_i\left(\frac{Z_R}{Z_L}\right)^{\frac12}. \tag{4.59}$$

At one loop, only the planar "up" and "down" tadpoles [8] contribute to the self-energy $\Sigma(m,n)$. We get a factor of symmetry of 2 so that

$$\Sigma(m,n)=-\lambda\sum_{r\in\mathbb{N}^2}(C_{mr}+C_{rn}), \tag{4.60}$$

where C_{mr} and C_{rn} are the bare propagators.

A direct calculation yields, with $r\in\mathbb{N}^2$,

$$Z_L=1-\lambda\sum_r\frac{1}{(pr+A)^2},\qquad Z_R=1-\lambda\sum_r\frac{1}{(qr+A)^2}. \tag{4.61}$$

Hence, at first order in λ,

$$\sqrt{\frac{Z_L}{Z_R}}=1-\frac12\lambda\sum_r\left[\frac{1}{(pr+A)^2}-\frac{1}{(qr+A)^2}\right]+O\big(\lambda^2\big). \tag{4.62}$$

The logarithmically divergent part of these sums governs the flows. In a slice corresponding to $r_1,r_2\in[M^{i-1},M^i]$, we have

$$\sum_{r_1,r_2=M^{i-1}}^{M^i}\frac{1}{(qr+A)^2}=\frac{1}{q^2}\sum_{r_1,r_2=M^{i-1}}^{M^i}\frac{1}{(r+\frac{A}{q})^2}$$
$$=\frac{1}{q^2}\kappa+O\big(M^{-i}\big), \tag{4.63}$$

where the constant κ is independent of i. We obtain at one loop,

$$q_{i-1}=q_i\left[1-\frac{\lambda_i}{2}\left(\frac{1}{p_i^2}-\frac{1}{q_i^2}\right)\kappa\right],$$
$$p_{i-1}=p_i\left[1-\frac{\lambda_i}{2}\left(\frac{1}{q_i^2}-\frac{1}{p_i^2}\right)\kappa\right] \tag{4.64}$$

from which the discrete flows are deduced

$$\frac{dq_i}{di}=\frac{\lambda_i}{2}q_i\left(\frac{1}{p_i^2}-\frac{1}{q_i^2}\right)\kappa,\qquad \frac{dp_i}{di}=\frac{\lambda_i}{2}p_i\left(\frac{1}{q_i^2}-\frac{1}{p_i^2}\right)\kappa. \tag{4.65}$$

This leads directly to

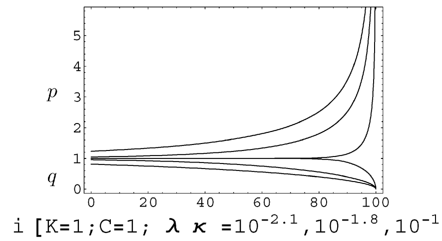

Fig. 5. RG flow of $q(i)$ and $p(i)$ versus i with cutoff $\Lambda=100$ and $p_{uv}=10^{-6}$.

$$\frac{dq_i}{q_idi}+\frac{dp_i}{p_idi}=0\quad\Leftrightarrow\quad q_ip_i=K, \tag{4.66}$$

where K is some positive constant. We substitute $q_i=K/p_i$ in (4.65) and find the solutions

$$p(i)^2=K\frac{e^{2\lambda_i\kappa(\Lambda-i)}(p_{uv}^2/K+1)+p_{uv}^2/K-1}{e^{2\lambda_i\kappa(\Lambda-i)}(p_{uv}^2/K+1)-(p_{uv}^2/K-1)}, \tag{4.67}$$

$$p_{uv}^2=-K\frac{e^{2\lambda_i\kappa(\Lambda-i)}(p(i)^2/K-1)+p(i)^2/K+1}{e^{2\lambda_i\kappa(\Lambda-i)}(p(i)^2/K-1)-(p(i)^2/K+1)}, \tag{4.68}$$

where Λ stands for the UV cutoff, and p_{uv} the bare value of p (see Appendix A). Similar expressions of q_i and q_{uv} follow. Graphic representations of $p(i)$ and $q(i)$ versus i for various values of the parameters are given in Fig. 5.

5. Conclusion

We have proved that the β-function governing the RG flow of the coupling constant of the complex GWm with magnetic field vanishes at all orders of perturbation. We have also computed at one loop the RG flows of the new wave function parameters (q,p). The non-Gaussian fixed point $p=q$ lies on the IR side rather than the UV one.

The motivation for studying these models in magnetic field comes from the quantum Hall effect physics, although this physics requires a different propagator and $2+1$ dimensions. We hope to describe the Hall plateaux as fixed points of a noncommutative RG flow. The results for the particular toy model considered here may not seem encouraging at first sight. Indeed the only infinite direction of such noncommutative RG is the UV one and that is where we do not find fixed points. However recall that the noncommutative interpretation of long and short distances is subtle, IR and UV in NC really referring to low versus high energy. The physics at small energies in a Hall fluid is well described in terms of anyons, whereas electrons appear as high energy particles [23]. Hence anyonic physics may be described by IR rather than UV fixed points.

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Appendix A

Consider the linear differential equation given by (4.65) after substituting $q_i=K/p_i$, $p_i=p(i)$ and $q_i=q(i)$,

$$\frac{dp(i)}{di}=\frac{\lambda_i}{2}p(i)\left(\frac{p(i)^2}{K^2}-\frac{1}{p^2(i)}\right)\kappa. \tag{A.1}$$

Separating variables and putting $u(i)=p(i)^2$, we get

$$K\frac{du}{u^2-1}=\lambda_i\kappa\,di \tag{A.2}$$

which can be easily integrated. Let us remark that $\lambda_i \simeq \lambda$, Theorem 3.1 having proved that the flow of λ_i is actually bounded. Between the ith slice and the UV cutoff Λ, $p(i)$ varies from $p(i)$ to its bare value p_{uv}, so that

$$\frac{1}{2}\left\{\ln\left(\frac{p_{uv}^2/K-1}{p_{uv}^2/K+1}\right)-\ln\left(\frac{p(i)^2/K-1}{p(i)^2/K+1}\right)\right\}=\lambda\kappa(\Lambda-i). \tag{A.3}$$

The bare and running values of p are related through

$$\frac{p(i)^2/K+1}{p(i)^2/K-1}=e^{2\lambda\kappa(\Lambda-i)}\frac{p_{uv}^2/K+1}{p_{uv}^2/K-1}, \tag{A.4}$$

and we recover (4.67) and (4.68).

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