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Symmetrisation of n-operads and compactification of real configuration spaces

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Abstract

It is well known that the forgetful functor from symmetric operads to nonsymmetric operads has a left adjoint Sym_1 given by product with the symmetric group operad. It is also well known that this functor does not affect the category of algebras of the operad. From the point of view of the author's theory of higher operads, the nonsymmetric operads are 1-operads and Sym_1 is the first term of the infinite series of left adjoint functors Sym_n, called symmetrisation functors, from n-operads to symmetric operads with the property that the category of one object, one arrow, ..., one $(n-1)$-arrow algebras of an n-operad A is isomorphic to the category of algebras of $Sym_n(A)$.

In this paper we consider some geometrical and homotopical aspects of the symmetrisation of n-operads. We follow Getzler and Jones and consider their decomposition of the Fulton–Macpherson operad of compactified real configuration spaces. We construct an n-operadic counterpart of this compactification which we call the Getzler–Jones operad. We study the properties of Getzler–Jones operad and find that it is contractible and cofibrant in an appropriate model category. The symmetrisation of the Getzler–Jones operad turns out to be exactly the operad of Fulton and Macpherson. These results should be considered as an extension of Stasheff's theory of 1-fold loop spaces to n-fold loop spaces $n \geqslant 2$. We also show that a space X with an action of a contractible n-operad has a natural structure of an algebra over an operad weakly equivalent to the little n-disks operad. A similar result holds for chain operads. These results generalise the classical Eckman–Hilton argument to arbitrary dimension.

Finally, we apply the techniques to the Swiss-Cheese type operads introduced by Voronov and prove analogous results in this case.

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Contents

1. Introduction

This is the second paper in a sequence of papers devoted to the relations between higher categories and n-fold loop space theory. In the first paper [4] we developed the necessary categorical techniques which allow us to go back and forth between n-operads and classical symmetric operads. The main goal of this paper is to clarify the geometric and homotopy theoretic aspects of this theory.

To do this we restrict ourselves to a class of so-called pruned $(n-1)$-terminal n-operads. This is a slightly smaller category of n-operads than we considered in [4] but it is big enough to include most applications we have in mind. The reason is that the functor of desymmetrisation from symmetric operads to n-operads [4] can be factorised through the category of pruned n-operads. Moreover, as in the unpruned case this desymmetrisation functor preserves endomorphism operads. This allows us to construct a theory of symmetrisation very much in parallel to the unpruned case. It turns out that pruned n-operads are easier to handle from the combinatorial point of view. We have a conjecture, however, that the main results of this paper are also true in the case of $(n-1)$-terminal n-operads but so far we have been unable to prove it in this generality.

We apply the categorical methods of [4] to the category of pruned $(n-1)$-terminal n-operads and to the even smaller class of reduced $(n-1)$-terminal n-operads. These methods immediately imply the existence of some categorical symmetric operads $\mathbf{ph}^n$ and $\mathbf{rh}^n$, which represent the theories of internal $(n-1)$-terminal pruned n-operads and internal $(n-1)$-terminal reduced n-operads inside categorical symmetric operads, in full analogy with the categorical operad $\mathbf{h}^n$ in [4]. Our first significant result here is Theorem 8.5 which asserts that the simplicial operads $N(\mathbf{ph}^n)$ and $N(\mathbf{rh}^n)$, where N is the nerve functor, are E_n-operads in the category of simplicial symmetric operads and reduced simplicial symmetric operads respectively.

Together with Theorems 3.2 and 4.3, these theorems show that once we have a space with an action of a contractible pruned $(n-1)$-terminal n-operad there is an action of an E_n-operad on this space. An analogous result holds for the reduced operads in chain complexes. As we conjectured in [4] this should give a very natural proof of Deligne's conjecture answering a question by Kontsevich [13].[2]

There are, however, other results in the present paper which we believe are significant. First we observe that the desymmetrisation of the operad of Fulton and Macpherson $\mathbf{fm}^n$ of compactified real configuration spaces [9,14] contains a contractible reduced $(n-1)$-terminal n-operad which we call the Getzler–Jones operad $\mathbf{GJ}^n$. Actually, this operad was discovered by Getzler and Jones in their remarkable preprint [9]. The apparatus of n-operads did not exist at that time and Getzler and Jones attempted to express the properties of $\mathbf{GJ}^n$ in terms of a natural subdivision of the operad $\mathbf{fm}^n$. It turned out, however, that Getzler–Jones subdivision does not give a cellular structure compatible with the operadic structure of $\mathbf{fm}^n$ as was first observed by D. Tamarkin (see [25] for an explanation of Tamarkin's counterexample). This counterexample implied considerable technical difficulties in the proof of Deligne's conjecture.

The second implicit appearance of $\mathbf{GJ}^n$ was in the Kontsevich and Soibelman paper [14]. For $n=2$ they considered closed contractible subsets X_T of $\mathbf{fm}^2$ where T is a finite set with two complementary orders on it (see Definition 2.2). They used some properties of X_T to prove Deligne's conjecture for A_∞-algebras. We show that the space $\mathbf{GJ}^n_T$ of arity T is indeed equal to the Kontsevich–Soibelman X_T for a pruned n-tree T (see Section 2 for the explanation of the connection between trees and complementary orders, and Section 7 for the definition of X_T for any n).

We show that $\mathbf{GJ}^n$ can be considered as a natural analogue of $\mathbf{fm}^n$ in the category of reduced $(n-1)$-terminal n-operads. In particular, set theoretically it is a free reduced $(n-1)$-terminal n-operad on the reduced n-collection of Getzler–Jones cells. This last result leads to the theorem that the symmetrisation of $\mathbf{GJ}^n$ is exactly $\mathbf{fm}^n$ which is the basis for the main results of our paper. We also think that this gives an interesting new insight to the geometry of $\mathbf{fm}^n$. We are going to continue this study in the next paper of this series.

The operad $\mathbf{GJ}^n$ is still not cellular in the strong sense that it is not a geometric realisation of a poset of cells of a regular CW-complex. Tamarkin's counterexample works well in this case too. Nevertheless, its unbased version $\mathbf{GJ}^n_\circ$ is a cellular object in the category of unbased reduced $(n-1)$-terminal n-operads in the model category theoretic sense, and in particular it is a cofibrant contractible operad. The term unbased here means that we forget about nullary operations of our operads. There is also an n-operad $\mathbf{RH}^n_\circ$, which is an unbased categorical reduced n-operad freely generated by its internal reduced n-operad. The geometric realisation of the nerve of this operad is cofibrant and contractible and is strongly homotopy equivalent to the operad $\mathbf{GJ}^n_\circ$. This implies a homotopy equivalence between the geometric realisation of $N(\mathbf{rh}^n_\circ)$ and $\mathbf{fm}^n_\circ$. This operad $\mathbf{rh}^n_\circ$ (the unbased categorical reduced symmetric operad freely generated by its internal reduced n-operad) plays, therefore, the role of the nonexistent poset operad of cells of the Getzler and Jones decomposition.

Perhaps, the most interesting result of the above study of the combinatorics of the Getzler–Jones operad is the following generalisation of Stasheff's classical theory of A_∞-spaces. The closure of a Getzler–Jones cell $K_T = cl(\mathrm{Mod}^n_T)$ inside $\mathbf{GJ}_T$ is a manifold with corners homeomorphic to the ball of dimension $E(T)-n-1$, where $E(T)$ is the number of edges of the

[2] During the preparation of this paper, D. Tamarkin informed me that he indeed obtained a proof of Deligne's conjecture by exhibiting a contractible 2-operad acting naturally on a 2-graph consisting of DG-categories, DG-functors and the complex of their derived natural transformations. In the particular case of a DG-category with one object and transformations of the identity functor, we obtain the Hochschild complex of an associative algebra and then we apply our Theorem 8.7 [22].

n-tree T. The cellularity of $\mathbf{GJ}^n_\circ$ means that an action of $\mathbf{GJ}^n_\circ$ on a pointed space X can be described as an inductive process of extension of higher homotopies from the boundary of K_T to its interior in exact analogy with Stasheff's description of A_∞-spaces. And our Theorem 7.3 states that the collection $K_\bullet$, where $\bullet$ runs over the set of pruned n-trees, gives full coherence conditions for E_n-spaces. If $n = 1$ the collection $K_\bullet$ is the sequence of associahedra [9] and we get Stasheff's theorem.

The difference between 1-dimensional and higher-dimensional cases appears in the existence of some cells (Tamarkin's cells) in $\mathbf{GJ}^n$ which are not completely on the boundary of K_T. This is not excessive information, however, just a defect of our language when we try to express the coherence laws in terms of an action of an $(n-1)$-terminal n-operad. There is a way of avoiding this problem but we will have to pay a price by using more sophisticated n-operads which have full source and target operations [3]. We will consider this subject in a forthcoming paper.

Finally, in the last section of this paper we apply our techniques to the case of Swiss-Cheese type operads [24]. The advantage of our categorical methods is that we have nothing to prove here once we put the right definitions of our main categories and functors in place. We deduce immediately a symmetrisation formula for Swiss-Cheese type n-operads[3] and other Swiss-Cheese analogues of the results for classical operads.

We hope that similar results can be obtained for symmetrisation of some other important coloured operads; for example, operads for morphisms between E_n-algebras. This should lead to a better understanding of coherence conditions for such morphisms.

Remark 1.1. We will freely use the terminology from [4] concerning n-trees, $(n-1)$-terminal n-operads and their algebras. Notice, however, that the main objects for us here are $(n-1)$-terminal n-algebras of our n-operads. Roughly speaking such an algebra is an object X of our basic symmetric monoidal category V together with an action $A_T \otimes X^{\otimes^k} \to X$ where k is the number of tips of T satisfying some natural conditions. An n-operad A may have, however, more complicated types of algebras which involve source and target operations but we do not need them here. So we will speak simply about category of algebras of A having in mind the subcategory of its $(n-1)$-terminal algebras. We refer the reader to [3,4] for more discussion about this issue.

2. Complementary orders and pruned n-trees

Here we consider a combinatorial techniques which will be used further to develop a theory of pruned n-operads. The machinery of bar-codes from [9] is an equivalent language but we prefer to work with the notion of complementary orders introduced by Kontsevich and Soibelman in [14].

Definition 2.1. A partial order on a set X is a transitive, antireflective relation on X. It is called linear if any two elements are comparable.

Antireflective here means that the diagonal is always in the complement relation. We choose this terminology just for technical reasons. Of course, we always can add the diagonal to our relations and work with reflective relations. So if we are given a partial order $<$ we will often use

[3] I am grateful to D. Tamarkin for encouraging me to look at the action of the Swiss-Cheese operad from the n-operadic point of view and for sending me a preliminary version of his papers concerning the action of Swiss-Cheese operads on associative algebras and their Hochschild complexes [23].

the notation $a \leqslant b$ to mean that $a < b$ or $a = b$. This does not lead to any trouble. Observe, also that a partial order in our sense is always antisymmetric.

Definition 2.2. (Kontsevich–Soibelman [14]) Let I be a set. Suppose we have an n-tuple of partial orders $\Xi = (<_0, \dots, <_{n-1})$ on I. We call them complementary orders provided any two elements $i, j \in I$ can be compared with respect to exactly one order $<_0, \dots, <_{n-1}$.

Definition 2.3. A set with a given n-tuple of complementary orders on it will be called an n-ordered set.

Lemma 2.1. *Let T be a pruned n-tree*

$$T = [k_n] \xrightarrow{\rho_{n-1}} [k_{n-1}] \xrightarrow{\rho_{n-2}} \cdots \xrightarrow{\rho_0} [1].$$

Then the relation: $i <_p j$ if and only if $i < j$ in $[k_n]$ and

$$\rho_{n-1} \cdot \dots \cdot \rho_p(i) = \rho_{n-1} \cdot \dots \cdot \rho_p(j)$$

but

$$\rho_{n-1} \cdot \dots \cdot \rho_{p+1}(i) \neq \rho_{n-1} \cdot \dots \cdot \rho_{p+1}(j);$$

defines n complementary orders on the set of tips of T.

Proof. The proof is obvious. $\square$

In fact we can characterise pruned n-trees in the following way:

Lemma 2.2. *Suppose we are given n-complementary orders $<_0, \dots, <_{n-1}$ on a finite set X such that*

- *if $i <_p j$ and $j <_r l$ then $i <_{\min(r,p)} j$*

then there exist a linear order on X and a pruned n-tree T such that the ordinal of its tips is $X \simeq [k_n]$ and the complementary orders $<_0, \dots, <_{n-1}$ are determined by T.

Proof. The linear order $<$ on X is defined by the requirement that $i < j$ if and only if there exists a p such that $i <_p j$. Let $[k_n]$ be the corresponding ordinal.

Suppose that there exists a triple $i < j < l$ from $[k_n]$ such that $i <_{n-1} < l$ but $i <_r j$ for some $r < n-1$ then $i <_r j$ so $i <_r l$; contradiction. Similarly for the other side. So the order $<_{n-1}$ determines a subdivision on k_{n-1} intervals of the ordinal $[k_n]$. This can be considered as a surjection of ordinals $[k_n] \to [k_{n-1}]$. Obviously we have $(n-1)$ complementary orders on $[k_{n-1}]$ which satisfy the conditions of the lemma. So we can proceed and construct a tree T. $\square$

Definition 2.4. We will call an n-ordered set X totally n-ordered if it satisfies the conditions of Lemma 2.2. If $X = \{1, \dots, k\}$ is totally n-ordered and the induced linear order makes it equal to the ordinal $[k]$ then we call X an n-ordinal. This includes the case of a terminal ordinal (one element set and empty n complementary orders) and of an initial ordinal (empty set).

Remark 2.1. We will often consider a special n-ordinal for which only one order $<_l$ is non-empty. We will use the notation M_l^k for such an ordinal on $\{1,\dots,k\}$ (see [4] for the picture of corresponding pruned tree with the same notation).

Definition 2.5. Let X and Y be two n-ordered sets. An order preserving map (or a map of n-ordered sets) from X to Y is a map $f:X\to Y$ such that $i<_p j$ in X implies that $f(i)\leqslant_r f(j)$ for some $r\geqslant p$ or $f(j)<_r f(i)$ for $r>p$.

Definition 2.6. Suppose we are given two n-tuples of complementary orders Ξ_1 and Ξ_2 on the same set X. We will say that Ξ_1 dominates Ξ_2 (notation $\Xi_2\triangleleft\Xi_1$) if $i<_p j$ in Ξ_1 implies $i<_r j$ for some $r\geqslant p$ or $j<_r i$ for $r>p$ in Ξ_2.

Of course, Ξ_1 dominates Ξ_2 if and only if the identity map $X\to X$ is a map of n-ordered sets.

Lemma 2.3. *Let*

$$T=[k_n]\xrightarrow{\rho_{n-1}}[k_{n-1}]\xrightarrow{\rho_{n-2}}\cdots\xrightarrow{\rho_0}[1]$$

and

$$S=[s_n]\xrightarrow{\xi_{n-1}}[s_{n-1}]\xrightarrow{\xi_{n-2}}\cdots\xrightarrow{\xi_0}[1]$$

be two pruned n-trees. Let $f:[k_n]\to[s_n]$ be a map. Then there exists a map of trees $\sigma:T\to S$ such that $\sigma_n=f$ if and only if f is a map of n-ordered sets.

Proof. If such a σ exists the order preserving property of f is obvious. Now we want to reconstruct a σ from f. We put $\sigma_n=f:[k_n]\to[s_n]$, of course. Now take a point a from $[k_{n-1}]$ then its preimage under ρ_{n-1} consists of an interval and we define $\sigma_{n-1}(a)=\xi_{n-1}(\sigma_n(i))$, where i is an arbitrary element from the preimage of a. Since f preserves order, this definition is correct. Indeed, if $i<_{n-1} j$ is another element from the preimage then $f(i)\leqslant_{n-1} f(j)$ and, hence, their images under ξ_{n-1} are equal. Now we can check quite easily that the constructed σ_{n-1} preserves the $(n-1)$ complementary orders determined by ∂S and ∂T and we can proceed with our construction. □

We assume that the only degenerate pruned n-tree is z^nU_0. We thus have

Theorem 2.1. *The category Ord_n of n-ordinals and their order preserving maps is isomorphic to the category of pruned trees and their morphisms.*

One can consider the poset $\mathbf{J}_X^n$ of all total complementary n-orders on a fixed set X with respect to the domination relation. If $X=\{1,\dots,k\}$ we will denote this poset by $\mathbf{J}_k^n$. The symmetric group Σ_k acts naturally on $\mathbf{J}_k^n$.

Let $\Upsilon_n(k)$ be the subcategory of Ω_n whose objects are pruned n-trees with k tips and whose morphisms are morphisms of trees which are bijections on tips. We call such a morphism *a quasibijection*. Theorem 2.1 implies

Corollary 2.1.1. *There is a natural isomorphism of categories*

$$\mathbf{J}_k^n/\Sigma_k\to\Upsilon_n(k).$$

Remark 2.2. The set $\mathbf{J}^n_k$ appeared many times in the literature [1,6,8,9]. It is isomorphic to the poset of cells of the classical Fox–Neuwirth stratification of configuration space $\mathrm{Conf}_k(\Re^n)$ (see Section 6); it is also isomorphic to the Getzler and Jones poset of bar-codes and to the *Milgram poset* of [1]. We use the last name in this paper.

The category $\Upsilon_n(k)$ is isomorphic to Berger's *shuffle category* [6], where it is also shown how to reconstruct $\mathbf{J}^n_k$ as a Grothendieck construction of a *shuffle functor* $\Upsilon_n(k) \to Cat$.

Definition 2.7. Let $f : T \to S$ be an order preserving map of n-ordinals. For an element $i \in S$, the preimage $f^{-1}(i)$ with its natural structure of an n-ordinal induced from T will be called the fiber of f over i.

Following [7,10] we define *a tree* to be an isomorphism class of finite connected acyclic graphs with a marked vertex v_0 called the root. A vertex which is not the root vertex and having valency more than 2 is called *an internal vertex*. The edges of this graph with one open end are called *leaves*. Every edge of a tree has, therefore, a target vertex and a source vertex provided this edge is not a leaf. Every vertex also has a set of incoming edges and one outcoming edge if the vertex is not a root. A monotone path in a tree is a sequence of edges such that the target of each edge in the sequence is equal to the source of the next edge. The length of this path is the number of edges in the sequence. The monotone paths are ordered by inclusion. Notice that there can be several maximal paths in the tree with respect to this order. The *length of a tree* is the maximal length of the maximal monotone paths in the tree.

If we choose an incoming edge e at a vertex v we can construct a subtree by considering all vertices and edges which can be connected to v by a monotone path. We call this subtree *a branch* corresponding to e.

A tree is called *labelled* by the set $\{1, \dots, k\}$ if there is given a bijection from the set of the leaves of the tree to $\{1, \dots, k\}$.

Definition 2.8. A labelled n-planar tree is a labelled tree such that for every internal vertex v the set of incoming edges is a totally n-ordered set.

Since every totally n-ordered set has a canonical linear order then every n-planar tree has a canonical structure of a planar tree. So we often will speak about n-planar trees as planar trees decorated by n-ordinals.

Every such tree determines a structure of n-ordered set on the set of its labels. For a vertex v of a planar tree τ and a label $i \in \{1, \dots, k\}$, let us define $\#_v(i)$ to be the last incoming edge (if it exists) in the monotone path which connects the outcoming edge with label i and v. Then we put $i <_p j$ if $\#_v(i) <_p \#_v(j)$ for a vertex $v \in \tau$, which always exists and is unique. The domination relation, therefore, induces a partial order on the set of n-planar trees labelled by the set $\{1, \dots, k\}$. We will use the same notation $\lhd$ for this relation.

3. Pruned $(n-1)$-terminal n-operads

Now we can easily give the definition of an n-operad based on n-ordinals and their morphisms. We will, however, show that this is just a subcategory of the category of n-operads which we will call the category of pruned $(n-1)$-terminal n-operads $PO^{(n-1)}_n(V)$, where V stands for a symmetric monoidal category over which we consider the operads.

We will call a morphism of trees $\sigma : T \to S$ a *full injection* if it is injective, and bijective on tips. Obviously every fiber of σ is equal to U_n. If A is an $(n-1)$-terminal n-operad in V and σ

is a full injection then we have the following composite morphism

$$A_S \xrightarrow{\simeq} A_S \otimes I \otimes \cdots \otimes I \to A_S \otimes A_{U_n} \otimes \cdots \otimes A_{U_n} \to A_T.$$

Definition 3.1. We call an n-operad *pruned* if this morphism is an identity for every full injection σ.

Remark 3.1. In the case of the empty set of fibers of σ (i.e. $S = zS'$), we require the morphism

$$A_S \to A_S \otimes I \otimes I \otimes \cdots \otimes I \to A_T$$

to be an identity. This means, that for any degenerate tree S there is an identification $A_S = A_{z^n U_0}$. Recall that the only degenerate pruned tree is $z^n U_0$.

Definition 3.2. A pruned ($(n-1)$-terminal) n-collection in V is a family of objects $A_T \in V$, where T runs over the set of pruned n-trees. They form a category $PColl_n(V)$ with respect to termwise morphisms of collections.

We will denote by $T^{(p)}$ the maximal pruned subtree of a tree T. Then we can reformulate the definition of the pruned n-operad in the following way:

Lemma 3.1. *A pruned $(n-1)$-terminal n-operad is given by a pruned $(n-1)$-terminal n-collection A equipped with*:

- *a morphism $e : I \to A_{U_n}$*;
- *a morphism*

$$m_\sigma : A_S \otimes A_{T_1^{(p)}} \otimes \cdots \otimes A_{T_k^{(p)}} \to A_T$$

for every morphism of trees $\sigma : T \to S$ between pruned trees in Ω_n.

They must satisfy the usual identities.

This makes it obvious that the category of pruned operads is isomorphic to the category of n-operads based on n-ordinals.

Proposition 3.1. *If V is a cocomplete symmetric monoidal category then the forgetful functor*

$$PU_n : PO_n^{(n-1)}(V) \to PColl_n(V)$$

has a left adjoint $\mathcal{PF}_n$ and is monadic.

The free pruned n-operad monad on the category of pruned n-collections in Set is finitary and Cartesian.

Remark 3.2. By slightly abusing notation we will denote the free pruned n-operad monad as well as its functor part by $\mathcal{PF}_n$.

Proof. The description of the required monad on a pruned collection X is analogous to the description of the free n-operad functor. It is given by an obvious inductive process.

Let us call a pruned tree T an *admissible* expression of arity T. We also have an admissible expression e of arity U_n. If $\sigma : T \to S$ is a morphism of pruned trees and the admissible expressions $x, x_1, \dots, x_k$ of arities $S, T_1^{(p)}, \dots, T_k^{(p)}$ respectively are already constructed then the expression $\mu_\sigma(x; x_1, \dots x_k)$ is also an admissible expression of arity T. We also introduce an obvious equivalence relation on the set of admissible expressions generated by pairs of composable morphisms of pruned trees and by two equivalences $T \sim \mu(T; e, \dots, e) \sim \mu(e; T)$ generated by the identity morphism of T and the unique morphism $T \to U_n$. Notice however, that there are morphisms of trees all of whose fibers (after the pruning operation) are equal to U_n. We can form an admissible expression $\mu_\sigma(S; e, \dots, e)$ corresponding to such a morphism but it is not equivalent to S, unless σ is equal to the identity.

Everything else is in complete analogy with the case of free $(n-1)$-terminal n-operads [4]. $\square$

Applying the general theory of internal algebras [4] we get

Corollary 3.1.1. *The* 2*-functor of internal pruned* n*-operads is representable by a pruned* n*-operad* $\mathbf{PH}^n$*. The object* a_T *of the canonical internal operad in* $\mathbf{PH}^n_T$ *is the terminal object in this category. The nerve of* $\mathbf{PH}^n$ *is obtained by a bar-construction on the terminal pruned* n*-operad.*

A more explicit description of $\mathbf{PH}^n$ will be given later.

The functor of desymmetrisation for general $(n-1)$-terminal n-operads factorises through the category of pruned $(n-1)$-terminal n-operads

$$SO(V) \xrightarrow{Des_n} PO_n^{(n-1)}(V) \hookrightarrow O_n^{(n-1)}(V) \tag{3.1}$$

where Des_n is defined by the formulas identical to the formulas for the desymmetrisation functor from [4]. If V is cocomplete the inclusion $PO_n^{(n-1)}(V) \hookrightarrow O_n(V)$ has a left adjoint L. It follows that the symmetrisation functor can also be factorised

$$SO(V) \xleftarrow{Sym_n} PO_n^{(n-1)}(V) \xleftarrow{L} O_n(V). \tag{3.2}$$

Notice that we use the same notations for pruned versions of symmetrisation and desymmetrisation functors as we used for the unpruned case in [4]. We believe that this does not lead to confusion since $PO_n^{(n-1)}(V)$ is a full subcategory of $O_n^{(n-1)}(V)$. Moreover, we will use the same notation in the reduced case in Section 4.

Again, as in the unpruned case [4], we get the following commutative square of adjunctions:

$$\begin{tikzcd}
SO(Set) \arrow[r, shift left, "Des_n"] \arrow[d, shift left, "U_\infty"] & PO_n^{(n-1)}(Set) \arrow[l, shift left, "Sym_n"] \arrow[d, shift left, "PU_n"] \\
Coll_1(Set) \arrow[r, shift left, "W_n"] \arrow[u, shift left, "\mathcal{F}_\infty"] & PColl_n(Set). \arrow[l, shift left, "C_n"] \arrow[u, shift left, "\mathcal{PF}_n"]
\end{tikzcd} \tag{3.3}$$

So, again, by the general theory of internal algebras, we have a representable 2-functor of internal pruned n-operads on the 2-category of symmetric *Cat*-operads [4]. We will denote by $\mathbf{ph}^n$ the symmetric categorical operad which represents this 2-functor.

The value of the composite $\mathcal{F}_\infty C_n$ on a pruned n-collection X is easy to describe. In arity k it consists of all labelled planar trees with label $\{1,\dots,k\}$ decorated by pruned n-trees and elements of X_T. So Theorem 2.3 from [4] provides us with the following description of the operad $\mathbf{ph}^n$.

The objects of $\mathbf{ph}^n$ are labelled planar trees decorated by pruned n-trees. Analogously to the categorical operad $\mathbf{h}^n$ in [4], the morphisms in $\mathbf{ph}^n$ are generated by simultaneous contractions of the input edges of a vertex provided there exists a corresponding morphism in Ω_n. We also can grow an internal edge by introducing a decoration by the linear tree U_n (see [4] for the analogous description of $\mathbf{h}^n$).

The description we gave for the n-operad $\mathbf{PH}^n$ in Corollary 3.1.1 is not very revealing. We are going to make it more accessible by using some more structured planar trees we will call *composable*.

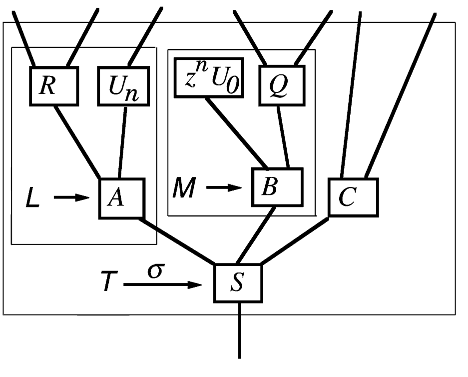
Recall from [4] that the square (3.3) induces a map

$$\beta : Obj\big(\mathbf{PH}^n\big) = \mathcal{PF}_n(1) \to Des_n\big(\mathcal{F}_\infty\big(C_n(1)\big)\big).$$

This is not an injective transformation but it shows that to every object of $\mathbf{PH}^n$ (an admissible expression) we can naturally associate a planar tree decorated by n-ordinals. We will call it the underlying tree of the admissible expression.

We define a notion of composable tree by induction. We call a corolla decorated by an n-tree T (or e) a decorated tree *composable to* T. Suppose we have already defined a notion of composable decorated tree for which their underlying trees have length less than or equal to l. Let τ be an equivalence class of admissible expressions whose underlying decorated tree has length $l+1$. Suppose, that its root vertex v_0 is decorated by a pruned tree S. We will call the tree *composable* to an n-tree T if it is equipped with a morphism of n-trees $\sigma : T \to S$ with fibers T_i, $1 \leqslant i \leqslant k$, and for each i the ith branch at v_0 is composable to $T_i^{(p)}$.

Example 3.1.



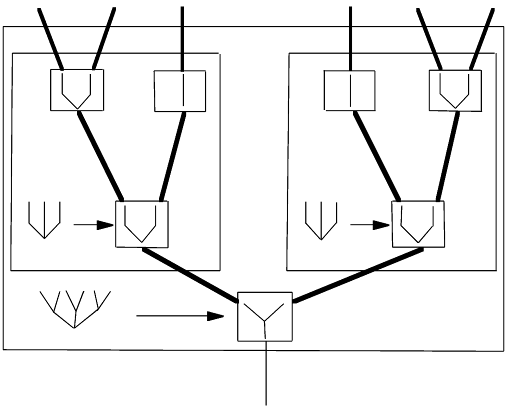
The following lemma is an n-operadic analogue of the trivial fact that, given a string of elements in a monoid, there is a canonical way to calculate its value by performing multiplication always starting from the most right pair of elements. The value, however, does not depend on the method of multiplication. We equally could choose as canonical, multiplication from the left end of the string. In the n-operad case, however, the situation is more subtle as Example 3.2 shows.

Lemma 3.2. *For every equivalence class of admissible expressions of arity T, there is a unique decorated tree composable to T representing this expression.*

Proof. The proof is by a routine induction. $\square$

So one can think of the objects of $\mathbf{PH}^n_T$ as trees composable to T. The morphisms are generated by composition of some nodes but in contrast with symmetric operads this operation can give a tree which lies outside $\mathbf{PH}^n_T$.

Example 3.2. The following tree is composable but we cannot produce the same composition if we start to compose it from its root to its leaves. This is actually the combinatorial 'raison d'être' of Tamarkin's counterexample to cellularity of the Getzler–Jones operad.



We also formulate some results about $\mathbf{ph}^n$ which can be proved following verbatim the proofs of the parallel results for $\mathbf{h}^n$ in [4]:

Theorem 3.1. *There is a natural isomorphism $Sym_n(\mathbf{PH}^n) \simeq \mathbf{ph}^n$. This isomorphism induces an isomorphism of nerves*

$$N\big(\mathbf{ph}^n\big) \to Sym_n\big(N\big(\mathbf{PH}^n\big)\big).$$

Theorem 3.2. *Let A be a cocomplete symmetric Cat-operad and a be an internal pruned n-operad in A. Then*

$$\big(Sym_n(a)\big)_k \simeq \operatorname*{colim}_{\mathbf{ph}^n_k} \tilde{a}_k$$

where $\tilde{a}_k : \mathbf{ph}^n_k \to A_k$ is the operadic functor generated by the operad a.

Finally, coming back to the adjunctions (3.1), (3.2) we find one more interesting categorical pruned n-operad in this picture, namely $L(\mathbf{H}^n)$. It comes together with a canonical morphism

$L(\mathbf{H}^n) \to \mathbf{PH}^n$. Obviously, $L(\mathbf{H}^n)$ is a categorical pruned n-operad freely generated by an internal (not pruned!) n-operad. From this characterisation we deduce a description of $L(\mathbf{H}^n)$. The objects are elements of the free pruned n-operad on the following pruned n-collection: for a pruned tree T it consists of all full injections $T \to S$. So, for any nonpruned n-tree S, we have an object, which we will denote by S as well, in the category $L(\mathbf{H}^n)_{S(p)}$ and these objects form an internal n-operad. The generators for morphisms correspond to the morphisms of nonpruned trees and have the form $\mu(S; T_1, \ldots, T_p) \to T$ for a morphism of n-trees $\sigma : T \to S$. Notice, however, that this morphism generates a morphism between admissible expressions which corresponds to $\sigma^{(p)} : T^{(p)} \to S^{(p)}$.

Conjecture 3.1. *The operad $N(L(\mathbf{H}^n))$ is a contractible simplicial operad.*

If this conjecture is true then we would be able to prove that the operad $N(\mathbf{h}^n)$ of [4] is equivalent to the little n-disks operad. All our efforts to prove this conjecture have failed so far because some morphisms in $L(\mathbf{H}^n)$ are going in opposite directions, which creates a lot of combinatorial difficulties in analysing the homotopy type of $L(\mathbf{H}^n)$.

4. Reduced $(n-1)$-terminal n-operads

We will call a symmetric operad A *reduced* if $A_0 = A_1 = I$ the unit for tensor product, and the operadic unit is given by the identity. Equivalently a reduced symmetric operad can be described as a contravariant functor A from the subcategory of nonempty ordinals and injective morphisms of the category Ω^s [4] such that $A([1]) = I$ (*a reduced symmetric collection*) plus operadic composition for any surjection of finite sets. Operadic composition has to be natural with respect to the injections. The maps of reduced symmetric operads (reduced symmetric collections) are the maps of operads (natural transformations) which induce identity morphisms in arity 0 and 1. We use the notation $RSO(V)$ for the category of reduced symmetric operads and $RColl_\infty(V)$ for the category of reduced symmetric collections.

Observe that our category of reduced operads is just a subcategory of the category of reduced operads of Berger and Moerdijk [6,7] since they do not require A_1 to be a unit of V.

We also have a category of reduced nonsymmetric collections which we will denote by $RColl_1(V)$. These are contravariant functors on injective maps of Δ^+ with the conditions $A([1]) = I$. Recall from [15], that Δ^+ is the full subcategory of Δ with nonempty ordinals as objects.

The n-operadic counterpart of $RSO(V)$ will be the category $RO_n^{(n-1)}(V)$ of reduced $(n-1)$-terminal n-operads.

Definition 4.1. A pruned $(n-1)$-terminal n-operad A is called reduced if

$$A_{z^n U_0} = A_{U_n} = I$$

and its unit is given by the identity. A morphism between two reduced n-operads is an n-operadic morphism which induces identity morphisms in arity $z^n U_0$ and U_n.

Definition 4.2. Let Λ_n^{inj} be the category of pruned nondegenerate n-trees and their injective morphisms. A reduced n-collection is a contravariant functor A from Λ_n^{inj} to V such that $A_{U_n} = I$.

The definition of the category of *reduced n-collections* $RCall_n(V)$ is obvious.

As for the reduced symmetric operads we have an equivalent characterisation of the reduced $(n-1)$-terminal operads.

Proposition 4.1. *A reduced $(n-1)$-terminal n-operad is given by a reduced n-collection A together with a multiplication*

$$\mu_\sigma : A_S \otimes A_{T_1^{(p)}} \otimes \cdots \otimes A_{T_k^{(p)}} \to A_T$$

for every surjection of n-trees $\sigma : T \to S$, satisfying associativity and unitary axioms and naturality with respect to injections of pruned trees.

Proposition 4.2. *If V is a cocomplete symmetric monoidal category then the forgetful functor*

$$RU_n : RO_n^{(n-1)}(V) \to RColl_n(V)$$

has a left adjoint $\mathcal{RF}_n$ and is monadic.

The free reduced n-operad monad on the category of reduced n-collections in Set is finitary and Cartesian.

The analogous statements hold for symmetric operads and reduced nonsymmetric collections.

Proof. Everything goes through in full analogy with the unpruned case. The only difference is that we use only surjections of pruned trees and we require some more identifications:

$$e \sim U_n.$$

This identification leads to the effect that we do not have an underlying n-planar tree with a vertex decorated by the linear tree U_n. $\square$

We also have a reduced desymmetrisation functor from the category $RSO(V)$ to the category $RO_n^{(n-1)}(V)$ which we again will denote by Des_n and from general considerations of [4], we have a commutative square of adjunctions

$$\begin{tikzcd} RSO(Set) \arrow[r, shift left, "Des_n"] \arrow[d, shift left, "RU_\infty"] & RO_n^{(n-1)}(Set) \arrow[l, shift left, "Sym_n"] \arrow[d, shift left, "RU_n"] \\ RColl_1(Set) \arrow[u, shift left, "\mathcal{RF}_\infty"] \arrow[r, shift left, "W_n"] & RColl_n(Set). \arrow[l, shift left, "C_n"] \arrow[u, shift left, "\mathcal{RF}_n"] \end{tikzcd} \tag{4.1}$$

Hence, we can develop the theory of internal reduced n-operads. We denote by $\mathbf{RH}^n$ the reduced categorical n-operad freely generated by an internal reduced n-operad and by $\mathbf{rh}^n$ the symmetric categorical operad freely generated by an internal reduced n-operad.

Definition 4.3. An n-tree is called reduced if it is pruned, nondegenerate and is not equal to U_n. A planar tree decorated by n-ordinals is called reduced if all the decorations are reduced.

As in the pruned case, we have a natural transformation

$$\gamma : \mathcal{RF}_n \to Des_n\big(\mathcal{RF}_\infty(C_n)\big)$$

and have a description of the free reduced n-operad functor in terms of composable reduced planar trees. There are, however, some changes in the definition of composable trees due to the fact that we have an identity $e = U_n$.

Again we define the composable trees by induction on the length of the underlying tree, but this time for $l = 1$ we call a tree composable to T when it is a corolla decorated by an n-tree S together with a quasibijection $T \to S$. Suppose we have already defined the notion of composable decorated tree for which the underlying trees have length less than or equal to l. Let τ be an equivalence class of admissible expressions whose underlying decorated tree has length $l + 1$. Suppose that its root vertex v_0 is decorated by a pruned tree S. We will call the tree *composable* to an n-tree T if it is equipped with a surjection of n-trees $\sigma : T \to S$ with fibers T_i, $1 \leqslant i \leqslant k$, and for each i the ith branch at v_0 is composable to $T_i^{(p)}$. Notice that a branch can be empty if the corresponding fiber of σ is the linear tree U_n.

Lemma 4.1. *For every equivalence class of admissible expressions of arity T there is a unique reduced tree composable to T representing this expression.*

A nice result is the following characterisation of the free reduced n-operad functor.

Theorem 4.1. *The natural transformation γ is injective. The image of the inclusion*

$$(\mathcal{RF}_n)_T(1) \subset Des_n\big(\mathcal{RF}_\infty\big(C_n(1)\big)_T\big)$$

consists of labelled reduced trees decorated by n-ordinals which are dominated by the n-ordinal T.

Proof. It is obvious that a labelled decorated tree from the image of γ is dominated by T. We are going to describe a procedure which reconstructs a unique composable reduced decorated tree from a labelled reduced planar n-tree dominated by T, so our theorem will follow from Lemma 4.1.

We construct a composable tree by induction. Let a labelled decorated tree have length equal to 1. So, this is a labelled corolla decorated by an n-ordinal S. Suppose we know that this tree is dominated by an n-ordinal T. The last property means that the identity map of labels can be extended to a quasibijection $\sigma : T \to S$ by Corollary 2.1.1. So we have a composable tree $\mu(S; e, \dots, e)$ of length 1.

Suppose we have already constructed a composable decorated tree of length l for every labelled reduced n-planar tree of length l dominated by T. Let τ be a labelled n-planar tree of length $l + 1$ such that at the root v_0 we have an n-ordinal S. Let us define a surjection $f : |T| \to |S|$ as follows:

$$f(i) = \#_{v_0}(i).$$

The condition $\tau \triangleleft T$ implies that if $i <_p j$ in T then either $f_n(i) = f_n(j)$ or if they are not equal then

$$f(i) = \#_{v_0}(i) <_r \#_{v_0}(j) = f(j)$$

for $r \geqslant p$ or

$$f(j) = \#_{v_0}(j) <_r \#_{v_0}(i) = f(i)$$

for $r > p$. So f is a surjection of ordinals.

Let $T_1, \dots, T_k$ be the list of fibers of f. It is not hard to see that the ith branch of τ at the vertex v_0 is dominated by the n-ordinal T_i. So we can proceed by induction and finish the proof. □

We now give a description of $\mathbf{rh}^n$. The objects of $\mathbf{rh}^n$ are labelled planar trees decorated by reduced n-trees. The morphisms are generated by simultaneous contractions of the input edges of a vertex provided there exists a corresponding surjection in Ω_n. From this description we immediately get the following

Lemma 4.2. *The categories* $\mathbf{rh}^n_k$ *and* $\mathbf{RH}^n_T$ *are finite.*

There are no morphisms for growing internal edge like in the $\mathbf{h}^n$ and $\mathbf{ph}^n$ cases. For this reason the operad $\mathbf{rh}^n$ is even a finite poset operad but we do not need this property here. This is quite an important property, however, and we are going to consider it in a separate paper.

Similarly to the pruned case we have the following.

Proposition 4.3. *The category* $\mathbf{RH}^n_T$ *is isomorphic to the comma-category of* $\mathbf{rh}^n$ *over its internal operad object* a_T. *The n-operad structure is given by the following 'convolution' product:*

Given a surjection of pruned trees $\sigma : T \to S$ *and objects in comma-categories*

$$x \to a_S, \quad x_1 \to a_{T_1^{(p)}}, \quad \dots, \quad x_k \to a_{T_k^{(p)}},$$

we define an object of the comma-category over a_T *by the composite:*

$$\pi(\sigma)^{-1} m(x; x_1, \dots, x_k) \to \pi(\sigma)^{-1} m(a_S; a_{T_1^{(p)}}, \dots, a_{T_k^{(p)}}) \to a_T$$

where m is multiplication in $\mathbf{rh}^n$ *and the last morphism is the structure morphism of the internal operad a in* $\mathbf{rh}^n$.

Theorem 4.2. *There is a natural isomorphism* $Sym_n(\mathbf{RH}^n) \simeq \mathbf{rh}^n$. *This isomorphism induces an isomorphism of nerves*

$$N(\mathbf{rh}^n) \to Sym_n(N(\mathbf{RH}^n)).$$

We also want to introduce an unbased version of reduced operads (we follow the terminology of [18]). These are reduced symmetric operads without nullary operations and with $A_1 = I$. Notice, that we do not require A_0 to be $\emptyset$ but simply forget about the 0-arity of our operads. This is, of course, a linguistic difference but it helps to express the results nicely.

Definition 4.4. An unbased reduced n-operad in a symmetric monoidal category V consists of a collection of objects $A_T \in V$, one for every reduced n-tree T, and $A_{U_n} = I$, together with a multiplication

$$\mu_\sigma : A_S \otimes A_{T_1^{(p)}} \otimes \dots \otimes A_{T_k^{(p)}} \to A_T$$

for every surjection of n-trees $\sigma : T \to S$, satisfying the usual associativity axiom and the unitarity axiom with respect to the identity morphism $I \to A_{U_n}$.

We will denote the category of unbased reduced symmetric operads by $URSO(V)$ and the category of unbased reduced n-operads by $URO_n^{(n-1)}(V)$. All the previous results about reduced operads can be carried over to the unbased case. In particular, we have a categorical unbased reduced operad $\mathbf{RH}^n_\circ$ representing internal unbased n-operads and a categorical unbased reduced symmetric operad $\mathbf{rh}^n_\circ$. It is not hard to see, however, that there are canonical operadic maps

$$\mathbf{RH}^n_\circ(+) \to \mathbf{RH}^n$$

and

$$\mathbf{rh}^n_\circ(+) \to \mathbf{rh}^n,$$

where $\mathbf{RH}^n_\circ(+)$ is obtained from $\mathbf{RH}^n_\circ$ by adding $\emptyset$ in the arity $z^n U_0$ ($\mathbf{rh}^n_\circ(+)$ is obtained from $\mathbf{rh}^n_\circ$ by adding $\emptyset$ in the arity 0) and these maps are isomorphisms in nondegenerate arities.

Finally, we want to produce a version of the symmetrisation formula. This formula for reduced operads admits a nice enhancement.

The opposite of the Milgram poset $(\mathbf{J}^n_k)^{\mathrm{op}}$ can be considered as a subcategory of $\mathbf{rh}^n_k$ which consists of labelled planar trees with only one decoration.

Lemma 4.3. *For every $\tau \in \mathbf{rh}^n_k$ the comma-category $\tau/(\mathbf{J}^n_k)^{\mathrm{op}}$ is nonempty and connected. So $(\mathbf{J}^n_k)^{\mathrm{op}}$ is a final subcategory of* $\mathbf{rh}^n_k$ [15].

Proof. For $n = 1$ the lemma is obviously true so we can assume that $n \geqslant 2$.

It is not hard to see that every object from $\mathbf{rh}^n_k$ is dominated by one of the objects of $(\mathbf{J}^n_k)^{\mathrm{op}}$. An object $\tau \in \mathbf{rh}^n_k$ is a labelled planar tree decorated by n-ordinals and, hence, determines a canonical linear order on the set of its labels $\{1, \dots, k\}$. Without loss of generality we can assume that this ordered set is the ordinal $[k]$. So it is dominated by the n-ordinal M^k_0 (see Section 2 for notation).

Let $\tau \in \mathbf{rh}^n_k$ be dominated by T' and T''.

Again without losing generality we can assume that $T'' = M^k_0$ and T' can be obtained from T'' by permuting the labels.

Let us construct a totally n-ordered set T which dominates τ and is dominated by both T' and T''.

To do this we apply the following reconstruction algorithm to T'. Let i be the first label in T' with respect to the linear order $<_0$ and j be the second. If $i < j$ in $[k]$ then we define $i <_0 j$ in $T^{(1)}$. If, however, $j < i$ in $[k]$ then we put $i <_1 j$ in $T^{(1)}$. We also put all the other labels in $T^{(1)}$ to be in the same order as in T'. So we have constructed an object of $(\mathbf{J}^n_k)^{\mathrm{op}}$ which is dominated by T' and also dominates τ.

Now we continue this process and do the same thing for the second and third consecutive labels in T' then for the third and fourth and so on. In this way we construct a sequence of n-ordered sets

$$\tau \lhd T = T^{(k-1)} \lhd T^{(k-2)} \lhd \cdots \lhd T^{(1)} \lhd T^{(0)} = T'.$$

Finally, we observe, that $i < j$ implies $i <_0 j$ or $i <_1 j$ or $j <_1 i$ in T by construction. This means that $T \lhd T''$ and we have finished the proof. $\square$

Theorem 4.3. *Let A be a cocomplete reduced symmetric Cat-operad and a be an internal reduced n-operad in A. Then*

$$Sym_n(a)_k \simeq \operatorname*{colim}_{\mathbf{rh}^n_k} \tilde{a}_k \simeq \operatorname*{colim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} \tilde{a}_k$$

where $\tilde{a}_k : \mathbf{rh}^n_k \to A_k$ is the operadic functor generated by a.

The analogous formula holds in the unbased case.

To be able to apply this theorem to the reduced operads in a symmetric monoidal category V, we have to exhibit V as a categorical reduced symmetric operad similar to what was done in [4] for the unreduced case. Our construction $V^{\bullet}$ from [4] produces only an unreduced operad. In the reduced case we define the reduced categorical symmetric operad $V^{\bullet\bullet}$ as follows

$$V^{\bullet\bullet}_k = \begin{cases} V & \text{if } k \geqslant 2, \\ 1 & \text{if } k = 0, 1 \end{cases}$$

with multiplication defined by iterated tensor product and trivial symmetric group action. It is obvious that there is an isomorphism of the categories of internal reduced operads in $V^{\bullet\bullet}$ and reduced operads in V (see [4] for this property in the unreduced case).

A comment has to be made about endomorphism operads in the reduced situation. First of all we have to consider only the pointed case; i.e. we consider an object X of our closed symmetric monoidal category V together with a fixed morphism $I \to X$. For every $k \geqslant 0$ one can consider the following pullback

$$\begin{array}{ccc} \bar{V}(X^k, X) & \longrightarrow & V(X^k, X) \\ \downarrow & & \downarrow \\ V(I^k, I) & \longrightarrow & V(I^k, X). \end{array}$$

This symmetric collection has an obvious structure of an operad and we define *the reduced endomorphism symmetric operad* $REnd(X)$ as follows:

$$REnd(X)_k = \bar{V}\big(X^k, X\big) \quad \text{for } k \neq 1$$

and

$$REnd(X)_1 = I.$$

An action of a reduced operad A on a pointed object X is then an operadic map $A \to REnd(X)$. Usually an action of an operad on a pointed object is defined as an operad map from A to its full endomorphism operad. It is obvious, however, that such an action can be factorised through $REnd(X)$, so our definition agrees with the usual one.

We give the analogous obvious definition for the reduced endomorphism n-operad $REnd_n(X)$. Now, we will have a canonical isomorphism

$$Des_n\big(REnd(X)\big) \simeq REnd_n(X)$$

as in the unreduced case. And, hence, similarly to the unreduced case we have

Theorem 4.4. *For a reduced $(n-1)$-terminal n-operad A, the categories of $Sym_n(A)$-algebras and A-algebras are isomorphic.*

5. Model structures on various categories of operads

Here we adapt the theory of [7] to the case of n-operads. Recall that one of the main technical points of this theory is the existence of a cofibrantly generated model structure on the category of reduced symmetric operads in a monoidal model category V. This structure is transferred along the free symmetric operad functor from a model structure on symmetric collections in V which in its turn is transferred along the free symmetric collection functor from the category of nonsymmetric collections. The term transferred means that the weak equivalences (fibrations) are precisely the morphisms which become weak equivalences (fibrations) after application of the right adjoint functor.

Theorem 5.1. (Theorem 3.1 [7]) *There exists a transferred model structure on the category of reduced symmetric operads in a monoidal model category V if*

- *V is cofibrantly generated and its unit I is cofibrant*;
- *the comma category V/I has a symmetric monoidal fibrant replacement functor*;
- *V admits a commutative Hopf interval.*

Theorem 5.2. (Theorem 3.2 [7]) *If V is a Cartesian closed model category then there is a transferred model structure on the category of all symmetric operads in V provided*

- *V is cofibrantly generated and the terminal object of V is cofibrant*;
- *V has a symmetric monoidal fibrant replacement functor.*

We already pointed out that our notion of reduced symmetric operad is stronger than that of [7]. It is, however, not too hard to check that their proof works well in our situation. We only have to take care of their construction of a cooperad TH from a commutative Hopf object H [7, Proposition 1.1]. If we put $T'H_1 = I$, $T'H_n = H^{\otimes n}$ then $T'H$ is still a cooperad and it works for the reduced operads in our sense like TH works in [7]. So Theorem 5.1 holds in our case without any changes.

Moreover, following the method of [7] in the case of n-operads we can prove:

Theorem 5.3. *There exists a transferred model structure on the category of reduced $(n-1)$-terminal n-operads in a monoidal model category V if*

- *V is cofibrantly generated and its unit I is cofibrant*;
- *the comma category V/I has a symmetric monoidal fibrant replacement functor*;
- *V admits a commutative Hopf interval.*

Moreover, the commutative square (4.2) *is a square of Quillen adjunctions.*

The analogous theorem holds in the unbased case.

Theorem 5.4. *If V is a Cartesian closed model category then there is a transferred model structure on the categories of $(n-1)$-terminal n-operads in V and pruned $(n-1)$-terminal n-operads in V provided*

- *V is cofibrantly generated and the terminal object of V is cofibrant*;
- *V has a symmetric monoidal fibrant replacement functor.*

The corresponding commutative squares of adjunctions are squares of Quillen adjunctions.

Examples of monoidal categories satisfying the conditions of these theorems are given in [7]. The most important for us are the categories of simplicial sets, the category *Top* of compactly generated topological spaces, and the category $Ch(R)$ of chain complexes over a commutative ring R with unit. We note that the category of chain complexes satisfies the hypothesis of Theorem 5.1 and the categories of simplicial sets and topological spaces satisfy the assumptions of Theorem 5.2.

The weak equivalences (fibrations) between topological or simplicial n-operads are therefore operadic maps which are termwise weak equivalences (fibrations) in simplicial sets and *Top*. The weak equivalences (fibrations) between chain n-operads are those operadic maps which are termwise quasi-isomorphisms (epimorphisms).

Since many of our categorical operads are given by a bar-construction we would like to investigate now what are the homotopy theoretic properties of the bar-construction for n-operads in general. We will do it for the general case of $(n-1)$-terminal n-operads. The pruned and reduced cases are similar.

In addition, suppose V is enriched in simplicial sets with simplicial *hom*-functor $V^S(-,-)$ satisfying

$$V^S(X,Y) = V^S\big(I, V(X,Y)\big) \tag{5.1}$$

where $V(X,Y)$ is the internal *hom*-functor in V. Then the categories of n-collections and n-operads become simplicially enriched. For two $(n-1)$-terminal n-collections X, Y, we define its simplicial set of morphisms as

$$Coll_n^S(X,Y) = \prod_{T \in Tr_n} V^S(X_T, Y_T).$$

As in [4], let $(\mathcal{F}_n, \mu, \epsilon)$ be the free $(n-1)$-terminal n-operad monad. Then for two n-operads A, B we define their simplicial set of morphisms as the equalizer

$$Oper_n^S(A,B) \longrightarrow Coll_n^S(A,B) \rightrightarrows Coll_n^S(\mathcal{F}_n A, B)$$

with the obvious horizontal morphisms generated by the operadic structures of A and B.

If V has tensors and cotensors with respect to simplicial sets then the categories $O_n^{(n-1)}(V)$ and $Coll_n^{(n-1)}(V)$ also have them. In particular, one can speak about the total object for a simplicial n-operad (n-collection) $A^\star$. By definition this is the coend

$$Tot(A^\star) = \int^{[n]\in\Delta^+} A^n \otimes \Delta^n,$$

where $\Delta^n = \Delta^+([n+1], -)$ is the standard simplex of dimension n.

Let X be an $(n-1)$-terminal n-operad in C. Then the bar-construction $B(\mathcal{F}_n, \mathcal{F}_n, X)$ of X is the total object of the simplicial operad

$$\mathcal{F}_n(X) \Lleftarrow \mathcal{F}_n^2(X) \Lleftarrow \mathcal{F}_n^3(X) \quad \cdots$$

Theorem 5.5. *Let V be a model category which satisfies the conditions of Theorem* 5.4*, has a simplicial enrichment $V^S(-,-)$ satisfying* (5.1)*, and which is a simplicial model category with respect to these structures* [11]*. Let X be an* $(n-1)$*-terminal n-operad in V with cofibrant underlying n-collection. Then the canonical operad morphism*

$$\rho : B(\mathcal{F}_n, \mathcal{F}_n, X) \to X$$

is a cofibrant replacement for X in the model category of $(n-1)$*-terminal n-operads.*

The analogous theorem holds in the pruned case and in the reduced and unbased reduced case if V satisfies the assumptions of Theorem 5.3.

Proof. It follows from the general properties of bar-construction [17] that the morphism ρ is a deformation retraction in $Coll_n^{(n-1)}$. Hence, ρ is a trivial fibration of operads.

It remains to prove that $B(\mathcal{F}_n, \mathcal{F}_n, X)$ is a cofibrant n-operad. Let $f : E \to B$ be a trivial fibration of n-operads.

We have to show that any operadic map $B(\mathcal{F}_n, \mathcal{F}_n, X) \to B$ can be lifted to E.

$$\begin{tikzcd} & E \arrow[d, "f"] \\ B(\mathcal{F}_n, \mathcal{F}_n, X) \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

By construction this amounts to the following lifting problem in the category of cosimplicial spaces

$$\begin{tikzcd} & Oper_n^S(\mathcal{F}_n^\star X, E) \arrow[d, "f^\star"] \\ \Delta^\star \arrow[r] \arrow[ur, dashed] & Oper_n^S(\mathcal{F}_n^\star X, B) \end{tikzcd}$$

where $\Delta^\star$ is the cosimplicial simplicial set consisting of standard simplices. Since $\Delta^\star$ is cofibrant in the Reedy model structure [11] it remains to show that $f^\star$ is a trivial fibration.

We follow a method developed in [2]. We have to prove that in the diagram

$$\begin{tikzcd}
Oper_n^S(\mathcal{F}_n^{i+1}X, E) \arrow[dr, "\omega_i"] \arrow[drr] \arrow[ddr] & & \\
& W_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M_i(Oper_n^S(\mathcal{F}_n^{\star}X, E)) \arrow[d, "M_i f^{\star}"] \\
& Oper_n^S(\mathcal{F}_n^{i+1}X, B) \arrow[r] & M_i(Oper_n^S(\mathcal{F}_n^{\star}X, B))
\end{tikzcd}$$

the canonical map ω_i to the pullback is a trivial fibration. In this diagram $M_i(-)$ is the ith matching object of the corresponding cosimplicial object [11].

According to Lemma 2.3 from [2] the diagram above is isomorphic to the diagram

$$\begin{tikzcd}
Coll_n^S(\mathcal{F}_n^{i}X, E) \arrow[dr, "\omega_i"] \arrow[drr, "\phi_i"] \arrow[ddr] & & \\
& W_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Coll_n^S(L_{i-1}\mathcal{F}_n^{\star-1}X, E) \arrow[d, "M_i f^{\star}"] \\
& Coll_n^S(\mathcal{F}_n^{i}X, B) \arrow[r, "\psi_i"] & Coll_n^S(L_{i-1}\mathcal{F}_n^{\star-1}X, B).
\end{tikzcd}$$

Here, $L_i(\mathcal{F}_n^{\star-1}(X))$ is the latching object [11] for the augmented cosimplicial object $\mathcal{F}_n^{\star-1}(X)$ in $Coll_n^S(V)$, and ϕ_i, ψ_i are generated by the canonical morphism

$$l_{i-1} : L_{i-1}\mathcal{F}_n^{\star-1}X \to \mathcal{F}_n^{i}X.$$

If we show that this morphism is a cofibration, then ω_i will be a trivial fibration by the axioms for a simplicial model category.

We will actually prove that l_{i-1} is an isomorphism onto a summand.

It was proved in [4, Theorem 9.1] that the kth iteration of the functor $\mathcal{F}_n$ is given by the following formula:

$$\mathcal{F}_n^k(X)_T = \coprod_{W_1 \xleftarrow{f_1} W_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{k-1}} W_k} \tilde{X}(W_k), \tag{5.2}$$

where $f_1, \ldots, f_{k-1}$ are morphisms in $\mathbf{H}_T^n$.

The coface operators in $\mathcal{F}_n^{\star-1}(X)$ are canonical inclusions on the summands corresponding to the operators of insertion of the identities to the chain

$$W_1 \xleftarrow{f_1} W_2 \xleftarrow{f_2} \cdots \xleftarrow{f_{k-1}} W_k.$$

The rest of the proof follows in complete analogy with Lemma 4.1 of [2].

The proof in the reduced and unbased cases are analogous. Notice that, in our version of the category of reduced operads, any reduced collection is automatically well pointed in the sense of [7] so we do not include this condition in the formulation of the reduced version of our theorem. □

Theorem 5.6. *The simplicial n-operads $N(\mathbf{H}^n)$, $N(\mathbf{PH}^n)$, $N(\mathbf{RH}^n_\circ)$ are cofibrant n-operads in the categories of $(n-1)$-terminal simplicial n-operads, $(n-1)$-terminal simplicial pruned n-operads and unbased reduced $(n-1)$-terminal simplicial pruned n-operads respectively.*

The simplicial symmetric operads $N(\mathbf{h}^n)$, $N(\mathbf{ph}^n)$ are cofibrant simplicial symmetric operads.

The unbased reduced symmetric operad $N(\mathbf{rh}^n_\circ)$ is cofibrant in the category of unbased reduced symmetric operads.

The same theorem is true for the geometric realisations of these operads and for the reduced operad of chain complexes of $\mathbf{RH}^n_\circ$ and $\mathbf{rh}^n_\circ$.

Proof. The nerves of the above n-operads are bar-constructions on the terminal n-collection in the corresponding categories. In these categories the terminal n-collection is obviously a cofibrant collection.

The cofibrantness of $N(\mathbf{h}^n)$, $N(\mathbf{ph}^n)$ and $N(\mathbf{rh}^n_\circ)$ follows from the corresponding Quillen adjunctions between n-operads and symmetric operads and the fact that in these cases the symmetrisation functor commutes with nerves.

The topological and chain versions of the theorem follow from the general considerations of [7]. □

Remark 5.1. The nerves of $\mathbf{RH}^n$ and $\mathbf{rh}^n$ are not cofibrant. The reason is that 1 is not a cofibrant reduced n-collection.

6. Fulton–Macpherson operad and Getzler–Jones decomposition

The operadic structure on compactified moduli space of configurations of points in $\Re^n$ was first observed by Getzler and Jones in [9]. Here we use an explicit approach of [14,19] to describe this compactification.

Let $\mathrm{mod}^n_{[k]}$ be the quotient of the configuration space

$$\mathrm{Conf}_k(\Re^n) = \{(x_1,\dots,x_k) \in (\Re^n)^k \mid x_i \neq x_j \text{ if } i \neq j\}$$

with respect to the obvious action of the $(n+1)$-dimensional Lie group G_n of affine transformations of the form $u \mapsto \lambda u + v$, where $\lambda > 0$ is a real number and v is a vector from $\Re^n$. The functions

$$u_{ij}(x_1,\dots,x_k) = \frac{x_j - x_i}{\|x_j - x_i\|}, \quad 1 \leqslant i, j \leqslant k,\ i \neq j,$$

$$d_{i,j,l}(x_1,\dots,x_k) = \frac{\|x_i - x_j\|}{\|x_i - x_l\|}, \quad 1 \leqslant i, j, l \leqslant k,\ i \neq j,\ j \neq l,\ i \neq l$$

embed $\mathrm{mod}^n_{[k]}$ into a compact space

$$\Gamma_k\big[\Re^n\big] = \prod_{\substack{1\leqslant i,j\leqslant k\\ i\neq j}} S^{n-1} \times \prod_{\substack{1\leqslant i,j,l\leqslant k\\ i\neq j,\, j\neq l,\, i\neq l}} [0,\infty].$$

The kth space of the Fulton–Macpherson operad $\mathbf{fm}^n$ is obtained as the closure of the moduli space $\mathrm{mod}^n_{[k]}$ inside $\Gamma_k[\Re^n]$. Notice, that we use here a reduced version of $\mathbf{fm}^n$ meaning that we put $\mathbf{fm}^n_0 = 1$. If we forget about nullary operations we will get the unbased version of the Fulton–Macpherson operad which we will denote by $\mathbf{fm}^n_\circ$. Notice also that in [9,14] the Fulton–Macpherson operad means $\mathbf{fm}^n_\circ$.

In [9,14,18,19] the following properties of $\mathbf{fm}^n$ were established:

- $\mathbf{fm}^n$ is a reduced symmetric operad weakly equivalent to the little n-cube operad;
- $\mathbf{fm}^n_\circ$ is an unbased reduced cofibrant operad weakly equivalent to the unbased little n-cube operad;
- set theoretically $\mathbf{fm}^n$ is the free reduced operad on the reduced symmetric collection $\mathrm{mod}^n_{[\bullet]}$;
- $\mathbf{fm}^n_k$ is a manifold with corners;
- $\mathbf{fm}^1$ is isomorphic to the Stasheff operad of associahedra.

Later in this paper we will use notations u_{ij} and d_{ijl} for the coordinates of points in $\mathbf{fm}^n$. The following lemma describes the behaviour of these coordinates under operadic multiplication.

Lemma 6.1. *Let $\sigma : [n] \to [k]$ be a surjection in Ω^s. Then*

$$u_{ij}\big(\mu(x; x_1,\ldots,x_k)\big) = \begin{cases} u_{ij}(x_l) & \text{if } \sigma(i)=\sigma(j)=l,\\ u_{\sigma(i')\sigma(j')}(x) & \text{if } \sigma(i)\neq\sigma(j)\end{cases}$$

where i', j' are images of i, j in the fiber of σ over l.

Similarly

$$d_{ijl}\big(\mu(x; x_1,\ldots,x_k)\big) = \begin{cases} d_{i'j'l'}(x_s) & \text{if } \sigma(i)=\sigma(j)=\sigma(l)=s,\\ 0 & \text{if } \sigma(i)=\sigma(j)\neq\sigma(l),\\ d_{\sigma(i)\sigma(j)\sigma(l)}(x) & \text{if } \sigma(i)\neq\sigma(j)\neq\sigma(l),\ \sigma(i)\neq\sigma(l).\end{cases}$$

All other values of $d_{ijl}(\mu(x; x_1,\ldots,x_k))$ can be deduced from the above table and the following relations between d_{ijk} [19]:

$$d_{ijk}d_{ikj} = d_{ijk}d_{ikl}d_{ilj} = d_{ijk}d_{jki}d_{kij} = 1.$$

Proof. The proof can be obtained using the explicit formulas for the operadic multiplication in $\mathbf{fm}^n$ from [16] or techniques from [19]. $\square$

Following Joyal [12] we give a definition of a generalised n-tree.

Definition 6.1. A generalised n-tree X is a chain of partially ordered sets and functions

$$R^n \xrightarrow{\rho_{n-1}} R^{n-1} \xrightarrow{\rho_{n-2}} \cdots \xrightarrow{\rho_1} R^1 \xrightarrow{\rho_0} R^0 = 1$$

such that the induced order on $\rho_i^{-1}(a)$ is linear for all $0 \leqslant i \leqslant n-1$ and $a \in R^i$.

A topological n-tree is an n-tree for which all R^i are endowed with a topology and all ρ_i are continuous functions.

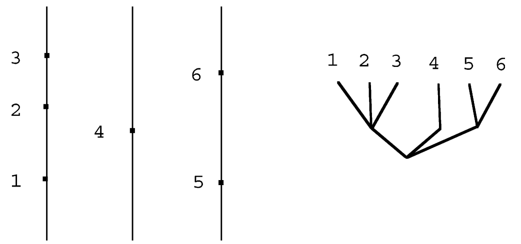
The definition of a morphism of generalised n-trees (also from [12]) coincides verbatim with the definition of morphism of finite n-trees in [4].

Consider the following topological n-tree $\Re^{\leqslant n}$:

$$\Re^n \to \Re^{n-1} \to \cdots \to \Re^1 \to \Re^0 = 0$$

where morphisms are projections on the first coordinates. We introduce the n-tree structure on $\Re^{\leqslant n}$ by ordering the fiber of each projection according to its natural order. Now, for every pruned n-tree T, one can consider the space of all injective n-tree morphisms from T to $\Re^{\leqslant n}$. This space is the classical *Fox–Neuwirth cell* FN_T corresponding to T [8]. We will consider it as an open submanifold of configurations of points with the labelling prescribed by the order in T [6,8,9,25].

Example 6.1.



The group G_n obviously acts on FN_T and we call the corresponding quotient space *a Getzler–Jones cell* Mod^n_T.

Recall that the functor $W_n : RColl_1 \to RColl_n$, $W_n(A)_T = A_{|T|}$ has a left adjoint

$$C_n : RColl_n \to RColl_1, \qquad C_n(B)_{[k]} = \coprod_{|T|=[k]} B_T.$$

The configuration space $\mathrm{Conf}_k(\Re^n)$ admits the classical Fox–Neuwirth decomposition

$$\mathrm{Conf}_k\big(\Re^n\big) = \bigcup_{|T|=[k],\, \pi \in \Sigma_k} \pi \mathrm{FN}_T.$$

This means that there is a bijective continuous map of collections

$$\varepsilon : S\big(C_n(\mathrm{FN}_\bullet)\big) \to \mathrm{Conf}_\bullet\big(\Re^n\big),$$

where $S : RColl_1 \to RColl_\infty$ is the functor of symmetrisation of reduced nonsymmetric collections. The inverse bijection is not continuous. Certainly ε is equivariant with respect to the action of G_n and so we have a stratification of the moduli space of configurations

$$\mathrm{mod}^n_{[k]} = \bigcup_{|T|=[k],\, \pi \in \Sigma_k} \pi \mathrm{Mod}^n_T$$

and a corresponding continuous bijection which we will denote by the same letter

$$\varepsilon : S\big(C_n\big(\mathrm{Mod}^n_{\bullet}\big)\big) \to \mathrm{mod}^n_{[\bullet]}. \tag{6.1}$$

The free reduced symmetric operad functor $\mathcal{RF}_\infty$ is factorized as $\mathcal{RF}_s \cdot S$, where $\mathcal{RF}_s$ is the free reduced symmetric operad functor on reduced symmetric collections. So we have a bijective continuous map of operads

$$\mathcal{RF}_\infty\big(C_n\big(\mathrm{Mod}^n_{\bullet}\big)\big) \simeq \mathcal{RF}_s\big(S\big(C_n\big(\mathrm{Mod}^n_{\bullet}\big)\big)\big) \to \mathcal{RF}_s\big(\mathrm{mod}^n_{[\bullet]}\big) \to \mathbf{fm}^n. \tag{6.2}$$

We would like to get a description of Mod^n_T in terms of the functions u_{ij}. Let $\mathring{S}^{n-p-1}_+$ denote the open $(n-p-1)$-hemisphere in $\Re^n$, $0 \leqslant p \leqslant n-1$:

$$\mathring{S}^{n-p-1}_+ = \left\{ (x_1, \dots, x_n) \in \Re^n \;\middle|\; \begin{array}{l} x_1^2 + \cdots + x_n^2 = 1, \\ x_{p+1} > 0 \text{ and } x_i = 0 \text{ if } 1 \leqslant i \leqslant p \end{array} \right\}$$

and let

$$\mathring{S}^{n-p-1}_- = \left\{ (x_1, \dots, x_n) \in \Re^n \;\middle|\; \begin{array}{l} x_1^2 + \cdots + x_n^2 = 1, \\ x_{p+1} < 0 \text{ and } x_i = 0 \text{ if } 1 \leqslant i \leqslant p \end{array} \right\}.$$

The closure of $\mathring{S}^{n-p-1}_+$ will be denoted S^{n-p-1}_+ and the closure of $\mathring{S}^{n-p-1}_-$ will be denoted S^{n-p-1}_-. Observe, that

$$\mathring{S}^{n-r-1}_+ \subset S^{n-p-1}_+ \quad \text{for } r \geqslant p \qquad \text{and} \qquad \mathring{S}^{n-r-1}_- \subset S^{n-p-1}_+ \quad \text{for } r > p.$$

Lemma 6.2. *For a pruned n-tree T, $|T| = k$, $k \geqslant 0$, the Getzler–Jones cell Mod^n_T is equal to the set*

$$\left\{ x \in \mathrm{mod}^n_{[k]} \;\middle|\; \begin{array}{ll} u_{ij}(x) \in \mathring{S}^{n-p-1}_+ & \text{if } i <_p j \text{ in } T, \\ u_{ij}(x) \in \mathring{S}^{n-p-1}_- & \text{if } j <_p i \text{ in } T \end{array} \right\}.$$

It is a contractible open manifold of dimension $E(T) - n - 1$ where $E(T)$ is the number of edges in the tree T.

Proof. Obvious from the definition of Mod^n_T. □

Definition 6.2. The dimension of an n-pruned tree $T \neq U_n$ is the integer

$$\dim(T) = E(T) - n - 1.$$

We also put $\dim(U_n) = 0$.

In virtue of (6.2) the decomposition (6.1) can be extended to a decomposition of $\mathbf{fm}^n$ [9]. Following [25] we will call it *the Getzler–Jones decomposition*. The cells of this decomposition are indexed by labelled reduced planar trees, i.e. by the objects of $\mathbf{rh}^n$ with vertices decorated by points of Getzler–Jones cells. We will call such a space indexed by an object $\tau \in \mathbf{rh}^n$ *a generalised Getzler–Jones cell* and will denote it Mod^n_τ. Since the generalised Getzler–Jones cells do not intersect each other we have a correctly defined map

$$\tau : \mathbf{fm}^n \to Obj\big(\mathbf{rh}^n\big).$$

Lemma 6.3. *Let $x \in \mathrm{Mod}^n_\tau$ then*

$$u_{ij}(x) \in \mathring{S}^{n-p-1}_{+}$$

if $i <_p j$ in $\tau(x)$ or

$$u_{ij}(x) \in \mathring{S}^{n-p-1}_{-}$$

if $j <_p i$ in $\tau(x)$.

Proof. The proof is easily obtained by induction on the length of τ and Lemmas 6.1, 6.2. $\square$

The Getzler–Jones decomposition is not a cellular decomposition or stratification of $\mathbf{fm}^n$ since the boundary of the closure of a Getzler–Jones cell may not be equal to the union of low-dimensional Getzler–Jones cells as was first observed by Tamarkin (see [25] for a description of Tamarkin's counterexample).

However, the following is true[4]:

Proposition 6.1. *The closure of the Getzler–Jones cell $K_T = cl(\mathrm{Mod}^n_T)$ is a manifold with corners homeomorphic to a ball of dimension* $\dim(T)$.

Proof. We have to use the original description of $\mathbf{fm}^n_k$ in terms of the iterated blow-up of $(\Re^n)^k$ along its fat diagonal [9,18,25]. Consider the closure of the Getzler–Jones cell Mod^n_T in $(\Re^n)^k$. The intersection of this subspace with each of the diagonals is a manifold, so restricting the Fulton–Macpherson blow-up procedure we get a manifold with corners homeomorphic to a ball since the blow-up does not change the topological type of the manifold. This manifold is homeomorphic to the closure of Mod^n_T in $\mathbf{fm}^n$. $\square$

7. Getzler–Jones operad

Let us take $\mathcal{RF}_n(\mathrm{Mod}^n_\bullet)$ to be the free reduced n-operad generated by the reduced n-collection $\mathrm{Mod}^n_\bullet$. Then we have a canonical inclusion of n-operads

$$\gamma : \mathcal{RF}_n\big(\mathrm{Mod}^n_\bullet\big) \to RDes_n\big(\mathcal{RF}_\infty\big(C_n\big(\mathrm{Mod}^n_\bullet\big)\big)\big).$$

Consider the composite Φ

$$\mathcal{RF}_n\big(\mathrm{Mod}^n_\bullet\big) \xrightarrow{\gamma} RDes_n\big(\mathcal{RF}_\infty\big(C_n\big(\mathrm{Mod}^n_\bullet\big)\big)\big) \to RDes_n\big(\mathcal{RF}_s\big(\mathrm{mod}^n_{[\bullet]}\big)\big) \to RDes_n\big(\mathbf{fm}^n\big)$$

which is an injective continuous map of n-operads.

[4] I am grateful to Ezra Getzler and Sasha Voronov who explained this fact to me.

Definition 7.1. The Getzler–Jones n-operad $\mathbf{GJ}^n$ is the image of Φ. This is a reduced $(n-1)$-terminal n-operad.

If we forget about operations with degenerate arity, we obtain *the unbased Getzler–Jones n-operad* $\mathbf{GJ}^n_\circ$.

Proposition 7.1. *Let $T \neq U_n$ be a nondegenerate pruned n-tree. The following topological spaces are equal*:

- $\mathbf{GJ}^n_T$;
- $\bigcup_{\tau \lhd T} \mathrm{Mod}^n_\tau$;
- *Kontsevich–Soibelman space* [14]

$$X_T = \left\{ x \in \mathbf{fm}^n_{|T|} \;\middle|\; \begin{array}{ll} u_{ij}(x) \in S^{n-p-1}_+ & \textit{if } i <_p j \textit{ in } T, \\ u_{ij}(x) \in S^{n-p-1}_- & \textit{if } j <_p i \textit{ in } T \end{array} \right\}.$$

Proof. The equality

$$\mathbf{GJ}^n_T = \bigcup_{\tau \lhd T} \mathrm{Mod}^n_\tau$$

readily follows from the definitions and Theorem 4.1.

Let us prove that

$$X_T = \bigcup_{\tau \lhd T} \mathrm{Mod}^n_\tau.$$

The inclusion

$$\bigcup_{\tau \lhd T} \mathrm{Mod}^n_\tau \subset X_T$$

is obvious from Lemma 6.3. Let us prove the inverse inclusion.

Let $x \in X_T$. We have to prove that $\tau(x) \lhd T$. Let $i <_p j$ in $\tau(x)$. Then, by Lemma 6.3, again $u_{ij}(x) \in \mathring{S}^{n-p-1}_+$. Since $x \in X_T$ two possibilities exist. Either $i <_r j$ in T for some r and then $u_{ij}(x) \in S^{n-r-1}_+$ or $j <_q i$ in T for some q and $u_{ij}(x) \in S^{n-q-1}_-$.

In the first case we have $x \in \mathring{S}^{n-p-1}_+ \cap S^{n-r-1}_+$ which is possible only if $r \leqslant p$. In the second case $x \in \mathring{S}^{n-p-1}_+ \cap S^{n-q-1}_-$ and this is possible only if $q < p$. So τ is dominated by T. □

Corollary 7.1.1. *The space $\mathbf{GJ}^n_T$ is a closed subspace of $\mathbf{fm}^n_{|T|}$.*

Now we will study the properties of $\mathbf{GJ}^n$ and its cousin $\mathbf{GJ}^n_\circ$.

Theorem 7.1. *The operad $\mathbf{GJ}^n$ is contractible.*

Proof. For $n = 2$ Kontsevich and Soibelman prove contractibility of X_T in [14, Proposition 7]. Unfortunately the details are left to the reader. So we will follow their idea and prove it for arbitrary n (but only for n-ordinals, not arbitrary complimentary orders as in [14]).

As in [14] we will use induction on the number of tips of a tree T. If T has only two tips then obviously the space X_T is S_+^{n-p-1}, where p is the index of the unique nonempty order on $\{1, 2\}$.

Suppose we have proved this theorem for all T with $|T| = [k]$. Let T be a tree with $|T| = [k+1]$, and let T' be an n-tree which is obtained from T by cutting off the most right branch. Then we have an injection

$$T' \to T$$

which induces a map $\pi : X_T \to X_{T'}$.

Let $a \in X_{T'}$ be a point. Let us suppose that a belongs to the cell $\mathrm{Mod}^n_{T'}$. We will show that the fiber of π is homeomorphic to a disk of dimension $n - p$. Here, p is such that $k <_p k + 1$ in T.

Since the fiber obviously depends only on configurations of points the labels of which belong to the most right branch of the tree T', then without loss of generality we can assume that T' is a suspension of some $(n-1)$-tree and $k <_0 k + 1$ in T. The manifold $\mathrm{Mod}^n_{T'}$ is diffeomorphic to the intersection of the space of configurations of points

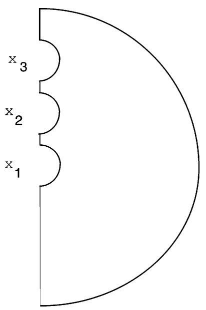
$$\left(x_1^1, \ldots, x_1^n\right), \left(x_2^1, \ldots, x_2^n\right), \ldots, \left(x_k^1, \ldots, x_k^n\right) \in \left(\Re^n\right)^k$$

which belong to the hyperplane $x^1 = 0$, with $x_1 = 0$ and $x_k^n = 1$ with the Fox–Neuwirth cell $\mathrm{FN}_{T'}$. Let $(x_1 = 0, x_2, \ldots, x_k)$ be the image of a in this space. In its turn Mod^n_T is diffeomorphic to the space of configurations of points $x_1, \ldots, x_{k+1}$ in $\Re^n$ such that $x_1, \ldots, x_k$ belongs to the previous intersection and x_{k+1} is in the open positive halfspace $x^1 > 0$.

Let r be a sufficiently small number such that the closed balls $B_r(x_i)$ of radius r with the centre x_i, $1 \leqslant i \leqslant k$, do not intersect each other. And let R be sufficiently big such that the union of all $B_r(x_i)$ belongs to the interior of the ball $B_R(0)$. Let $C(a, r, R)$ be the manifold with corners

$$C(a,r,R) = \left\{ x \in \Re^n \;\middle|\; x^1 \geqslant 0,\ x \in B_R(0),\ x \in \Re^n \Big\backslash \left(\bigcup_{i=1}^k int\big(B_r(x_i)\big) \right) \right\}.$$

Example 7.1.



We claim, that there is a homeomorphism from $C(a, r, R)$ to the fiber of π over a.

To construct such a homeomorphism we first choose an $r_1 > r$ and $R_1 < R$ such that the $B_{r_1}(x_i)$ still do not intersect and their union is still in the interior of the ball $B_{R_1}(0)$. For a fixed $1 \leqslant i \leqslant k$ let

$$\psi_i(y) = \frac{r(\|y\| - r)}{r_1(r_1 - r)}(y - x_i) + x_i,$$

for $y \in C(a, r, R)$ such that $r \leqslant \|y - x_i\| \leqslant r_1$. And let

$$\psi_\infty(y) = \frac{R(R_1 - R)y}{(R - \|y\|)R_1}$$

for $y \in C(a, r, R)$ such that $R_1 \leqslant \|y\| < R$.

We can construct a map F from $C(a, r, R)$ to the positive halfspace $x_1 \geqslant 0$ with the following properties:

- the restriction of F on $C(a, r_1, R_1)$ is a homeomorphism of manifolds with corners;
- $F(y) = \psi_i(y)$ if $r \leqslant \|y - x_i\| \leqslant r_1$;
- $F(y) = \psi_\infty(y)$ if $R_1 \leqslant \|y\| < R$.

Now we construct a map F' from $C(a, r, R)$ to $\pi^{-1}(a)$ to be equal at a point $y \in C(a, r, R)$ to

$$\lim_{m\to\infty}\bigl(u_{ij}\bigl(x_1, \ldots, x_k, F(y_m)\bigr), d_{ijl}\bigl(x_1, \ldots, x_k, F(y_m)\bigr)\bigr)$$

where $\{y_m\}$ is an arbitrary sequence from the interior of $C(a, r, R)$ which converges to y.

Obviously, F' is correctly defined and continuous. It also maps injectively the interior of $C(a, r, R)$ to the intersection of $\pi^{-1}(a)$ with Mod^n_T. Points from the boundaries of $B(x_i, r)$ are mapped injectively to the points of intersections of $\pi^{-1}(a)$ and Mod^n_c where c is the Getzler–Jones cell corresponding to a map of trees $\sigma : T \to T'$, $\sigma(k+1) = i$, $\sigma(j) = j$, $j \leqslant k$. This intersection is homeomorphic to the hemisphere $\mathring{S}^{n-1}_+$.

Points from the hyperplane $x_1 = 0$ are mapped under F' to the points of the Getzler–Jones cell corresponding to the configuration $(x_1, \ldots, x_k, y)$ and, finally, points from the outer hemisphere boundary of $C(a, r, R)$ are mapped to the Getzler–Jones cell corresponding to the map of trees $\sigma : T \to M^2_0$, $\sigma(i) = 1$ if $1 \leqslant i \leqslant k$ and $\sigma(k+1) = 2$.

It is not difficult to check that F' is bijective and so it is a homeomorphism since both spaces are compact.

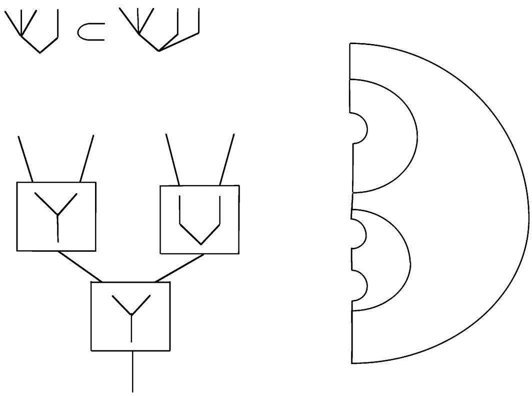
Suppose now a belongs to a generalized Getzler–Jones cell in $X_{T'}$ then the fiber $\pi^{-1}(a)$ can be glued from the manifolds $C(b_t, r_t, R_t)$, $t \in \mathit{Vertex}(\tau(a))$, by the following inductive procedure. First we construct $C(b_0, r_0, R_0)$ where b_0 is the projection of a to the configuration which decorates the root vertex v_0 of $\tau(a)$. Then we consider the vertices $v_1, \ldots, v_s$ which can be connected to v_0 by exactly one edge. We then construct $C(b_l, r_l, R_l)$, $1 \leqslant l \leqslant s$, where b_l is the corresponding projection of a. By scaling up or down the configurations b_l if necessary we always can make $R_1 = R_2 = \cdots = R_s = r_0$. So we construct the next manifold by gluing the outer hemisphere $C(b_l, r_l, R_l)$ to the inner hemisphere of $C(b_0, r_0, R_0)$ in the place l.

Example 7.2. The following example illustrates the proof. Here

$$T = [5] \xrightarrow{\rho} [3] \to [1],$$
$$\rho(1) = \rho(2) = \rho(3) = 1, \qquad \rho(4) = 2, \qquad \rho(5) = 3$$

and the Getzler–Jones cell corresponds to the map $\sigma : T' \to S$ of 2-trees

$$S = [2] \to [1] \to [1], \qquad \sigma(1) = \sigma(2) = 1, \qquad \sigma(3) = \sigma(4) = 2.$$



Obviously, this inductive procedure stops after a finite number of steps and produces a contractible manifold. □

Theorem 7.2. *The operad* $\mathbf{GJ}^n_\circ$ *is a cellular object in* $URO_n^{(n-1)}(Top)$; *in particular, it is cofibrant.*

Proof. First recall some terminology [7]. We will work in the category of unbased reduced n-operads and unbased reduced n-collections and we will call them simply n-operads and n-collections. Let $A \in URO_n^{(n-1)}(Top)$. Let

$$V_n : URO_n^{(n-1)}(Top) \to URColl_n(Top)$$

be the forgetful functor with left adjoint $\mathcal{RF}_n^\circ$. Let $\omega : V_n(A) \to K$ be a cofibration of n-collections. *The cellular extension* $A \to A[\omega]$ *generated by* ω is the following pushout in $URO_n^{(n-1)}(Top)$:

$$\begin{tikzcd} \mathcal{RF}_n^\circ V_n(A) \arrow[r] \arrow[d] & \mathcal{RF}_n^\circ(K) \arrow[d] \\ A \arrow[r] & A[\omega] \end{tikzcd}$$

where the left vertical map is the counit of the adjunction.

We will prove that $\mathbf{GJ}^n_\circ$ can be obtained as a sequential colimit of cellular extensions starting from the initial n-operad.

Consider a filtration of the n-collection $\mathrm{Mod}^n_\bullet$ by the following subcollections $\mathrm{Mod}^n_\bullet(m)$, $m \geqslant 0$.

$$\mathrm{Mod}^n_T(m) = \begin{cases} \mathrm{Mod}^n_T & \text{if } \dim(T) \leqslant m, \\ \emptyset & \text{if } \dim(T) > m. \end{cases}$$

We have an inclusion for every m

$$\Phi_m : \mathcal{RF}^\circ_n\big(\mathrm{Mod}^n_\bullet(m)\big) \to \mathcal{RF}^\circ_n\big(\mathrm{Mod}^n_\bullet\big) \to \mathbf{GJ}^n_\circ.$$

The image $\mathbf{GJ}^n_\circ(m)$ of this inclusion is closed and, hence, a compact suboperad of $\mathbf{GJ}^n_\circ$. Moreover,

$$\mathbf{GJ}^n_\circ \simeq \operatorname*{colim}_m \mathbf{GJ}^n_\circ(m).$$

We want to show that $\mathbf{GJ}^n_\circ(m) \subset \mathbf{GJ}^n_\circ(m+1)$ is a cellular extension generated by a cofibration. Indeed, let us consider the following n-collection $k(m+1)$,

$$k_T(m+1) = \begin{cases} V_n(\mathbf{GJ}^n_\circ(m))_T \cup \mathrm{Mod}^n_T & \text{if } \dim(T) = m+1, \\ V_n(\mathbf{GJ}^n_\circ(m))_T & \text{if } \dim(T) \neq m+1. \end{cases}$$

For every n-tree T, $k_T(m+1)$ is a closed subspace of $V_n(\mathbf{GJ}^n_\circ)(m+1)_T$. Moreover, if $\dim(T) = m+1$ then Proposition 6.1 implies that there exists a homeomorphism

$$S^{\dim(T)} \to Bd(K_T) = Bd\big(cl\big(\mathrm{Mod}^n_T\big)\big)$$

such that we have the following pushout in *Top*:

$$\begin{tikzcd} \coprod_{T,\dim(T)=m+1} S^{\dim(T)} \arrow[r] \arrow[d] & V_n(\mathbf{GJ}^n_\circ(m))_T \arrow[d] \\ \coprod_{T,\dim(T)=m+1} B^{\dim(T)} \arrow[r] & k_T(m+1). \end{tikzcd}$$

Hence, the inclusion

$$\omega_{m+1} : V_n\big(\mathbf{GJ}^n_\circ(m)\big) \subset k(m+1)$$

is a cofibration in the category of unbased reduced n-collections.

Besides that, we have a map of n-operads

$$\mathcal{RF}^\circ_n\big(k(m+1)\big) \to \mathbf{GJ}^n_\circ(m+1)$$

generated by the obvious map of collections $k(m+1) \to \mathbf{GJ}^n_\circ(m+1)$.

We also can construct another n-collection

$$l_T(m+1) = \begin{cases} V_n(\mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m)))_T \amalg \mathrm{Mod}_T^n & \text{if } \dim(T) = m+1, \\ V_n(\mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m)))_T & \text{if } \dim(T) \neq m+1 \end{cases}$$

with a bijective continuous map of n-collections

$$l(m+1) \to k(m+1),$$

a cofibration

$$\varsigma_{m+1} : W_n\big(\mathcal{RF}_n^\circ\big(\mathrm{Mod}_\bullet^n(m)\big)\big) \subset L(m+1), \tag{7.1}$$

and a map

$$\mathcal{RF}_n^\circ\big(l(m+1)\big) \to \mathcal{RF}_n^\circ\big(\mathrm{Mod}_\bullet^n(m+1)\big).$$

The cofibration (7.1) is actually a canonical coprojection into a coproduct.

All these maps of operads can be organised into a commutative cube

$$\begin{tikzcd}
 & \mathcal{RF}_n^\circ(V_n(\mathbf{GJ}_\circ^n(m))) \arrow[rr] \arrow[dd] & & \mathcal{RF}_n^\circ(l(m+1)) \arrow[dd] \\
\mathcal{RF}_n^\circ(V_n(\mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m)))) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{RF}_n^\circ(l(m+1)) \arrow[ur] \arrow[dd] & \\
 & \mathbf{GJ}_\circ^n(m) \arrow[rr] & & \mathbf{GJ}_\circ^n(m+1) \\
\mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m)) \arrow[ur] \arrow[rr] & & \mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m+1)). \arrow[ur] &
\end{tikzcd}$$

The front square of this cube is a pushout in the category of n-operads because ς_{m+1} is a coprojection and the free operad functor preserves colimits. All maps from the front square to the back square are continuous bijections. So the induced morphism from $\mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m+1))$ to the pushout P of the back square must be a continuous bijection as well, and, hence, the induced map $\alpha : P \to \mathbf{GJ}_\circ^n(m+1)$ is a continuous bijection.

Moreover, the vertical map $\mathcal{RF}_n^\circ(l(m+1)) \to \mathcal{RF}_n^\circ(\mathrm{Mod}_\bullet^n(m+1))$ admits a section. It follows that $\mathcal{RF}_n^\circ(k(m+1))_T \to P_T$ is epi. Since $\mathcal{RF}_n^\circ(l(m+1))_T$ is a finite coproduct of compact spaces the space P_T is compact. Therefore, α is an isomorphism and we have proved our theorem. $\square$

By definition we have an inclusion $\mathbf{GJ}^n \to \mathit{Des}_n(\mathbf{fm}^n)$, and, by adjunction, a canonical morphism

$$\mathit{Sym}_n\big(\mathbf{GJ}^n\big) \to \mathbf{fm}^n.$$

Theorem 7.3. *This canonical map from $Sym_n(\mathbf{GJ}^n)$ to $\mathbf{fm}^n$ is an isomorphism. The analogous result holds in the unbased case.*

Proof. The canonical map

$$\mathcal{RF}_n\big(\mathrm{Mod}^n_\bullet\big) \to \mathbf{GJ}^n$$

is a continuous bijection. Hence, after application of Sym_n to this map we have a composite of continuous bijections

$$\mathcal{RF}_\infty\big(C_n\big(\mathrm{Mod}^n_\bullet\big)\big) \to Sym_n\big(\mathcal{RF}_n\big(\mathrm{Mod}^n_\bullet\big)\big) \to Sym_n(\mathbf{GJ}).$$

We also have the following commutative diagram

$$\begin{tikzcd}
\mathcal{RF}_\infty(C_n(\mathrm{Mod}^n_\bullet))_k \arrow[rr] \arrow[d] & & Sym_n(\mathcal{RF}_n(\mathrm{Mod}^n_\bullet)) \arrow[d] \\
\mathcal{F}_s(\mathrm{mod}^n_{[k]}) \arrow[r] & \mathbf{fm}^n & Sym_n(\mathbf{GJ}^n) \arrow[l]
\end{tikzcd}$$

where the composite of the left vertical and horizontal maps is a continuous bijection by (6.2). Hence, the canonical map from $Sym_n(\mathbf{GJ}^n)$ to $\mathbf{fm}^n$ is a continuous bijection, as well. By formula (4.3) the space $Sym_n(\mathbf{GJ}^n)_k$ is a finite colimit of compact spaces, hence, it is compact and therefore the bijection between $Sym_n(\mathbf{GJ}^n)$ and $\mathbf{fm}^n$ is a homeomorphism. □

8. n-operads and E_n-operads

We finally are able to consider some applications of the results obtained.

Theorem 8.1. *The operad $\mathbf{fm}^n_\circ$ is a cellular object in the category of unbased reduced symmetric operads. In particular, it is cofibrant.*

Proof. This is an easy consequence of the fact that Sym_n is a left Quillen functor. □

Remark 8.1. The cofibrantness of $\mathbf{fm}^n_\circ$ was first claimed in [9] without a proof. To the best of our knowledge the proof first appeared in [18] and uses a comparison between $\mathbf{fm}^n_\circ$ and its Boardman–Vogt W-construction.

For a categorical operad A we will denote by $|A|$ the geometric realisation of the nerve of A.

Theorem 8.2. *The operad $|\mathbf{rh}^n_\circ|$ is strongly homotopy equivalent to $\mathbf{fm}^n_\circ$.*

Proof. Both n-operads $|\mathbf{RH}^n_\circ|$ and $\mathbf{GJ}^n_\circ$ are fibrant, cofibrant and contractible. Hence, they are strongly homotopy equivalent. So are their images under Sym_n. □

Remark 8.2. The Getzler–Jones decomposition of $\mathbf{fm}^n_\circ$ is not a regular CW-decomposition so we cannot take the poset of its cells and form a categorical operad as was proposed in [9]. However, the previous corollary shows that $\mathbf{rh}^n_\circ$ is an appropriate substitute for this nonexistent poset operad.

Theorem 8.3. *There is a chain of weak operadic equivalences between* $|\mathbf{rh}^n|$ *and* $\mathbf{fm}^n$.

Proof. The operad $|\mathbf{RH}^n|$ is the bar-construction on the terminal reduced n-operad. So, we have a zig–zag

$$B(\mathcal{RF}_n, \mathcal{RF}_n, 1) \leftarrow B(\mathcal{RF}_n, \mathcal{RF}_n, \mathbf{GJ}^n) \rightarrow \mathbf{GJ}^n$$

which, after symmetrization, gives the following zig–zag of morphisms of symmetric operads

$$|\mathbf{rh}^n| \leftarrow Sym_n B(\mathcal{RF}_n, \mathcal{RF}_n, \mathbf{GJ}^n) \rightarrow \mathbf{fm}^n \tag{8.1}$$

by Theorem 4.2. If we forget about nullary operations, we get the zig–zag of weak operadic equivalences

$$|\mathbf{rh}^n_\circ| \leftarrow Sym_n B(\mathcal{RF}^\circ_n, \mathcal{RF}^\circ_n, \mathbf{GJ}^n_\circ) \rightarrow \mathbf{fm}^n_\circ$$

since it can be obtained by symmetrisation from the zig–zag of equivalences of fibrant cofibrant operads

$$|\mathbf{RH}^n_\circ| \leftarrow B(\mathcal{RF}^\circ_n, \mathcal{RF}^\circ_n, \mathbf{GJ}^n_\circ) \rightarrow \mathbf{GJ}^n_\circ.$$

Hence, the zig–zag (8.1) consists of weak equivalences. □

Recall that the iterated monoidal category operad $\mathbf{m}^n$ contains an internal n-operad [4]. It is easy to see that $\mathbf{m}^n$ is a reduced categorical symmetric operad and its internal n-operad is also a reduced internal n-operad. So we have a canonical map of operads

$$k^n : \mathbf{rh}^n \rightarrow \mathbf{m}^n.$$

Recall also that the inclusion of the Milgram poset

$$(\mathbf{J}^n_k)^{\mathrm{op}} \rightarrow \mathbf{m}^n_k$$

induces a weak equivalence on the nerves [1].

The following theorem provides an alternative proof of Theorem 8.3.

Theorem 8.4. *The map of operads* $N(k^n) : N(\mathbf{rh}^n) \rightarrow N(\mathbf{m}^n)$ *is a weak equivalence.*

Proof. By Theorems 4.2 and 4.3, we have

$$N(\mathbf{rh}^n_k) \simeq \operatorname*{colim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N(\widetilde{\mathbf{RH}^n})_T.$$

It is trivial to check that $N(\widetilde{\mathbf{RH}^n})_{(-)}$ is a Reedy cofibrant functor on $(\mathbf{J}^n_k)^{\mathrm{op}}$ because

$$\operatorname*{colim}_{T\leftarrow T',\, T\neq T'} N\big(\widetilde{\mathbf{RH}^n}\big)_{T'} \to N\big(\widetilde{\mathbf{RH}^n}\big)_T = N\big(\mathbf{RH}^n\big)_T$$

is a monomorphism. Therefore,

$$\operatorname*{hocolim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N\big(\widetilde{\mathbf{RH}^n}\big)_T \to \operatorname*{colim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N\big(\widetilde{\mathbf{RH}^n}\big)_T$$

is a weak equivalence. But $N(\mathbf{RH}^n_T)$ is contractible, hence, we have a weak equivalence

$$\operatorname*{hocolim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N\big(\mathbf{J}^n_k/T\big) \to \operatorname*{hocolim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N\big(\widetilde{\mathbf{RH}^n}\big)_T.$$

So in the commutative square

$$\begin{tikzcd}
\mathrm{hocolim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N(\mathbf{J}^n_k/T) \arrow[r] \arrow[d] & N(\mathbf{J}^n_k) \arrow[d] \\
\mathrm{hocolim}_{(\mathbf{J}^n_k)^{\mathrm{op}}} N(\widetilde{\mathbf{RH}^n})_T \arrow[r] & N(\mathbf{rh}^n_k)
\end{tikzcd}$$

both horizontal and left vertical arrows are weak equivalences and, therefore, the right vertical arrow is a weak equivalence.

Finally, the functor $(\mathbf{J}^n_k)^{\mathrm{op}} \to \mathbf{m}^n_k$ is factorised as

$$\big(\mathbf{J}^n_k\big)^{\mathrm{op}} \to \mathbf{rh}^n_k \to \mathbf{m}^n_k$$

and the statement of our theorem follows. □

Corollary 8.4.1. *The canonical map $|\mathbf{rh}^n_\circ| \to |\mathbf{m}^n_\circ|$ provides a cofibrant replacement for $|\mathbf{m}^n_\circ|$ where $\mathbf{m}^n_\circ$ is the unbased version of $\mathbf{m}^n$.*

There is a canonical map $\psi : \mathbf{ph}^n \to \mathbf{rh}^n$ since $\mathbf{rh}^n$ contains an internal pruned operad. We actually can restrict ψ to a subcategory of $\mathbf{ph}^n$ which consists of objects which do not contain degenerate decorations. Obviously, the last subcategory is a deformation retract of $\mathbf{ph}^n$. The deformation retraction is given by the following composite (dropping off of dead leaves):

$$\begin{aligned}
&\mu\big(T; z^n U_0, x_1, \ldots, x_k\big) \to \mu\big(T; z^n U_0, \mu(U_n, x_1), \ldots, \mu(U_n, x_k)\big) \\
&\quad = \mu\big(\mu\big(T; z^n U_0, U_n, \ldots, U_n\big); x_1, \ldots, x_k\big) \to \mu\big(T'; z^n U_0, x_1, \ldots, x_k\big),
\end{aligned}$$

where T' is obtained from T by dropping off a branch which corresponds to the degenerate fiber.

So we will consider this subcategory as the domain of ψ but abusing notation will call it $\mathbf{ph}^n$.

Let $\tau \in \mathbf{rh}^n_k$ be a labelled tree decorated by pruned n-trees. Obviously, an object from the fiber of ψ over τ will be a labelled planar tree decorated by pruned n-trees such that the reduction of it (i.e. the deletion of all vertices of valency 2 together with their decorations) is τ. The morphisms are insertion of a vertex of valency 2 decorated by U_n and deletion of such decorations. The following lemma is obvious from this description.

Lemma 8.1. *The reduced decorated tree τ considered as an object of the fiber $\psi^{-1}(\tau)$ is a terminal object of this fiber.*

Lemma 8.2. *The operadic morphism ψ has a (nonoperadic) section*

$$s : \mathbf{rh}^n \to \mathbf{ph}^n$$

which maps an object τ to the terminal object τ in ψ^{-1}.

Proof. To define s on morphisms we have to define it on generators and then check correctness. We define it on a generator corresponding to a surjection $\sigma : T \to S$ equal to the composite

$$\mu\big(S; T_1^{(p)}, \dots, T_k^{(p)}\big) \to \mu\big(S; T_1^{(p)}, \dots, U_n, \dots, T_k^{(p)}\big) \to T$$

where we insert $e \to U_n$ in the places where σ has one-tip fibers. It is routine to check the relations. $\Box$

From these two lemmas we have

Theorem 8.5. *The operadic functor ψ induces a weak equivalence of simplicial operads*

$$N(\psi) : N\big(\mathbf{ph}^n\big) \to N\big(\mathbf{rh}^n\big);$$

so all three operads $N(\mathbf{ph}^n)$, $N(\mathbf{rh}^n)$, $N(\mathbf{m}^n)$ are E_n-operads.

A pruned topological n-operad A will be called *contractible* provided the unique map to the terminal n-operad is a weak equivalence i.e. every A_T is a contractible topological space.

Theorem 8.6. *Let A be a contractible pruned n-operad in the category of compactly generated Hausdorff spaces such that every A_T is a cofibrant topological space and let X be an algebra of A. Then X has a structure of an E_n-space, so up to group completion X is an n-fold loop space.*

Proof. Since X is an algebra of A it is also an algebra of $Sym_n(A)$ and, by Theorem 3.2, $Sym_n(A) \simeq \operatorname{colim}_{\mathbf{ph}_k^n} \tilde{A}_k$. It is not hard to check that the sequence $\operatorname{hocolim}_{\mathbf{ph}_k^n} \tilde{A}_*$ has the structure of an operad and, moreover, the canonical map

$$\operatorname*{hocolim}_{\mathbf{ph}_k^n} \tilde{A}_k \to \operatorname*{colim}_{\mathbf{ph}_k^n} \tilde{A}_k \tag{8.2}$$

is operadic. But $\operatorname{hocolim}_{\mathbf{ph}_k^n} \tilde{A}_k$ has the same homotopy type as $\mathbf{ph}_k^n$ because of contractibility of A_T. So X is an algebra of a E_n-operad $\operatorname{hocolim}_{\mathbf{ph}_k^n} \tilde{A}_k$. $\Box$

Example 8.1. It is still possible for X from the previous theorem to be an E_m-space for $m > n$. For example, if B is any E_∞-operad, then $A = Des_n(B)$ is contractible but $Sym_n(A) \simeq B$.

A similar theorem holds for reduced n-operads in the category $Ch(R)$ of chain complexes over a commutative ring with unit R.

Let M be a reduced n-operad in $Ch(R)$ which has $M_T = R$ for every pruned tree. A reduced n-operad in $Ch(R)$ equipped with an augmentation $A \to M$ will be called *contractible* provided its augmentation is a weak equivalence.

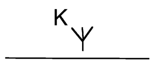
The method used in the proof of Theorem 8.6 can be used without change to prove the following.

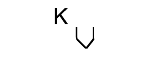
Theorem 8.7. *Let A be a contractible reduced n-operad in $Ch(R)$ such that A_T is a chain complex of projective R-modules for every T. Let X be an algebra of A. Then X admits an action of a symmetric reduced operad weakly equivalent to the operad of R-chains of the little disk operad.*

Finally, Theorems 7.2 and 7.3 imply that we can obtain the full solution of the coherence problem of n-fold loop spaces in the spirit of Stasheff's original work [20,21] using cells K_T for higher n-trees instead of associahedra. This was first claimed by Getzler and Jones but some doubts appeared since Tamarkin came up with his counterexample. Our Theorems 7.2 and 7.3 do confirm that the $\mathbf{fm}^n_\circ$-algebra structure is equivalent to the existence of a sequence of inductive extensions of higher homotopies from the boundaries of K_T to their interior even though $\mathbf{GJ}_T$ is not always a PL-ball. The exact combinatorics of K_T will be discussed in a future paper. Here we just give a few examples of the manifolds K_T which can be drawn on paper.

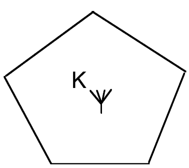
If $n = 1$ and $T = [m]$ then $K_T = K_m$, the Stasheff associahedron as was said before.

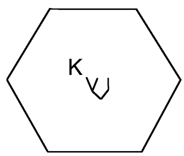
If $n = 2$ we have two one-dimensional manifolds commonly known as associator and braiding.

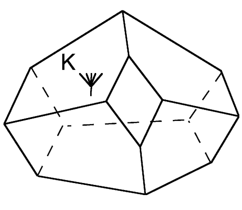


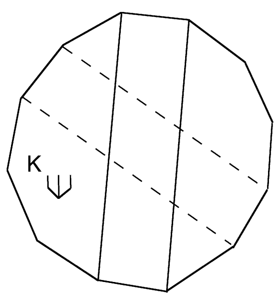


We also have three 2-dimensional polytopes: a pentagon and two hexagons, and two 3-dimensional polytopes, the associahedron K_5 and a 3-dimensional polytope well known in the theory of Yang–Baxter operators.

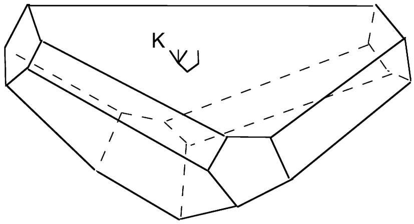


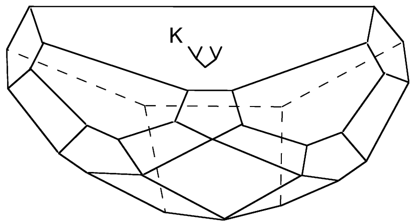






Finally we have three 3-dimensional polytopes (there should be a copy of the first one we do not show here) which can be found in the paper of Bar-Natan [5].





It is recommended to the reader to try to draw up all the polytopes K_T of dimension less than or equal to 3 with $n = 3$ (there will be only three of them which are different from the above polytopes) and $n = 4$ (only one new polytope).

9. Swiss-Cheese type *n*-operads and their symmetrisation

This is a short section in which we define reduced (both based and unbased) Swiss-Cheese type n-operads and show that the techniques developed in the previous sections are powerful enough to easily carry out the main results for classical operads to the case of Swiss-Cheese type operads.

The Swiss-Cheese type operads (*SC-operads or SC type operads* for short) were introduced by Voronov in [24] with motivation to describe a finite-dimensional model of the moduli space of genus-zero Riemann surfaces from open-closed string theory. The importance of this class of operads was also understood by Kontsevich in [13] who explained that the category of algebras of Voronov's Swiss-Cheese operad is a natural place for developing a theory of deformation complexes and its higher-dimensional generalisation.

Recall first [24] that a symmetric reduced SC-operad in a symmetric monoidal category V is a special symmetric coloured operad with two colours and consists of a collection $A_{k,l}$ of objects of V indexed by pairs of natural numbers with an action of the product of the symmetric groups $\Sigma_k \times \Sigma_l$. We also require that $A_{0,0} = A_{1,0} = A_{0,1} = I$ for a reduced operad. The operadic multiplication is represented by a family of morphisms

$$A_{k,l} \otimes (A_{a_1,b_1} \otimes \cdots \otimes A_{a_k,b_k}) \otimes (A_{0,c_1} \otimes \cdots \otimes A_{0,c_l})$$
$$\to A_{a_1+\cdots+a_k,\, b_1+\cdots+b_k+c_1+\cdots+c_l}$$

which must satisfy some natural associativity, equivariancy and unitarity conditions.

For the unbased case (which is actually considered in [13,24]) we forget about the space $A_{0,0}$.

Notice that we have two symmetric operads here: $A_{\bullet,0}$ and $A_{0,\bullet}$. An algebra of such an operad consists of a pair of objects (X_1, X_2) such that $A_{\bullet,0}$ acts on X_1 and $A_{0,\bullet}$ acts on X_2 and all the other spaces $A_{k,l}$ provide the interplay between these two actions.

The main example of an unbased reduced symmetric SC-operad is Voronov's Swiss-Cheese operad $\mathbf{SC}^n$ whose (k,l)-space is the space of l disjoint n-disks and k disjoint n-semidisks in a big n-semidisk

$$B^n_+ = \{(x_1, \ldots, x_n) \in \Re^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1,\ x_1 \geqslant 0\}$$

Notice that our presentation of $\mathbf{SC}^n_{k,l}$ is different from Voronov's picture in [24]. He defines B^n_+ by requiring $x_n \geqslant 0$. It is more convenient for us to ask $x_1 \geqslant 0$, however, because it makes the relations between $\mathbf{SC}_{k,l}$ and the Getzler–Jones decomposition more evident.

Definition 9.1. A coloured pruned n-tree is a pruned n-tree T equipped with a map of trees

$$\xi_T : T \to M_0^2$$

such that the induced n-ordinal structure on $\xi^{-1}(1)$ is a suspension over $(n-1)$-tree or a degenerate n-tree.

One can think of a coloured n-tree as an n-tree with a distinguished (coloured by 1) most left branch, which can be empty however.

Definition 9.2. A coloured morphism between coloured trees $\sigma : T \to S$ is a morphism of underlying trees such that $\xi_S(\sigma(a)) = 1$ for $a \in \xi^{-1}(1)$.

In other words σ sends the distinguished branch of T to the distinguished branch of S. Every coloured morphism between n-trees can be restricted to the distinguished branches of these trees. Since these branches are suspensions of n-trees, one can consider the restriction of a morphism as a morphism between $(n-1)$-trees.

A coloured morphism between coloured n-trees is *an injection* if the underlying morphism of trees is an injection. A coloured morphism between coloured n-trees is *a surjection* if the underlying morphism of trees is a surjection. One can take a fiber of a coloured surjection between n-trees and then the fibers obtain canonical colouring.

Definition 9.3. A reduced SC type n-operad in a symmetric monoidal category V is a collection $A_T \in V$ where T runs over all coloured pruned n-trees such that

$$A_{U_n^1} = A_{U_n^2} = I \quad \text{and} \quad A_{z^n U_0} = I,$$

where U_n^1 is a linear tree with its unique tip coloured by 1 and U_n^2 is a linear tree with its tip coloured by 2. This collection is equipped with a morphism

$$m_\sigma : A_S \otimes A_{T_1^{(p)}} \otimes \cdots \otimes A_{T_k^{(p)}} \to A_T$$

for every coloured morphism of trees $\sigma : T \to S$ between coloured pruned n-trees. They must satisfy the obvious associativity and unitarity conditions.

For an unbased reduced SC-operad we use collections without degenerate trees and define multiplication only with respect to surjections of coloured trees.

Immediately from the definition we see that as in the symmetric case an SC type n-operad A gives rise to two operads $A_{z^n U_n, \bullet}$ and $A_{\bullet, z^n U_n}$ if we consider its restriction to the n-trees with no branches with colour 1 or to the trees with no branches coloured 2. But unlike in the symmetric case these two operads are of different types: $A_{z^n U_n, \bullet}$ is an n-operad but $A_{\bullet, z^n U_n}$ is an $(n-1)$-operad.

The reduced SC type n-operads form a category and we have a desymmetrisation functor from symmetric reduced SC-operads to the reduced SC type n-operads. All the machinery of Section 4 is then applicable to the Swiss-Cheese case. We denote by $SCSym_n$ the corresponding functor of symmetrisation and by $\mathbf{scrh}^n$ the free reduced symmetric SC-operad freely generated by an internal reduced SC type n-operad. We have the following analogue of Theorem 4.3

Theorem 9.1. *Let A be a cocomplete reduced symmetric categorical n-operad of SC type and x be an internal reduced SC type n-operad in A then*

$$\big(SCSym_n(a)\big)_{k,l} \simeq \operatorname*{colim}_{\mathbf{scrh}^n_{k,l}} \tilde{x}_{k,l}$$

where $\tilde{x}_{k,l} : \mathbf{scrh}^n_{k,l} \to A_{k,l}$ is the operadic functor representing the operad x.

In addition,

$$\big(SCSym_n(a)\big)_{0,\bullet} \simeq Sym_n(A_{z^n U_n,\bullet}),$$
$$\big(SCSym_n(a)\big)_{\bullet,0} \simeq Sym_{n-1}(A_{\bullet,z^n U_n}).$$

The analogous formula holds in the unbased case.

The unbased Swiss-Cheese analogue of the Fulton–Macpherson operad $\mathbf{scfm}^n_\circ$ was defined by Voronov in [24]. The definition of its based version $\mathbf{scfm}^n$ is obvious. It is also obvious now how to define the analogue of the Getzler and Jones operad $\mathbf{SCGJ}^n$. A little gift for us is that, for a pruned coloured n-tree T, the spaces $\mathbf{SCGJ}^n_T$ and $\mathbf{GJ}^n_T$ coincide. So we have the proof of our next theorem almost for free:

Theorem 9.2. *The following analogues of Theorems* 7.1–7.3, 8.1–8.3 *hold*:

- *The operad $\mathbf{SCGJ}^n$ is contractible and $\mathbf{SCGJ}^n_\circ$ is a cellular and contractible unbased reduced SC type n-operad.*
- *The canonical map*

$$SCSym_n\big(\mathbf{SCGJ}^n\big) \to \mathbf{scfm}^n$$

  is an isomorphism. The analogous result holds in the unbased case.
- *The operad $\mathbf{scfm}^n_\circ$ is cellular.*
- *The operad $|\mathbf{scrh}^n_\circ|$ is strongly homotopy equivalent to $\mathbf{scfm}^n_\circ$.*
- *There is a chain of weak operadic equivalences between $|\mathbf{scrh}^n|$ and $\mathbf{scfm}^n$.*

It is not hard now to give the definitions of the endomorphism SC n-operad of a pair of objects $X_1, X_2 \in V$ and of the category of algebras of an SC n-operad. The desymmetrisation functor preserves the endomorphism operad and, hence, symmetrisation preserves the category of algebras. Analogously to Theorems 8.6 and 8.7 we have

Theorem 9.3. *Let A be a contractible reduced SC type n-operad in the category of compactly generated Hausdorff spaces such that every A_T is a cofibrant topological space and let (X_1, X_2)*

be an algebra of A. Then (X_1, X_2) admits an action of a symmetric SC-operad weakly equivalent to Voronov's reduced Swiss-Cheese operad.

Theorem 9.4. *Let A be a contractible reduced SC type n-operad in $Ch(R)$ such that A_T is a chain complex of projective R-modules for every T. Let (X_1, X_2) be an algebra of A. Then (X_1, X_2) admits an action of a symmetric reduced SC-operad weakly equivalent to the operad of R-chains of the Voronov's reduced Swiss-Cheese operad.*

Analogous theorems hold in the unbased case.

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