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Sharp L^p–L^q estimates for generalized k-plane transforms

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Abstract

In this paper, optimal L^p–L^q estimates are obtained for operators which average functions over polynomial submanifolds, generalizing the k-plane transform. An important advance over previous work (e.g., [P. Gressman, L^p-improving properties of X-ray like transforms, Math. Res. Lett. 13 (5) (2006) 787–803]) is that full L^p–L^q estimates are obtained by methods which have traditionally yielded only restricted weak-type estimates. In the process, one is led to make coercivity estimates for certain functionals on L^p for $p < 1$.

Keywords: X-ray transform; Radon transform; k-plane transform; Polynomial hypersurfaces; Geometric combinatorics

1. Introduction

1.1. Background

The object of study in this paper is the family of operators which integrate a function f over all submanifolds given by polynomials in some appropriate coordinate system. On $\mathbb{R}^2$, for example, such an operator would map a polynomial p in a single variable to the integral of a function $f(x, y)$ over the graph $y = p(x)$. To be more precise, fix any positive integers n, n', and d. Let $M_{n,d}$ be the set of all multiindices of length n and degree at most d (recall that a multiindex is simply an n-tuple of nonnegative integers, the degree of a multiindex $\alpha = (\alpha_1, \dots, \alpha_n)$ is denoted

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by $|\alpha|$ and equals $\sum_{j=1}^{n} \alpha_j$, and $t^\alpha := \prod_{j=1}^{n} t_j^{\alpha_j}$ as well as $\alpha! := \prod_{j=1}^{n} \alpha_j!$). Let $T_{n,n',d}$ be the operator mapping functions on $\mathbb{R}^n \times \mathbb{R}^{n'}$ (written $f(x,y)$ where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^{n'}$) to functions on $(\mathbb{R}^{n'})^{M_{n,d}}$ (thought of as the space of coefficients of an n'-tuple of polynomials in n variables of degree at most d) given by

$$T_{n,n',d} f(u) := \int_{\mathbb{R}^n} f\Bigg(t, \sum_{\alpha \in M_{n,d}} u_\alpha t^\alpha\Bigg)\, dt \tag{1}$$

(i.e., $u_\alpha \in \mathbb{R}^{n'}$ for all $\alpha \in M_{n,d}$ and $u = (u_\alpha)_{\alpha \in M_{n,d}}$). The purpose of this paper is to establish the L^p mapping properties of the family (1).

This family (1), when $d = 1$, generalizes the classical k-plane transform. To see this, let $u_0, \ldots, u_k \in \mathbb{R}^{n-k}$ be vectors, and consider the following mapping into $M_{k,n}$, the space of all affine k-planes in $\mathbb{R}^n$:

$$\sigma(u_0, u_1, \ldots, u_k) := \big\{(t, u_0 + u_1 t_1 + \cdots + u_k t_k) \in \mathbb{R}^n \;\big|\; t = (t_1, \ldots, t_k) \in \mathbb{R}^k\big\}.$$

Provided that $\|u_1\| + \cdots + \|u_k\|$ is small, the pull-back of the natural measure on $M_{k,n}$ is comparable to the Lebesgue measure on $\mathbb{R}^{(n-k)(k+1)}$; furthermore, the pull-back of the Lebesgue measure on the k-plane $\sigma(u)$ is comparable to dt, that is,

$$C^{-1} T_{k,n-k,1} f(u) \leqslant \int_{\sigma(u)} f \leqslant C T_{k,n-k,1} f(u)$$

(where f is a nonnegative function). To obtain global inequalities, one simply averages over all rotations $f^\theta(x) := f(\theta \cdot x)$ for $\theta \in SO(n)$.

The L^p-boundedness (in both standard and mixed-norm spaces) of the classical k-plane transform (including the Radon transform as a special case) was established in the 1980s in various papers including, for example, the works of Christ [4], Drury [6–8], and Oberlin and Stein [11]. The classical estimate to be generalized by this paper is that the k-plane transform maps $L^p(\mathbb{R}^n)$ to $L^q(M_{k,n})$ when $p = \frac{n+1}{k+1}$ and $q = n+1$ (the restricted weak-type version was established by Drury [6], and the full estimate by Christ [4]). By the remarks of the preceding paragraph, this estimate is a special case of Theorem 1 after performing the prescribed average over rotations.

When $d > 1$, the corresponding operators are largely new. The family $T_{1,n-1,d}$ arose in earlier work of the author [9] as examples of overdetermined, one-dimensional averaging operators. The significance of the family $T_{1,n-1,d}$ in that paper is that such operators are, in some sense, less degenerate than the classical X-ray transform; for example, the operators $T_{1,n-1,d}$ map L^p_{comp} to L^q_{loc} for a larger set of indices (p,q) than does the classical X-ray transform. The main result of that paper [9] was a family of restricted weak-type estimates; in this paper, the corresponding strong estimates follow from Theorem 1 as well.

The Fourier integral operator realization of (1) has nondegenerate canonical relation, so earlier theorems concerning overdetermined averaging operators, including recent work of Brandolini, Greenleaf, and Travaglini [2] and Ricci and Travaglini [12], can be applied. The proofs of these theorems are heavily concerned with the behavior of the operators (1) near L^2 (and rely on oscillatory integral estimates in one form or another). Such theorems give suboptimal results in general, meaning that they are restricted to the study of $L^p \to L^{p'}$ estimates for conjugate exponents p, p'. Unlike these earlier results, this paper approaches the question from the standpoint of geometric combinatorics (pioneered by Christ [5]), and is able to establish complete results.

It is useful to note that the operators (1) possess a variety of symmetries. First and foremost is an $(n+n')$-dimensional family of dilation symmetries: taking $f_{\delta,\delta'}(x,y) := f(\delta_1 x_1, \ldots, \delta_n x_n, \delta'_1 y_1, \ldots, \delta'_{n'} y_{n'})$ for arbitrary positive numbers $\delta_1, \ldots, \delta_n, \delta'_1, \ldots, \delta'_{n'}$ induces a scaling $u^{\delta,\delta'}$ on the space $(\mathbb{R}^{n'})^{M_{n,d}}$ by requiring $T_{n,n',d} f_{\delta,\delta'}(u) = T_{n,n',d} f(u^{\delta,\delta'})$. This family will appear explicitly in Section 5. Likewise, the translations $f_{h,h'}(x,y) := f(x+h, y+h')$ induce a family of translation operators $\tau^{h,h'}$ on $(\mathbb{R}^{n'})^{M_{n,d}}$ by (again) requiring that $T_{n,n',d} f_{h,h'}(u) = T_{n,n',d} f(\tau^{h,h'}(u))$. Although this family $\tau^{h,h'}$ is not the usual family of translations on Euclidean space, it does possess many of the same properties, including that $\tau^{h,h'}$ is measure-preserving with respect to the Lebesgue measure du. Finally, functions in the range of $T_{n,n',d}$ satisfy a family of PDEs (a fact first observed by F. John [10]). Let $\partial_{\alpha,j}$ be differentiation with respect to the jth component of u_α for $\alpha \in M_{n,d}$. For any j and k between 1 and n' (inclusive) and any $\alpha, \beta, \tilde{\alpha}, \tilde{\beta} \in M_{n,d}$ satisfying $\alpha + \beta = \tilde{\alpha} + \tilde{\beta}$,

$$[\partial_{\alpha,j}\partial_{\beta,k} - \partial_{\tilde{\alpha},j}\partial_{\tilde{\beta},k}] T_{n,n',d} f(u) = 0$$

(in the sense of distributions) for any f.

1.2. Main theorems

The main theorems of this paper establish sharp L^p–L^q boundedness of (1) and related generalizations. As already mentioned, an important feature (not found previously) of these theorems is that full endpoint estimates are obtained, not simply restricted weak-type estimates. The first theorem deals with the global and local L^p–L^q mapping properties of the family (1):

Theorem 1. *The operator $T_{n,n',d}$ maps $L^p(\mathbb{R}^n \times \mathbb{R}^{n'})$ to $L^q((\mathbb{R}^{n'})^{M_{n,d}})$ if and only if $p = 1 + \frac{n'd}{n+1}$ and $q = |M_{n,d}|p$, where $|M_{n,d}| = \binom{n+d}{d} := \frac{(n+d)!}{n!d!}$ is the number of multiindices of length n and degree at most d. Furthermore, $T_{n,n',d}$ maps $L^p_{\mathrm{comp}} \to L^q_{\mathrm{loc}}$ if and only if (p^{-1}, q^{-1}) is in the closed convex hull of the points $(0,0)$, $(1,1)$, $(0,1)$ and*

$$\left(\frac{n+1}{n+n'j+1}, \binom{n+j}{j}^{-1} \frac{(n+1)}{(n+n'j+1)}\right)$$

for $j = 1, \ldots, d$.

As indicated by the theorem (see Fig. 1 for an illustration of the Riesz diagram of a typical operator), the local mapping properties of $T_{n,n',d}$ are far more complex than a non-overdetermined averaging operator with nonvanishing rotational curvature (for example). If one moves to the scale of mixed-norm spaces, the boundedness properties of $T_{n,n',d}$ become even more complex because of natural "factorizations" which occur. For example, to prove the $L^p_{\mathrm{comp}} \to L^q_{\mathrm{loc}}$ estimates, it suffices to prove *only* global estimates. This is because an $L^p \to L^q$ estimate for $T_{n,n',j}$ implies mixed-norm boundedness of the form $L^p \to L^\infty(L^q)$ for $T_{n,n',d}$ when $d > j$, where

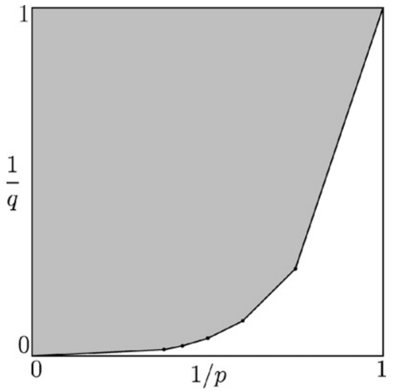

Fig. 1. The operator $T_{2,1,5}$ maps L^p_{comp} to L^q_{loc} precisely when (p^{-1}, q^{-1}) is in the shaded polygon shown above. The nontrivial vertices are marked by dots.

the L^q-norm is taken over the variables u_α for $|\alpha| \leqslant j$. To see this, let π be the natural projection from $(\mathbb{R}^{n'})^{M_{n,d}}$ to $(\mathbb{R}^{n'})^{M_{n,j}}$ and let $\tilde{\pi}$ be the corresponding projection onto the orthogonal complement of $(\mathbb{R}^{n'})^{M_{n,j}}$. It follows that

$$T_{n,n',d} f(u) = T_{n,n',j} f^{\tilde{\pi}(u)}\big(\pi(u)\big) \tag{2}$$

where $f^{\tilde{\pi}(u)}(t, x) := f(t, x + \sum_{|\alpha|>j} u_\alpha t^\alpha)$. Thus

$$\int\limits_{\tilde{\pi}(u)=v} \big|T_{n,n',d} f(u)\big|^q \leqslant C \big\| f^v \big\|_p^q = C \| f \|_p^q$$

uniformly in v when $T_{n,n',j}$ maps $L^p \to L^q$.

To identify many of the "trivial" $L^\infty(L^q)$ estimates satisfied by the operators (1), it is useful to restrict $T_{n,n',d}$ to somewhat general hyperplanes. In this paper, attention is fixed on coordinate hyperplanes. Such hyperplanes will be identified by the coordinate axes they contain; the axes themselves will be identified with elements in $M_{n,d} \times \{1, \dots, n'\}$, so coordinate hyperplanes are identified with subsets $\mathcal{A} \subset M_{n,d} \times \{1, \dots, n'\}$. For each $j = 1, \dots, n'$, let $\mathcal{A}_j$ be the collection of multiindices α for which $(\alpha, j) \in \mathcal{A}$. Then the restriction of $T_{n,n',d}$ to the coordinate hyperplane given by $\mathcal{A}$ will be denoted $T_{\mathcal{A}}$; the explicit formula is simply

$$T_{\mathcal{A}} f(u) := \int\limits_{\mathbb{R}^n} f\left(t, \sum_{\alpha \in \mathcal{A}_1} u_{(\alpha,1)} t^\alpha, \dots, \sum_{\alpha \in \mathcal{A}_{n'}} u_{(\alpha,n')} t^\alpha\right) dt; \tag{3}$$

for convenience, the following shorthand will be used in the future:

$$\pi_{\mathcal{A}}(t, u) := \left(t, \sum_{\alpha \in \mathcal{A}_1} u_{(\alpha,1)} t^\alpha, \dots, \sum_{\alpha \in \mathcal{A}_{n'}} u_{(\alpha,n')} t^\alpha\right).$$

Of course, not all coordinate hyperplanes $\mathcal{A}$ will give rise to restricted operators $T_{\mathcal{A}}$ which have nontrivial L^p–L^q boundedness properties. In particular, if $T_{\mathcal{A}}$ is to be bounded from any L^p

to some L^q, it must be the case that the following conditions are satisfied (the proof of necessity will be taken up in Section 5):

(1) (Dimensionality) There exists an integer $\#\mathcal{A}$ such that, for any $j = 1, \dots, n'$, the cardinality of $\mathcal{A}_j$ is $\#\mathcal{A}$.
(2) (Scaling) There exists an integer $|\mathcal{A}|$ such that $\sum_j \sum_{\alpha \in \mathcal{A}_j} \alpha = |\mathcal{A}|\mathbf{1}$, where $\mathbf{1} := (1, \dots, 1)$.
(3) (Spanning) The multiindices $\bigcup_j \mathcal{A}_j$ span $\mathbb{R}^n$ as vectors.

Throughout this paper, the collection $\mathcal{A}$ will be called *admissible* when it satisfies the dimensionality and scaling conditions, along with a slightly stronger form of the spanning condition (to be addressed in Section 2.2) and a further "nondegeneracy" condition:

(4) (Nondegeneracy) For each $j = 1, \dots, n'$, $\mathbf{0} \in \mathcal{A}_j$, where $\mathbf{0} = (0, \dots, 0)$.

With these definitions, the second main theorem of this paper is:

Theorem 2. *Let $\mathcal{A}$ be admissible. Then $T_{\mathcal{A}}$ maps $L^{\frac{|\mathcal{A}|+\#\mathcal{A}}{\#\mathcal{A}}} \to L^{|\mathcal{A}|+\#\mathcal{A}}$.*

This theorem implies the global result of Theorem 1, since the collection $\mathcal{A} = M_{n,d} \times \{1, \dots, n'\}$ (which corresponds to the trivial restriction of $T_{n,n',d}$ to the whole space on which it is defined) is readily checked to be an admissible collection.

1.3. Examples

Theorems 1 and 2 have several corollaries which are interesting in their own right. The first of these worth mentioning is similar to an earlier result of Ricci and Travaglini [12]:

Corollary 1. *Given an n'-tuple of polynomials $r := (r_1, \dots, r_{n'})$ in n variables of degree at most d (for $d \geqslant 2$), let μ_r be the measure on the graph of r (as a subset of $\mathbb{R}^n \times \mathbb{R}^{n'}$) given by*

$$\int \varphi(x, y)\, d\mu_r(x, y) := \int_{\mathbb{R}^n} \varphi\big(t, r_1(t), \dots, r_{n'}(t)\big)\, dt. \tag{4}$$

Then given any $f \in L^p(\mathbb{R}^n \times \mathbb{R}^{n'})$, $p := 1 + \frac{n'd}{n+1}$,

$$\| f \star \mu_r \|_{L^q(\mathbb{R}^n \times \mathbb{R}^{n'})} < \infty \tag{5}$$

for almost every r, where $q := \binom{n+d}{d} p$.

Proof. The idea of the proof is to express integration over $y \in \mathbb{R}^{n'}$ and $x \in \mathbb{R}^n$ as integration over coefficients of polynomials and thereby reduce (5) to Theorem 1. Understanding y is straightforward, but the x variable has more subtle properties which must be understood. The idea is that the family of polynomials $r(t + x)$ (with parameter x) is quite often, though not always, an n-dimensional hypersurface in the space of polynomials. When this is the case, standard change-of-variables arguments may be employed.

By definition of convolution and an elementary change of variables, one has the formula

$$f \star \mu_r(x,y) = \int f\big(-t, y - \Phi_x(r)(t)\big)\,dt,$$

where $\Phi.$ is the group of transformations of polynomials given by $\Phi_x(r)(t) := r(t+x)$. Fix any multiindices $\alpha_1,\dots,\alpha_n$ of degree $d-1$ and integers $j_1,\dots,j_n$, each in $1,\dots,n'$, and consider the mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ given by $x \mapsto (\partial^{\alpha_1}[\Phi_x(r)]_{j_1}(0),\dots,\partial^{\alpha_n}[\Phi_x(r)]_{j_n}(0))$ (here the derivatives are in t, then t is set equal to zero). This mapping is an affine linear function of x. Furthermore, the entries in the Jacobian matrix each depend linearly on the coefficients of the degree d terms of r. Standard results from algebraic geometry dictate that the corresponding mapping $x \mapsto (\partial^{\alpha_1}[\Phi_x(r)]_{j_1}(0),\dots,\partial^{\alpha_n}[\Phi_x(r)]_{j_n}(0))$ will be either degenerate for all r or invertible for almost every r (comprising an open set).

Consider the choice $\alpha_1 := (d-1,0,\dots,0),\dots,\alpha_n := (0,0,\dots,0,d-1)$ and $j_1 = \dots = j_n = 1$. For any n'-tuple of polynomials r with $r_1(t) = \sum_{j=1}^n t_j^d$, the mapping considered in the previous paragraph is simply $x \mapsto d(x_1,\dots,x_n)$, which is manifestly invertible. Therefore, for this particular choice of α's and j's, the mapping $x \mapsto (\partial^{\alpha_1}[\Phi_x(r)]_{j_1}(0),\dots,\partial^{\alpha_n}[\Phi_x(r)]_{j_n}(0))$ must be invertible for any r in some open set containing almost every n'-tuple of polynomials; call the set of such r's the set of invertibility. The standard change-of-variables argument gives that for any sufficiently small ball B centered at an r in the set of invertibility, there is a constant C such that, for any function g depending on n'-tuples of polynomials,

$$\iiint_B g\big(\Phi_x(\tilde r)\big)\,dx\,dy\,d\tilde r \leqslant C\|g\|_1;$$

here $d\tilde r$ is the Lebesgue integral over the coefficients of all n'-tuples of polynomials in B. To obtain this inequality, one changes the order of integration so that the integrals in x and y are performed before the α_kth coefficient of the j_kth polynomial for each $k = 1,\dots,n$ and before the constant coefficients $\alpha = \mathbf{0}$ of each polynomial as well. If the *remaining* coefficients (comprising $\tilde r$) are collectively referred to as v, the mapping $(x,y,v) \mapsto \Phi_x(r)$ has a constant, nonzero Jacobian determinant (for each value of the frozen coefficients); after the change-of-variables $(x,y,v) \mapsto \Phi_x(r)$, the integral $dx\,dy\,dv$ is simply an integral over the space of (n'-tuples of) polynomials. Taking $g(r) := (\int f(-t, y - r(t))\,dt)^q$ gives

$$\int_B \|f \star \mu_{\tilde r}\|_q^q\,d\tilde r \leqslant C\|T_{n,n',d}f\|_q^q,$$

which, by Theorem 1, implies (5) for almost every $\tilde r$ near r. $\square$

Another important corollary of Theorem 2 deals with restrictions $T_{\mathcal{A}}$ which are not overdetermined at all. This happens, for example, when $n' = 1$ and $\#\mathcal{A} = n$. In this case, Theorem 2 reduces to the following:

Corollary 2. *Let $\alpha_1,\dots,\alpha_n$ be multiindices on $\mathbb{R}^n$ which are linearly independent as vectors and sum to $(\sigma,\dots,\sigma)$ for some integer σ. Then the averaging operator on $\mathbb{R}^n \times \mathbb{R}$ given by*

$$Rf(y_0,\dots,y_n) := \int f\big(t, y_0 + y_1 t^{\alpha_1} + \dots + y_n t^{\alpha_n}\big)\,dt$$

is bounded from $L^{\frac{n+1+\sigma}{n+1}}$ to $L^{n+1+\sigma}$.

1.4. About the proof

As mentioned earlier, the proof is based on combinatorial tools introduced by Christ [5] and expanded upon by Tao and Wright [13] and many others. As in these works, the idea is to consider the bilinear form induced by (1) and iterate kernel flows of the corresponding projection operators. Because of the concrete nature of the operators (1), a coordinate dependent approach will be used, and much of the general geometry found, for example, in Tao and Wright [13] or earlier work of the author [9], will be suppressed.

As in [9], an essential feature of the proof is that the kernel flows are "lifted" to a higher-dimensional space in order to make the necessary change-of-variables arguments. A new feature introduced here is that all inequalities are shown to behave well with respect to tensor products (meaning that the "lifted" inequalities are again lifted to product spaces of arbitrarily high dimension). This allows one to deduce strong-type inequalities from the tensored restricted weak-type inequalities, as in earlier work of Bennett, Carbery, and Wright [1] and Carbery [3]. Unlike these earlier situations, however, there is no natural "tensor-invariance" to exploit. Instead, there are several new, nontrivial estimates which must be established to reproduce the earlier argument in the current, more general, case.

The rest of this paper proceeds as follows: Section 2 is devoted to establishing a number of inequalities concerning the Lebesgue measure (and certain L^p-spaces for $0 < p < 1$) which are essentially combinatorial in nature. These inequalities will be necessary to establish the main theorem using the standard approach of geometric combinatorics. Section 3 is concerned with the introduction of tensor-product inequalities. In particular, it is demonstrated in this section how one deduces strong-type inequalities from tensored restricted weak-type inequalities, and the tensor-product behavior of certain inequalities from Section 2 is addressed as well. Section 4 gives the proof of Theorem 2 (and hence Theorem 1 as well), and Section 5 establishes the necessity of various conditions of the main theorems.

2. Measure inequalities

2.1. Interpolation of monomial-weight measures

The first result of this section is an interpolation inequality for measures on $\mathbb{R}^n$ which are equal to the Lebesgue measure times a monomial weight. Let $d\mu(x) := |x_1|^{-1} \cdots |x_n|^{-1}\, dx_1 \cdots dx_n$, and for any $s \in \mathbb{R}^n_+$, let

$$|E|_s := \int_E \left|x^s\right| d\mu(x) := \int_E \prod_{j=1}^n |x_j|^{s_j - 1}\, dx.$$

To prove an interpolation inequality for these measures $|\cdot|_s$, the first step is to determine the measure-theoretic properties of certain extremal sets. Since it is relatively straightforward, the sharp constants are given both in Proposition 1 and Lemma 1.

Proposition 1. *Let $v_1, \ldots, v_n \in \mathbb{R}^n$ and $s \in \mathbb{R}^n_+$ be such that $v_1, \ldots, v_n$ are linearly independent and s is in the interior of the convex cone generated by $v_1, \ldots, v_n$. Let $\sigma \in \mathbb{R}^n_+$ be such that*

$s = \sum_{j=1}^n \sigma_j v_j$. Then for any $a \in \mathbb{R}^n_+$, the set $E^a_v := \{x \in \mathbb{R}^n \mid \sum_{i=1}^n a_i^{-1} |x|^{v_i} \leqslant 1\}$ satisfies

$$\left|E^a_v\right|_s = 2^n \frac{a^\sigma}{V} \frac{\prod_{j=1}^n \Gamma(\sigma_j)}{\Gamma(1+|\sigma|)} \tag{6}$$

where V is the absolute value of the determinant of the matrix with columns $v_1, \ldots, v_n$.

Proof. In the integral $\int \chi_{E^a_v}(x) \prod_{j=1}^n |x_j|^{s_j - 1} \, dx$, make the change of variables $u_i = a_i^{-1} x^{v_i}$. The linear independence of the v_i's guarantees that this map is one-to-one and onto on $\mathbb{R}^n_+$ (by symmetry, this is the only orthant on which the integral need be computed). The (i, j)-entry of the Jacobian matrix of this change is precisely $a_i^{-1} v_{i,j} x^{v_i} x_j^{-1}$, so the absolute value of the determinant is $V \prod_{j=1}^n x_j^{-1} \prod_{i=1}^n u_i$. It follows that $x^s \, d\mu(x) = a^\sigma V^{-1} u^\sigma \, d\mu(u)$ and hence

$$\left|E^a_v\right|_s = 2^n a^\sigma V^{-1} \int_T u^\sigma \, d\mu(u)$$

where T is the simplex $\{u \in \mathbb{R}^n_+ \mid \sum u_i \leqslant 1\}$. A straightforward induction argument (using Euler's identity for the Beta function) computes this integral and gives (6). $\square$

Lemma 1. *Let $w_1, \ldots, w_n \in \mathbb{R}^n_+$ be linearly independent, and let $s \in \mathbb{R}^n_+$ be in the interior of the convex hull of the w_j's and 0. Let $\theta_0, \ldots, \theta_n$ be such that $\sum_{j=0}^n \theta_j = 1$ and $s = \sum_{j=1}^n \theta_j w_j$. Then*

$$|E|_s \leqslant 2^{n\theta_0} \prod_{j=0}^n \theta_j^{-\theta_j} \left[\frac{\prod_{j=1}^n \Gamma(\theta_j \theta_0^{-1})}{W \Gamma(\theta_0^{-1})}\right]^{\theta_0} \prod_{j=1}^n |E|_{w_j}^{\theta_j} \tag{7}$$

where W is the absolute value of the determinant of the matrix with columns $w_1, \ldots, w_n$.

Proof. Let $a_j := |E|_{w_j}$ and $g(x) := \sum_{j=1}^n \theta_j a_j^{-1} |x|^{w_j - s}$. By construction of the function g, $\int_E g(x)|x|^s \, d\mu(x) = 1 - \theta_0$. Let $G_\lambda := \{x \in \mathbb{R}^n \mid g(x) < \lambda\}$. The quantity $|G_\lambda|_s$ can be evaluated using Proposition 1: simply take $v_j := w_j - s$ and $\sigma_j := \theta_j \theta_0^{-1}$. Elementary computations give that $V = \theta_0 W$ (the matrix of v's "factors" as the w-matrix and a matrix involving only the θ's). Combining these observations, Proposition 1 dictates that

$$|G_\lambda|_s^{\theta_0} = 2^{n\theta_0} \lambda^{1-\theta_0} \prod_{j=0}^n \theta_j^{-\theta_j} \left[\frac{\prod_{j=1}^n \Gamma(\theta_j \theta_0^{-1})}{W \Gamma(\theta_0^{-1})}\right]^{\theta_0} \prod_{j=1}^n |E|_{w_j}^{\theta_j}. \tag{8}$$

Along these same lines, $\int_{G_\lambda} g(x)|x|^s \, d\mu(x) = \lambda |G_\lambda|_s (1 - \theta_0)$; this is because $|G|_{w_j} = \lambda |G|_s |E|_{w_j}$ by Proposition 1 as well. Now choose λ so that $|G_\lambda|_s = |E|_s$. Then $1 - \theta_0 = \int_E g(x)|x|^s \, d\mu(x) \geqslant \int_{G_\lambda} g(x)|x|^s \, d\mu(x)$. To see this, simply observe that the integral over $E \setminus G_\lambda$ is necessarily greater than the integral over $G_\lambda \setminus E$ (and therefore the value of the integral decreases if all the parts of E outside G_λ are moved inside G_λ). Since $\int_{G_\lambda} g(x)|x|^s \, d\mu(x) = \lambda |E|_s (1 - \theta_0)$, it follows that $\lambda |E|_s \leqslant 1$. Multiplying both sides of (8) by $|E|_s^{1-\theta_0}$, recalling that $|G_\lambda|_s = |E|_s$ and using the inequality $\lambda |E|_s \leqslant 1$ give (7). $\square$

In practice, the following corollary of Lemma 1 will be more useful than Lemma 1 itself. At this point, accounting for constants becomes a chore and will be neglected.

Corollary 3. *Let $w_1, \dots, w_N \in \mathbb{R}^n_+$ for $N \geqslant n$ have a sub-n-tuple which is linearly independent. Then for any positive $\theta_0, \dots, \theta_N$ satisfying $\sum_{j=0}^N \theta_j = 1$, there exists a constant $C < \infty$ such that*

$$|E|_s \leqslant C \prod_{j=1}^N |E|_{w_j}^{\theta_j} \tag{9}$$

where $s = \sum_{j=1}^N \theta_j w_j$.

Proof. Suppose that $w_1, \dots, w_n$ are linearly independent. If $N = n$, then Lemma 1 gives precisely the desired conclusion; assume, then, that $N > n$. Let $\varphi := \sum_{j=n+1}^N \theta_j$. Hölder's inequality immediately gives that

$$|E|_{\varphi^{-1}\sum_{j=n+1}^N \theta_j w_j} \leqslant \prod_{j=n+1}^N |E|_{w_j}^{\varphi^{-1}\theta_j}. \tag{10}$$

On the other hand, Lemma 1 gives that

$$|E|_{(1-\varphi)^{-1}\sum_{j=1}^n \theta_j w_j} \leqslant C \prod_{j=1}^n |E|_{w_j}^{(1-\varphi)^{-1}\theta_j} \tag{11}$$

because the w's satisfy the independence condition, and

$$(1-\varphi)^{-1} \sum_{j=1}^n \theta_j = (1-\varphi)^{-1}(1-\varphi-\theta_0) < 1.$$

But Hölder's inequality also dictates that

$$|E|_s \leqslant |E|^{\varphi}_{\varphi^{-1}\sum_{j=n+1}^N \theta_j w_j} |E|^{1-\varphi}_{(1-\varphi)^{-1}\sum_{j=1}^n \theta_j w_j},$$

so taking a convex combination of (10) and (11) gives (9). □

2.2. *Vandermonde means*

Suppose $\mathcal{A}$ is an admissible subset of $M_{n,d} \times \{1, \dots, n'\}$. Let $x_j \in \mathbb{R}^n$ for $j = 1, \dots, \#\mathcal{A}$, and for $k = 1, \dots, n'$, let $V_k(x)$ be the determinant of the $\#\mathcal{A} \times \#\mathcal{A}$ matrix whose j,m-entry is $x_j^{\alpha_m}$ (where $\alpha_1, \dots, \alpha_{\#\mathcal{A}}$ is an enumeration of $\mathcal{A}_k$). The product of these functions V_k will be called the *Vandermonde polynomial associated to* $\mathcal{A}$, and denoted $V_{\mathcal{A}}$, that is:

$$V_{\mathcal{A}}(x) := \prod_{k=1}^{n'} V_k(x). \tag{12}$$

If, for example, $n = n' = 1$ and $\mathcal{A} = 0, 1, \ldots, d$, then $V_{\mathcal{A}}$ is exactly the dth classical Vandermonde polynomial (modulo, of course, a factor of ± 1).

To establish Theorem 2, it will be necessary to have an estimate for the expectation of $|V_{\mathcal{A}}(x)|$ when the x_j's ($j = 1, \ldots, \#\mathcal{A}$) are randomly chosen points. It will suffice for the purposes here to prove that

$$\int \left|V_{\mathcal{A}}(x)\right| \prod_{j=1}^{\#\mathcal{A}} \left|f_j(x_j)\right| dx_1 \cdots dx_{\mathcal{A}} \geqslant c \prod_{j=1}^{\#\mathcal{A}} \|f_j\|_{L^p(\mathbb{R}^n)} \tag{13}$$

where $p := \frac{\#\mathcal{A}}{\#\mathcal{A}+|\mathcal{A}|} < 1$ (many more such inequalities are, in fact, true, but will not be needed here). Along the way, the strengthened spanning condition for $\mathcal{A}$ will be encountered.

The proof of (13) begins with a definition. Given a Lebesgue-measurable set $E \subset \mathbb{R}^n$, let $S^j(E)$ be the set given by

$$\left\{(y_1, \ldots, y_n) \in \mathbb{R}^n \;\middle|\; |y_j| < \frac{1}{2} \int \chi_E(y_1, \ldots, y_{j-1}, s, y_{j+1}, \ldots, y_n)\, ds\right\}.$$

By Fubini's theorem, this set is well defined (up to a set of measure zero); it is called the *Steiner symmetrization* with respect to the hyperplane $P_j := \{(y_1, \ldots, y_n) \in \mathbb{R}^n \mid y_j = 0\}$. Observe that the intersection of $S^j(E)$ with any line ℓ pointing in the jth coordinate direction is simply a line segment with center in P_j. Moreover, the measure of $S^j(E)$ is the same as the measure of E, and if $f(y)$ is any nonnegative measurable function which does not depend on y_j, then $\int_E f(y)\,dy = \int_{S^j(E)} f(y)\,dy$ (both of these facts follow almost directly from Fubini's theorem). The following propositions illustrate how Steiner symmetrization will be useful here. Heuristically, if one wants to estimate the integral of a function $|f|$ on a set E, the function f may be replaced by a simpler function if, in exchange, the set E is replaced by a Steiner-symmetrized version of itself.

Proposition 2. *Let $I \subset \mathbb{R}$ be some (possibly infinite) interval, and let f be a function in $C^k(I)$ (for some fixed $k \geqslant 1$) which satisfies the inequality $f^{(k)}(t) \geqslant 1$ for all $t \in I$. There exists a constant $c_k > 0$ such that, for any measurable set $E \subset I$,*

$$\int_E \left|f(t)\right| dt \geqslant c_k \int_{S(E)} \frac{|t|^k}{k!}\, dt. \tag{14}$$

Proof. The first step is to establish the inequality (14) in the case when E is an interval. Let $E := [a, b] \subset I$, $\delta := \frac{b-a}{k+1}$, and $a_j := a + j\delta$ for $j = 0, \ldots, k+1$. By the fundamental theorem of calculus (and an elementary induction argument on k), the following identity holds for all $f \in C^k(I)$ when $[a, b] \subset I$:

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \int_{a_j}^{a_{j+1}} f(t)\, dt = \int_0^{\delta} \cdots \int_0^{\delta} f^{(k)}(a + t_1 + \cdots + t_{k+1})\, dt_1 \cdots dt_{k+1}.$$

The right-hand side is at least δ^{k+1} by virtue of the pointwise estimate for $f^{(k)}$; majorizing the left-hand side in the standard way gives that

$$\binom{k}{j_0}\int_a^b \left|f(t)\right| dt \geqslant \frac{|b-a|^{k+1}}{(k+1)^{k+1}} \tag{15}$$

where $j_0 = \frac{k}{2}$ or $\frac{k+1}{2}$ when k is even or odd, respectively. This inequality (15) is precisely (14) by virtue of the fact that $|E| = |b-a|$.

Let $F_\lambda := \{t \in I \mid |f(t)| \leqslant \lambda\}$. The set F_λ is a union of no more than k disjoint, closed intervals (between any two connected components of F_λ, f' must vanish; but f' can vanish only $k-1$ times by Rolle's theorem). The inequality (15) can thus be used to estimate $\int_{F_\lambda} |f(t)|\,dt$ by restricting to subintervals and summing. The right-hand side of a k-fold sum of (15) is minimized (given $|F_\lambda|$) when each subinterval has measure $\frac{1}{k}|F_\lambda|$; thus

$$\int_{F_\lambda} \left|f(t)\right| dt \geqslant \frac{j_0!(k-j_0)!}{k!k^k(k+1)^{k+1}}|F_\lambda|^{k+1}. \tag{16}$$

Now $|F_\lambda|$ is clearly a nondecreasing function of λ; as $\lambda \to \infty$, it must also be the case that $|F_\lambda| \to |I|$ (because $|f|$ is bounded on any finite subinterval of I). Inner and outer regularity of the Lebesgue measure, coupled with the fact that $|\{t \in I \mid |f(t)| = \lambda\}| = 0$ (solutions to $f(t) = \pm\lambda$ are isolated thanks to the derivative inequality), give that $|F_\lambda|$ is a continuous function of λ as well. Therefore, given an arbitrary measurable set $E \subset I$, there exists a unique minimal $\lambda > 0$ such that $|E| = |F_\lambda|$. But

$$\int_E \left|f(t)\right| dt \geqslant \int_{E\cap F_\lambda} \left|f(t)\right| dt + \lambda|E \setminus F_\lambda|$$

since $|f(t)| > \lambda$ outside F_λ. Since $|E| = |F_\lambda|$, $|E \setminus F_\lambda| = |F_\lambda \setminus E|$; then

$$\lambda|E \setminus F_\lambda| \geqslant \int_{F_\lambda\setminus E} \left|f(t)\right| dt.$$

Therefore, among all measurable sets in I with measure $|E|$, the integral on the left-hand side of (14) is minimized for F_λ. Furthermore, $\int_{S(E)} |t|^k\,dt = 2^{-k}|E|^{k+1}/(k+1)$ (by definition of the Steiner symmetrization). Combining with (16) gives

$$\int_E \left|f(t)\right| dt \geqslant 2^k \frac{j_0!(k-j_0)!}{k^k(k+1)^k} \int_{S(E)} \frac{|t|^k}{k!}\,dt,$$

which is precisely the desired inequality. $\square$

At this point, one must impose an ordering on the set of multiindices. Throughout the remainder of this section, the dictionary order on multiindices will be used; that is, given two multiindices α, β, of length n, one says that $\alpha \leqslant \beta$ if and only if the smallest index i for which

$\alpha_i \neq \beta_i$ (if it exists) satisfies $\alpha_i < \beta_i$. It is an elementary exercise to check that this does, in fact, define a total ordering on the set of multiindices of a given length, and consequently, any finite subset $\mathcal{A}_j$ of such multiindices has a maximal element.

The next proposition contains the main inequality which will be needed to establish (13). In essence, it is a generalization of Proposition 2 to higher dimensions. As before the trade-off is that integrals of (somewhat arbitrary) functions are replaced by integrals of monomials over a symmetrized set—in this case, the symmetrization is an n-fold symmetrization with respect to the n coordinate directions (and the order of the symmetrizations is determined by the order on multiindices that was just chosen).

Proposition 3. *For any positive integer n', and for $j = 1, \dots, n'$, let $\mathcal{A}_j$ be a collection of multi-indices (of length n) such that $\#\mathcal{A}_j = \#\mathcal{A}_1 < \infty$ for all j. For each j, let $\max(\mathcal{A}_j)$ be the maximal element of $\mathcal{A}_j$, and let $\mathcal{A}'_j := \mathcal{A}_j \setminus \{\max(\mathcal{A}_j)\}$. Then there exists a constant c depending only on the $\max(\mathcal{A}_j)$'s such that, for any measurable set $E \subset \mathbb{R}^n$ and any $x' \in (\mathbb{R}^n)^{\#\mathcal{A}_1 - 1}$,*

$$\int_E \left| \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx \geqslant c \left| \prod_{j=1}^{n'} V_{\mathcal{A}'_j}(x') \right| \int_{E^*} \left| x^{\sum_{j=1}^{n'} \max(\mathcal{A}_j)} \right| dx,$$

where $\prod_j V_{\mathcal{A}_j}$ is the Vandermonde polynomial associated to $\mathcal{A}$, and $E^* := S^n(\cdots(S^1(E))\cdots)$.

Proof. This lemma follows from repeated application of the sublevel inequality of Proposition 2. For any $k > [\max(\mathcal{A}_j)]_1$, it must be true that $\frac{\partial^k}{\partial x_1^k} V_{\mathcal{A}_j}(x, x') = 0$. This is because $V_{\mathcal{A}_j}$ is linear in the monomials x^α for $\alpha \in \mathcal{A}_j$; if the derivative did not vanish, it would contradict the maximality of $\max(\mathcal{A}_j)$. Let $\beta := \sum_{j=1}^{n'} \max(\mathcal{A}_j)$. Suppose $x = (x_1, \dots, x_n)$; by Proposition 2,

$$\int \chi_E(x) \left| \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx_1 \geqslant c_{\beta_1} \int \chi_{S^1(E)}(x) \frac{|x_1^{\beta_1}|}{\beta_1!} \left| \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx_1$$

since the differentiated quantity on the right-hand side is independent of x_1. Repeating for $x_2, \dots, x_n$, it must be the case that

$$\int_E \left| \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx \geqslant \prod_{i=1}^{n} c_{\beta_i} \int_{E^*} \frac{|x^\beta|}{\beta!} \left| \left(\frac{\partial}{\partial x} \right)^\beta \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx.$$

The differentiated quantity on the right-hand side is independent of x entirely; moreover, by the Leibniz rule,

$$\left(\frac{\partial}{\partial x} \right)^\beta \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') = \sum \cdots \sum_{\gamma^1 + \cdots + \gamma^{n'} = \beta} \left[\prod_{j=1}^{n'} \frac{\beta!}{\gamma^j!} \left(\frac{\partial}{\partial x} \right)^{\gamma^j} V_{\mathcal{A}_j}(x, x') \right].$$

Since the multiindices γ^j sum to β, and $\partial_x^{\gamma^j} V_{\mathcal{A}_j} \equiv 0$ if $\gamma^j > \max(\mathcal{A}_j)$, it follows that all terms on the right-hand side vanish except for the term $\gamma^j = \max(\mathcal{A}_j)$, $j = 1, \dots, n'$, and hence

$$\int_E \left| \prod_{j=1}^{n'} V_{\mathcal{A}_j}(x, x') \right| dx \geqslant \prod_{i=1}^{n} c_{\beta_i} \left| \prod_{j=1}^{n'} \left(\frac{\partial}{\partial x} \right)^{\max(\mathcal{A}_j)} \frac{V_{\mathcal{A}_j}(x, x')}{(\max(\mathcal{A}_j))!} \right| \int_{E^*} \left| x^\beta \right| dx.$$

Cramer's rule dictates that the product in absolute values on the right-hand side is precisely $|\prod_j V_{\mathcal{A}'_j}(x')|$, completing the proof. $\square$

With the aid of Proposition 3, the desired inequality (13) concerning Vandermonde means is quickly established. This is precisely the purpose of the following theorem. In the process, the strengthened spanning condition (referred to in Section 1.2) will be employed. Recall that the spanning condition already stated is that, for any admissible $\mathcal{A}$, the monomials in $\mathcal{A}_1, \dots, \mathcal{A}_{n'}$ collectively span $\mathbb{R}^n$ as vectors. The strengthened spanning condition goes as follows. Let $\alpha_{1,k} < \alpha_{2,k} < \cdots < \alpha_{\#\mathcal{A},k}$ be the ordered enumeration of $\mathcal{A}_k$ relative to the dictionary order, and let $\beta_j := \sum_{j=1}^{n'} \alpha_{j,k}$ for $j = 1, \dots, \#\mathcal{A}$. The strengthened spanning condition holds when $\beta_1, \dots, \beta_{\#\mathcal{A}}$ span $\mathbb{R}^n$ as vectors. After the proof of Theorem 3, this condition will be examined more closely, but first comes the main result of this section:

Theorem 3. *Let $\mathcal{A}$ be an admissible subset of $M_{n,d} \times \{1, \dots, n'\}$ which satisfies the strengthened spanning condition given above. Then* (13) *holds, that is, there exists a constant $c > 0$ such that*

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \left| f_j(x_j) \right| dx_1 \cdots dx_{\#\mathcal{A}} \geqslant c \prod_{j=1}^{\#\mathcal{A}} \| f_j \|_{L^p(\mathbb{R}^n)}$$

for any measurable functions f_j, where $p := \frac{\#\mathcal{A}}{\#\mathcal{A} + |\mathcal{A}|} < 1$.

Proof. By repeated application of Proposition 3, it follows that there exists some $c > 0$ such that, for all measurable sets $E_j \subset \mathbb{R}^n$,

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \chi_{E_j}(x_j)\, dx \geqslant c \prod_{j=1}^{\#\mathcal{A}} \left| E_j^* \right|_{\mathbf{1}+\beta_j}$$

(using the notation of Section 2.1; here $\mathbf{1} := (1, \dots, 1)$). Because the left-hand side is symmetric, the sets $E_2, \dots, E_{\#\mathcal{A}}$ may be permuted to obtain a family of inequalities *while E_1 remains fixed.* Taking the geometric mean of all these inequalities gives that

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \chi_{E_j}(x_j)\, dx \geqslant c \left| E_1^* \right| \left(\prod_{j'=2}^{\#\mathcal{A}} \prod_{j=2}^{\#\mathcal{A}} \left| E_j^* \right|_{\mathbf{1}+\beta_{j'}} \right)^{\frac{1}{\#\mathcal{A}-1}}$$

(where the observation that $|E_1|_{\mathbf{1}+\beta_1} = |E_1|_{\mathbf{1}} = |E_1|$ has been quietly exploited). Since $\mathcal{A}$ is admissible, $|\mathcal{A}|\mathbf{1} = \sum_{j=2}^{\#\mathcal{A}} \beta_j$, so $\sum_{j=2}^{\#\mathcal{A}} \mathbf{1} + \beta_j = (\#\mathcal{A} - 1 + |\mathcal{A}|)\mathbf{1}$ as well. For any scalars $c_2, \dots, c_{\#\mathcal{A}}$,

$$\sum_{j=2}^{\#\mathcal{A}} c_j(\mathbf{1} + \beta_j) = \sum_{j=2}^{\#\mathcal{A}} \left(c_j + |\mathcal{A}|^{-1} \sum_{k=2}^{\#\mathcal{A}} c_k \right) \beta_j.$$

As vectors in $\mathbb{R}^n$, the right-hand side can assume any value for appropriate choice of the c_j's (since the matrix with off-diagonal entries $|\mathcal{A}|^{-1}$ and diagonal entries $1 + |\mathcal{A}|^{-1}$ is invertible). Therefore, the vectors $\mathbf{1} + \beta_j$ span $\mathbb{R}^n$ as well, and so one may employ Lemma 1 (or more precisely, inequality (9)) to conclude that

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \chi_{E_j}(x_j)\, dx \geqslant c \left| E_1^* \right| \prod_{j=2}^{\#\mathcal{A}} \left| E_j^* \right|^{1+\frac{|\mathcal{A}|}{\#\mathcal{A}-1}}$$

(for some new constant c). Recalling $|E_j^*| = |E_j|$, this inequality becomes a restricted weak-type estimate for (13). By standard machinery, this estimate may be summed to obtain Lorentz space inequalities. In this case, the result is that

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \left| f_j(x_j) \right| dx \geqslant c \| f_1 \|_{L^1(\mathbb{R}^n)} \prod_{j=2}^{\#\mathcal{A}} \| f_j \|_{L^{p_1,1}(\mathbb{R}^n)}$$

where $p_1^{-1} := 1 + \frac{|\mathcal{A}|}{\#\mathcal{A}-1}$. Now the f_j's are permuted again, this time including f_1. The geometric mean of these permuted inequalities is precisely

$$\int \left| V_{\mathcal{A}}(x) \right| \prod_{j=1}^{\#\mathcal{A}} \left| f_j(x_j) \right| dx \geqslant c \prod_{j=1}^{\#\mathcal{A}} \| f_j \|_{L^1}^{\frac{1}{\#\mathcal{A}}} \| f_j \|_{L^{p_1,1}}^{1-\frac{1}{\#\mathcal{A}}}.$$

By the standard convexity inequalities for Lorentz space quasi-norms, the proof is complete. $\square$

Concerning the strengthened spanning condition: it should be noted that the strengthened spanning condition depends on the order which is imposed on multiindices. The β_j's themselves may change if, for example, one takes a different dictionary ordering on $M_{n,d}$ (reordering the coordinate axes, for example). If, however, one is in the situation that $\mathcal{A}_j = \mathcal{A}_k$ for all j, k, the strengthened spanning condition reduces to the spanning condition mentioned in the introduction. This is because each β_j is necessarily equal to $n'\alpha_j$ for some $\alpha_j \in \mathcal{A}_j$, and changing the ordering on $M_{n,d}$ simply reorders the β_j's. Situations in which this occurs (and, hence, the two spanning conditions are equivalent) include that of Theorem 1 and Corollary 2, as well as the case of codimension 1 averaging operators ($n' = 1$).

3. Tensor inequalities

In this section, two propositions are established concerning the relationship (first observed by Carbery [3]) between strong-type estimates for an operator and restricted weak-type estimates for the tensor products of that operator. Propositions 4 and 5 each shed a small amount of light on this relationship from different perspectives. The overarching idea is that weak-L^p norms do not naturally behave well under tensor products, e.g., the weak-$L^p(\mathbb{R}^N)$ norm of $\prod_{j=1}^N f(x_j)$ is in general greater than the Nth power of the weak-$L^p(\mathbb{R})$ norm of f. If, by chance, there is some control on the growth of these norms as $N \to \infty$, then one can gain information about the regularity of f. Conversely, if one has extra information about f it can be possible to control the growth as $N \to \infty$. Propositions 4 and 5 each establish this principle in one direction; as noted in the statements themselves, the implications go both ways. To prove the converse of Proposition 5, one mimics the proof of Proposition 4 and so on. The converses have been omitted only because they are not necessary here.

To simplify notation throughout the rest of this paper, bold will be used to indicate objects in a product space. For example, the variable u will, from here on, denote a point in $\mathbb{R}^{\mathcal{A}}$. The bold version, $\mathbf{u}$, is to be understood as an element of $(\mathbb{R}^{\mathcal{A}})^N$. If an operation (which is *not* bold) is performed on a tensor (bold) variable, it is to be understood as a component-wise operation. For example,

$$\pi_{\mathcal{A}}(\mathbf{t}, \mathbf{u}) := \big(\pi_{\mathcal{A}}\big(t^1, u^1\big), \dots, \pi_{\mathcal{A}}\big(t^N, u^N\big)\big)$$

where, as noted, $\mathbf{u} := (u^1, \dots, u^N) \in (\mathbb{R}^{\mathcal{A}})^N$ and $\mathbf{t} := (t^1, \dots, t^N) \in (\mathbb{R}^n)^N$. Functions which are not meant to be applied component-wise (i.e., functions which have nontrivial behavior on product spaces) and sets in product spaces will also be bold.

Proposition 4. *The operator $T_{\mathcal{A}}$ maps $L^p(\mathbb{R}^n \times \mathbb{R}^{n'})$ to $L^q(\mathbb{R}^{\mathcal{A}})$ $(1 < p, q < \infty)$ if (and only if) there exists a constant C such that, for all positive integers N,*

$$\int_{(\mathbb{R}^{\mathcal{A}})^N} \mathbf{T}_{\mathcal{A}}(\chi_{\mathbf{F}})(\mathbf{u})\chi_{\mathbf{G}}(\mathbf{u})\,d\mathbf{u} \leqslant C^N |\mathbf{F}|^{\frac{1}{p}} |\mathbf{G}|^{1-\frac{1}{q}} \tag{17}$$

for all measurable $\mathbf{F} \subset (\mathbb{R}^n \times \mathbb{R}^{n'})^N$ and $\mathbf{G} \subset (\mathbb{R}^{\mathcal{A}})^N$, where $\mathbf{T}_{\mathcal{A}}$ is the N-fold tensor product of $T_{\mathcal{A}}$, i.e.,

$$\mathbf{T}_{\mathcal{A}}\mathbf{f}(\mathbf{u}) = \int_{(\mathbb{R}^n)^N} \mathbf{f}\big(\pi_{\mathcal{A}}(\mathbf{t}, \mathbf{u})\big)\,d\mathbf{t}.$$

Proof. Let $f \in L^p(\mathbb{R}^n \times \mathbb{R}^{n'})$ and $g \in L^{q'}(\mathbb{R}^{\mathcal{A}})$; for each integer j, let $F_j := \{x \in \mathbb{R}^n \times \mathbb{R}^{n'} \mid 2^{j-1} \leqslant |f(x)| < 2^j\}$ and likewise for G_j. Now, for each positive integer M, let $f_M(x) := \sum_{|j| \leqslant M} 2^j \chi_{F_j}(x)$ and so on for g_M. The functions f_M and g_M converge monotonically as $M \to \infty$ to functions majorizing $|f(x)|$ and $|g(x)|$, respectively. Therefore (by the monotone convergence theorem)

$$\left|\int T_{\mathcal{A}} f(u) g(u)\,du\right| \leqslant \sup_M \int T_{\mathcal{A}} f_M(u) g_M(u)\,du.$$

For each fixed M and every positive integer N,

$$\left(\int T_{\mathcal{A}} f_M(u) g_M(u)\,du\right)^N = \sum_{|j|\leqslant NM} \sum_{|j'|\leqslant NM} 2^{j+j'} \int \mathbf{T}_{\mathcal{A}} \chi_{\mathbf{F}_j}(\mathbf{u}) \chi_{\mathbf{G}_{j'}}(\mathbf{u})\,d\mathbf{u}$$

where $\mathbf{F}_j := \bigcup \{F_{j_1} \times \cdots \times F_{j_N} \mid j_1 + \cdots + j_n = j,\ |j_1|, \ldots, |j_N| \leqslant M\}$ (and likewise for $\mathbf{G}_{j'}$). To see that this is true, simply write the left-hand side as an N-fold product of integrals and group the terms accordingly. By the hypothesis of this proposition, then,

$$\left(\int T_{\mathcal{A}} f_M(u) g_M(u)\,du\right)^N \leqslant C^N \sum_{|j|\leqslant NM} \sum_{|j'|\leqslant NM} 2^{j+j'} |\mathbf{F}_j|^{\frac{1}{p}} |\mathbf{G}_{j'}|^{\frac{1}{q'}}.$$

Applying Jensen's inequality to each sum on the right-hand side gives that the right-hand side is itself dominated by

$$C^N (2NM+1)^{\frac{1}{p'}+\frac{1}{q}} \left(\sum_{|j|\leqslant NM} 2^{jp} |\mathbf{F}_j|\right)^{\frac{1}{p}} \left(\sum_{|j'|\leqslant NM} 2^{j'q'} |\mathbf{G}_{j'}|\right)^{\frac{1}{q'}}.$$

Next observe that the sums over j and j' (*inside* the parentheses) are nothing other than $\|f_M\|_p^{Np}$ and $\|g_M\|_{q'}^{Nq'}$. Taking Nth roots and letting $N \to \infty$, it must be the case that

$$\left|\int T_{\mathcal{A}} f(u) g(u)\,du\right| \leqslant \sup_M C \|f_M\|_p \|g_M\|_{q'}.$$

But $f_M(x) \leqslant 2|f(x)|$ and $g_M(x) \leqslant 2|g(x)|$, so the right-hand side is dominated by $\|f\|_p \|g\|_{q'}$. Therefore the operator $T_{\mathcal{A}}$ must be bounded. $\square$

Proposition 5. *Let V be any C^∞ function on $(\mathbb{R}^k)^m$, and let $0 < p_j < 1$ for $j = 1, \ldots, m$. Then there exists a constant $c > 0$ such that*

$$\int \left|\mathbf{V}(\mathbf{x}_1, \ldots, \mathbf{x}_m)\right| \prod_{j=1}^m \chi_{\mathbf{E}_j}(\mathbf{x}_j)\,d\mathbf{x}_1 \cdots d\mathbf{x}_m \geqslant c^N \prod_{j=1}^m |\mathbf{E}_j|^{\frac{1}{p_j}} \tag{18}$$

(where $|\mathbf{V}(\mathbf{x}_1, \ldots, \mathbf{x}_m)|$ is the component-wise product $\prod_{j=1}^N |V(x_1^j, \ldots, x_m^j)|$) for all measurable sets $\mathbf{E}_j \subset (\mathbb{R}^k)^N$ if (and only if) there exists a constant $c > 0$ such that

$$\int \left|\mathbf{V}(x_1, \ldots, x_m)\right| \prod_{j=1}^m \left|f_j(x_j)\right| dx_1 \cdots dx_m \geqslant c \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbb{R}^k)} \tag{19}$$

for all functions f_j on $\mathbb{R}^k$.

Proof. On the left-hand side of (18), consider first the integral $dx_1^1 \cdots dx_m^1$. By (19), there exists a constant c such that

$$\begin{aligned}&\int \left|\mathbf{V}(\mathbf{x}_1,\ldots,\mathbf{x}_m)\right| \prod_{j=1}^{m} \chi_{\mathbf{E}_j}(\mathbf{x}_j)\, dx_1^1 \cdots dx_m^1 \\ &\quad \geqslant c \prod_{i=2}^{N} \left|V\left(x_1^i,\ldots,x_m^i\right)\right| \prod_{j=1}^{m} \left(\int \left(\chi_{\mathbf{E}_j}(\mathbf{x}_j)\right)^{p_j} dx_j^1 \right)^{\frac{1}{p_j}}.\end{aligned}$$

This inequality may be integrated with respect to $x_1^2,\ldots,x_m^2$; again (19) may be applied. Using the identity

$$\left\| \left(\int \left(\chi_{\mathbf{E}_j}(\mathbf{x}_j)\right)^{p_j} dx_j^1 \right)^{\frac{1}{p_j}} \right\|_{L^{p_j}(x_j^2)} = \left(\int \left(\chi_{\mathbf{E}_j}(\mathbf{x}_j)\right)^{p_j} dx_j^1\, dx_j^2 \right)^{\frac{1}{p_j}},$$

one proceeds by induction on N, arriving at (18). □

4. Proof of Theorem 2

With the machinery of the previous sections in hand, the proof of Theorem 2 may now be undertaken. By Proposition 4, it suffices to show that the uniform estimate (17) holds. This section is devoted to the proof of (17).

Throughout this section, the variable t will represent a point in $\mathbb{R}^n$, and u will represent a point in $\mathbb{R}^{\mathcal{A}}$. For any $s \in \mathbb{R}^n$, let $\varphi_l^s : \mathbb{R}^n \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^n \times \mathbb{R}^{\mathcal{A}}$ be given by

$$\varphi_l^s(t,u) := (t+s,u). \tag{20}$$

Let $\mathcal{A}^\circ := \mathcal{A} \setminus \{(\mathbf{0},j)\}_{j=1,\ldots,n'}$. For $x \in \mathbb{R}^{\mathcal{A}^\circ}$, let $\hat{\varphi}_r^x$ be the function from $\mathbb{R}^n \times \mathbb{R}^{\mathcal{A}}$ to $\mathbb{R}^{\mathcal{A}}$ with components

$$\hat{\varphi}_r^x(t,u)_{(\alpha,j)} := \begin{cases} u_{(\mathbf{0},j)} - \sum_{\beta \in \mathcal{A}_j^\circ} x_\beta t^\beta, & \alpha = \mathbf{0}, \\ u_{(\alpha,j)} + x_{(\alpha,j)}, & \alpha \neq \mathbf{0}, \end{cases}$$

for any $(\alpha,j) \in \mathcal{A}$. Similarly, let $\varphi_r^x : \mathbb{R}^n \times \mathbb{R}^{\mathcal{A}} \to \mathbb{R}^n \times \mathbb{R}^{\mathcal{A}}$ be given by

$$\varphi_r^x(t,u) := \left(t, \hat{\varphi}_r^x(t,u)\right). \tag{21}$$

These maps φ_l^s and φ_r^x are nothing more than the kernel flow maps which appear in the work of Tao and Wright [13], for example. An important feature which distinguishes this proof from Tao and Wright's earlier work is that the flows φ_l^s and φ_l^x may be multidimensional flows.

Just as in the work of Christ [5], one of the major components of this proof is a change-of-variables argument involving a Jacobian determinant of a repeated composition of flow maps. In that case, the structure of the composition was fairly straightforward (arising from the repeated composition $(T^*T)^{n/2}$). Here, in contrast, the flows are more complicated, and the change-of-variables argument takes place in a "lifted" space (as also occurs for X-ray like transforms [9]). For any measurable set $F \subset \mathbb{R}^n \times \mathbb{R}^{n'}$, let

$$I_{\mathcal{A}}[F](t,u) := \int \left|V_{\mathcal{A}}(t,t+s)\right| \prod_{j=1}^{\#\mathcal{A}-1} \chi_F\left(\pi_{\mathcal{A}} \varphi_l^{s_j} \varphi_r^x(t,u)\right) ds\, dx \tag{22}$$

where $|V_{\mathcal{A}}(t,t+s)| := |V_{\mathcal{A}}(t,t+s_1,\dots,t+s_{\#A-1})|$. This functional is the centerpiece of the change-of-variables argument, as demonstrated by the following proposition:

Proposition 6. *For any $(t,u) \in \mathbb{R}^n \times \mathbb{R}^{\mathcal{A}}$ and any measurable $F \subset \mathbb{R}^n \times \mathbb{R}^{n'}$,*

$$I_{\mathcal{A}}[F](t,u) = |F|^{\#\mathcal{A}-1}.$$

Furthermore, for any $(\mathbf{t},\mathbf{u}) \in (\mathbb{R}^n)^N \times (\mathbb{R}^{\mathcal{A}})^N$ and any measurable $\mathbf{F} \subset (\mathbb{R}^n \times \mathbb{R}^{n'})^N$,

$$\mathbf{I}_{\mathcal{A}}[\mathbf{F}](\mathbf{t},\mathbf{u}) = |\mathbf{F}|^{\#\mathcal{A}-1},$$

where $\mathbf{I}_{\mathcal{A}}$ is the N-fold tensor product of $I_{\mathcal{A}}$.

Proof. For each $k = 1,\dots,n'$, let $\phi^{k,s,t,u} : \mathbb{R}^{\mathcal{A}_k^\circ} \to \mathbb{R}^{\#A-1}$ be the map with components

$$\phi_j^{k,s,t,u}(x) := u_{(\mathbf{0},k)} + \sum_{\alpha \in \mathcal{A}_k^\circ} x_{(\alpha,k)}\big[(t+s_j)^\alpha - t^\alpha\big] + u_{(\alpha,k)}(t+s_j)^\alpha$$

for $j = 1,\dots,\#\mathcal{A}-1$. The dimensionality constraint on $\mathcal{A}$ guarantees that $\phi^{k,s,t,u}$ is a map between spaces of the same dimension when s, t, and u are fixed. The Jacobian matrix of $\phi^{k,s,t,u}$ has as its $(j,(\alpha,k))$-entry $(t+s_j)^\alpha - t^\alpha$ (that is, in the jth row and the column corresponding to (α,k)). The absolute value of the determinant is precisely $|V_k(t,t+s)|$, defined at the beginning of Section 2.2; to see this, simply note that the former determinant can be obtained from the latter by subtracting the t-row from all remaining rows. Now, in the x-integral appearing in (22), make the changes of variables $y_k := \phi^{k,s,t,u}(x)$. This is permitted for almost every s since $V_{\mathcal{A}}$ vanishes on a closed set of measure zero (and each $\phi^{k,s,t,u}$ depends on different x-variables). Direct computation shows that

$$\pi_A \varphi_l^{s_j} \varphi_r^x(t,u) = \big(t+s_j, \big(\phi_j^{k,s,t,u}(x)\big)_{k=1,\dots,n'}\big).$$

It follows that

$$I_{\mathcal{A}}[F](t,u) = \int \prod_{j=1}^{\#\mathcal{A}-1} \chi_F\big(t+s_j,(y_{1j},\dots,y_{n'j})\big)\,ds\,dy.$$

After a trivial change of variables, the right-hand side is easily seen to equal $|F|^{\#\mathcal{A}-1}$.

As for the functional $\mathbf{I}_{\mathcal{A}}$, the exact same changes of variables can be performed on each of the factors, giving the same conclusion as in the single factor case. □

Before proceeding, it is useful to recall the isoperimetric formulation of restricted weak-type estimates as introduced by Tao and Wright [13]. Given measurable sets $\mathbf{F} \subset (\mathbb{R}^n \times \mathbb{R}^{n'})^N$ and $\mathbf{G} \subset (\mathbb{R}^{\mathcal{A}})^N$, let $\mathbf{\Omega}$ be the subset of $(\mathbb{R}^n \times \mathbb{R}^{n'} \times \mathbb{R}^{\mathcal{A}})^N$ given by

$$\chi_{\mathbf{\Omega}}(\mathbf{t},\mathbf{u}) := \chi_{\mathbf{F}}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\chi_{\mathbf{G}}(\mathbf{u}).$$

By Proposition 4, to prove Theorem 2, it suffices to show that there is a constant $C < \infty$ independent of N, $\mathbf{F}$, $\mathbf{G}$, and $\boldsymbol{\Omega}$ such that

$$|\boldsymbol{\Omega}| \leqslant C^N |\mathbf{F}|^{\frac{\#\mathcal{A}}{\#\mathcal{A}+|\mathcal{A}|}} |\mathbf{G}|^{1-\frac{1}{\#\mathcal{A}+|\mathcal{A}|}}. \tag{23}$$

This inequality will be established via a careful analysis of the functional $\mathbf{I}_{\mathcal{A}}$. By Proposition 6, one has the identity

$$|\mathbf{F}|^{\#\mathcal{A}-1}|\boldsymbol{\Omega}| = \int \mathbf{I}_{\mathcal{A}}[\mathbf{F}](\mathbf{t},\mathbf{u})\chi_{\boldsymbol{\Omega}}(\mathbf{t},\mathbf{u})\, d\mathbf{t}\, d\mathbf{u}.$$

Using the definition of $\mathbf{I}_{\mathcal{A}}$ and Fubini's theorem, the order of integration can be changed so that the integration in $\mathbf{t}$ and $\mathbf{u}$ comes before the integrals in the $\mathbf{s}_j$'s and $\mathbf{x}$. Next make the change of variables $(\mathbf{t},\mathbf{u}) \to \varphi_r^{-\mathbf{x}}(\mathbf{t},\mathbf{u})$. This is simply a translation of the u's, so the Jacobian determinant must equal exactly 1.Thus $|\mathbf{F}|^{\#\mathcal{A}-1}|\boldsymbol{\Omega}|$ is exactly equal to

$$\int \left[\int |\mathbf{V}_{\mathcal{A}}(\mathbf{t},\mathbf{t}+\mathbf{s})| \prod_{j=1}^{\#\mathcal{A}-1} \chi_{\mathbf{F}}\big(\pi_{\mathcal{A}}\varphi_l^{\mathbf{s}_j}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{s}\right]\left[\int \chi_{\boldsymbol{\Omega}}\big(\varphi_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{x}\right] d\mathbf{t}\, d\mathbf{u} \tag{24}$$

(where the integrals have again been reordered and the change $\mathbf{x} \to -\mathbf{x}$ has been made). The following proposition will be used to estimate the second term in brackets in (24) so that Theorem 3 can be applied via Proposition 5:

Proposition 7. *There exists a nonempty subset* $\mathbf{F}' \subset \mathbf{F}$ *such that*

$$\left[\int \chi_{\boldsymbol{\Omega}}\big(\varphi_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{x}\right] \geqslant \frac{1}{2}\frac{|\boldsymbol{\Omega}|}{|\mathbf{F}|}\chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big) \tag{25}$$

and, for any $p \geqslant 1$,

$$\int \left(\int \chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{t}\right)^p \chi_{\mathbf{G}}(\mathbf{u})\, d\mathbf{u} \geqslant \left(\frac{1}{2}\frac{|\boldsymbol{\Omega}|}{|\mathbf{G}|}\right)^p |\mathbf{G}|. \tag{26}$$

Proof. By definition of $\boldsymbol{\Omega}$ and the fact that $\pi_{\mathcal{A}}\varphi_r^x(t,u) = \pi_{\mathcal{A}}(t,u)$, it must be the case that, for each pair $(\mathbf{t},\mathbf{u})$,

$$\int \chi_{\boldsymbol{\Omega}}\big(\varphi_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{x} = \chi_{\mathbf{F}}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\int \chi_{\mathbf{G}}\big(\hat{\varphi}_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{x}.$$

Let $\mathbf{F}'$ be the subset of $\mathbf{F}$ such that

$$\chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\int \chi_{\mathbf{G}}\big(\hat{\varphi}_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\, d\mathbf{x} \geqslant \frac{1}{2}\frac{|\boldsymbol{\Omega}|}{|\mathbf{F}|}\chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big); \tag{27}$$

$\mathbf{F}'$ is well defined because the integral over $\chi_{\mathbf{G}}$ depends only on $\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})$, not on $(\mathbf{t},\mathbf{u})$ itself. Because $\mathbf{F}' \subset \mathbf{F}$, the left-hand side of (25) is greater than the left-hand side of (27); thus (25) is vacuously true in this case. On the other hand, it must also be the case that

$$\chi_{\mathbf{F}\setminus\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\int \chi_{\mathbf{G}}\big(\hat{\varphi}_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})\big)\,d\mathbf{x} \leqslant \frac{1}{2}\frac{|\mathbf{\Omega}|}{|\mathbf{F}|}\chi_{\mathbf{F}\setminus\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big).$$

Now integrate both sides with respect to $\mathbf{t}$ and $\mathbf{u}_{(\mathbf{0},1)},\ldots,\mathbf{u}_{(\mathbf{0},n')}$ (and *only* these particular $\mathbf{u}$'s; the rest are left fixed). A change of variables can now be made on both sides. On the left, the change to be made is $(\mathbf{t}',\mathbf{u}') := \varphi_r^{\mathbf{x}}(\mathbf{t},\mathbf{u})$ (that is, $\mathbf{t}',\mathbf{u}'$ depend on $\mathbf{x}$, $\mathbf{t}$ and the $\mathbf{u}_{(0,j)}$'s); on the right, the change is $\mathbf{y} := \pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})$. Both changes are volume preserving, giving

$$\int \chi_{\mathbf{F}\setminus\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t}',\mathbf{u}')\big)\chi_{\mathbf{G}}(\mathbf{u}')\,d\mathbf{t}'\,d\mathbf{u}' \leqslant \frac{1}{2}\frac{|\mathbf{\Omega}|}{|\mathbf{F}|}|\mathbf{F}\setminus\mathbf{F}'|.$$

Subtracting both sides from $|\mathbf{\Omega}|$ gives (26) for $p=1$. Applying Jensen's inequality gives all remaining p. $\square$

The proof of Theorem 2 concludes as follows: combining the previous proposition with the identity (24), it follows that $|\mathbf{F}|^{\#\mathcal{A}-1}|\mathbf{\Omega}|$ is greater than or equal to

$$\frac{1}{2}\frac{|\mathbf{\Omega}|}{|\mathbf{F}|}\int\left[\int\big|\mathbf{V}_{\mathcal{A}}(\mathbf{t},\mathbf{t}+\mathbf{s})\big|\chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\prod_{j=1}^{\#\mathcal{A}-1}\chi_{\mathbf{F}}\big(\pi_{\mathcal{A}}\varphi_l^{\mathbf{s}_j}(\mathbf{t},\mathbf{u})\big)\,d\mathbf{s}\,d\mathbf{t}\right]\chi_{\mathbf{G}}(\mathbf{u})\,d\mathbf{u}.$$

Observe that $\pi_{\mathcal{A}}\varphi_l^{\mathbf{s}_j}(\mathbf{t},\mathbf{u}) = \pi_{\mathcal{A}}(\mathbf{t}+\mathbf{s}_j,\mathbf{u})$; thus by Theorem 3 and Proposition 5, there is a constant $c>0$ which is independent of $\mathbf{F}$, $\mathbf{G}$, $\mathbf{\Omega}$, and N such that the quantity in brackets is at least

$$c^N\left(\int \chi_{\mathbf{F}'}\big(\pi_{\mathcal{A}}(\mathbf{t},\mathbf{u})\big)\,d\mathbf{t}\right)^{\#\mathcal{A}+|\mathcal{A}|}$$

for each value of $\mathbf{u}$ (since $\mathbf{F}$ may be everywhere replaced by $\mathbf{F}'$ with impunity). Now by (26), it follows that

$$|\mathbf{F}|^{\#\mathcal{A}-1}|\mathbf{\Omega}| \geqslant \frac{c^N}{2}\frac{|\mathbf{\Omega}|}{|\mathbf{F}|}\left(\frac{1}{2}\frac{|\mathbf{\Omega}|}{|\mathbf{G}|}\right)^{\#\mathcal{A}+|\mathcal{A}|}|\mathbf{G}|,$$

which is precisely of the desired form (23) after elementary manipulations.

5. Necessity

5.1. Admissibility criteria

An important condition of Theorem 2 is that the set $\mathcal{A}$ be admissible. As mentioned in the introduction, many of the admissibility criteria are, in fact, necessary for L^p boundedness to hold in any form at all. By scaling, it is fairly straightforward to see that the dimensionality and scaling assumptions are necessary, and that only one global $L^p \to L^q$ estimate can hold. Let $\delta \in \mathbb{R}^n_+$ and $\delta' \in \mathbb{R}^{n'}_+$. For any function f on $\mathbb{R}^n \times \mathbb{R}^{n'}$, the (δ, δ') dilation of f is defined to be

$$f_{\delta,\delta'}(s,w) := f\big(t_1\delta_1, \dots, t_n\delta_n, w_1\delta'_1, \dots, w_{n'}\delta'_{n'}\big).$$

Likewise, for any function g on the parameter space, let

$$g^{\delta,\delta'}(u) := \delta^{-\mathbf{1}} g\big(\big\{\delta'_j \delta^{-\alpha} u_{(\alpha,j)}\big\}_{(\alpha,j)\in\mathcal{A}}\big),$$

where $\mathbf{1}$ is the multiindex $(1,\dots,1)$. The standard change-of-variables argument shows that $T_A(f_{\delta,\delta'}) = (T_{\mathcal{A}} f)^{\delta,\delta'}$. Furthermore,

$$\|f_{\delta,\delta'}\|_p = \delta^{-\frac{1}{p}\mathbf{1}} \delta'^{-\frac{1}{p}\mathbf{1}} \|f\|_p \quad \text{and} \quad \big\|g^{\delta,\delta'}\big\|_q = \delta^{-\mathbf{1}+\frac{1}{q}v} \delta'^{-\frac{1}{q}v'} \|g\|_q,$$

where $v := \sum_{(\alpha,j)\in\mathcal{A}} \alpha$ and $v' := (v'_1, \dots, v'_{n'})$ with v_j equal to the cardinality of $\mathcal{A}_j$. By the usual arguments, for any $L^p \to L^q$ estimate to hold, it must be the case that

$$\frac{v}{q} = \left(1 - \frac{1}{p}\right)\mathbf{1} \quad \text{and} \quad \frac{v'}{q} = \frac{\mathbf{1}}{p}.$$

Thus the scaling and dimensionality conditions on $\mathcal{A}$ are necessary for $T_{\mathcal{A}}$ to be bounded at all, and when satisfied, $T_{\mathcal{A}}$ can map $L^p \to L^q$ only when $p = \frac{|\mathcal{A}| + \#\mathcal{A}}{\#\mathcal{A}}$ and $q = |\mathcal{A}| + \#\mathcal{A}$.

Next, consider what happens when $\mathcal{A}$ fails to satisfy the (weak) spanning condition; that is, suppose that the monomials in $A := \bigcup_j \mathcal{A}_j$ span only some subspace of $\mathbb{R}^n$ (when interpreted as vectors). Let $\beta_1, \dots, \beta_m$ be linearly independent monomials which span the same subspace as $\bigcup_j \mathcal{A}_j$, and let $\beta_{m+1}, \dots, \beta_n$ be linearly independent vectors such that $\beta_1, \dots, \beta_n$ span $\mathbb{R}^n$. Now let $E_R \subset \mathbb{R}^n$ be the set on which $1 \leqslant |x^{\beta_j}| \leqslant 2$ for $j = 1, \dots, m$ and $1 \leqslant |x^{\beta_j}| \leqslant R$ for $j = m+1, \dots, n$. To compute the measure of this set use the change-of-variables $y_j := x^{\beta_j}$, as in Proposition 1. One obtains $|E_R| = C|\ln R|^{n-m}$. Now let f_R be the characteristic function on $\mathbb{R}^n \times \mathbb{R}^{n'}$ of the set E_R times the ball of radius 1 centered at the origin in $\mathbb{R}^{n'}$. For all u sufficiently near zero, $T_{\mathcal{A}} f_R(u)$ also grows like $|\ln R|^{n-m}$ (since $\sum u_{(\alpha,j)} t^\alpha$ will be bounded for all $t \in E_R$). If it is to be the case that $\|T_{\mathcal{A}} f_R\|_q \leqslant C\|f_R\|_p$, taking $R \to \infty$ shows that p must be less than q. This, however, cannot happen because of the dimensionality and scaling conditions.

5.2. Local L^p estimates of Theorem 1

To conclude, consider the necessity claims of Theorem 1. The necessary constraints on global boundedness of $T_{n,n',d}$ are easily established by the same scaling argument used to demonstrate the necessity of the dimensionality and scaling conditions. The only new feature of Theorem 1 not present in Theorem 2 is the claim concerning local L^p estimates.

For each $\delta > 0$, let $F_\delta \subset \mathbb{R}^n \times \mathbb{R}^{n'}$ and $G_\delta \subset (\mathbb{R}^{n'})^{M_{n,d}}$ be given by $F_\delta := \{(t,s) \mid |t_j| \leqslant \delta, |s_k| \leqslant C\delta^l\}$ and $G_\delta := \{u \mid |u_{(\alpha,j)}| \leqslant \delta^{\max\{l-|\alpha|,0\}}\}$ (for some fixed constant C). Elementary counting shows that $|F_\delta| = C^{n'}\delta^{n+n'l}$ and $|G_\delta| = \delta^K$, where $K := n'\binom{n+l}{n+1}$. If C is fixed suitably large, it will be the case that $\int_{G_\delta} T_{n,n',d}\chi_{F_\delta} = \delta^n |G_\delta|$ since $|\sum_{|\alpha|\leqslant d} u_{(\alpha,j)} t^\alpha| \leqslant C\delta^l$ for all $u \in G_\delta$. If T is to be bounded from L^p to L^q, it must therefore be the case that $\delta^n|G_\delta| \leqslant C'|F_\delta|^{1/p}|G_\delta|^{1/q'}$. Letting $\delta \to 0$, the inequality can hold for some C' only when the exponent of δ on the left-hand side is greater than the exponent on the right. This gives the necessary inequalities

$$n + \frac{n'}{q}\binom{n+l}{n+1} \geqslant \frac{n + ln'}{p} \tag{28}$$

for $l = 1, \ldots, d$.

Finally, let $F'_\delta := \{(t, s) \mid |t_j| \leqslant 1, |s_k| \leqslant C\delta\}$ and $G'_\delta := \{u \mid |u_{(\alpha,j)}| \leqslant \delta\}$. In this case, $|F'_\delta| = C^{n'}\delta^{n'}$ and $|G'_\delta| = \delta^{K'}$ where $K' := n'\binom{n+d}{d}$. Proceeding as before, it follows that $\int_{G'_\delta} T_{n,n',d}\chi_{F'_\delta} = |G'_\delta|$, and after computing exponents, that

$$\frac{n'}{q}\binom{n+d}{d} \geqslant \frac{n'}{p}. \tag{29}$$

The constraints (28) combined with (29) give precisely the necessity conditions of Theorem 1, illustrated in Fig. 1.

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References

[1] J. Bennett, A. Carbery, J. Wright, A non-linear generalization of the Loomis–Whitney inequality and applications, Math. Res. Lett. 12 (2005) 443–457.
[2] L. Brandolini, A. Greenleaf, G. Travaglini, L^p–$L^{p'}$ estimates for overdetermined Radon transforms, Trans. Amer. Math. Soc. 359 (2007) 2559–2575.
[3] A. Carbery, A multilinear generalisation of the Cauchy–Schwarz inequality, Proc. Amer. Math. Soc. 132 (11) (2004) 3141–3152 (electronic).
[4] M. Christ, Estimates for the k-plane transform, Indiana Univ. Math. J. 33 (6) (1984) 891–910.
[5] M. Christ, Convolution, curvature, and combinatorics: A case study, Int. Math. Res. Not. 1998 (19) (1998) 1033–1048.
[6] S.W. Drury, L^p estimates for the X-ray transform, Illinois J. Math. 27 (1) (1983) 125–129.
[7] S.W. Drury, Generalizations of Riesz potentials and L^p estimates for certain k-plane transforms, Illinois J. Math. 28 (3) (1984) 495–512.
[8] S.W. Drury, An endpoint estimate for certain k-plane transforms, Canad. Math. Bull. 29 (1) (1986) 96–101.
[9] P. Gressman, L^p-improving properties of X-ray like transforms, Math. Res. Lett. 13 (5) (2006) 787–803.
[10] F. John, The ultrahyperbolic differential equation with four independent variables, Duke Math. J. 4 (2) (1938) 300–322.
[11] D.M. Oberlin, E.M. Stein, Mapping properties of the Radon transform, Indiana Univ. Math. J. 31 (5) (1982) 641–650.
[12] F. Ricci, G. Travaglini, Convex curves, Radon transforms and convolution operators defined by singular measures, Proc. Amer. Math. Soc. 129 (6) (2001) 1739–1744.
[13] T. Tao, J. Wright, L^p improving bounds for averages along curves, J. Amer. Math. Soc. 16 (3) (2003) 605–638.