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Frobenius manifolds from regular classical W-algebras

Yassir Ibrahim Dinar [a,b,*]

[a] *Faculty of Mathematical Sciences, University of Khartoum, Sudan*
[b] *The Abdus Salam International Centre for Theoretical Physics (ICTP), Italy*

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Abstract

We obtain polynomial Frobenius manifolds from classical W-algebras associated to regular nilpotent elements in simple Lie algebras using the related opposite Cartan subalgebras.

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* Address for correspondence: Faculty of Mathematical Sciences, University of Khartoum, Sudan.
E-mail addresses: dinar@ictp.it, yassird@gmail.com.

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1. Introduction

This work is a continuation of [4] where we began to develop a construction of algebraic Frobenius manifolds from Drinfeld–Sokolov reduction to support a Dubrovin conjecture.

A Frobenius manifold is a manifold M with the structure of Frobenius algebra on the tangent space T_t at any point $t \in M$ with certain compatibility conditions [7]. We say M is semisimple or massive if T_t is semisimple for generic t. This structure locally corresponds to a potential satisfying a system of partial differential equations known in topological field theory as the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. We say M is algebraic if, in the flat coordinates, the potential is an algebraic function. Dubrovin conjecture is stated as follows: Semisimple irreducible algebraic Frobenius manifolds with positive degrees correspond to quasi-Coxeter (primitive) conjugacy classes in finite Coxeter groups. We discussed in [4] how the examples of algebraic Frobenius manifolds constructed from Drinfeld–Sokolov reduction support this conjecture.

Let e be a **regular nilpotent element** in a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ of rank r. By definition a nilpotent element is called regular if its centralizer in $\mathfrak{g}$ is of dimension r. We fix, by using the Jacobson–Morozov theorem, a semisimple element h and a nilpotent element f such that $\mathcal{A} = \{e, h, f\}$ is an sl_2-triple. Then $\mathcal{A}$ is called **regular sl_2-triple**. Let $\kappa + 1$ be the Coxeter number of $\mathfrak{g}$. We prove the following

Theorem 1.1. *The **Slodowy slice***

$$Q' := e + \ker \operatorname{ad} f \tag{1.1}$$

has a natural structure of polynomial Frobenius manifold of degree $\frac{\kappa-1}{\kappa+1}$.

By natural structure we mean that it can be formulated entirely in terms of the representation theory of the regular sl_2-triple $\mathcal{A}$ along with the closely related opposite Cartan subalgebra. Let us recall some structures related to $\mathcal{A}$. The element $h \in \mathcal{A}$ defines a $\mathbb{Z}$-grading on $\mathfrak{g}$ called the Dynkin grading given as follows

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_i = \{q \in \mathfrak{g}\colon \operatorname{ad} h(q) = iq\}. \tag{1.2}$$

Fix nonzero element $a \in \mathfrak{g}_{-2\kappa}$. Then it follows from the work of Kostant [17] that $y_1 = e + a$ is regular semisimple. The Cartan subalgebra $\mathfrak{h}' = \ker \operatorname{ad} y_1$ is called **the opposite Cartan subalgebra** and it is one of the main ingredients in our work. Let

$$1 = \eta_1 \leqslant \eta_2 \leqslant \cdots \leqslant \eta_{r-1} < \eta_r = \kappa \tag{1.3}$$

be the exponents of $\mathfrak{g}$. Then the element y_1 can be completed to a basis $y_1, \ldots, y_r$ for $\mathfrak{h}'$ having the form

$$y_i = v_i + u_i, \quad u_i \in \mathfrak{g}_{2\eta_i}, \; v_i \in \mathfrak{g}_{2\eta_i - 2(\kappa+1)}. \tag{1.4}$$

We define the coordinates $(z^1, \ldots, z^r)$ on Q' by setting for $q \in Q'$

$$z^i(q) := \langle q | u_i \rangle. \tag{1.5}$$

Here $\langle .|. \rangle$ is the invariant bilinear form on $\mathfrak{g}$ normalized such that $\langle e | f \rangle = 1$.

Our main idea is to use the theory of local bihamiltonian structures on a loop space to construct the polynomial Frobenius manifold on Q'. Recall that a bihamiltonian structure on a manifold M is two compatible Poisson brackets on M. It is well known that the dispersionless limit of a local bihamiltonian structure on the loop space $\mathfrak{L}(M)$ of a finite dimensional manifold M (if it exists) always gives a bihamiltonian structure of hydrodynamic type on $\mathfrak{L}(M)$:

$$\begin{aligned} \{t^i(x), t^j(y)\}_1 &= g_1^{ij}\big(t(x)\big)\delta'(x-y) + \Gamma_{1;k}^{ij}\big(t(x)\big)t_x^k\delta(x-y), \\ \{t^i(x), t^j(y)\}_2 &= g_2^{ij}\big(t(x)\big)\delta'(x-y) + \Gamma_{2;k}^{ij}\big(t(x)\big)t_x^k\delta(x-y). \end{aligned} \tag{1.6}$$

This gives a flat pencil of metrics $g_{1,2}^{ij}$ on M provided that the two matrices g_1^{ij} and g_2^{ij} are non-degenerate. A flat pencil of metrics, under the quasihomogeneity and the regularity conditions, corresponds to a Frobenius structure on M [9] (see Section 2.1 for details).

We obtain a bihamiltonian structure on the affine loop space

$$Q = e + \mathfrak{L}(\ker \operatorname{ad} f) \tag{1.7}$$

by using the Drinfeld–Sokolov reduction [6] (see also [4] or [15]). This reduction depends only on the representation theory of $\mathcal{A}$. It begins by defining a bihamiltonian structure P_1 and P_2 in $\mathfrak{L}(\mathfrak{g})$. The Poisson structure P_2 is the standard Lie–Poisson structure and P_1 depends on the adjoint action of a. Main while, the space Q will be transversal to an action of the adjoint group of $\mathfrak{L}(\mathfrak{n})$ on a suitable affine subspace

$$S := e + \mathfrak{L}(\mathfrak{b}) \tag{1.8}$$

of $\mathfrak{L}(\mathfrak{g})$. Here

$$\mathfrak{n} := \bigoplus_{i \leqslant -2} \mathfrak{g}_i, \qquad \mathfrak{b} := \bigoplus_{i \leqslant 0} \mathfrak{g}_i. \tag{1.9}$$

It turns out that the space of local functionals with densities in the ring R of invariant differential polynomials of this action is closed under P_1 and P_2. This defines the Drinfeld–Sokolov bihamiltonian structure P_1^Q and P_2^Q on Q since the coordinates $z^i(x)$ of Q can be interpreted as generators of the ring R. The second reduced Poisson structure on Q is known in the literature as **classical W-algebras** associated to principal nilpotent elements in $\mathfrak{g}$. Therefore, we call it **regular classical W-algebras**. For a general definition for classical W-algebras see [16]. In [1] they proved the Drinfeld–Sokolov reduction of P_2 on Q is the same as Dirac reduction of P_2 to Q. In particular, they obtained the following

Proposition 1.2. *(See [1].) The second Poisson bracket on Q take the form*

$$\begin{aligned} \{z^1(x), z^1(y)\}_2^Q &= \epsilon\delta'''(x-y) + 2z^1(x)\delta'(x-y) + z_x^1\delta(x-y), \\ \{z^1(x), z^i(y)\}_2^Q &= (\eta_i+1)z^i(x)\delta'(x-y) + \eta_i z_x^i\delta(x-y). \end{aligned} \tag{1.10}$$

We use this result and some facts about the structure of Lie–Poisson brackets on $\mathfrak{g}$ to prove the following

Theorem 1.3. *The Drinfeld–Sokolov bihamiltonian structure on Q admits a dispersionless limit. The corresponding bihamiltonian structure of hydrodynamic type gives a flat pencil of metrics on the Slodowy slice Q'.*

A large portion of this work is devoted to prove the nondegeneracy condition for the matrices g_1^{ij} and g_2^{ij} obtained from the dispersionless limit of P_1^Q and P_2^Q, respectively. For this end we use mainly two facts. First, the basis y_i for $\mathfrak{h}'$ can be normalized in such away that the elements u_i in (1.4) are the highest weight vectors for irreducible $\mathcal{A}$-submodules V^i satisfying

$$\mathfrak{g} = \bigoplus_{i=1}^{r} V^i, \qquad \langle V^i | V^j \rangle = 0 \text{ if } i \neq j. \tag{1.11}$$

Using this decomposition we introduce a basis

$$X_I^i; \quad i = 1, \ldots, r; \qquad I = -\eta_i, -\eta_i+1, \ldots, \eta_i \tag{1.12}$$

for $\mathfrak{g}$ compatible with the adjoint action of $\mathcal{A}$. Second, in the coordinates corresponding to this basis X_I^i, it is very easy to obtain the linear terms of the generators $z^i(x)$ written as differential polynomials in the coordinates of S. In the end we are able to prove

Proposition 1.4. *The matrix g_1^{ij} is nondegenerate and its determinant is equal to the determinant of the matrix $A_{ij} = \langle y_i | y_j \rangle$.*

The nondegeneracy condition for g_2^{ij} will follow from a certain differential relation between the entries of two matrices. Namely we have

$$\partial_{z^r} g_2^{ij} = g_1^{ij}. \tag{1.13}$$

The quasihomogeneity and the regularity conditions for the flat pencil of metrics fellows from Proposition 1.2 and the quasihomogeneity of the entries of g_2^{ij} when we assign degree $2\eta_i + 2$ to z^i. Finally we get the promised polynomial Frobenius manifold by using the work of [9].

We mention that from the work of Dubrovin [8] and Hertling [13] semisimple polynomial Frobenius manifolds with positive degrees are already classified. They correspond to Coxeter conjugacy classes in Coxeter groups. Dubrovin constructed all these polynomial Frobenius manifolds on the orbit spaces of Coxeter groups using the results of [20]. There is another method to obtain the classical W-algebra associated to regular nilpotent elements known in the literature as Miura type transformation [6]. It was used in [12] (see also [5]) to prove that the dispersionless limit of the Drinfeld–Sokolov bihamiltonian structure gives the polynomial Frobenius manifold defined on the orbit space of the corresponding Weyl group [8]. The proof depends also on the invariant theory of Coxeter groups. In the present work we give a new method to obtain polynomial Frobenius manifolds from the Drinfeld–Sokolov reduction which depending only on the representation theory of principal sl_2-triples.

2. Preliminaries

2.1. Frobenius manifolds and local bihamiltonian structures

Starting we want to recall some definitions and review the construction of Frobenius manifolds from local bihamiltonian structure of hydrodynamics type.

A **Frobenius manifold** is a manifold M with the structure of Frobenius algebra on the tangent space T_t at any point $t \in M$ with certain compatibility conditions [7]. This structure locally corresponds to a potential $\mathbb{F}(t^1, \dots, t^r)$ satisfying the WDVV equations

$$\partial_{t^i} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^n} \mathbb{F}(t) = \partial_{t^n} \partial_{t^j} \partial_{t^k} \mathbb{F}(t) \eta^{kp} \partial_{t^p} \partial_{t^q} \partial_{t^i} \mathbb{F}(t) \tag{2.1}$$

where $(\eta^{-1})_{ij} = \partial_{t^r} \partial_{t^i} \partial_{t^j} \mathbb{F}(t)$ is a constant matrix. Here we assume that the quasihomogeneity condition takes the form

$$\sum_{i=1}^{r} d_i t_i \partial_{t^i} \mathbb{F}(t) = (3-d) \mathbb{F}(t) \tag{2.2}$$

where $d_r = 1$. This condition defines **the degrees** d_i and **the charge** d of the Frobenius structure on M. If $\mathbb{F}(t)$ is an algebraic function we call M an **algebraic Frobenius manifold**.

Let $\mathfrak{L}(M)$ denote the loop space of M, i.e. the space of smooth maps from the circle to M. A local Poisson bracket $\{.,.\}_1$ on $\mathfrak{L}(M)$ can be written in the form [11]

$$\{u^i(x), u^j(y)\}_1 = \sum_{k=-1}^{\infty} \epsilon^k \{u^i(x), u^j(y)\}_1^{[k]}. \tag{2.3}$$

Here ϵ is just a parameter and

$$\{u^i(x), u^j(y)\}_1^{[k]} = \sum_{s=0}^{k+1} A_{k,s}^{i,j} \delta^{(k-s+1)}(x-y), \tag{2.4}$$

where $A_{k,s}^{i,j}$ are homogeneous polynomials in $\partial_x^j u^i(x)$ of degree s (we assign degree j to $\partial_x^j u^i(x)$) and $\delta(x-y)$ is the Dirac delta function defined by

$$\int_{S^1} f(y) \delta(x-y)\, dy = f(x).$$

The first terms can be written as follows

$$\{u^i(x), u^j(y)\}_1^{[-1]} = F_1^{ij}(u(x))\delta(x-y), \tag{2.5}$$

$$\{u^i(x), u^j(y)\}_1^{[0]} = g_1^{ij}(u(x))\delta'(x-y) + \Gamma_{1k}^{ij}(u(x))u_x^k\delta(x-y). \tag{2.6}$$

Here the entries $g_1^{ij}(u)$, $F_1^{ij}(u)$ and $\Gamma_{1k}^{ij}(u)$ are smooth functions on the finite dimension space M. We note that, under the change of coordinates on M the matrices $g_1^{ij}(u)$, $F_1^{ij}(u)$ change as a $(2,0)$-tensors.

The matrix $F_1^{ij}(u)$ defines a Poisson structure on M. If $F_1^{ij}(u(x)) = 0$ and $\{u^i(x), u^j(y)\}_1^{[0]} \neq 0$ we say the Poisson bracket admits a **dispersionless limit**. If the Poisson bracket admits a dispersionless limit then $\{u^i(x), u^j(y)\}_1^{[0]}$ defines a Poisson bracket on $\mathfrak{L}(M)$ known as **Poisson bracket of hydrodynamic type**. By nondegenerate Poisson bracket of hydrodynamic type we mean those with the matrix g_1^{ij} is nondegenerate. In this case the matrix $g_1^{ij}(u)$ defines a contravariant flat metric on the cotangent space T^*M and $\Gamma_{1k}^{ij}(u)$ is its contravariant Levi-Civita connection [10].

Assume there are two Poisson structures $\{.,.\}_2$ and $\{.,.\}_1$ on $\mathfrak{L}(M)$ which form a bihamiltonian structure, i.e. $\{.,.\}_\lambda := \{.,.\}_2 + \lambda\{.,.\}_1$ is a Poisson structure on $\mathfrak{L}(M)$ for every λ. Consider the notations for the leading terms of $\{.,.\}_1$ given above and write the leading terms of $\{.,.\}_2$ in the form

$$\{u^i(x), u^j(y)\}_2^{[-1]} = F_2^{ij}(u(x))\delta(x-y), \tag{2.7}$$

$$\{u^i(x), u^j(y)\}_2^{[0]} = g_2^{ij}(u(x))\delta'(x-y) + \Gamma_{2k}^{ij}(u(x))u_x^k\delta(x-y). \tag{2.8}$$

Suppose that $\{.,.\}_1$ and $\{.,.\}_2$ admit a dispersionless limit. In addition, assume the corresponding Poisson brackets of hydrodynamics type are nondegenerate as well as the dispersionless limit of $\{.,.\}_\lambda$ for generic λ. Then by definition $g_1^{ij}(u)$ and $g_2^{ij}(u)$ form what is called **flat pencil of metrics** [9], i.e. $g_\lambda^{ij}(u) := g_2^{ij}(u) + \lambda g_1^{ij}(u)$ defines a flat metric on T^*M for generic λ and its Levi-Civita connection is given by $\Gamma_{\lambda k}^{ij}(u) = \Gamma_{2k}^{ij}(u) + \lambda\Gamma_{1k}^{ij}(u)$.

Definition 2.1. A contravariant flat pencil of metrics on a manifold M defined by the matrices g_1^{ij} and g_2^{ij} is called **quasihomogeneous of degree** d if there exists a function τ on M such that the vector fields

$$\begin{aligned} E &:= \nabla_2\tau, \qquad E^i = g_2^{is}\partial_s\tau, \\ e &:= \nabla_1\tau, \qquad e^i = g_1^{is}\partial_s\tau \end{aligned} \tag{2.9}$$

satisfy the following properties

(i) $[e, E] = e$.
(ii) $\mathfrak{L}_E(\ ,\ )_2 = (d-1)(\ ,\ )_2$.
(iii) $\mathfrak{L}_e(\ ,\ )_2 = (\ ,\ )_1$.
(iv) $\mathfrak{L}_e(\ ,\ )_1 = 0$.

Here for example $\mathfrak{L}_E$ denote the Lie derivative along the vector field E and $(\,,\,)_1$ denote the metric defined by the matrix g_1^{ij}. In addition, the quasihomogeneous flat pencil of metrics is called **regular** if the $(1,1)$-tensor

$$R_i^j = \frac{d-1}{2}\delta_i^j + \nabla_{1i}E^j \tag{2.10}$$

is nondegenerate on M.

The connection between the theory of Frobenius manifolds and flat pencil of metrics is encoded in the following theorem

Theorem 2.2. *(See [9].) A contravariant quasihomogeneous regular flat pencil of metrics of degree d on a manifold M defines a Frobenius structure on M of the same degree.*

It is well known that from a Frobenius manifold we always have a flat pencil of metrics but it does not necessary satisfy the regularity condition (2.10). In the notations of (2.1) from a Frobenius structure on M, the flat pencil of metrics is found from the relations

$$\begin{aligned} \eta^{ij} &= g_1^{ij}, \\ g_2^{ij} &= (d-1+d_i+d_j)\eta^{i\alpha}\eta^{j\beta}\partial_{t^\alpha}\partial_{t^\beta}\mathbb{F}. \end{aligned} \tag{2.11}$$

This flat pencil of metric is quasihomogeneous of degree d with $\tau = t^1$. Furthermore we have

$$E = \sum_i d_i t^i \partial_{t^i}, \qquad e = \partial_{t^r}. \tag{2.12}$$

2.2. Regular nilpotent element and opposite Cartan subalgebra

We review some facts about regular nilpotent elements in simple Lie algebra we need to perform the Drinfeld–Sokolov reduction. In particular, we recall the concept of the opposite Cartan subalgebra and we introduce a particular basis for $\mathfrak{g}$ compatible with the action of a given regular sl_2-triple.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r. We fix a regular nilpotent element $e \in \mathfrak{g}$. By definition a nilpotent element is called regular if $\mathfrak{g}^e := \ker \operatorname{ad} e$ has dimension equals to r. Using the Jacobson–Morozov theorem we fix a semisimple element h and a nilpotent element f in $\mathfrak{g}$ such that $\{e, h, f\}$ generate sl_2 subalgebra $\mathcal{A} \subset \mathfrak{g}$, i.e.

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h. \tag{2.13}$$

Then $\mathcal{A}$ is called regular sl_2-triple. We normalize the invariant bilinear from $\langle .|. \rangle$ on $\mathfrak{g}$ such that $\langle e|f \rangle = 1$. The affine space

$$Q' = e + \mathfrak{g}^f \tag{2.14}$$

is called the **Slodowy slice**. Let

$$1 = \eta_1 < \eta_2 \leqslant \eta_3 \leqslant \cdots \leqslant \eta_{r-1} < \eta_r \tag{2.15}$$

the exponents of the Lie algebra $\mathfrak{g}$. We will refer to the number η_r by κ. Recall that $\kappa + 1$ is the Coxeter number of $\mathfrak{g}$ and the exponents satisfy the relation

$$\eta_i + \eta_{r-i+1} = \kappa + 1. \tag{2.16}$$

We also recall that for all simple Lie algebras the exponents are different except for the Lie algebra of type D_{2n} the exponent $n-1$ appears twice.

Consider the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathcal{A}$. Under this restriction $\mathfrak{g}$ decomposes to irreducible $\mathcal{A}$-submodules

$$\mathfrak{g} = \bigoplus V^i \tag{2.17}$$

with $\dim V^i = 2\eta_i + 1$ [14]. We normalize this decomposition by using the following proposition

Proposition 2.3. *There exists a decomposition of $\mathfrak{g}$ into a sum of irreducible $\mathcal{A}$-submodules $\mathfrak{g} = \bigoplus_{i=1}^r V^i$ in such a way that there is a basis X_I^i, $I = -\eta_i, -\eta_i + 1, \ldots, \eta_i$ in each V^i, $i = 1, \ldots, r$ satisfying the following relations*

$$X_I^i = \frac{1}{(\eta_i + I)!} \operatorname{ad} e^{\eta_i + I} X_{-\eta_i}^i, \qquad I = -\eta_i, -\eta_i + 1, \ldots, \eta_i \tag{2.18}$$

and

$$\langle X_I^i, X_J^j \rangle = \delta_{i,j} \delta_{I,-J} (-1)^{\eta_i - I + 1} \binom{2\eta_i}{\eta_i - I}. \tag{2.19}$$

Furthermore

$$\begin{aligned} \operatorname{ad} h X_I^i &= 2I X_I^i, \\ \operatorname{ad} e X_I^i &= (\eta_i + I + 1) X_{I+1}^i, \\ \operatorname{ad} f X_I^i &= (\eta_i - I + 1) X_{I-1}^i. \end{aligned} \tag{2.20}$$

Proof. The proof that one could compose the Lie algebra as irreducible $\mathcal{A}$-submodules satisfying (2.18) and (2.20) is standard and can be found in [14] or [17]. Let $\mathfrak{g} = \bigoplus_{i=1}^r V^i$ be such decomposition. It is easy to prove $\langle V^i | V^j \rangle = 0$ in the case $\eta_i \neq \eta_j$ by applying the step operators $\operatorname{ad} e$ and using the invariance of the bilinear form. Hence the proof is reduced to the case of irreducible $\mathcal{A}$-submodules of the same dimension. But there is at most two irreducible submodules of the same dimension. Assume V^{i_1} and V^{i_2} have the same dimension and denote the corresponding basis $X_I^{i_1}$ and $X_J^{i_1}$, respectively. Then one can prove by using the step operator $\operatorname{ad} e$ that the subspaces V^{i_1} and V^{i_2} are orthogonal if and only if $\langle X_0^{i_1} | X_0^{i_2} \rangle = 0$. But it obvious that the restriction of the invariant bilinear form to $X_0^{i_1}$ and $X_0^{i_2}$ is nondegenerate. Hence by applying the Gram–Schmidt procedure we can assume that $\langle X_0^{i_1} | X_0^{i_2} \rangle = 0$. Therefore, we can assume that the given decomposition satisfying $\langle V^i|V^j\rangle = 0$ if $i \neq j$. It remains to obtain the normalization (2.19). From the invariance of the bilinear form we have

$$\langle h.X_I^i \big| X_J^i\rangle = (2I)\langle X_I^i \big| X_J^i\rangle \tag{2.21}$$

while

$$-\langle X_I^i \big| h.X_J^i\rangle = -(2J)\langle X_I^i \big| X_J^i\rangle. \tag{2.22}$$

Therefore $\langle X_I^i|X_J^j\rangle = 0$ if $I + J \neq 0$. We calculate using the step operator $\operatorname{ad} e$ where $I \geqslant 0$ the value

$$\begin{aligned}
\langle X_I^i \big| X_{-I}^i\rangle &= \frac{1}{(\eta_i - I)}\langle X_I^i \big| e.X_{-I-1}^i\rangle \\
&= \frac{-1}{\eta_i - I}\langle e.X_I^i \big| X_{-I-1}^i\rangle \\
&= \frac{(-1)(\eta_i - I + 1)}{\eta_i - I}\langle X_{I+1}^i \big| X_{-I-1}^i\rangle \\
&= \frac{(-1)^{\eta_i - I}(\eta_i - I + 1)(\eta_i - I + 2)\dots 2\eta_i}{(\eta_i - I)(\eta_i - I - 1)\dots(1)}\langle X_{\eta_i}^i \big| X_{-\eta_i}^i\rangle \\
&= (-1)^{\eta_i - I}\binom{2\eta_i}{\eta_i - I}\langle X_{\eta_i}^i \big| X_{-\eta_i}^i\rangle.
\end{aligned} \tag{2.23}$$

The result follows by multiplying X_I^i by the value of $-\langle X_{\eta_i}^i|X_{-\eta_i}^i\rangle^{-1}$. We note that the formula (2.19) will give the same result when replacing I with $-I$. This ends the proof. □

Note that the normalized basis for V^1 are $X_1^1 = -e$, $X_0^1 = h$, $X_{-1}^1 = f$ since it is isomorphic to $\mathcal{A}$ as a vector subspace.

The element h defines a $\mathbb{Z}$-grading on $\mathfrak{g}$ called the Dynkin grading given as follows

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i, \quad \mathfrak{g}_i = \{q \in \mathfrak{g}\colon \operatorname{ad} h(q) = iq\}. \tag{2.24}$$

It is well known that $\mathfrak{g}_i = 0$ if i is odd and

$$\mathfrak{b} = \bigoplus_{i\leqslant 0} \mathfrak{g}_i \tag{2.25}$$

is a Borel subalgebra with

$$\mathfrak{n} = \bigoplus_{i\leqslant -2} \mathfrak{g}_i = [\mathfrak{b}, \mathfrak{b}] \tag{2.26}$$

is a nilpotent subalgebra. Note that the subalgebra $\mathfrak{g}^f$ has a basis $X_{-\eta_i}^i$, $i = 1, \dots, r$ and

$$\mathfrak{b} = \mathfrak{g}^f \oplus \operatorname{ad} e(\mathfrak{n}). \tag{2.27}$$

Hence Q' is transversal to the orbit of e under the adjoint group action.

In order to introduce the concept of opposite Cartan subalgebra we need to summarize Kostant results about the relation between the regular nilpotent element e and Coxeter conjugacy class in Weyl group of $\mathfrak{g}$.

Theorem 2.4. *(See [17].) The element $y_1 = e + X^r_{-2\kappa}$ is regular semisimple. Denote $\mathfrak{h}'$ the Cartan subalgebra containing y_1, i.e. $\mathfrak{h}' := \ker \operatorname{ad} y_1$ and consider the adjoint group element w defined by $w := \exp \frac{\pi \mathbf{i}}{\kappa+1} \operatorname{ad} h$. Then w acts on $\mathfrak{h}'$ as a representative of the Coxeter conjugacy class in the Weyl group acting on $\mathfrak{h}'$. Furthermore, the element y_1 can be completed to a basis y_i, $i = 1, \dots, r$ for $\mathfrak{h}'$ having the form*

$$y_i = v_i + u_i, \quad u_i \in \mathfrak{g}_{2\eta_i}, \ v_i \in \mathfrak{g}_{2\eta_i - 2(\kappa+1)}$$

and such that y_i is an eigenvector of w with eigenvalue $\exp \frac{2\pi \mathbf{i} \eta_i}{\kappa+1}$.

Remark 2.5. Kostant proved this theorem by writing the regular nilpotent element e as the sum of the root vectors corresponding to simple roots. It will follow then $X^r_{-2\kappa}$ is a constant multiplication of the root vector corresponding to the negative of the maximum root. These assumptions will follow easily if we choose the root vectors with respect to the Cartan subalgebra $\mathfrak{h}$ contains h and ordering the roots with respect to h [2].

Let a denote the element $X^r_{-2\kappa}$. The element $y_1 = e + a$ is called a **cyclic element** and the Cartan subalgebra $\mathfrak{h}' = \ker \operatorname{ad} y_1$ is called the **opposite Cartan subalgebra**. We fix a basis y_i for $\mathfrak{h}'$ satisfying the theorem above. It is easy to see that $u_i, i = 1, \dots, r$ form a homogeneous basis for $\mathfrak{g}^e$. We assume the basis y_i is normalized such that

$$u_i = -X^i_{\eta_i}. \tag{2.28}$$

Form construction this normalization does not effect y_1.

Let us define the matrix of the invariant bilinear form on $\mathfrak{h}'$

$$A_{ij} := \langle y_i | y_j \rangle = -\langle X^i_{\eta_i} | v_j \rangle - \langle v_i | X^i_{\eta_j} \rangle, \quad i, j = 1, \dots, r. \tag{2.29}$$

The following proposition summarizes some useful properties we need in the following sections.

Proposition 2.6. *The matrix A_{ij} is nondegenerate and antidiagonal with respect to the exponents η_i, i.e. $A_{ij} = 0$, if $\eta_i + \eta_j \neq \kappa + 1$. Moreover, the commutators of a and $X^i_{\eta_i}$ satisfy the relations*

$$\frac{\langle [a, X^i_{\eta_i}] | X^j_{\eta_j - 1} \rangle}{2\eta_j} + \frac{\langle [a, X^j_{\eta_j}] | X^i_{\eta_i - 1} \rangle}{2\eta_i} = A_{ij} \tag{2.30}$$

for all $i, j = 1, \dots, r$.

Proof. The matrix A_{ij} is nondegenerate since the restriction of the invariant bilinear form to a Cartan subalgebra is nondegenerate. The fact that it is antidiagonal with respect to the exponents follows from the identity

$$\langle y_i|y_j\rangle = \langle wy_i|wy_j\rangle = \exp\frac{(\eta_i+\eta_j)\pi\mathbf{i}}{\kappa+1}\langle y_i|y_j\rangle \tag{2.31}$$

where $w := \exp\frac{\pi\mathbf{i}}{\kappa+1}\operatorname{ad}h$. For the second part of the proposition we note that the commutator of $y_1 = e + a$ and $y_i = v_i - X^i_{\eta_i}$ gives the relation

$$[e, v_i] = \bigl[a, X^i_{\eta_i}\bigr], \quad i = 1, \dots, r. \tag{2.32}$$

Which in turn gives the following equality for every $i, j = 1, \dots, r$

$$\begin{aligned}\bigl\langle\bigl[a, X^i_{\eta_i}\bigr]\big|X^j_{\eta_j-1}\bigr\rangle &= \bigl\langle[e, v_i]\big|X^j_{\eta_j-1}\bigr\rangle = -\bigl\langle v_i\big|\bigl[e, X^j_{\eta_j-1}\bigr]\bigr\rangle \\ &= -2\eta_j\bigl\langle v_i\big|X^j_{\eta_j}\bigr\rangle\end{aligned} \tag{2.33}$$

but then

$$\frac{\langle[a, X^i_{\eta_i}]|X^j_{\eta_j-1}\rangle}{2\eta_j} + \frac{\langle[a, X^j_{\eta_j}]|X^i_{\eta_i-1}\rangle}{2\eta_i} = -\bigl\langle v_i\big|X^j_{\eta_j}\bigr\rangle - \bigl\langle v_j\big|X^i_{\eta_i}\bigr\rangle = A_{ij}. \qquad \square \tag{2.34}$$

3. Drinfeld–Sokolov reduction

We will review the standard Drinfeld–Sokolov reduction associated with the regular nilpotent element [6] (see also [4]).

We introduce the following bilinear form on the loop algebra $\mathfrak{L}(\mathfrak{g})$:

$$(u|v) = \int_{S^1}\bigl\langle u(x)\big|v(x)\bigr\rangle\,dx, \quad u, v \in \mathfrak{L}(\mathfrak{g}), \tag{3.1}$$

and we identify $\mathfrak{L}(\mathfrak{g})$ with $\mathfrak{L}(\mathfrak{g})^*$ by means of this bilinear form. For a functional $\mathcal{F}$ on $\mathfrak{L}(\mathfrak{g})$ we define the gradient $\delta\mathcal{F}(q)$ to be the unique element in $\mathfrak{L}(\mathfrak{g})$ such that

$$\frac{d}{d\theta}\mathcal{F}(q+\theta\dot{s})\bigg|_{\theta=0} = \int_{S^1}\langle\delta\mathcal{F}|\dot{s}\rangle\,dx \quad \text{for all } \dot{s} \in \mathfrak{L}(\mathfrak{g}). \tag{3.2}$$

Recall that we fixed an element $a \in \mathfrak{g}$ such that $y_1 = e + a$ is a cyclic element. Let us introduce a bihamiltonian structure on $\mathfrak{L}(\mathfrak{g})$ by means of Poisson tensors

$$\begin{aligned}P_2(v)\bigl(q(x)\bigr) &= \frac{1}{\epsilon}\bigl[\epsilon\partial_x + q(x), v(x)\bigr], \\ P_1(v)\bigl(q(x)\bigr) &= \frac{1}{\epsilon}\bigl[a, v(x)\bigr].\end{aligned} \tag{3.3}$$

It is well known fact that these define a bihamiltonian structure on $\mathfrak{L}(\mathfrak{g})$ [18].

We consider the gauge transformation of the adjoint group G of $\mathfrak{L}(\mathfrak{g})$ given by

$$q(x) \to \exp \operatorname{ad} s(x)\big(\partial_x + q(x)\big) - \partial_x \tag{3.4}$$

where $s(x), q(x) \in \mathfrak{L}(\mathfrak{g})$. Following Drinfeld and Sokolov [6], we consider the restriction of this action to the adjoint group $\mathcal{N}$ of $\mathfrak{L}(\mathfrak{n})$.

Proposition 3.1. *(See [4,19].) The action of $\mathcal{N}$ on $\mathfrak{L}(\mathfrak{g})$ with Poisson tensor*

$$P_\lambda := P_2 + \lambda P_1 \tag{3.5}$$

is Hamiltonian for all λ. It admits a momentum map J to be the projection

$$J : \mathfrak{L}(\mathfrak{g}) \to \mathfrak{L}(\mathfrak{n})^+$$

where $\mathfrak{n}^+$ is the image of $\mathfrak{n}$ under the Killing map. Moreover, J is Ad^*-equivariant.

We take e as regular value of J. Then

$$S := J^{-1}(e) = \mathfrak{L}(\mathfrak{b}) + e, \tag{3.6}$$

since $\mathfrak{b}$ is the orthogonal complement to $\mathfrak{n}$. It follows from the Dynkin grading that the isotropy group of e is $\mathcal{N}$. Let R be the ring of invariant differential polynomials of S under the action of $\mathcal{N}$. Then the set $\mathcal{R}$ of functionals on S which have densities in the ring R is closed under P_2 and P_1. Another proof of this result can be found in [6].

Recall that the space Q is defined as

$$Q := e + \mathfrak{L}\big(\mathfrak{g}^f\big). \tag{3.7}$$

The following proposition identified $S/\mathcal{N}$ with the space Q.

Proposition 3.2. *(See [6].) The space Q is a cross section for the action of $\mathcal{N}$ on S, i.e. for any element $q(x) + e \in S$ there is a unique element $s(x) \in \mathfrak{L}(\mathfrak{n})$ such that*

$$z(x) + e = \big(\exp \operatorname{ad} s(x)\big)\big(\partial_x + q(x)\big) - \partial_x \in Q. \tag{3.8}$$

The entries of $z(x)$ are generators of the ring R of differential polynomials on S invariant under the action of $\mathcal{N}$.

Hence we have an isomorphism between the set of functionals on Q and the set $\mathcal{R}$. Therefore, Q has a bihamiltonian structure P_1^Q and P_2^Q from P_1 and P_2, respectively. The reduced Poisson structure P_2^Q is known as **classical W-algebra** associated to the regular nilpotent element e. For a formal definition of classical W-algebras see [16].

The generators of the invariant ring R will have nice properties when we use the normalized basis we developed in last section. Let us begin by writing the equation of gauge fixing (3.8) after introducing a parameter τ as follows

$$q(x)+e=\tau\sum_{i=1}^{r}\sum_{I=0}^{\eta_i}q_i^I(x)X_{-I}^i+e\in S,$$

$$z(x)+e=\tau\sum_{i=1}^{r}z^i(x)X_{-\eta_i}^i+e\in Q,$$

$$s(x)=\tau\sum_{i=1}^{r}\sum_{I=1}^{\eta_i}s_i^I(x)X_{-I}^i\in\mathfrak{L}(\mathfrak{n}).$$

Then Eq. (3.8) expands to

$$\begin{aligned}&\sum_{i=1}^{r}z^i(x)X_{-\eta_i}^i+\sum_{i=1}^{r}\sum_{I=1}^{\eta_i}(\eta_i-I+1)s_i^IX_{-I+1}^i\\&\quad=\sum_{i=1}^{r}\sum_{I=0}^{\eta_i}q_i^I(x)X_{-I}^i-\sum_{i=1}^{r}\sum_{I=1}^{\eta_i}\partial_x s_i^I(x)X_{-I}^i+\mathcal{O}(\tau).\end{aligned}\tag{3.9}$$

It obvious that any invariant $z^i(x)$ has the form

$$\begin{aligned}z^i(x)&=q_i^{\eta_i}(x)-\partial_x s_i^{\eta_i}+\mathcal{O}(\tau)\\&=q_i^{\eta_i}(x)-\partial_x q_i^{\eta_i-1}+\mathcal{O}(\tau).\end{aligned}\tag{3.10}$$

That is, we obtained the linear terms of each invariant $z^i(x)$. Furthermore, since $\langle e|f\rangle=1$ then $z^1(x)$ has the expression

$$\begin{aligned}z^1(x)=q_1^1(x)-\partial_x s_1^1&+\tau\big\langle e\big|\big[s_i^1(x)X_{-1}^i,q_i^0X_0^i\big]\big\rangle\\&+\frac{1}{2}\tau\big\langle e\big|\big[s_i^1(x)X_{-1}^i,\big[s_i^I(x)X_{-1}^i,e\big]\big]\big\rangle.\end{aligned}\tag{3.11}$$

Which is simplified by using the identity

$$\big[s_i^1(x)X_{-1}^i,\big[s_i^I(x)X_{-1}^i,e\big]\big]=-\big[s_i^1(x)X_{-1}^i,q_i^0(x)X_0^i\big]\tag{3.12}$$

and

$$\big\langle e\big|\big[s_i^1(x)X_{-1}^i,q_i^0X_0^i\big]\big\rangle=-\big\langle\big[s_i^1(x)X_{-1}^i,e\big]\big|q_i^0(x)X_0^i\big\rangle=\big(q_i^0(x)\big)^2\big\langle X_0^i\big|X_0^i\big\rangle\tag{3.13}$$

with $s_1^1(x)=q_1^0(x)$ to the expression

$$z^1(x)=q_1^1(x)-\partial_x q_1^0(x)+\frac{1}{2}\tau\sum_i\big(q_i^0(x)\big)^2\big\langle X_0^i\big|X_0^i\big\rangle.\tag{3.14}$$

The invariant $z^1(x)$ is called a **Virasoro density** and the expression above agree with [1].

We observe that the reduced bihamiltonian structure can be calculated as follows. We write the coordinates of Q as differential polynomials in the coordinates of S by means of Eq. (3.9) and then apply the Leibnitz rule. For $u, v \in R$ the Leibnitz rule have the following form

$$\big\{u(x), v(y)\big\}_\lambda = \frac{\partial u(x)}{\partial (q_i^I)^{(m)}}\partial_x^m\left(\frac{\partial v(y)}{\partial (q_j^J)^{(n)}}\partial_y^n\big(\big\{q_i^I(x), q_j^J(y)\big\}_\lambda\big)\right). \tag{3.15}$$

Our analysis will relay on the quasihomogeneity of the invariants $z^i(x)$ in the coordinates of $q(x) \in \mathfrak{L}(\mathfrak{b})$ and their derivatives. This property is summarized in the following corollary

Corollary 3.3. *If we assign degree $2J + 2l + 2$ to $\partial_x^l(q_i^J(x))$ then $z^i(x)$ will be quasihomogeneous of degree $2\eta_i + 2$. Furthermore, each invariant $z^i(x)$ depends linearly only on $q_i^{\eta_i}(x)$ and $\partial_x q_i^{\eta_i - 1}(x)$. In particular, $z^i(x)$ with $i < n$ does not depend on $\partial_x^l q_r^{\eta_r}(x)$ for any value l.*

Let us fix the following notations for the leading terms of the Drinfeld–Sokolov bihamiltonian structure on Q

$$\begin{aligned}\big\{z^i(x), z^j(y)\big\}_1^Q &= \sum_{k=-1}^{\infty}\epsilon^k\big\{z^i(x), z^j(y)\big\}_1^{[k]},\\ \big\{z^i(x), z^j(y)\big\}_2^Q &= \sum_{k=-1}^{\infty}\epsilon^k\big\{z^i(x), z^j(y)\big\}_2^{[k]},\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}\big\{z^i(x), z^j(y)\big\}_1^{[-1]} &= F_1^{ij}\big(z(x)\big)\delta(x-y),\\ \big\{z^i(x), z^j(y)\big\}_1^{[0]} &= g_1^{ij}\big(z(x)\big)\delta'(x-y) + \Gamma_{1k}^{ij}\big(z(x)\big)z_x^k\delta(x-y),\\ \big\{z^i(x), z^j(y)\big\}_2^{[-1]} &= F_2^{ij}\big(z(x)\big)\delta(x-y),\\ \big\{z^i(x), z^j(y)\big\}_2^{[0]} &= g_2^{ij}\big(z(x)\big)\delta'(x-y) + \Gamma_{2k}^{ij}\big(z(x)\big)z_x^k\delta(x-y).\end{aligned} \tag{3.17}$$

3.1. The nondegeneracy condition

In this section we find the antidiagonal entries of the matrix g_1^{ij} with respect to the exponents of $\mathfrak{g}$, i.e. the entry g_1^{ij} with $\eta_i + \eta_j = \kappa + 1$. Our goal is to prove this matrix is nondegenerate.

Let Ξ_I^i denote the value $\langle X_I^i | X_I^i\rangle$ and we set

$$\big[a, X_I^i\big] = \sum_j \Delta_I^{ij} X_{I-\eta_r}^j.$$

By definition, for a functional $\mathcal{F}$ on $\mathfrak{g}$

$$\delta\mathcal{F}(x) = \sum_i\sum_{I=0}^{\eta_i}\frac{1}{\Xi_I^i}\frac{\delta\mathcal{F}}{\delta q_i^I(x)}X_I^i \tag{3.18}$$

and the Poisson brackets of two functionals $\mathcal{I}$ and $\mathcal{F}$ on $\mathfrak{g}$ reads

$$\{\mathcal{I},\mathcal{F}\}_1=\langle\delta\mathcal{I}(x)\big|\big[a,\delta\mathcal{F}(x)\big]\rangle=\sum_i\sum_{I=0}^{\eta_i}\sum_j\frac{\Delta_I^{ij}}{\Xi_I^i}\frac{\delta\mathcal{I}}{\delta q_j^{\kappa-I}(x)}\frac{\delta\mathcal{F}}{\delta q_i^I(x)}. \tag{3.19}$$

Therefore, the Poisson brackets in coordinates have the form

$$\big\{q_j^{\kappa-I}(x),q_i^I(y)\big\}_1=\frac{\Delta_I^{ij}}{\Xi_I^i}\delta(x-y). \tag{3.20}$$

Recall that the Poisson bracket $\{v(x),u(y)\}_1^Q$ of elements $u,v\in R$ is obtained by the Leibnitz rule which expands as

$$\begin{aligned}\big\{v(x),u(y)\big\}_1^Q&=\sum_{i,I;j}\sum_{l,h}\frac{\Delta_I^{ij}}{\Xi_I^i}\frac{\partial v(x)}{\partial(q_j^{\kappa-I})^{(l)}}\partial_x^l\left(\frac{\partial u(y)}{\partial(q_i^I)^{(h)}}\partial_y^h\big(\delta(x-y)\big)\right)\\&=\sum_{i,I;j}\sum_{l,h,m,n}(-1)^h\binom{h}{m}\binom{l}{n}\frac{\Delta_I^{ij}}{\Xi_I^i}\frac{\partial v(x)}{\partial(q_j^{\kappa-I})^{(l)}}\left(\frac{\partial u(x)}{\partial(q_i^I)^{(h)}}\right)^{m+n}\delta^{h+l-m-n}(x-y).\end{aligned}$$

Here we omitted the ranges of the indices since no confusion can arise. Let $\mathbb{A}(v,u)$ denote the coefficient of $\delta'(x-y)$

$$\mathbb{A}(v,u)=\sum_{i,I,J}\sum_{h,l}(-1)^h(l+h)\frac{\Delta_I^{ij}}{\Xi_I^i}\frac{\partial v(x)}{\partial(q_j^{\kappa-I})^{(l)}}\left(\frac{\partial u(x)}{\partial(q_i^I)^{(h)}}\right)^{h+l-1}. \tag{3.21}$$

Obviously, we obtain the entry g_1^{ij} from $\mathbb{A}(z^i,z^j)$.

Lemma 3.4. *If $\eta_i+\eta_j<\kappa+1$ then $\mathbb{A}(z^i,z^j)=0$. In particular, the matrix g_1^{ij} is lower antidiagonal with respect to the exponents of $\mathfrak{g}$ and the antidiagonal entries are constants.*

Proof. We note that if $v(x)$ and $u(x)$ are in R and quasihomogeneous of degree θ and ξ, respectively, then $\mathbb{A}(v,u)$ will be quasihomogeneous of degree

$$\theta+\xi-(2\kappa+2)-4.$$

The proof is completed by observing that the generators $z^i(x)$ of the ring R is quasihomogeneous of degree $2\eta_i+2$. □

Proposition 3.5. *The matrix g_1^{ij} is nondegenerate and its determinant is equal to the determinant of the matrix A_{ij} defined in Proposition* 2.6.

Proof. From the last lemma we need only to consider the expression $\mathbb{A}(z^n,z^m)$ with $\eta_n+\eta_m=\kappa+1$. Here

$$\mathbb{A}\big(z^n, z^m\big) = \sum_{i,I,J}\sum_{h,l}(-1)^h(l+h)\frac{\Delta_I^{ij}}{\Xi_I^i}\frac{\partial z^n(x)}{\partial(q_j^{\kappa-I})^{(l)}}\left(\frac{\partial z^m(x)}{\partial(q_i^I)^{(h)}}\right)^{h+l-1} \tag{3.22}$$

where z^m and z^n are quasihomogeneous of degree $2\eta_m + 2$ and $2\kappa - 2\eta_m + 4$, respectively. The expression $\frac{\partial z^m(x)}{\partial(q_i^I)^{(h)}}$ gives the constrains

$$\begin{aligned} 2I + 2 &\leqslant 2\eta_m + 2, \\ 2\kappa - 2I + 2 &\leqslant 2\kappa - 2\eta_m + 4, \end{aligned} \tag{3.23}$$

which implies

$$\eta_m - 1 \leqslant I \leqslant \eta_m.$$

Therefore the only possible values for the index I in the expression of $\mathbb{A}(z^n, z^m)$ that make sense are η_m and $\eta_m - 1$. Consider the partial summation of $\mathbb{A}(z^n, z^m)$ when $I = \eta_m$. The degree of z^m yields $h = 0$ and that z^m depends linearly on $q_i^{\eta_m}$. But then Eq. (3.10) implies i is fixed and equals to m. A similar argument on $z^n(x)$ we find that the indices l and j are fixed and equal to 1 and n, respectively. But then the partial summation when $I = \eta_m$ gives the value

$$\frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m}\frac{\partial z^n(x)}{\partial(q_n^{\kappa-\eta_m})^{(1)}}\frac{\partial z^m(x)}{\partial(q_m^{\eta_m})^{(0)}} = -\frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m}.$$

We now turn to the partial summation of $\mathbb{A}(z^n, z^m)$ when $I = \eta_m - 1$. The possible values for h are 1 and 0. When $h = 0$ we get zero since l and h can only be zero. When $h = 1$ we get, similar to the above calculation, the value

$$(-1)\frac{\Delta_{\eta_m-1}^{mn}}{\Xi_I^i}\frac{\partial z^n(x)}{\partial(q_n^{\kappa-\eta_m})^{(0)}}\frac{\partial z^m(x)}{\partial(q_m^{\eta_m-1})^{(1)}} = \frac{\Delta_{\eta_m-1}^{mn}}{\Xi_{\eta_m-1}^m}.$$

Hence we end with the expression

$$\begin{aligned} \mathbb{A}\big(z^n, z^m\big) &= \frac{\Delta_{\eta_m-1}^{mn}}{\Xi_{\eta_m-1}^m} - \frac{\Delta_{\eta_m}^{mn}}{\Xi_{\eta_m}^m} \\ &= \frac{\langle [a, X_{\eta_n}^n] | X_{\eta_m-1}^m\rangle}{2\eta_m} + \frac{\langle [a, X_{\eta_m}^m] | X_{\eta_n-1}^n\rangle}{2\eta_n} = A_{mn} \end{aligned}$$

where we derive the last equality in Proposition 2.6. Hence the determinate of g_1^{ij} equals to the determinant of A_{mn} which is nondegenerate. □

3.2. *Differential relation*

We want to observe a differential relation between the first and the second Poisson brackets. This relation is a consequence of the fact that $z^r(x)$ is the only generator of the ring R which depends explicitly on $q_r^\kappa(x)$ and this dependence is linear.

Proposition 3.6. *The entries of matrices of the reduced bihamiltonian structure on Q satisfy the relations*

$$\partial_{z^r} F_2^{ij} = F_1^{ij}, \qquad \partial_{z^r} g_2^{ij} = g_1^{ij}. \tag{3.24}$$

Proof. The fact that we calculate the reduced Poisson structure by using Leibnitz rule and $z^r(x)$ depends on $q_r^\kappa(x)$ linearly, means that the invariant $z^r(x)$ will appear on the reduced Poisson bracket $\{z^i(x), z^j(y)\}_2^Q$ only as a result of the following "brackets"

$$\left[q_j^{\kappa-I}(x), q_i^I(y)\right] := q_r^\kappa(x)\frac{\Delta_I^{ij}}{\Xi_I^i}\delta(x-y) \tag{3.25}$$

which are the terms of the second Poisson bracket on $\mathfrak{L}(\mathfrak{g})$ depending explicitly on $q_r^\kappa(x)$. We expand the "brackets" $[z^i(x), z^j(y)]$ by imposing the Leibnitz rule. We find the coefficient of $\delta(x-y)$ and $\delta'(x-y)$ are, respectively,

$$\mathbb{B} = \sum_{i,I,J}\sum_{h,l}(-1)^h \frac{\Delta_I^{ij}}{\Xi_I^i} q_r^\kappa(x)\frac{\partial z^i(x)}{\partial (q_j^{\kappa-I})^{(l)}}\left(\frac{\partial z^j(x)}{\partial (q_i^I)^{(h)}}\right)^{h+l},$$

$$\mathbb{D} = \sum_{i,I,J}\sum_{h,l}(-1)^h (l+h)\frac{\Delta_I^{ij}}{\Xi_I^i} q_r^\kappa(x)\frac{\partial z^i(x)}{\partial (q_j^{\kappa-I})^{(l)}}\left(\frac{\partial z^j(x)}{\partial (q_i^I)^{(h)}}\right)^{h+l-1}. \tag{3.26}$$

Obviously, we have $\partial_{z^r} F_2^{ij}$ from $\partial_{q_r^\kappa}\mathbb{B}$ and $\partial_{z^r} g_2^{ij}$ from $\partial_{q_r^\kappa}\mathbb{D}$. But we see that $\partial_{q_r^\kappa}\mathbb{D}$ is just the coefficient $\mathbb{A}(z^i, z^j)$ of $\delta'(x-y)$ of $\{z^i(x), z^j(y)\}_1^Q$. This proves that

$$\partial_{z^r} g_2^{ij} = g_1^{ij}.$$

A similar argument shows that

$$\partial_{z^r} F_2^{ij} = F_1^{ij}. \qquad \square$$

4. Some results from Dirac reduction

We recall that the Poisson bracket $\{.,.\}_2^Q$ can be obtained by performing the Dirac reduction of $\{.,.\}_2$ on Q. We derive from this some facts concerning the dispersionless limit of the bihamiltonian structure on Q. Let $\mathbf{n}$ denote the dimension of $\mathfrak{g}$.

Let ξ_I, $I = 1, \ldots, \mathbf{n}$ be a total order of the basis X_I^i such that

1. The first r are

$$X^1_{-\eta_1} < X^2_{-\eta_2} < \cdots < X^r_{-\eta_r}. \tag{4.1}$$

2. The matrix

$$\langle \xi_I | \xi_J \rangle, \quad I, J = 1, \dots, \mathbf{n} \tag{4.2}$$

is antidiagonal.

Let ξ_I^* denote the dual basis of ξ_I under $\langle .|. \rangle$. Note that if $\xi_I \in \mathfrak{g}_\mu$ then $\xi_I^* \in \mathfrak{g}_{-\mu}$.

We extend the coordinates on Q to all $\mathfrak{L}(\mathfrak{g})$ by setting

$$z^I\big(b(x)\big) := \big\langle b(x) - e \big| \xi_I^* \big\rangle, \quad I = 1, \dots, \mathbf{n}. \tag{4.3}$$

Let us fix the following notations for the structure constants and the bilinear form on $\mathfrak{g}$

$$\big[\xi_I^*, \xi_J^*\big] := \sum c_K^{IJ} \xi_K^*, \qquad \widetilde{g}^{IJ} = \big\langle \xi_I^* \big| \xi_J^* \big\rangle. \tag{4.4}$$

Now consider the following matrix differential operator

$$\mathbb{F}^{IJ} = \epsilon \widetilde{g}^{IJ} \partial_x + \widetilde{F}^{IJ}. \tag{4.5}$$

Here

$$\widetilde{F}^{IJ} = \sum_K \big(c_K^{IJ} z^K(x)\big).$$

Then the Poisson brackets of P_2 will have the form

$$\big\{z^I(x), z^J(y)\big\}_2 = \mathbb{F}^{IJ} \frac{1}{\epsilon} \delta(x-y). \tag{4.6}$$

Proposition 4.1. *(See [1].) The second Poisson bracket* $\{.,.\}_2^Q$ *can be obtained by performing Dirac reduction of* $\{.,.\}_2$ *on* Q.

A consequence of this proposition is the following

Proposition 4.2. *(See [1].)*

$$\begin{aligned} \big\{z^1(x), z^1(y)\big\}_2 &= \epsilon \delta'''(x-y) + 2z^1(x)\delta'(x-y) + z_x^1 \delta(x-y), \\ \big\{z^1(x), z^i(y)\big\}_2 &= (\eta_i + 1) z^i(x) \delta'(x-y) + \eta_i z_x^i \delta(x-y). \end{aligned} \tag{4.7}$$

For the rest of this section we consider three types of indices which have different ranges; capital letters $I, J, K, \ldots = 1, \ldots, \mathbf{n}$, small letters $i, j, k, \ldots = 1, \ldots, r$ and Greek letters $\alpha, \beta, \delta, \ldots = r+1, \ldots, \mathbf{n}$. Recall that the space Q is defined by $z^\alpha = 0$.

We note that the matrix $\widetilde{F}^{IJ}$ defines the finite Lie–Poisson structure on $\mathfrak{g}$. It is well known that the symplectic subspaces of this structure are the orbit spaces of $\mathfrak{g}$ under the adjoint group action and we have r global Casimirs [18]. Since the Slodowy slice $Q' = e + \mathfrak{g}^f$ is transversal to the orbit of e, the minor matrix $\widetilde{F}^{\alpha\beta}$ is nondegenerate. Let $\widetilde{F}_{\alpha\beta}$ denote its inverse.

Proposition 4.3. *(See [4].) The Dirac formulas for the leading terms of* $\{.,.\}_2^Q$ *are given by*

$$F_2^{ij} = \left(\widetilde{F}^{ij} - \widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{F}^{\alpha j}\right), \tag{4.8}$$

$$g_2^{ij} = \widetilde{g}^{ij} - \widetilde{g}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{F}^{\alpha j} + \widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{g}^{\alpha\varphi}\widetilde{F}_{\varphi\gamma}\widetilde{F}^{\gamma j} - \widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{g}^{\alpha j}. \tag{4.9}$$

Now we are able to prove the following

Proposition 4.4. *The Drinfeld–Sokolov bihamiltonian structure on* Q *admits a dispersionless limit. The corresponding bihamiltonian structure of hydrodynamic type gives a flat pencil of metrics on the Slodowy slice* Q'.

Proof. We note that (4.8) is the formula of the Dirac reduction of the Lie–Poisson brackets of $\mathfrak{g}$ to the finite space Q'. The fact that Slodowy slice is transversal to the orbit space of the nilpotent element and this orbit has dimension $\mathbf{n} - r$ yield F_2^{ij} is trivial. From Proposition 4.2 it follows that g_2^{ij} is not trivial. This proves that the brackets $\{.,.\}_2^Q$ admit a dispersionless limit. From Propositions 3.5 and 3.6 it follows that $\{.,.\}_1^Q$ admits a dispersionless limit and the matrix g_2^{ij} is nondegenerate. Therefore, the two matrices g_1^{ij} and g_2^{ij} define a flat pencil of metrics on Q'. □

Now we want to study the quasihomogeneity of the entries of the matrix g_2^{ij}. We assign the degree $\mu_I + 2$ to $z^I(x)$ if $\xi_I^* \in \mathfrak{g}_{\mu_I}$. These degrees agree with those given in Corollary 3.3. We observe that degree $z^{\mathbf{n}-I+1}$ equal to $-\mu_I + 2$ from our order of the basis, and an entry $\widetilde{F}^{IJ}$ is quasihomogeneous of degree $\mu_I + \mu_J + 2$ since $[\mathfrak{g}_{\mu_I}, \mathfrak{g}_{\mu_J}] \subset \mathfrak{g}_{\mu_I+\mu_J}$.

The following proposition proved in [3]

Proposition 4.5. *The matrix* $\widetilde{F}_{\beta\alpha}$ *restricted to* Q *is polynomial and the entry* $\widetilde{F}_{\beta\alpha}$ *is quasihomogeneous of degree* $-\mu_\beta - \mu_\alpha - 2$.

Proposition 4.6. *The entry* g_2^{ij} *is quasihomogeneous of degree* $2\eta_i + 2\eta_j$.

Proof. We will derive the quasihomogeneity from the expression (4.9). We know that the matrix $\widetilde{g}^{IJ}$ is constant antidiagonal, i.e. $g^{IJ} = C^I \delta^I_{\mathbf{n}-J+1}$ where C^I are nonzero constants. In particular $g^{ij} = 0$. Now for a fixed i we have

$$\widetilde{g}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{F}^{\alpha j} = C^i \widetilde{F}_{\mathbf{n}-i+1,\alpha}\widetilde{F}^{\alpha j}.$$

But then the left hand sight is quasihomogeneous of degree

$$\mu_j + \mu_\alpha + 2 - \mu_\alpha - (-\mu_i) - 2 = \mu_j + \mu_i = 2\eta_i + 2\eta_j.$$

A similar argument shows that $\widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{g}^{\alpha j}$ is quasihomogeneous of degree $2\eta_i + 2\eta_j$. Let us consider

$$\widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{g}^{\alpha\varphi}\widetilde{F}_{\varphi\gamma}\widetilde{F}^{\gamma j} = \sum_\alpha C^\alpha \widetilde{F}^{i\beta}\widetilde{F}_{\beta\alpha}\widetilde{F}_{\mathbf{n}-\alpha+1,\gamma}\widetilde{F}^{\gamma j}.$$

Then any term in this summation will have the degree

$$\mu_i + \mu_\beta + 2 - \mu_\beta - \mu_\alpha - 2 - \mu_{\mathbf{n}-\alpha+1} - \mu_\gamma - 2 + \mu_\gamma + \mu_j + 2 = 2\eta_i + 2\eta_j.$$

This completes the proof. □

5. Polynomial Frobenius manifold

Let us consider the finite dimension manifold Q' defined by the coordinates $z^1, \dots, z^n$. We will obtain a natural polynomial Frobenius structure on Q'.

The proof of the following proposition depends only on the quasihomogeneity of the matrix g_1^{ij}.

Proposition 5.1. *(See [8].) There exist quasihomogeneous polynomials coordinates of degree d_i in the form*

$$t^1 = \frac{1}{\kappa+1} z^1$$

and

$$t^i = z^i + T^i\big(z^1, \dots, z^{i-1}\big), \quad i > 1$$

such that the matrix $g_1^{ij}(t)$ is constant antidiagonal.

For the remainder of this section, we fix coordinates $(t^1, \dots, t^n)$ satisfying the proposition above. The following proposition emphasis that under this change of coordinates some entries of the matrix g_2^{ij} remain invariant.

Proposition 5.2. *The second metric $g_2^{ij}(t)$ have the following entries*

$$g_2^{1,n}(t) = \frac{\eta_i + 1}{\kappa + 1} t^i. \tag{5.1}$$

Proof. We know from Proposition 4.2 that in the coordinates z^i the matrix $g_2^{ij}(z)$ has the following entries

$$g_2^{1,n}(z) = (\eta_i + 1) z^i. \tag{5.2}$$

Let E' denote the Euler vector field give by

$$E' = \sum_i \frac{\eta_i + 1}{\kappa + 1} z^i \partial_{z^i}. \tag{5.3}$$

Then from the quasihomogeneity of t^i we have $E'(t^i) = \frac{\eta_i+1}{\kappa+1} t^i$. The formula for change of coordinates and the fact that $t^1 = \frac{1}{\kappa+1} z^1$ give the following

$$g^{1j}(t) = \partial_{z^a} t^1 \partial_{z^b} t^j \ g_2^{ab}(z) = E'\big(t^j\big) = \frac{\eta_j + 1}{\kappa + 1} t^j. \qquad \Box \tag{5.4}$$

We arrive to our basic result

Theorem 5.3. *The flat pencil of metrics on the Slodowy slice Q' obtained from the dispersionless limit of Drinfeld–Sokolov bihamiltonian structure on Q (see Theorem* 4.4*) is regular quasihomogeneous of degree $\frac{\kappa-1}{\kappa+1}$.*

Proof. In the notations of Definition 2.1 we take $\tau = t^1$ then

$$E = g_2^{ij}\partial_{t^j}\tau\partial_{t^i} = \frac{1}{\kappa+1}\sum_i(\eta_i+1)t^i\partial_{t^i},$$
$$e = g_1^{ij}\partial_{t^j}\tau\partial_{t^i} = \partial_{t^r}. \tag{5.5}$$

We see immediately that

$$[e, E] = e.$$

The identity

$$\mathfrak{L}_e(\,,\,)_2 = (\,,\,)_1 \tag{5.6}$$

follows from and the fact that $\partial_{t^r} = \partial_{z^r}$ and Proposition 3.6. The fact that

$$\mathfrak{L}_e(\,,\,)_1 = 0 \tag{5.7}$$

is a consequence from the quasihomogeneity of the matrix g_1^{ij} (see Lemma 3.4). We also obtain from Proposition 4.6

$$\mathfrak{L}_E(\,,\,)_2 = (d-1)(\,,\,)_2 \tag{5.8}$$

since

$$\mathfrak{L}_E(\,,\,)_2\big(dt^i, dt^j\big) = E\big(g_2^{ij}\big) - \frac{\eta_i+1}{\kappa+1}g_2^{ij} - \frac{\eta_j+1}{\kappa+1}g_2^{ij} = \frac{-2}{\kappa+1}g_2^{ij}. \tag{5.9}$$

The $(1,1)$-tensor

$$R_i^j = \frac{d-1}{2}\delta_i^j + \nabla_{1i}E^j = \frac{\eta_i}{\kappa+1}\delta_i^j \tag{5.10}$$

is obviously nondegenerate. This completes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. It follows from Theorems 5.3 and 2.2 that Q' has a Frobenius structure of degree $\frac{\kappa-1}{\kappa+1}$ from the dispersionless limit of Drinfeld–Sokolov bihamiltonian structure. This Frobenius structure is polynomial since in the coordinates t^i the potential $\mathbb{F}$ is constructed from Eqs. (2.11) and we know from Proposition 4.5 that the matrix g_2^{ij} is polynomial. □

5.1. Conclusions and remarks

The results of the present work can be generalized to a certain class of distinguished nilpotent elements in simple Lie algebras. In particular, we notice that the existence of opposite Cartan subalgebras is the main reason behind the examples of algebraic Frobenius manifolds constructed in [4] which are associated to distinguished nilpotent elements in the Lie algebra of type F_4. In [4] we discussed how these examples support Dubrovin conjecture. Our goal is to develop a method to uniformize the construction of all algebraic Frobenius manifolds that can be obtained from distinguished nilpotent elements in simple Lie algebras by performing the generalized Drinfeld–Sokolov reduction. Similar treatment of the present work for algebraic Frobenius manifolds that can be obtained from subregular nilpotent elements in simple Lie algebras is now under preparation.

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