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Inverse spectral problems for differential pencils on the half-line with turning points ☆

V. Yurko

Department of Mathematics, Saratov University, Astrakhanskaya 83, 410026 Saratov, Russia

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Abstract

The inverse spectral problem of recovering pencils of second-order differential operators on the half-line with turning points is studied. We establish properties of the spectral characteristics, give a formulation of the inverse problem, prove a uniqueness theorem and provide a constructive procedure for the solution of the inverse problem.

Keywords: Differential equations; Turning points; Inverse spectral problems; Method of spectral mappings

1. Introduction

We consider the differential equation

$$y''(x) + \big(\rho^2 r(x) + i\rho q_1(x) + q_0(x)\big)y(x) = 0, \quad x \geqslant 0, \tag{1}$$

on the half-line with nonlinear dependence on the spectral parameter ρ. Let $a, \omega > 0$, and let

$$r(x) = \begin{cases} -\omega^2 & \text{for } x \in [0, a), \\ 1 & \text{for } x \geqslant a, \end{cases}$$

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E-mail address: yurkova@info.sgu.ru.

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i.e., the weight-function $r(x)$ changes the sign in an interior point, which is called the turning point. The functions $q_j(x)$ are complex-valued, $q_1(x)$ is absolutely continuous, and $(1+x)q_j^{(\nu)}(x) \in \mathcal{L}(0,\infty)$ for $0 \leqslant \nu \leqslant j \leqslant 1$.

Differential equations with nonlinear dependence on the spectral parameter and with turning points arise in various problems of mathematics as well as in applications (see [1–9] for details). In this paper we study the inverse problem for singular non-selfadjoint indefinite pencil (1). Inverse problems of spectral analysis consist in recovering operators from their spectral characteristics. For the classical Sturm–Liouville operator the inverse problem has been studied fairly completely (see [10–16] and the references therein). Some aspects of the inverse problem theory for differential pencils without turning points were studied in [17–23] and other works. In [24–29] the inverse problem was investigated for differential equations with turning points but with linear dependence on the spectral parameter.

Indefinite differential pencils with turning point produce essential qualitative modifications in the investigation of the inverse problem. To study the inverse problem in this paper we use the method of spectral mappings [30] connected with ideas of the contour integral method. In Section 2 we obtain properties of the spectral characteristics of boundary value problems for pencil (1). In Section 3 we give a formulation of the inverse problem and prove the uniqueness theorem for the solution of this inverse problem. In Section 4 we provide a constructive procedure for the solution of the inverse problem considered.

2. Properties of the spectral characteristics

We consider the boundary value problem L for Eq. (1) on the half-line $x>0$ with the boundary condition

$$U(y) := y'(0) + (\beta_1\rho + \beta_0)y(0) = 0, \tag{2}$$

where the coefficients β_1 and β_0 are complex numbers and $\beta_1 \neq \pm\omega$. The last condition excludes from the consideration Regge-type problems [31] which require a separate investigation (see [32]).

Let $\varphi(x,\rho)$ and $S(x,\rho)$ be solutions of Eq. (1) under the initial conditions $\varphi(0,\rho)=1$, $U(\varphi)=0$, $S(0,\rho)=0$, $S'(0,\rho)=1$. For each fixed $x \geqslant 0$, the functions $\varphi^{(m)}(x,\rho)$ and $S^{(m)}(x,\rho)$, $m=0,1$, are entire in ρ. Moreover,

$$\bigl\langle \varphi(x,\rho), S(x,\rho)\bigr\rangle \equiv 1, \tag{3}$$

where $\langle y,z\rangle := yz' - y'z$, since by virtue of Liouville's formula for the Wronskian [33] $\langle \varphi(x,\rho), S(x,\rho)\rangle$ does not depend on x.

Denote $\Pi_\pm := \{\rho\colon \pm\operatorname{Im}\rho > 0\}$, $\Pi_0 := \{\rho\colon \operatorname{Im}\rho = 0\}$. By the well-known method (see, for example, [1,34,35]) we get that for $x \geqslant a$, $\rho \in \Pi_\pm$, there exists a solution $e(x,\rho)$ of Eq. (1) (which is called the Jost-type solution) with the following properties:

1° For each fixed $x \geqslant a$, the functions $e^{(m)}(x,\rho)$, $m=0,1$, are holomorphic for $\rho \in \Pi_+$ and $\rho \in \Pi_-$ (i.e., they are piecewise holomorphic).

2° The functions $e^{(m)}(x,\rho)$, $m=0,1$, are continuous for $x \geqslant a$, $\rho \in \overline{\Pi_+}$ and $\rho \in \overline{\Pi_-}$ (we differ the sides of the cut Π_0). In other words, for real ρ, there exist the finite limits

$$e_{\pm}^{(m)}(x,\rho) = \lim_{z\to\rho,\, z\in\Pi_{\pm}} e^{(m)}(x,z).$$

Moreover, the functions $e^{(m)}(x,\rho)$, $m=0,1$ are continuously differentiable with respect to $\rho \in \overline{\Pi_+} \setminus \{0\}$ and $\rho \in \overline{\Pi_-} \setminus \{0\}$.

3° For $x \to \infty$, $\rho \in \overline{\Pi_{\pm}} \setminus \{0\}$, $m=0,1$,

$$e^{(m)}(x,\rho) = (\pm i\rho)^m \exp\big(\pm\big(i\rho x - Q(x)\big)\big)\big(1+o(1)\big), \tag{4}$$

where

$$Q(x) = \frac{1}{2}\int_0^x q_1(t)\,dt. \tag{5}$$

4° For $|\rho| \to \infty$, $\rho \in \overline{\Pi_{\pm}}$, $m=0,1$, uniformly in $x \geqslant a$,

$$e^{(m)}(x,\rho) = (\pm i\rho)^m \exp\big(\pm\big(i\rho x - Q(x)\big)\big)[1], \tag{6}$$

where $[1] := 1 + O(\rho^{-1})$.

We extend $e(x,\rho)$ to the segment $[0,a]$ as a solution of Eq. (1) which is smooth for $x \geqslant 0$, i.e.,

$$e^{(m)}(a-0,\rho) = e^{(m)}(a+0,\rho), \quad m=0,1. \tag{7}$$

Then the properties 1°–2° remain true for $x \geqslant 0$.

The Jost-type solution $e(x,\rho)$, $x \geqslant 0$ is a generalization of the classical Jost solution for the Sturm–Liouville equation (see [10–12]).

Denote

$$\Delta(\rho) := U\big(e(x,\rho)\big). \tag{8}$$

The function $\Delta(\rho)$ is called the characteristic function for the boundary value problem L. The function $\Delta(\rho)$ is holomorphic in Π_+ and Π_-, and for real ρ there exist the finite limits

$$\Delta_{\pm}(\rho) = \lim_{z\to\rho,\ z\in\Pi_{\pm}} \Delta(z).$$

Moreover, the function $\Delta(\rho)$ is continuously differentiable for $\rho \in \overline{\Pi_{\pm}} \setminus \{0\}$.

Lemma 1. *For $|\rho| \to \infty$, $\rho \in \overline{\Pi_{\pm}}$, the following asymptotical formula holds*:

$$\begin{aligned}\Delta(\rho) = \frac{\rho}{2}\exp\big(\pm\big(i\rho a - Q(a)\big)\big)\big((\beta_1-\omega)(1\mp i/\omega)\exp\big(\omega\rho a - iQ(a)/\omega\big)[1] \\ + (\beta_1+\omega)(1\pm i/\omega)\exp\big(-\omega\rho a + iQ(a)/\omega\big)[1]\big).\end{aligned} \tag{9}$$

Proof. Denote $\Pi_{\pm}^1 := \{\rho\colon \pm\operatorname{Re}\rho > 0\}$. Let $\{y_k(x,\rho)\}_{k=1,2}$, $x \in [0,a]$, $\rho \in \overline{\Pi_{\pm}^1}$, be the Birkhoff-type fundamental system of solutions of Eq. (1) on the interval $[0,a]$ with the asymptotics for $|\rho| \to \infty$, $m=0,1$,

$$y_k^{(m)}(x,\rho) = \big((-1)^k\omega\rho\big)^m \exp\big((-1)^k\big(\omega\rho x - iQ(x)/\omega\big)\big)[1] \tag{10}$$

(see [1,34,35]). Then

$$e(x,\rho)=b_1(\rho)y_1(x,\rho)+b_2(\rho)y_2(x,\rho),\quad x\in[0,a]. \tag{11}$$

Using (6), (7) and (10), we obtain for $\rho\in\overline{\Pi_\pm}$, $k=0,1$,

$$(-1)^k\exp\bigl(-\omega\rho a+iQ(a)/\omega\bigr)[1]b_1(\rho)+\exp\bigl(\omega\rho a-iQ(a)/\omega\bigr)[1]b_2(\rho)$$
$$=(\pm i/\omega)^k\exp\bigl(\pm\bigl(i\rho a-Q(a)\bigr)\bigr)[1].$$

Calculating $b_1(\rho)$ and $b_2(\rho)$ from this algebraic system and substituting the result into (11), we arrive at the following asymptotical formula for $|\rho|\to\infty$, $\rho\in\overline{\Pi_\pm}$, $m=0,1$, $x\in[0,a]$,

$$\begin{aligned}e^{(m)}(x,\rho)={}&\frac{(\omega\rho)^m}{2}\exp\bigl(\pm\bigl(i\rho a-Q(a)\bigr)\bigr)\\&\times\bigl((-1)^m(1\mp i/\omega)\exp\bigl(\omega\rho a-iQ(a)/\omega\bigr)\exp\bigl(-\omega\rho x+iQ(x)/\omega\bigr)[1]\\&+(1\pm i/\omega)\exp\bigl(-\omega\rho a+iQ(a)/\omega\bigr)\exp\bigl(\omega\rho x-iQ(x)/\omega\bigr)[1]\bigr).\end{aligned} \tag{12}$$

Together with (2) and (8) this yields (9). Lemma 1 is proved. □

Similarly one can calculate

$$\begin{aligned}e(0,\rho)={}&\frac{1}{2}\exp\bigl(\pm\bigl(i\rho a-Q(a)\bigr)\bigr)\bigl((1\mp i/\omega)\exp\bigl(\omega\rho a-iQ(a)/\omega\bigr)[1]\\&+(1\pm i/\omega)\exp\bigl(-\omega\rho a+iQ(a)/\omega\bigr)[1]\bigr),\end{aligned} \tag{13}$$

$$\begin{aligned}\dot\Delta(\rho)={}&\frac{\rho}{2}\bigl(\pm i\omega a^2\bigr)\exp\bigl(\pm\bigl(i\rho a-Q(a)\bigr)\bigr)\\&\times\bigl((\beta_1-\omega)(1\mp i/\omega)\exp\bigl(\omega\rho a-iQ(a)/\omega\bigr)[1]\\&-(\beta_1+\omega)(1\pm i/\omega)\exp\bigl(-\omega\rho a+iQ(a)/\omega\bigr)[1]\bigr)\end{aligned} \tag{14}$$

as $|\rho|\to\infty$, $\rho\in\overline{\Pi_\pm}$, where $\dot\Delta(\rho):=\frac{d}{d\rho}\Delta(\rho)$.

It follows from (9) that for sufficiently large $|\rho|$, the function $\Delta(\rho)$ has simple zeros of the form

$$\rho_k=\frac{1}{\omega a}(k\pi i+iQ/\omega+\kappa\pm\kappa_1)+O\left(\frac{1}{k}\right), \tag{15}$$

where $Q:=Q(a)$, and

$$\kappa=\frac{1}{2}\ln\frac{\beta_1+\omega}{\beta_1-\omega},\qquad \kappa_1=\frac{1}{2}\ln\frac{i+\omega}{i-\omega}. \tag{16}$$

Here $\ln z:=\ln|z|+i\arg z$, $\arg z\in[0,2\pi)$. Denote

$$\Lambda'_\pm=\bigl\{\rho\in\Pi_\pm:\ \Delta(\rho)=0\bigr\},\qquad \Lambda'=\Lambda'_+\cup\Lambda'_-,$$
$$\Lambda''_\pm=\bigl\{\rho\in\mathbf{R}:\ \Delta_\pm(\rho)=0\bigr\},\qquad \Lambda''=\Lambda''_+\cup\Lambda''_-,$$
$$\Lambda_\pm=\Lambda'_\pm\cup\Lambda''_\pm,\qquad \Lambda=\Lambda_+\cup\Lambda_-.$$

Obviously, $\Lambda=\Lambda'\cup\Lambda''$, Λ' is a countable unbounded set, and Λ'' is a bounded set.

We put

$$\Phi(x,\rho)=\frac{e(x,\rho)}{\Delta(\rho)}. \tag{17}$$

The function $\Phi(x,\rho)$ is a solution of Eq. (1), and on account of (2), (4) and (8), also the conditions $U(\Phi)=1$, $\Phi(x,\rho)=O(\exp(\pm i\rho x))$, $x\to\infty$, $\rho\in\Pi_\pm$ (while $\Delta(\rho)\neq 0$). In particular, $\lim_{x\to\infty}\Phi(x,\rho)=0$. Denote

$$M(\rho):=\Phi(0,\rho). \tag{18}$$

We will call $M(\rho)$ the Weyl-type function for L, since it is a generalization of the concept of the Weyl function for the classical Sturm–Liouville operator (see [36]). It follows from (17) and (18) that

$$M(\rho)=\frac{e(0,\rho)}{\Delta(\rho)}. \tag{19}$$

Using the conditions at the point $x=0$ we get

$$\Phi(x,\rho)=S(x,\rho)+M(\rho)\varphi(x,\rho). \tag{20}$$

It follows from (3), (17) and (20) that

$$\bigl\langle\varphi(x,\rho),\Phi(x,\rho)\bigr\rangle\equiv 1, \tag{21}$$

$$\bigl\langle\varphi(x,\rho),e(x,\rho)\bigr\rangle\equiv \Delta(\rho). \tag{22}$$

Theorem 1. *The Weyl-type function $M(\rho)$ is holomorphic in $\Pi_\pm\setminus\Lambda'_\pm$ and continuously differentiable in $\overline{\Pi_\pm}\setminus\Lambda_\pm$. The set of singularities of $M(\rho)$ (as an analytic function) coincides with the set $\mathbf{R}\cup\Lambda$. For $|\rho|\to\infty$, $\rho\in\Pi^1_\pm$,*

$$M(\rho)=\frac{1}{\rho(\beta_1\mp\omega)}[1]. \tag{23}$$

Theorem 1 follows from (19) and from properties of the functions $\Delta(\rho)$ and $e(0,\rho)$.

Definition 1. The set of singularities of the Weyl-type function $M(\rho)$ is called the spectrum of L (and is denoted by $\sigma(L)$). The values of the parameter ρ, for which Eq. (1) has nontrivial solutions satisfying (2) and the condition $y(\infty)=0$ (i.e., $\lim_{x\to\infty}y(x)=0$), are called eigenvalues of L, and the corresponding solutions are called eigenfunctions of L.

Thus, $\sigma(L)=\mathbf{R}\cup\Lambda$. The set Λ is the discrete spectrum, and $\mathbf{R}$ is the continuous spectrum. Note that $\mathbf{C}\setminus\sigma(L)$ is the resolvent set of L.

Theorem 2. *L has no eigenvalues for real $\rho\neq 0$.*

Proof. For real $\rho\neq 0$, the functions $e_+(x,\rho)$ and $e_-(x,\rho)$ are solutions of Eq. (1), and in view of (4),

$$e_\pm(x,\rho)\sim\exp\bigl(\pm\bigl(i\rho x-Q(x)\bigr)\bigr),\quad \text{as } x\to\infty. \tag{24}$$

Using (24) and Liouville's formula for the Wronskian we calculate

$$\langle e_+(x,\rho), e_-(x,\rho)\rangle = -2i\rho. \tag{25}$$

Suppose that a real number $\rho_0 \neq 0$ is an eigenvalue, and let $y_0(x)$ be a corresponding eigenfunction. By virtue of (25), the functions $\{e_+(x,\rho_0), e_-(x,\rho_0)\}$ form a fundamental system of solutions for Eq. (1), and consequently $y_0(x) = C_1 e_+(x,\rho_0) + C_2 e_-(x,\rho_0)$. As $x \to \infty$, $y_0(x) \sim 0$, $e_\pm(x,\rho_0) \sim \exp(\pm(i\rho_0 x - Q(x))$. But this is possible only if $C_1 = C_2 = 0$, i.e., $y_0 \equiv 0$. Theorem 2 is proved. □

Theorem 3. *The countable set Λ' coincides with the set $\{\rho_k\}$ of all non-zero eigenvalues of L. For $\rho_k \in \Lambda'$, the functions $e(x,\rho_k)$ and $\varphi(x,\rho_k)$ are eigenfunctions, and*

$$e(x,\rho_k) = \gamma_k \varphi(x,\rho_k), \quad \gamma_k \neq 0. \tag{26}$$

For the eigenvalues $\{\rho_k\}$ the asymptotical formula (15) *holds.*

Proof. Let $\rho_k \in \Lambda'$. Then $U(e(x,\rho_k)) = \Delta(\rho_k) = 0$ and, by virtue of (4), $\lim_{x\to\infty} e(x,\rho_k) = 0$. Thus, $e(x,\rho_k)$ is an eigenfunction, and ρ_k is an eigenvalue. Moreover, it follows from (22) that $\langle \varphi(x,\rho_k), e(x,\rho_k)\rangle = 0$, and consequently (26) is valid.

Conversely, let $\rho_k \in \Pi_+ \cup \Pi_-$ be an eigenvalue, and let $y_k(x)$ be a corresponding eigenfunction. Clearly, $y_k(0) \neq 0$, $U(y_k(x)) = 0$. Then $y_k(x) = \beta_k^0 \varphi(x,\rho_k)$. Since $\lim_{x\to\infty} y_k(x) = 0$, one gets $y_k(x) = \beta_k^1 e(x,\rho_k)$. This yields (26). Consequently, $\Delta(\rho_k) = U(e(x,\rho_k)) = 0$, and $\varphi(x,\rho_k)$ and $e(x,\rho_k)$ are eigenfunctions. □

Remark 1. We note that $(\Lambda''_+ \setminus \{0\}) \cap (\Lambda''_- \setminus \{0\}) = \emptyset$, i.e., for real $\rho \neq 0$ the functions $\Delta_+(\rho)$ and $\Delta_-(\rho)$ are not equal to zero simultaneously. Indeed, it follows from (8) and (25) that for real $\rho \neq 0$, one has

$$\begin{aligned} 0 \neq \langle e_+(x,\rho), e_-(x,\rho)\rangle &= e_+(0,\rho)e'_-(0,\rho) - e'_+(0,\rho)e_-(0,\rho) \\ &= e_+(0,\rho)\Delta_-(\rho) - e_-(0,\rho)\Delta_+(\rho). \end{aligned}$$

For brevity, we confine ourselves to the case of a simple spectrum in the following sense.

Definition 2. We shall say that L has simple spectrum if all zeros of $\Delta(\rho)$ are simple, have no finite limit points, and $\rho M(\rho) = m_\pm + o(1)$ as $\rho \to 0$, $\rho \in \overline{\Pi_\pm}$, $m_\pm \in \mathbf{C}$.

Let L have simple spectrum. Then Λ'' is a finite set, and $\Lambda = \Lambda' \cup \Lambda''$ is a countable set:

$$\Lambda = \{\rho_k\}_{k\in\omega}.$$

Here $\omega = \omega_0 \cup \omega^0$, where ω_0 is a finite set, $\omega^0 = \{k \in \mathbf{Z}\colon\ |k| > k_0\}$ for some k_0, and the numbers ρ_k have the form (15) for $k \in \omega^0$. Each element of Λ' is an eigenvalue of L. According to Theorem 2, the points of $\Lambda'' \setminus \{0\}$ are not eigenvalues of L, they are called *spectral singularities* of L. Thus, the discrete spectrum Λ consists of two parts: the set of eigenvalues, and the set of spectral singularities.

Denote

$$M_k = \frac{e(0,\rho_k)}{\dot{\Delta}(\rho_k)}, \quad \rho_k \in \Lambda \setminus \{0\}. \tag{27}$$

Obviously, $M_k \neq 0$, and

$$\lim_{\rho\to\rho_k,\, \rho\in\overline{\Pi_\pm}} (\rho-\rho_k)M(\rho) = M_k, \quad \rho_k \in \Lambda_\pm \setminus \{0\}. \tag{28}$$

Let

$$\alpha_k := \begin{cases} M_k & \text{for } \rho_k \in \Lambda', \\ \frac{1}{2}M_k & \text{for } \rho_k \in \Lambda'' \setminus \{0\}, \end{cases} \tag{29}$$

$$V(\rho) := \frac{1}{2\pi i}\big(M^-(\rho) - M^+(\rho)\big), \quad \rho \in \Pi := \mathbf{R} \setminus \Lambda'', \tag{30}$$

where

$$M^\pm(\rho) := \lim_{z\to 0,\ z\in\Pi_\pm} M(\rho \pm iz) = \frac{e_\pm(0,\rho)}{\Delta_\pm(\rho)}.$$

Put $\alpha_0 = (m_+ + m_-)/(\pi i)$ for $\rho_0 = 0$. Using (13)–(15), (27) and (29) we calculate

$$\alpha_k = \mp\frac{\omega}{k\pi a(\beta_1^2 - \omega^2)} + O\left(\frac{1}{k^2}\right), \quad k \to \pm\infty. \tag{31}$$

By virtue of (19) and (30),

$$V(\rho) = \frac{1}{2\pi i}\left(\frac{e_-(0,\rho)}{\Delta_-(\rho)} - \frac{e_+(0,\rho)}{\Delta_+(\rho)}\right), \quad \rho \in \Pi.$$

Taking (8) and (25) into account we infer

$$V(\rho) = \frac{\rho}{\pi}\cdot\frac{1}{\Delta_-(\rho)\Delta_+(\rho)}, \quad \rho \in \Pi. \tag{32}$$

Definition 3. The data $S := (\{V(\rho)\}_{\rho\in\Pi}, \{\rho_k, \alpha_k\}_{k\in\omega})$ are called the *spectral data* of L.

The spectral data describe the behavior of the spectrum; $\{V(\rho)\}$ is connected with the continuous spectrum, and $\{\rho_k, \alpha_k\}_{k\in\omega}$ describe the discrete spectrum. Using the results obtained above we arrive at the following statement.

Theorem 4. *The spectral data $S := (\{V(\rho)\}_{\rho\in\Pi}, \{\rho_k, \alpha_k\}_{k\in\omega})$ have the following properties*:

- (i_1) *$\rho_k \neq \rho_s$ for $k \neq s$; moreover, $(\Lambda''_+ \setminus \{0\}) \cap (\Lambda''_- \setminus \{0\}) = \emptyset$;*
- (i_2) *as $k \to \pm\infty$, the asymptotical formulas* (15) *and* (31) *are valid*;
- (i_3) *the function $V(\rho)$ is continuously differentiable for $\rho \in \Pi$, and for $\rho_k \in \Lambda''$ there exist finite limits $V_k := \lim_{\rho\to\rho_k}(\rho-\rho_k)V(\rho)$; moreover,*

$$V_k = \mp\frac{\alpha_k}{\pi i} \quad \textit{for } \rho_k \in \Lambda''_\pm \setminus \{0\}; \tag{33}$$

(i_4) *as* $\rho \to \pm\infty$,

$$V(\rho) = \frac{4[1]}{\pi\rho(\beta_1 \mp \omega)^2(1 + 1/\omega)^2} \exp \mp(2\omega\rho a - 2iQ/\omega). \tag{34}$$

The asymptotics (34) follows from (32) and (9). Notice that relation (33) gives us a connection between $V(\rho)$, which describes the continuous spectrum, and $\{\rho_k, \alpha_k\}$, which describe the discrete spectrum.

3. Formulation of the inverse problem. The uniqueness theorem

Let us go on to studying the inverse problem for the boundary value problem L. The inverse problem is formulated as follows.

Inverse Problem 1. Given the spectral data S, construct the coefficients of the pencil (1)–(2).

In this section we prove the uniqueness theorem for the solution of this inverse problem. For this purpose together with L we will consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{r}(x)$, $\tilde{p}(x)$, $\tilde{q}(x)$, $\tilde{\beta}_1$, $\tilde{\beta}_0$. We agree that if a certain symbol α denotes an object related to L, then $\tilde{\alpha}$ will denote the analogous object related to $\tilde{L}$, and $\hat{\alpha} = \alpha - \tilde{\alpha}$.

Theorem 5. *If* $S = \tilde{S}$, *then* $r(x) = \tilde{r}(x)$, $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ *for* $x > 0$, $\beta_1 = \tilde{\beta}_1$ *and* $\beta_0 = \tilde{\beta}_0$. *Thus, the specification of the spectral data uniquely determines the coefficients of the pencil* (1)–(2).

Proof. Fix $\delta > 0$. Let $\kappa_\delta^0(\rho_k) := \{\rho\colon\ \rho \in [\rho_k - \delta, \rho_k + \delta]\}$, $\rho_k \in \Lambda''$. Denote by $\xi_\delta := \mathbf{R} \setminus (\bigcup_{\rho_k \in \Lambda''} \kappa_\delta^0(\rho_k))$ the real axis without δ-neighbourhoods of the points of Λ''.

Let us show that the specification of the spectral data S uniquely determines the Weyl-type function $M(\rho)$ via the formula

$$M(\rho) = \sum_{\rho_k \in \Lambda} \frac{\alpha_k}{\rho - \rho_k} + \int_{-\infty}^{\infty} \frac{V(\mu)}{\rho - \mu}\, d\mu, \quad \rho \notin \sigma(L), \tag{35}$$

where the integral is understood in the principal value sense: $\int_{-\infty}^{\infty} := \lim_{\delta \to 0} \int_{\xi_\delta}$.

Indeed, fix $\delta > 0$ and denote $G_\delta := \{\rho \in \mathbf{C}\colon\ |\rho - \rho_k| \geqslant \delta,\ \rho_k \in \Lambda\}$. It follows from (9), (13) and (19) that

$$\bigl|\Delta(\rho)\bigr| \geqslant C|\rho| \exp\bigl(|\sigma|\omega a\bigr) \exp\bigl(-|\tau|a\bigr), \quad \bigl|M(\rho)\bigr| \leqslant C|\rho|^{-1}, \quad \rho \in G_\delta, \tag{36}$$

where $\sigma := \operatorname{Re}\rho$, $\tau := \operatorname{Im}\rho$, i.e., $\rho = \sigma + i\tau$. According to (34), the integral in (35) converges absolutely at infinity. Moreover, in view of (15) and (31), the series in (35) converges absolutely too.

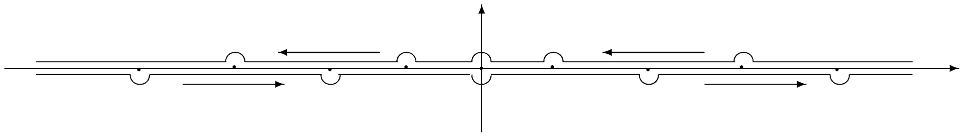

Fig. 1.

Take positive numbers $R_N = \frac{N\pi}{\omega a} + \chi$ such that the circles $\theta_N := \{\rho\colon |\rho| = R_N\}$ lie in G_δ for sufficiently small $\delta > 0$. Fix $\rho \notin \sigma(L)$, and take $\delta > 0$ and N such that $\rho \in G_\delta \cap \operatorname{int}\theta_N$. Consider the contour integral

$$I_N(\rho) := \frac{1}{2\pi i}\int\limits_{\theta_N} \frac{M(\mu)}{\rho - \mu}\,d\mu \tag{37}$$

with counterclockwise circuit. It follows from (36) that

$$\lim_{N\to\infty} I_N(\rho) = 0. \tag{38}$$

For each $\rho_k \in \Lambda''_\pm$ we take a semicircle $\kappa_\delta(\rho_k) := \{\rho\colon |\rho - \rho_k| = \delta, \rho \in \Pi_\pm\}$. Let Π_δ be the two-sided cut Π_0 without the δ-neighbourhoods of the points of Λ'', and let $\Gamma_\delta := \Pi_\delta \cup (\bigcup_{\rho_k\in\Lambda''}\kappa_\delta(\rho_k))$ be the contour with counterclockwise circuit (see Fig. 1). Denote $\Gamma_{\delta,N} := \Gamma_\delta \cap \theta_{N,0}$, where $\theta_{N,0} = \{\rho\colon |\rho| \leqslant R_N\}$. Contracting the contour in (37) to the real axis through the poles of Λ' and using (19), (27), (29) and the residue theorem, we get

$$M(\rho) = \sum_{\substack{\rho_k\in\Lambda' \\ |\rho_k|<R_N}} \frac{\alpha_k}{\rho - \rho_k} + \frac{1}{2\pi i}\int\limits_{\Gamma_{\delta,N}} \frac{M(\mu)}{\rho - \mu}\,d\mu - I_N(\rho).$$

By virtue of (38) this yields as $N \to \infty$:

$$M(\rho) = \sum_{\rho_k\in\Lambda'} \frac{\alpha_k}{\rho - \rho_k} + \frac{1}{2\pi i}\int\limits_{\Gamma_\delta} \frac{M(\mu)}{\rho - \mu}\,d\mu. \tag{39}$$

Taking (28)–(30) into account we calculate

$$\lim_{\delta\to 0}\sum_{\rho_k\in\Lambda''} \frac{1}{2\pi i}\int\limits_{\kappa_\delta(\rho_k)} \frac{M(\mu)}{\rho - \mu}\,d\mu = \sum_{\rho_k\in\Lambda''} \frac{\alpha_k}{\rho - \rho_k},$$

$$\frac{1}{2\pi i}\int\limits_{\Pi_\delta} \frac{M(\mu)}{\rho - \mu}\,d\mu = \int\limits_{\xi_\delta} \frac{V(\mu)}{\rho - \mu}\,d\mu.$$

Therefore, from (39) as $\delta \to 0$ we arrive at (35).

Furthermore, it follows from (6) and (12) that for $\rho \in \overline{\Pi_\pm}$, $m = 0, 1$,

$$\left.\begin{aligned} &\bigl|e^{(m)}(x,\rho)\bigr| \leqslant C|\rho|^m \exp\bigl(|\sigma|\omega(a - x)\bigr)\exp\bigl(-|\tau|a\bigr) \quad \text{for } x \leqslant a,\\ &\bigl|e^{(m)}(x,\rho)\bigr| \leqslant C|\rho|^m \exp\bigl(-|\tau|x\bigr) \quad \text{for } x \geqslant a. \end{aligned}\right\} \tag{40}$$

Using (17), (36) and (40) we conclude that for $\rho \in G_\delta$, $m = 0, 1$,

$$\left.\begin{aligned} &\bigl|\Phi^{(m)}(x,\rho)\bigr| \leqslant C|\rho|^{m-1} \exp\bigl(-|\sigma|\omega x\bigr) \quad \text{for } x \leqslant a,\\ &\bigl|\Phi^{(m)}(x,\rho)\bigr| \leqslant C|\rho|^{m-1} \exp\bigl(-|\sigma|\omega a\bigr)\exp\bigl(-|\tau|(x - a)\bigr) \quad \text{for } x \geqslant a. \end{aligned}\right\} \tag{41}$$

Now we need to study the asymptotic behavior of the solution $\varphi(x,\rho)$ as $|\rho|\to\infty$. Using the Birkhoff-type fundamental system of solutions $\{y_k(x,\rho)\}_{k=1,2}$ of Eq. (1) on the interval $[0,a]$, one has

$$\varphi(x,\rho)=a_1(\rho)y_1(x,\rho)+a_2(\rho)y_2(x,\rho),\quad x\in[0,a]. \tag{42}$$

Let $\{Y_k(x,\rho)\}_{k=1,2}$, $x\geqslant a$, $\rho\in\overline{\Pi_\pm}$, be the Birkhoff-type fundamental system of solutions of Eq. (1) on the interval $[a,\infty)$, with the asymptotics for $|\rho|\to\infty$, $m=0,1$,

$$Y_k^{(m)}(x,\rho)=\big((-1)^{k-1}i\rho\big)^m\exp\big((-1)^{k-1}\big(i\rho x-Q(x)\big)\big)[1] \tag{43}$$

(see [1,34,35]). Then

$$\varphi(x,\rho)=A_1(\rho)Y_1(x,\rho)+A_2(\rho)Y_2(x,\rho),\quad x\geqslant a. \tag{44}$$

Taking (10) and the initial conditions $\varphi(0,\rho)=1$, $\varphi'(0,\rho)=\beta_1\rho+\beta_0$ into account, we calculate

$$a_1(\rho)[1]+a_2(\rho)[1]=1,\qquad (\beta_1-\omega)a_1(\rho)[1]+(\beta_1+\omega)a_2(\rho)[1]=0,$$

and consequently,

$$a_1(\rho)=\frac{\omega+\beta_1}{2\omega}[1],\quad a_2(\rho)=\frac{\omega-\beta_1}{2\omega}[1],\quad |\rho|\to\infty. \tag{45}$$

Substituting (10) and (45) into (42) we obtain the asymptotical formula for $\varphi^{(m)}(x,\rho)$, $m=0,1$ as $|\rho|\to\infty$, uniformly in $x\in[0,a]$:

$$\begin{aligned}\varphi^{(m)}(x,\rho)=\frac{1}{2\omega}\big((-\omega\rho)^m(\omega+\beta_1)\exp\big(-\omega\rho x+iQ(x)/\omega\big)[1]\\ +(\omega\rho)^m(\omega-\beta_1)\exp\big(\omega\rho x-iQ(x)/\omega\big)[1]\big).\end{aligned} \tag{46}$$

In order to calculate the coefficients $A_k(\rho)$, $k=1,2$, we use (43), (44), (46) and the smooth conditions $\varphi^{(m)}(a-0,\rho)=\varphi^{(m)}(a+0,\rho)$, $m=0,1$. This yields for $|\rho|\to\infty$:

$$\begin{aligned}&A_1(\rho)\exp(i\rho a-Q)[1]+(-1)^mA_2(\rho)\exp(-i\rho a+Q)[1]=(i\rho)^{-m}\varphi^{(m)}(a,\rho),\\ &\quad m=0,1,\end{aligned}$$

where the asymptotics for $\varphi^{(m)}(a,\rho)$ is taken from (46). Calculating $A_k(\rho)$ from this algebraic system and substituting the result and (43) into (44) we get for $x\geqslant a$, $|\rho|\to\infty$:

$$\begin{aligned}\varphi^{(m)}(x,\rho)={}&\frac14\exp\big(i\rho(x-a)-Q_a(x)\big)\big((\omega+\beta_1)(1/\omega+i)\exp(-\omega\rho a+iQ/\omega)[1]\\ &+(\omega-\beta_1)(1/\omega-i)\exp(\omega\rho a-iQ/\omega)[1]\big)\\ &+\frac14\exp\big(-i\rho(x-a)+Q_a(x)\big)\\ &\times\big((\omega+\beta_1)(1/\omega-i)\exp(-\omega\rho a+iQ/\omega)[1]\\ &+(\omega-\beta_1)(1/\omega+i)\exp(\omega\rho a-iQ/\omega)[1]\big),\\ Q_a(x):={}&\int_a^x q_1(t)\,dt.\end{aligned} \tag{47}$$

It follows from (46) and (47) that

$$\left.\begin{aligned} &\left|\varphi^{(m)}(x,\rho)\right| \leqslant C|\rho|^m \exp\bigl(|\sigma|\omega x\bigr) \quad \text{for } x \leqslant a,\\ &\left|\varphi^{(m)}(x,\rho)\right| \leqslant C|\rho|^m \exp\bigl(|\sigma|\omega a\bigr)\exp\bigl(|\tau|(x-a)\bigr) \quad \text{for } x \geqslant a. \end{aligned}\right\} \tag{48}$$

By the assumption of Theorem 5, $S = \tilde{S}$. Hence, in view of (35),

$$M(\rho) \equiv \tilde{M}(\rho). \tag{49}$$

Using (15), (23), (31) and (49) we infer

$$\beta_1 = \tilde{\beta}_1, \qquad \omega = \tilde{\omega}, \qquad a = \tilde{a}, \qquad Q = \tilde{Q}. \tag{50}$$

Let us now define the matrix $\mathcal{P}(x,\rho) = [\mathcal{P}_{jk}(x,\rho)]_{j,k=1,2}$ by the formula

$$\mathcal{P}(x,\rho)\begin{bmatrix} \tilde{\varphi}(x,\rho) & \tilde{\Phi}(x,\rho)\\ \tilde{\varphi}'(x,\rho) & \tilde{\Phi}'(x,\rho) \end{bmatrix} = \begin{bmatrix} \varphi(x,\rho) & \Phi(x,\rho)\\ \varphi'(x,\rho) & \Phi'(x,\rho) \end{bmatrix}. \tag{51}$$

By virtue of (21) this yields

$$\left.\begin{aligned} &\mathcal{P}_{j1}(x,\rho) = \varphi^{(j-1)}(x,\rho)\tilde{\Phi}'(x,\rho) - \Phi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho),\\ &\mathcal{P}_{j2}(x,\rho) = \Phi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{\Phi}(x,\rho), \end{aligned}\right\} \tag{52}$$

$$\left.\begin{aligned} &\varphi(x,\rho) = \mathcal{P}_{11}(x,\rho)\tilde{\varphi}(x,\rho) + \mathcal{P}_{12}(x,\rho)\tilde{\varphi}'(x,\rho),\\ &\Phi(x,\rho) = \mathcal{P}_{21}(x,\rho)\tilde{\Phi}(x,\rho) + \mathcal{P}_{22}(x,\rho)\tilde{\Phi}'(x,\rho). \end{aligned}\right\} \tag{53}$$

It follows from (41), (48) and (52) that for $x \geqslant 0$, $\rho \in G_\delta$,

$$\left|\mathcal{P}_{11}(x,\rho)\right| \leqslant C, \qquad \left|\mathcal{P}_{12}(x,\rho)\right| \leqslant C|\rho|^{-1}. \tag{54}$$

Using (20) and (52) we calculate

$$\begin{aligned} \mathcal{P}_{j1}(x,\rho) &= \varphi^{(j-1)}(x,\rho)\tilde{S}'(x,\rho) - S^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho)\\ &\quad + \bigl(\tilde{M}(\rho) - M(\rho)\bigr)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}'(x,\rho),\\ \mathcal{P}_{j2}(x,\rho) &= S^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho) - \varphi^{(j-1)}(x,\rho)\tilde{S}(x,\rho)\\ &\quad + \bigl(M(\rho) - \tilde{M}(\rho)\bigr)\varphi^{(j-1)}(x,\rho)\tilde{\varphi}(x,\rho). \end{aligned}$$

Taking (49) into account we conclude that the functions $\mathcal{P}_{jk}(x,\rho)$ are entire in ρ for each fixed $x \geqslant 0$. Together with (54) this yields $\mathcal{P}_{12}(x,\rho) \equiv 0$, $\mathcal{P}_{11}(x,\rho) \equiv \mathcal{P}_1(x)$, i.e., the function $\mathcal{P}_{11}$ does not depend on ρ. By virtue of (53) we have for all x and ρ:

$$\mathcal{P}_1(x)\tilde{\varphi}(x,\rho) \equiv \varphi(x,\rho), \qquad \mathcal{P}_1(x)\tilde{\Phi}(x,\rho) \equiv \Phi(x,\rho). \tag{55}$$

Let $x \in [0,a]$. Using (9), (12), (17), (46) and (50) we get as $|\rho| \to \infty$, $\arg\rho \in (0,\pi/2)$:

$$\begin{aligned} &\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = \exp\bigl(-i\bigl(Q(x) - \tilde{Q}(x)\bigr)/\omega\bigr)[1],\\ &\frac{\Phi(x,\rho)}{\tilde{\Phi}(x,\rho)} = \exp\bigl(i\bigl(Q(x) - \tilde{Q}(x)\bigr)/\omega\bigr)[1]. \end{aligned} \tag{56}$$

Since $\mathcal{P}_1(x)$ does not depend on ρ, it follows from (55) and (56) that

$$\mathcal{P}_1(x) \equiv \exp\bigl(-i\bigl(Q(x) - \tilde{Q}(x)\bigr)/\omega\bigr), \qquad \mathcal{P}_1(x) \equiv \exp\bigl(i\bigl(Q(x) + \tilde{Q}(x)\bigr)/\omega\bigr),$$

and consequently, $Q(x) \equiv \tilde{Q}(x)$, $\mathcal{P}_1(x) \equiv 1$ for $x \in [0, a]$.

Let $x \geqslant a$. Using (6), (9), (17), (47) and (50) we have as $|\rho| \to \infty$, $\arg\rho \in (0, \pi/2)$:

$$\frac{\varphi(x,\rho)}{\tilde{\varphi}(x,\rho)} = \exp\bigl(\hat{Q}_a(x) - i\hat{Q}\bigr)[1], \qquad \frac{\Phi(x,\rho)}{\tilde{\Phi}(x,\rho)} = \exp\bigl(-\bigl(\hat{Q}_a(x) - i\hat{Q}\bigr)\bigr)[1]. \tag{57}$$

Since $\mathcal{P}_1(x)$ does not depend on ρ, and $Q = \tilde{Q}$, it follows from (55) and (57) that

$$\mathcal{P}_1(x) \equiv \exp\bigl(\hat{Q}_a(x)\bigr), \qquad \mathcal{P}_1(x) \equiv \exp\bigl(-\hat{Q}_a(x)\bigr),$$

and consequently, $Q_a(x) \equiv \tilde{Q}_a(x)$, $\mathcal{P}_1(x) \equiv 1$ for $x \geqslant a$. Thus, $\mathcal{P}_1(x) \equiv 1$ and $q_1(x) \equiv \tilde{q}_1(x)$ for all $x \geqslant 0$. According to (55) this yields $\tilde{\varphi}(x,\rho) \equiv \varphi(x,\rho)$, $\tilde{\Phi}(x,\rho) \equiv \Phi(x,\rho)$. Hence, $q_0(x) = \tilde{q}_0(x)$ a.e. on $(0,\infty)$, and $\beta_0 = \tilde{\beta}_0$. Theorem 5 is proved. □

Corrollary 1. *If $M(\rho) \equiv \tilde{M}(\rho)$, then $L = \tilde{L}$.*

It follows from the proof of Theorem 5 that the last assertion is also valid for pencil (1)–(2) with arbitrary behavior of the spectrum.

4. Solution of the inverse problem

In this section we give a constructive procedure for the solution of the inverse problem. The central role here is played by the so-called main equation of the inverse problem which connects spectral characteristics with the corresponding solutions of the differential equation. We give a derivation of the main equation which is a linear equation in a suitable Banach space. Moreover, we prove the unique solvability of the main equation. Using the solution of the main equation we provide an algorithm for the solution of the inverse problem considered.

Let the spectral data S of the boundary value problem L be given. Our goal is to calculate the coefficients a, ω, β_1, β_0, $q_1(x)$ and $q_0(x)$.

First, using (23) and (35) one can find ω and β_1 by the formula

$$\beta_1 \mp \omega = \lim \rho M(\rho), \quad |\rho| \to \infty, \rho \in \Pi^1_{\pm}. \tag{58}$$

Then, taking (15) into account we calculate a via

$$a = \frac{1}{\omega} \lim_{k\to\pm\infty} \frac{k\pi i}{\rho_k}, \tag{59}$$

and construct κ and κ_1 by (16). Using (15) again we find $Q := Q(a)$:

$$Q = -i\omega \lim_{k\to\infty} (\omega a \rho_k - k\pi i - \kappa - \kappa_1). \tag{60}$$

Take a boundary value problem $\tilde{L}$ such that

$$\tilde{a} = a, \qquad \tilde{\omega} = \omega, \qquad \tilde{\beta}_1 = \beta_1, \qquad \tilde{Q} = Q, \tag{61}$$

and $\tilde{L}$ is arbitrary in the rest. Let $\tilde{S} := (\{\tilde{V}(\rho)\}, \{\tilde{\rho}_k, \tilde{\alpha}_k\})$ be the spectral data of $\tilde{L}$. Denote $\rho_{k0} = \rho_k$, $\rho_{k1} = \tilde{\rho}_k$, $\alpha_{k0} = \alpha_k$, $\alpha_{k1} = \tilde{\alpha}_k$, $\varphi_{kj}(x) = \varphi(x, \rho_{kj})$, $\varphi_\rho(x) = \varphi(x,\rho)$,

$$D(x,\rho,\mu) = \frac{\langle \varphi_\rho(x), \varphi_\mu(x)\rangle}{\rho - \mu}. \tag{62}$$

Since

$$\frac{d}{dx}\langle\varphi_\rho(x),\varphi_\mu(x)\rangle=(\rho-\mu)\big(\rho+\mu+iq_1(x)\big)\varphi_\rho(x)\varphi_\mu(x),$$
$$\langle\varphi_\rho(x),\varphi_\mu(x)\rangle\big|_{x=0}=\beta_1(\rho-\mu),$$

it follows that

$$D(x,\rho,\mu)=\beta_1+\int_0^x\big(\rho+\mu+iq_1(s)\big)\varphi_\rho(s)\varphi_\mu(s)\,ds. \tag{63}$$

Let for definiteness $x\leqslant a$. Taking (15), (48), (62) and (63) into account we obtain

$$\left.\begin{aligned}&\big|\varphi^{(m)}(x,\rho)\big|\leqslant C|\rho|^m\exp\big(|\sigma|\omega x\big),\qquad \big|\varphi_{kj}^{(m)}(x)\big|\leqslant C|k|^m,\quad m=0,1,\\ &\big|D(x,\rho,\mu)\big|\leqslant C\frac{|\rho|+|\mu|+1}{|\rho-\mu|+1}\exp\big(|\sigma|\omega x\big)\exp\big(|\theta|\omega x\big),\end{aligned}\right\} \tag{64}$$

where $\sigma:=\operatorname{Re}\rho$, $\theta:=\operatorname{Re}\mu$. Similarly one can get

$$\left|\frac{\partial}{\partial\rho}\varphi(x,\rho)\right|\leqslant C\exp\big(|\sigma|\omega x\big), \tag{65}$$

$$\left|\frac{\partial^{s+p}}{\partial\rho^s\partial\mu^p}D(x,\rho,\mu)\right|\leqslant C\frac{|\rho|+|\mu|+1}{|\rho-\mu|+1}\exp\big(|\sigma|\omega x\big)\exp\big(|\theta|\omega x\big),\quad s,p=0,1. \tag{66}$$

Denote

$$P_{\rho,\mu}(x)=D(x,\rho,\mu)\hat V(\mu),\qquad P_{\rho,kj}(x)=D(x,\rho,\rho_{kj})\alpha_{kj},$$
$$P_{ni,\mu}(x)=D(x,\rho_{ni},\mu)\hat V(\mu),\qquad P_{ni,kj}(x)=D(x,\rho_{ni},\rho_{kj})\alpha_{kj}.$$

We define $\tilde\varphi_{kj}$, $\tilde D$, $\tilde P_{\rho,\mu}$, $\tilde P_{\rho,kj}$, $\tilde P_{ni,\mu}$, $\tilde P_{ni,kj}$ by the same formulas but with $\tilde\varphi$, $\tilde D$ instead of φ, D. If $\omega_0\neq\tilde\omega_0$, then we define the corresponding functions identically zero (for example, if $n\in\omega_0\setminus\tilde\omega_0$, then $\varphi_{n1}=P_{n1,\mu}=P_{n1,kj}=P_{\rho,n1}=P_{kj,n1}=0$, and the same for functions with tilde). Let $\omega':=\omega\cup\tilde\omega$, and let ω_1 be a set of indices $v=(k,j)$, where $k\in\omega'$, $j=0,1$. Denote by

$$\xi'_\delta:=\mathbf{R}\Big\backslash\bigg(\Big(\bigcup_{\rho_k\in\Lambda''}\kappa_\delta^0(\rho_k)\Big)\cup\Big(\bigcup_{\tilde\rho_k\in\tilde\Lambda''}\kappa_\delta^0(\tilde\rho_k)\Big)\bigg)$$

the real axis without δ-neighbourhoods of the points of $\Lambda''\cup\tilde\Lambda''$.

Lemma 2. *The following relations hold for $x\in[0,a]$:*

$$\Omega(x)\tilde\varphi_\rho(x)=\varphi_\rho(x)+\int_{-\infty}^{\infty}\tilde P_{\rho,\mu}(x)\varphi_\mu(x)\,d\mu$$
$$+\sum_{k\in\omega'}\big(\tilde P_{\rho,k0}(x)\varphi_{k0}(x)-\tilde P_{\rho,k1}(x)\varphi_{k1}(x)\big), \tag{67}$$

$$D(x,\rho,\mu)-\tilde D(x,\rho,\mu)+\int_{-\infty}^{\infty}\tilde P_{\rho,\xi}(x)D(x,\xi,\mu)\,d\xi$$
$$+\sum_{s\in\omega'}\big(\tilde P_{\rho,s0}(x)D(x,\rho_{s0},\mu)-\tilde P_{\rho,s1}(x)D(x,\rho_{s1}\mu)\big)=\Lambda(x)\tilde\varphi_\rho(x)\varphi_\mu(x), \quad (68)$$

where

$$\left.\begin{aligned}\Omega(x)&=\frac12\big(\exp\big(i\hat Q(x)/\omega\big)+\exp\big(-i\hat Q(x)/\omega\big)\big),\\ \Lambda(x)&=-\frac{\omega}{2}\big(\exp\big(i\hat Q(x)/\omega\big)-\exp\big(-i\hat Q(x)/\omega\big)\big),\end{aligned}\right\} \quad (69)$$

and the integrals are understood in the principal value sense: $\int_{-\infty}^{\infty}:=\lim_{\delta\to0}\int_{\xi'_\delta}$.

Proof. It follows from (9), (12), (41), (46), (48), (52) and (61) that

$$\big|\mathcal P_{kj}(x,\rho)\big|\leqslant C|\rho|^{k-j},\quad \rho\in G_\delta\cap\tilde G_\delta,\ k,j=1,2, \quad (70)$$

$$\mathcal P_{kk}(x,\rho)=\Omega(x)+O\left(\frac1\rho\right),$$
$$\mathcal P_{21}(x,\rho)=-\rho\Lambda(x)+O(1),\quad |\rho|\to\infty,\ \arg\rho=\theta\neq\pm\frac{\pi}{2}. \quad (71)$$

Take positive numbers $R_N=\frac{N\pi}{\omega a}+\chi$ such that the circles $\theta_N:=\{\rho:\ |\rho|=R_N\}$ lie in $G_\delta\cap\tilde G_\delta$ for sufficiently small $\delta>0$. Fix $\rho\notin\sigma(L)\cup\sigma(\tilde L)$, and take $\delta>0$ and N such that $\rho\in G_\delta\cap\tilde G_\delta\cap\operatorname{int}\theta_N$. Consider the contour integral (with counterclockwise circuit)

$$J_{N,k}(x,\rho)=\frac{1}{2\pi i}\int_{\theta_N}\frac{\mathcal P_{1k}(x,\mu)-\Omega(x)\delta_{1k}}{\rho-\mu}\,d\mu,\quad k=1,2, \quad (72)$$

where δ_{jk} is the Kronecker delta. In view of (70) and (71), $\lim_{N\to\infty}J_{N,k}(x,\rho)=0$. Moving the contour in (72) through the pole $\mu=\rho$, we get

$$\mathcal P_{1k}(x,\rho)-\Omega(x)\delta_{1k}=\frac{1}{2\pi i}\int_{\gamma_N}\frac{\mathcal P_{1k}(x,\mu)}{\rho-\mu}\,d\mu+J_{N,k}(x,\rho), \quad (73)$$

where the contour γ_N (with counterclockwise circuit) is such that $(\sigma(L)\cup\sigma(\tilde L))\cap\operatorname{int}\theta_N\subset\operatorname{int}\gamma_N$, and $\rho\notin\overline{\operatorname{int}\theta_N}$. It follows from (53) and (73) that

$$\varphi(x,\rho)=\Omega(x)\tilde\varphi(x,\rho)+\frac{1}{2\pi i}\int_{\gamma_N}\big(\tilde\varphi(x,\rho)\mathcal P_{11}(x,\mu)+\tilde\varphi'(x,\rho)\mathcal P_{12}(x,\mu)\big)\frac{d\mu}{\rho-\mu}$$
$$+J_N(x,\rho), \quad (74)$$

where $J_N(x,\rho)=J_{N,1}(x,\rho)\tilde\varphi(x,\rho)+J_{N,2}(x,\rho)\tilde\varphi'(x,\rho)$, and consequently,

$$\lim_{N\to\infty}J_N(x,\rho)=0.$$

Substituting (52) into (74) we calculate

$$\varphi(x,\rho)=\Omega(x)\tilde{\varphi}(x,\rho)+\frac{1}{2\pi i}\int\limits_{\gamma_N}\Big(\tilde{\varphi}(x,\rho)\big(\varphi(x,\mu)\tilde{\Phi}'(x,\mu)-\Phi(x,\mu)\tilde{\varphi}'(x,\mu)\big)$$
$$+\tilde{\varphi}'(x,\rho)\big(\Phi(x,\mu)\tilde{\varphi}(x,\mu)-\varphi(x,\mu)\tilde{\Phi}(x,\mu)\big)\Big)\frac{d\mu}{\rho-\mu}+J_N(x,\lambda).$$

In view of (20) this yields

$$\Omega(x)\tilde{\varphi}_\rho(x)=\varphi_\rho(x)+\frac{1}{2\pi i}\int\limits_{\gamma_N}\tilde{D}(x,\rho,\mu)\hat{M}(\mu)\varphi_\mu(x)\,d\mu-J_N(x,\lambda), \tag{75}$$

since the terms with $S(x,\mu)$ vanish by Cauchy's theorem.

Furthermore, we consider the contour integral

$$J_{N,jk}(x,\rho,\mu):=\frac{1}{2\pi i}\int\limits_{\theta_N}\frac{\mathcal{P}_{jk}(x,\xi)+\xi\Lambda(x)\delta_{j2}\delta_{k1}}{(\rho-\xi)(\xi-\mu)}\,d\xi,\quad \rho,\mu\in\operatorname{int}\theta_N. \tag{76}$$

In view of (70) and (71), $\lim_{N\to\infty}J_{N,jk}(x,\rho,\mu)=0$. Since

$$\frac{1}{(\rho-\xi)(\xi-\mu)}=\frac{1}{\rho-\mu}\left(\frac{1}{\rho-\xi}-\frac{1}{\mu-\xi}\right),$$

by similar arguments we infer from (76)

$$\frac{\mathcal{P}_{jk}(x,\rho)-\mathcal{P}_{jk}(x,\mu)}{\rho-\mu}=-\Lambda(x)\delta_{j2}\delta_{k1}+\frac{1}{2\pi i}\int\limits_{\gamma_N}\frac{\mathcal{P}_{jk}(x,\xi)\,d\xi}{(\rho-\xi)(\xi-\mu)}$$
$$+J_{N,jk}(x,\rho,\mu), \tag{77}$$

for $\rho,\mu\notin\operatorname{int}\gamma_N$. Let $y(x)$ be an arbitrary smooth function. Then, by virtue of (77),

$$\frac{\mathcal{P}(x,\rho)-\mathcal{P}(x,\mu)}{\rho-\mu}\begin{bmatrix}y(x)\\y'(x)\end{bmatrix}$$
$$=\begin{bmatrix}0\\-\Lambda(x)y(x)\end{bmatrix}+\frac{1}{2\pi i}\int\limits_{\gamma_N}\mathcal{P}(x,\xi)\begin{bmatrix}y(x)\\y'(x)\end{bmatrix}\frac{d\xi}{(\rho-\xi)(\xi-\mu)}+\varepsilon_N(x,\rho,\mu), \tag{78}$$

where $\lim_{N\to\infty}\varepsilon_N(x,\rho,\mu)=0$. According to (52),

$$\mathcal{P}(x,\rho)\begin{bmatrix}y(x)\\y'(x)\end{bmatrix}=\big\langle y(x),\tilde{\Phi}(x,\rho)\big\rangle\begin{bmatrix}\varphi(x,\rho)\\\varphi'(x,\rho)\end{bmatrix}-\big\langle y(x),\tilde{\varphi}(x,\rho)\big\rangle\begin{bmatrix}\Phi(x,\rho)\\\Phi'(x,\rho)\end{bmatrix},$$

and equality (78) takes the following form:

$$\frac{\mathcal{P}(x,\rho)-\mathcal{P}(x,\mu)}{\rho-\mu}\begin{bmatrix}y(x)\\y'(x)\end{bmatrix}$$
$$=\begin{bmatrix}0\\-\Lambda(x)y(x)\end{bmatrix}+\frac{1}{2\pi i}\int\limits_{\gamma_N}\left(\big\langle y(x),\tilde{\Phi}(x,\xi)\big\rangle\begin{bmatrix}\varphi(x,\xi)\\\varphi'(x,\xi)\end{bmatrix}\right.$$
$$\left.-\big\langle y(x),\tilde{\varphi}(x,\xi)\big\rangle\begin{bmatrix}\Phi(x,\xi)\\\Phi'(x,\xi)\end{bmatrix}\right)\frac{d\xi}{(\rho-\xi)(\xi-\mu)}+\varepsilon_N(x,\rho,\mu).$$

Hence, for $y(x)=\tilde\varphi(x,\rho)$ we have

$$\det\left(\frac{\mathcal{P}(x,\rho)-\mathcal{P}(x,\mu)}{\rho-\mu}\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\mu)\\ \varphi'(x,\mu)\end{bmatrix}\right)$$
$$=-\Lambda(x)\tilde\varphi(x,\rho)\varphi(x,\mu)+\frac{1}{2\pi i}\int\limits_{\gamma_N}\left(\frac{\langle\tilde\varphi(x,\rho),\tilde\Phi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\varphi(x,\xi),\varphi(x,\mu)\rangle}{\xi-\mu}\right.$$
$$\left.-\frac{\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\xi)\rangle}{\rho-\xi}\cdot\frac{\langle\Phi(x,\xi),\varphi(x,\mu)\rangle}{\xi-\mu}\right)d\xi+\varepsilon_N^1(x,\rho,\mu),\tag{79}$$

where $\lim_{N\to\infty}\varepsilon_N^1(x,\rho,\mu)=0$. By virtue of (51),

$$\mathcal{P}(x,\rho)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix}=\begin{bmatrix}\varphi(x,\rho)\\ \varphi'(x,\rho)\end{bmatrix},$$

and consequently,

$$\det\left(\mathcal{P}(x,\rho)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\mu)\\ \varphi'(x,\mu)\end{bmatrix}\right)=\big\langle\varphi(x,\rho),\varphi(x,\mu)\big\rangle.\tag{80}$$

Using (21) and (52) we obtain

$$\left.\begin{aligned}&\mathcal{P}_{11}(x,\rho)\varphi'(x,\rho)-\mathcal{P}_{21}(x,\rho)\varphi(x,\rho)=\tilde\varphi'(x,\rho),\\ &\mathcal{P}_{22}(x,\rho)\varphi(x,\rho)-\mathcal{P}_{12}(x,\rho)\varphi'(x,\rho)=\tilde\varphi(x,\rho),\end{aligned}\right\}$$

and hence,

$$\det\left(\mathcal{P}(x,\mu)\begin{bmatrix}\tilde\varphi(x,\rho)\\ \tilde\varphi'(x,\rho)\end{bmatrix},\begin{bmatrix}\varphi(x,\mu)\\ \varphi'(x,\mu)\end{bmatrix}\right)$$
$$=\tilde\varphi(x,\rho)\big(\mathcal{P}_{11}(x,\mu)\varphi'(x,\mu)-\mathcal{P}_{21}(x,\mu)\varphi(x,\mu)\big)$$
$$-\tilde\varphi'(x,\rho)\big(\mathcal{P}_{22}(x,\mu)\varphi(x,\mu)-\mathcal{P}_{12}(x,\mu)\varphi'(x,\mu)\big)$$
$$=\big\langle\tilde\varphi(x,\rho),\tilde\varphi(x,\mu)\big\rangle.\tag{81}$$

Substituting (20), (80) and (81) into (79) and taking (62) into account we obtain

$$D(x,\rho,\mu)-\tilde D(x,\rho,\mu)+\frac{1}{2\pi i}\int\limits_{\gamma_N}\tilde D(x,\rho,\xi)\hat M(\xi)D(x,\xi,\mu)\,d\xi$$
$$=\Lambda(x)\tilde\varphi(x,\rho)\varphi(x,\mu)+\varepsilon_N^1(x,\rho,\mu),\tag{82}$$

since the terms containing $S(x,\xi)$ and $\tilde S(x,\xi)$ vanish by Cauchy's theorem.

Let Π'_δ be the two-sided cut Π_0 without the δ-neighbourhoods of the points of $\Lambda''\cup\tilde\Lambda''$, and let $\Gamma'_\delta:=\Pi'_\delta\cup(\bigcup_{\rho_k\in\Lambda''}\kappa_\delta(\rho_k))\cup(\bigcup_{\tilde\rho_k\in\tilde\Lambda''}\kappa_\delta(\tilde\rho_k))$ be the contour with counterclockwise circuit (see Fig. 1). Denote $\Gamma'_{\delta,N}:=\Gamma'_\delta\cap\theta_{N,0}$, where $\theta_{N,0}:=\{\rho\colon |\rho|\leqslant R_N\}$. Contracting the contour γ_N in (75) to the real axis through the poles of $\Lambda'\cup\tilde\Lambda'$ and using the residue theorem, we get

$$\Omega(x)\tilde{\varphi}_\rho(x)=\varphi_\rho(x)+\frac{1}{2\pi i}\int\limits_{\Gamma'_{\delta,N}}\tilde{D}(x,\rho,\mu)\hat{M}(\mu)\varphi_\mu(x)\,d\mu$$
$$+\sum_{\substack{\rho_{k0}\in\Lambda'\\|\rho_{k0}|<R_N}}\tilde{D}(x,\rho,\rho_{k0})\alpha_{k0}\varphi_{k0}(x)-\sum_{\substack{\rho_{k1}\in\tilde{\Lambda}'\\|\rho_{k1}|<R_N}}\tilde{D}(x,\rho,\rho_{k1})\alpha_{k1}\varphi_{k1}(x)$$
$$-J_N(x,\rho).$$

As $N\to\infty$ this yields

$$\Omega(x)\tilde{\varphi}_\rho(x)=\varphi_\rho(x)+\frac{1}{2\pi i}\int\limits_{\Gamma'_\delta}\tilde{D}(x,\rho,\mu)\hat{M}(\mu)\varphi_\mu(x)\,d\mu$$
$$+\sum_{\rho_{k0}\in\Lambda'}\tilde{D}(x,\rho,\rho_{k0})\alpha_{k0}\varphi_{k0}(x)-\sum_{\rho_{k1}\in\tilde{\Lambda}'}\tilde{D}(x,\rho,\rho_{k1})\alpha_{k1}\varphi_{k1}(x). \tag{83}$$

Since

$$\lim_{\delta\to 0}\frac{1}{2\pi i}\int\limits_{\kappa_\delta(\rho_{kj})}\tilde{D}(x,\rho,\mu)\hat{M}(\mu)\varphi_\mu(x)\,d\mu=(-1)^j\tilde{P}_{\rho,kj}(x)\alpha_{kj}\varphi_{kj}(x),$$
$$\rho_{kj}\in\Lambda''\cup\tilde{\Lambda}'',$$
$$\frac{1}{2\pi i}\int\limits_{\Pi'_\delta}\tilde{D}(x,\rho,\mu)\hat{M}(\mu)\varphi_\mu(x)\,d\mu=\int\limits_{\xi'_\delta}\tilde{P}_{\rho,\mu}(x)\varphi_\mu(x)\,d\mu,$$

from (83) as $\delta\to 0$ we arrive at (67). Analogously, leaning on (82) we deduce (68). Lemma 2 is proved. □

Similarly one can prove that

$$\Omega(x)\tilde{\Phi}_\rho(x)=\Phi_\rho(x)+\int\limits_{-\infty}^{\infty}\tilde{p}_{\rho,\mu}(x)\varphi_\mu(x)\,d\mu$$
$$+\sum_{k\in\omega'}\big(\tilde{p}_{\rho,k0}(x)\varphi_{k0}(x)-\tilde{p}_{\rho,k1}(x)\varphi_{k1}(x)\big), \tag{84}$$

where

$$\tilde{p}_{\rho,\mu}(x)=\tilde{d}(x,\rho,\mu)\hat{V}(\mu),\qquad \tilde{p}_{\rho,kj}(x)=\tilde{d}(x,\rho,\rho_{kj})\alpha_{kj},$$
$$\tilde{d}(x,\rho,\mu):=\frac{\langle\tilde{\Phi}(x,\rho),\tilde{\varphi}(x,\mu)\rangle}{\rho-\mu}.$$

Remark 2. For each fixed x, relation (67) can be considered as a linear equation with respect to $\varphi_\rho(x)$ for $\rho\in\sigma(L)\cup\sigma(\tilde{L})$. But the series in (67) converges only “with brackets,” and the integral is understood in the principal value sense. Therefore, it is not convenient to use (67) as a main equation of the inverse problem. Below we will transfer (67) to a linear equation in a suitable Banach space (see (94)).

Denote $\xi_k := |\rho_k - \tilde\rho_k| + |\alpha_k - \tilde\alpha_k|\langle k\rangle$, where $\langle k\rangle := |k|$ for $k \in \omega^0$, and $\langle k\rangle := 1$, otherwise. It follows from (15), (16), (31), (34) and (61) that

$$\xi_k = O\left(\frac{1}{k}\right), \quad |k| \to \infty; \qquad \hat V(\rho) = O\big(|\rho|^{-2}\exp\big(-2\omega|\rho|a\big)\big), \quad |\rho| \to \infty.$$

For $(n,i), (k,j) \in \omega_1$ and $\rho, \mu \in \mathbf{R}$, we introduce the functions

$$\begin{aligned}
&\psi_\rho(x) = \varphi_\rho(x)\exp\big(-|\rho|\omega x\big),\\
&\psi_{k0}(x) = \big(\varphi_{k0}(x) - \varphi_{k1}(x)\big)\chi_k, \quad \psi_{k1}(x) = \varphi_{k1}(x),\\
&H_{\rho,\mu}(x) = P_{\rho,\mu}(x)\exp\big(-|\rho|\omega x\big)\exp\big(|\mu|\omega x\big),\\
&H_{\rho,k0}(x) = P_{\rho,k0}(x)\xi_k\exp\big(-|\rho|\omega x\big),\\
&H_{\rho,k1}(x) = \big(P_{\rho,k0}(x) - P_{\rho,k1}(x)\big)\exp\big(-|\rho|\omega x\big),\\
&H_{n0,\mu}(x) = \big(P_{n0,\mu}(x) - P_{n1,\mu}(x)\big)\chi_n\exp\big(|\mu|\omega x\big),\\
&H_{n1,\mu}(x) = P_{n1,\mu}(x)\exp\big(|\mu|\omega x\big),\\
&H_{n0,k0}(x) = \big(P_{n0,k0}(x) - P_{n1,k0}(x)\big)\chi_n\xi_k,\\
&H_{n1,k1}(x) = P_{n1,k0}(x) - P_{n1,k1}(x),\\
&H_{n0,k1}(x) = \big(P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x)\big)\chi_n,\\
&H_{n1,k0}(x) = P_{n1,k0}(x)\xi_k,
\end{aligned}$$

where $\chi_k = \xi_k^{-1}$ for $\xi_k \neq 0$, and $\chi_k = 0$ for $\xi_k = 0$. Analogously we define $\tilde\psi_\rho(x)$, $\tilde\psi_{kj}(x)$, $\tilde H_{\rho,\mu}(x)$, $\tilde H_{\rho,kj}(x)$, $\tilde H_{ni,\mu}(x)$ and $\tilde H_{ni,kj}(x)$. It follows from (34), (64)–(66) and Schwarz's lemma that for $(n,i), (k,j) \in \omega_1$, $\nu = 0, 1$, $x \in [0,a]$, $\rho, \mu \in \mathbf{R}$, $|\mu| \geqslant \mu^*$, the following estimates hold:

$$\left.\begin{aligned}
&\left|\frac{\partial^\nu}{\partial\rho^\nu}\psi_\rho(x)\right| \leqslant C, \quad \big|\psi_{kj}(x)\big| \leqslant C,\\
&\left|\frac{\partial^\nu}{\partial\rho^\nu}H_{\rho,\mu}(x)\right| \leqslant \frac{C}{|\mu|^2}\cdot\frac{|\rho|+|\mu|+1}{|\rho-\mu|+1}\exp\big(-2|\mu|\omega(a-x)\big),\\
&\left|\frac{\partial^\nu}{\partial\rho^\nu}H_{\rho,kj}(x)\right| \leqslant \frac{C\xi_k}{\langle k\rangle},\\
&\big|H_{ni,\mu}(x)\big| \leqslant \frac{C}{|\mu|^2}\exp\big(-2|\mu|\omega(a-x)\big),\\
&\big|H_{ni,kj}(x)\big| \leqslant \frac{C\xi_k}{\langle k\rangle}\frac{|n|+|k|+1}{|n-k|+1}.
\end{aligned}\right\} \tag{85}$$

The same estimates are also valid for $\tilde\psi_\rho(x)$, $\tilde\psi_{kj}(x)$, $\tilde H_{\rho,\mu}(x)$, $\tilde H_{\rho,kj}(x)$, $\tilde H_{ni,\mu}(x)$ and $\tilde H_{ni,kj}(x)$.

Consider the Banach space m of bounded sequences $\beta = [\beta_v]_{v\in\omega_1}$ with the norm $\|\beta\|_m = \sup_{v\in\omega_1} |\beta_v|$. Define the vectors

$$\psi(x) = \big[\psi_v(x)\big]_{v\in\omega_1} = \begin{bmatrix} \psi_{k0}(x) \\ \psi_{k1}(x) \end{bmatrix}_{k\in\omega'}, \qquad \tilde\psi(x) = \big[\tilde\psi_v(x)\big]_{v\in\omega_1} = \begin{bmatrix} \tilde\psi_{k0}(x) \\ \tilde\psi_{k1}(x) \end{bmatrix}_{k\in\omega'}.$$

It follows from (85) that for each fixed x, $\psi(x), \tilde\psi(x) \in m$. Let $B := C^1(-\infty,0] \oplus C^1[0,\infty)$ be the Banach space of continuously differentiable on $(-\infty,0]$ and $[0,\infty)$ functions $\rho \to f(\rho)$ such that $f(\rho)$ and $\frac{\partial}{\partial\rho} f(\rho)$ are bounded, with the norm $\|f\|_B = \max_{\nu=0,1} \sup_{\rho\in\mathbf{R}} |\frac{\partial^\nu}{\partial\rho^\nu} f(\rho)|$. It follows from (85) that for each fixed x, $\psi_\rho(x), \tilde\psi_\rho(x) \in B$. Consider the Banach space $\mathcal{B}$ of vectors

$$F = \begin{bmatrix} f \\ \beta \end{bmatrix},$$

where $f \in B$, $\beta = [\beta_v]_{v\in\omega_1} \in m$, with the norm $\|F\|_{\mathcal{B}} = \max(\|f\|_B, \|\beta\|_m)$. Denote

$$\Psi(x) = \begin{bmatrix} \psi_\rho(x) \\ \psi(x) \end{bmatrix}, \qquad \tilde\Psi(x) = \begin{bmatrix} \tilde\psi_\rho(x) \\ \tilde\psi(x) \end{bmatrix}.$$

Clearly, $\Psi(x), \tilde\Psi(x) \in \mathcal{B}$ for each fixed $x \in [0,a]$. For a fixed $x \in [0,a]$, we define the operator $\tilde H = \tilde H(x) : \mathcal{B} \to \mathcal{B}$ by the formulas

$$\tilde F = \tilde H F, \qquad F = \begin{bmatrix} f \\ \beta \end{bmatrix} \in \mathcal{B}, \qquad \tilde F = \begin{bmatrix} \tilde f \\ \tilde\beta \end{bmatrix} \in \mathcal{B},$$

$$\left.\begin{aligned} \tilde f(\rho) &= \int_{-\infty}^{\infty} \tilde H_{\rho,\mu}(x) f(\mu)\,d\mu + \sum_{v\in\omega_1} \tilde H_{\rho,v}(x)\beta_v, \\ \tilde\beta_u &= \int_{-\infty}^{\infty} \tilde H_{u,\mu}(x) f(\mu)\,d\mu + \sum_{v\in\omega_1} \tilde H_{u,v}(x)\beta_v, \end{aligned}\right\} \tag{86}$$

$\rho, \mu \in \mathbf{R}$; $u = (n,i)$, $v = (k,j)$; $n,k \in \omega'$; $i,j = 0,1$. Analogously we define the operator $H = H(x)$.

Lemma 3. *For each fixed x, the operators $H(x)$ and $\tilde H(x)$ are linear bounded operators acting from $\mathcal{B}$ to $\mathcal{B}$.*

Proof. For definiteness, we consider here only one of the blocks in (86), the remaining blocks are studied similarly. Let $f(\rho) \in B$, and let

$$f^*(\rho) := \int_{-\infty}^{\infty} \tilde H_{\rho,\mu}(x) f(\mu)\,d\mu.$$

We will show that

$$f^*(\rho) \in B, \quad \|f^*\|_B \leqslant C\|f\|_B. \tag{87}$$

Since $\Lambda'' \cup \tilde{\Lambda}'' := \{\rho_k^*\}_{k=\overline{1,p}}$ is a finite set, there exist numbers $\{d_j\}_{j=\overline{-1,p+1}}$ such that $-\infty = d_{-1} < d_0 < \rho_1^* < d_1 < \cdots < \rho_p^* < d_p < d_{p+1} = \infty$. Then

$$f^*(\rho) = \sum_{k=0}^{p+1} f_k(\rho), \quad \text{where } f_k(\rho) := \int_{d_{k-1}}^{d_k} \tilde{H}_{\rho,\mu}(x) f(\mu)\, d\mu.$$

By virtue of (85),

$$\left| \frac{\partial^\nu}{\partial\rho^\nu} f_{p+1}(\rho) \right| \leqslant C \int_{d_p}^{\infty} \exp\bigl(-2|\mu|\omega(a-x)\bigr) \frac{|\rho| + |\mu| + 1}{|\rho - \mu| + 1} \cdot \frac{d\mu}{|\mu|^2} \leqslant C,$$

and consequently, $f_{p+1}(\rho) \in B$ and $\|f_{p+1}\|_B \leqslant C\|f\|_B$. Similarly, $f_0(\rho) \in B$ and $\|f_0\|_B \leqslant C\|f\|_B$. For $k = \overline{1,p}$ we denote

$$w_k(\mu) := \hat{V}(\mu)\bigl(\mu - \rho_k^*\bigr), \quad \tilde{D}_1(x,\rho,\mu) = \tilde{D}(x,\rho,\mu) \exp\bigl(-|\rho|\omega x\bigr) \exp\bigl(|\mu|\omega x\bigr).$$

Clearly, $w_k(\mu) \in C[d_{k-1}, d_k]$. Then

$$\begin{aligned} f_k(\rho) &= \int_{d_{k-1}}^{d_k} \tilde{D}_1(x,\rho,\mu) \hat{V}(\mu) f(\mu)\, d\mu \\ &= \int_{d_{k-1}}^{d_k} \tilde{D}_1(x,\rho,\mu) w_k(\mu) \frac{f(\mu) - f(\rho_k^*)}{\mu - \rho_k^*}\, d\mu \\ &\quad + f\bigl(\rho_k^*\bigr) \int_{d_{k-1}}^{d_k} \frac{\tilde{D}_1(x,\rho,\mu) - \tilde{D}_1(x,\rho,\rho_k^*)}{\mu - \rho_k^*} w_k(\mu)\, d\mu \\ &\quad + f\bigl(\rho_k^*\bigr) \tilde{D}_1(x,\rho,\rho_k^*) \int_{d_{k-1}}^{d_k} \hat{V}(\mu)\, d\mu, \end{aligned} \tag{88}$$

where the last integral is understood in the principal value sense. Using (88) and (66) it is easy to verify that $f_k(\rho) \in B$ and $\|f_k\|_B \leqslant C\|f\|_B$ for $k = \overline{1,p}$. Thus, (87) holds. Lemma 3 is proved. □

Theorem 6. *For each fixed $x \in [0,a]$, the following relations hold*:

$$\Omega(x)\tilde{\Psi}(x) = \bigl(E + \tilde{H}(x)\bigr)\Psi(x), \tag{89}$$

$$\Omega(x)\Psi(x) = \bigl(E - H(x)\bigr)\tilde{\Psi}(x), \tag{90}$$

$$\bigl(E + \tilde{H}(x)\bigr)\bigl(E - H(x)\bigr)Y = Y - K(x)\tilde{\Psi}(x), \tag{91}$$

$$\bigl(E - H(x)\bigr)\bigl(E + \tilde{H}(x)\bigr)Y = Y - \tilde{K}(x)\Psi(x), \tag{92}$$

where E is the identity operator,

$$Y = \begin{bmatrix} y_\rho \\ y \end{bmatrix} \in \mathcal{B} \quad \bigl(i.e.,\ y_\rho \in B,\, y = [y_{ni}]_{(n,i)\in\omega_1} \in m\bigr),$$

and

$$K(x) = \Lambda(x)\Biggl(\int_{-\infty}^{\infty} \psi_\mu(x)\exp\bigl(2|\mu|\omega x\bigr)\hat V(\mu)y_\mu\,d\mu + \sum_{k\in\omega'}\bigl(\varphi_{k0}(x)\alpha_{k0}\xi_k y_{k0} + \bigl(\varphi_{k0}(x)\alpha_{k0} - \varphi_{k1}(x)\alpha_{k1}\bigr)y_{k1}\bigr)\Biggr),$$

$$\tilde K(x) = \Lambda(x)\Biggl(\int_{-\infty}^{\infty} \tilde\psi_\mu(x)\exp\bigl(2|\mu|\omega x\bigr)\hat V(\mu)y_\mu\,d\mu + \sum_{k\in\omega'}\bigl(\tilde\varphi_{k0}(x)\alpha_{k0}\xi_k y_{k0} + \bigl(\tilde\varphi_{k0}(x)\alpha_{k0} - \tilde\varphi_{k1}(x)\alpha_{k1}\bigr)y_{k1}\bigr)\Biggr).$$

Proof. Taking into account our notations we rewrite (67) in the following form:

$$\Omega(x)\tilde\psi_\rho(x) = \psi_\rho(x) + \int_{-\infty}^{\infty} \tilde H_{\rho,\mu}(x)\psi_\mu(x)\,d\mu + \sum_{(k,j)\in\omega_1} \tilde H_{\rho,kj}(x)\psi_{kj}(x), \quad \rho\in\mathbf{R},$$

$$\Omega(x)\tilde\psi_{ni}(x) = \psi_{ni}(x) + \int_{-\infty}^{\infty} \tilde H_{ni,\mu}(x)\psi_\mu(x)\,d\mu + \sum_{(k,j)\in\omega_1} \tilde H_{ni,kj}(x)\psi_{kj}(x), \quad (n,i)\in\omega_1,$$

which is equivalent to (89). It follows from (68) that

$$\bigl(\tilde H(x) - H(x) - \tilde H(x)H(x)\bigr)Y = K(x)\tilde\Psi(x),$$

and consequently, (91) holds. Interchanging places for L and $\tilde L$, symmetrically to (89) and (91), we obtain (90) and (92). Theorem 6 is proved. □

Thus, for each fixed $x\in[0,a]$, the vector $\Psi(x)\in\mathcal{B}$ is the solution of Eq. (89) in the Banach space $\mathcal{B}$.

Remark 3. We note that in the particular case of a Sturm–Liouville equation (when $q_1(x)\equiv 0$) we have $\Omega(x)\equiv 1$, $\Lambda(x)\equiv 0$, and consequently, the operators $E+\tilde H(x)$ and $E-H(x)$ are inverse of each other.

We set $\Omega_1 := \{x\in[0,a]:\ \Omega(x)\neq 0\}$, $\Omega_0 := \{x\in[0,a]:\ \Omega(x)=0\}$.

Theorem 7.

(1) *Let* $x \in \Omega_1$. *Then the homogeneous equation*

$$\big(E + \tilde{H}(x)\big)Y = 0, \quad Y \in \mathcal{B}, \tag{93}$$

has only the trivial solution $Y = 0$.

(2) *Let* $x \in \Omega_0$. *Then the solutions of the homogeneous equation* (93) *form the one-dimensional subspace* $Y = C\Psi(x)$, $C = const$.

Proof. (1) Let $x \in \Omega_1$, and let $Y \in \mathcal{B}$ be a solution of (93). Then $(E - H(x))(E + \tilde{H}(x))Y = 0$. On the other hand, in accordance with (91), $(E - H(x))(E + \tilde{H}(x))Y = Y - \tilde{K}(x)\Psi(x)$, and therefore, $Y = \tilde{K}(x)\Psi(x)$. Applying the operator $E + \tilde{H}(x)$ to both parts of this equality we obtain $\tilde{K}(x)(E + \tilde{H}(x))\Psi(x) = 0$. By virtue of (89) this yields $\tilde{K}(x)\Omega(x)\tilde{\Psi}(x) = 0$. Consequently, $\tilde{K}(x) = 0$, i.e., $Y = 0$.

(2) Fix $x \in \Omega_0$. Then, in view of (89), $(E + \tilde{H}(x))\Psi(x) = 0$, and consequently, the vectors $Y = C\Psi(x)$, $C = const$, are solutions of Eq. (93). On the other hand, let Y^0 be a solution of (93). Then $(E - H(x))(E + \tilde{H}(x))Y^0 = 0$. Applying (92) we obtain $0 = (E - H(x))(E + \tilde{H}(x))Y^0 = Y^0 - \tilde{K}(x)\Psi(x)$, i.e., $Y^0 = C\Psi(x)$, $C = const$. Theorem 7 is proved. □

Let L and $\tilde{L}$ be such that $\Omega_0 = \emptyset$. For example, this always holds if $q_1(x)$ is real valued. For each fixed $x \in [0, a]$, we consider in $\mathcal{B}$ the linear equation

$$\tilde{\Psi}(x) = \big(E + \tilde{H}(x)\big)Z(x) \tag{94}$$

with respect to $Z(x)$. Equation (94) is called the *main equation* of the inverse problem. The following result is an obvious consequence of Theorems 6 and 7.

Theorem 8. *For each fixed* $x \in [0, a]$, *Eq.* (94) *has a unique solution, namely* $Z(x) = (\Omega(x))^{-1}\Psi(x)$.

Using the main equation we can get an algorithm for the solution of Inverse Problem 1. For this purpose we introduce the functions $z_1(x, \rho) = \varphi(x, \rho)/\Omega(x)$, $z_2(x, \rho) = \Phi(x, \rho)/\Omega(x)$. Since $\varphi(x, \rho)$ and $\Phi(x, \rho)$ satisfy Eq. (1), it follows that

$$z_k'' + a(x)z_k' + \big(\rho^2 r(x) + i\rho q_1(x) + h(x)\big)z_k = 0, \quad k = 1, 2,$$

where

$$a(x) = 2\frac{\Omega'(x)}{\Omega(x)}, \qquad h(x) = q_0(x) + \frac{a'(x)}{2} + \frac{a^2(x)}{4}. \tag{95}$$

Therefore

$$a(x)z_k' + \big(i\rho q_1(x) + h(x)\big)z_k = -z_k'' + \omega^2\rho^2 z_k, \quad k = 1, 2. \tag{96}$$

Moreover, by (84),

$$z_2(x, \rho) = \frac{\tilde{\Phi}(x, \rho)}{\Omega(x)} - \int_{-\infty}^{\infty} \tilde{p}_{\rho,\mu}(x)z_1(x, \mu)\, d\mu + \sum_{k \in \omega'} \big(\tilde{p}_{\rho,k0}(x)z_1(x, \rho_{k0}) - \tilde{p}_{\rho,k1}(x)z_1(x, \rho_{k1})\big), \tag{97}$$

and equality (21) takes the form

$$\langle z_1(x,\rho), z_2(x,\rho)\rangle \equiv \Omega^2(x). \tag{98}$$

Using the results obtained above we arrive at the following procedure for the solution of Inverse Problem 1.

Algorithm 1. Let the spectral data S of the boundary value problem L be given.

(1) Construct the coefficients ω, β_1, a, κ, κ_1 and Q by (15), (58), (59) and (60).
(2) Take a boundary value problem $\tilde{L}$ such that (61) holds.
(3) Calculate $\tilde{\Psi}(x)$ and $\tilde{H}(x)$, $x \in [0,a]$.
(4) Find $Z(x) = [z_\rho(x), z(x)] \in \mathcal{B}$ by solving the main equation (94).
(5) Construct $z_1(x,\rho) = z_\rho(x)$ and $z_2(x,\rho)$ by (97).
(6) Calculate $\Omega^2(x)$ via (98).
(7) Find $a(x)$, $q_1(x)$ and $h(x)$ by solving the system of linear algebraic equations (96) with the determinant $-\Omega^2(x) \neq 0$.
(8) Construct $\Omega(x)$, $q_1(x)$, $q_0(x)$ for $x \in [0,a]$ and β_0 using (2), (5), (69) and (95).

Thus, we have constructed $q_1(x)$ and $q_0(x)$ for the interval $x \in [0,a]$. For the interval $x \in (a,\infty)$, the arguments are similar.

Remark 4. In order to construct $q_j(x)$, $j = 0,1$, for $x \in (a,\infty)$ we can act also in another way. Suppose that, using Algorithm 1, we have constructed ω, a, β_j and $q_j(x)$ for $j = 0,1$, $x \in [0,a]$. Consequently, the solutions $\varphi(x,\rho)$ and $S(x,\rho)$ are known for $x \in [0,a]$. By virtue of (20), the solution $\Phi(x,\rho)$ is also known for $x \in [o,a]$. Denote

$$M_1(\rho) := \frac{\Phi(a,\rho)}{\Phi'(a,\rho)}. \tag{99}$$

The function $M_1(\rho)$ is the Weyl function for Eq. (1) on the interval (a,∞). Thus, we can reduce our problem to the inverse problem for (a,∞). In this interval the weight-function $r(x) \equiv 1$. This inverse problem was solved in [23].

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