NOTE

Where Are the Nodes of “Good” Interpolation Polynomials on the Real Line?

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It is shown that the interval where the nodes of a “good” interpolation polynomial
are situated is strongly connected with the Mhaskar–Rahmanov–Saff number.

Let \( w \) be a Freud-type weight on \( \mathbb{R} \). For a formal definition of Freud-
type weights see, e.g., [LL]. Here we mention only the archetypal example
\( w(x) = e^{-|x|^a}, \ x > 1 \). Let \( P_n \) denote the set of algebraic polynomials of
degree at most \( n \), and let \( \| \cdot \| \) denote the supremum norm over \( \mathbb{R} \). It is
known that with each Freud-type weight and natural integer \( n \), one can
associate a positive number \( a_n \) such that

\[
\|q_n\| = \max_{|x| \leq a_n} \|q_n(x)\|
\] (1)

for all weighted polynomials \( q_n \) of degree at most \( n \), that is, for all \( q_n \) such
that \( q_n/w \in P_n \) (see [MS]). The quantity \( a_n \) is often called the Mhaskar–
Rahmanov–Saff number which tells us where the norm of a weighted poly-
nomial “lives.”

Now consider the Lagrange interpolation on arbitrary nodes \( x_0 < x_1 < \cdots < x_n \). The weighted Lebesgue constant plays an important role in the
theory of weighted convergence of Lagrange interpolation in certain classes
of functions. It is defined as the supremum norm on \( \mathbb{R} \) of the weighted
Lebesgue function

\[
\lambda_n(x) = w(x) \sum_{k=0}^{n} \frac{|\xi_k(x)|}{w(x_k)},
\]

for all \( x \) in the interval where the nodes of a “good” interpolation polynomial
are situated.
where
\[ l_k(x) = \prod_{i=0}^{n} \frac{x-x_i}{x_k-x_i}, \quad k = 0, \ldots, n, \]
are the fundamental polynomials of Lagrange interpolation. In this note we establish a relation between the size of the nodes and the weighted Lebesgue constant.

**Proposition 1.** For every system of nodes we have
\[ \max_{0 \leq k \leq n} |x_k| \leq a_n \left( 1 + c \left( \frac{\log \| \lambda_n \|}{n} \right)^{2/3} \right) \]
with some constant \( c > 0 \) depending only on \( w \).

**Proof.** If \( q_n/w \in \Pi_n \) is arbitrary, then
\[ |q_n(x)| \leq e^{e^{U_{a_n}(x/a_n)}} \| q_n \|, \quad x \in \mathbb{R} \]
where \( U_{a_n}(x) \) is a so-called “majorizing function” (cf. [LL, Lemma 7.1] applied with \( R = a_n \) and combined with (1)). Using [LL, inequality (7.14)] with \( R = a_n \) and \( \epsilon = (x/a_n) - 1 \) we obtain
\[ |q_n(x)| \leq e^{-c_1 n(x/a_n) - 1} \| q_n \|, \quad |x| \geq a_n, \quad (2) \]
where \( c_1 > 0 \) depends only on \( w \).

Now let \( y_n \in \mathbb{R} \) be such that \( \| \lambda_n \| = \lambda_n(y_n) \), and consider the weighted polynomial
\[ q_n(x) := w(x) \sum_{k=0}^{n} \frac{\ell_k(x) \text{sgn} \ell_k(y_n)}{w(x_k)} \]
of degree at most \( n \). Evidently
\[ |q_n(x)| \leq \lambda_n(x) \leq \| \lambda_n \| = q_n(y_n), \quad x \in \mathbb{R}, \]
that is, \( \| q_n \| = \| \lambda_n \| \).

Suppose \( |x_j| \geq a_n \). Then applying (2) to this \( q_n \) with \( x = x_j \) and using \( |q_n(x_j)| = 1 \) we obtain
\[ 1 \leq e^{-c_1 n(x_j/a_n) - 1} \| \lambda_n \|. \]
Hence, a simple rearrangement yields the statement of the proposition with \( c = 1/c_1^{2/3} \).

In particular, if the Lebesgue constant is optimal, that is, \( \lambda_n = O(\log n) \), then Proposition 1 gives

\[
\max_{0 \leq k \leq n} |x_k| \leq a_n \left( 1 + c_2 \left( \frac{\log \log n}{n} \right)^{2/3} \right).
\]

For the construction of such system of nodes see [S].

In some situations, it is interesting to consider the case when the weighted fundamental polynomials

\[
\frac{w(x)}{w(x_k)} f_k(x)
\]

are uniformly bounded. This is the case when we consider Hermite–Fejér interpolation with bounded norm (see [S]), or we want to construct convergent Lagrange interpolation polynomials of degree at most \( n(1 + \varepsilon) \) (see [V]). Then, similarly to the above considerations, we obtain

**Proposition 2.** If the weighted fundamental function (3) belonging to the node of largest absolute value is uniformly bounded, then

\[
\max_{0 \leq k \leq n} |x_k| \leq a_n \left( 1 + c_2 \frac{n^{2/3}}{n^{2/3}} \right).
\]

**REFERENCES**


