JOURNAL OF COMBINATORIAL THEORY, Series A 23, 275-290 (1977)

Central and Local Limit Theorems for the Coefficients of Polynomials of Binomial Type

E. RODNEY CANFIELD*

Department of Statistics and Computer Science, University of Georgia, Athens, Georgia 30602

Communicated by the Managing Editors

Received June 28, 1976

We introduce the problem of establishing a central limit theorem for the coefficients of a sequence of polynomials $P_n(x)$ of binomial type; that is, a sequence $P_n(x)$ satisfying $\exp(xg(u)) = \sum_{n=0}^{\infty} P_n(x)(u^n/n!)$ for some (formal) power series $g(u)$ lacking constant term. We give a complete answer in the case when $g(u)$ is a polynomial, and point out the widest known class of nonpolynomial power series $g(u)$ for which the corresponding central limit theorem is known true. We also give the least restrictive conditions known for the coefficients of $P_n(x)$ which permit passage from a central to a local limit theorem, as well as a simple criterion for the generating function $g(u)$ which assures these conditions on the coefficients of $P_n(x)$. The latter criterion is a new and general result concerning log concavity of doubly indexed sequences of numbers with combinatorial significance. Asymptotic formulas for the coefficients of $P_n(x)$ are developed.

1. ASYMPTOIC NORMALITY AND POLYNOMIALS OF BINOMIAL TYPE

We say that a doubly indexed sequence of nonnegative numbers $s(n, k)$ is *asymptotically normal* (or, *satisfies a central limit theorem*), if there are numbers μ_n and σ_n such that for the probabilities $p_n(k) = s(n, k)/\sum_k s(n, k)$,

$$\lim_{n\to\infty} \sup_x \left| \sum_{k\leqslant\mu_n+x\sigma_n} p_n(k) - (1/(2\pi)^{1/2}) \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = 0. \qquad (1.1)$$

Because the limit distribution $N(x) = (1/(2\pi)^{1/2}) \int_{-\infty}^{x} e^{-t^2/2}\, dt$ is continuous, condition (1.1) is equivalent to pointwise convergence. We also say that the numbers $s(n, k)$ are asymptotically normal *with mean* μ_n *and variance* σ_n^2. Such a central limit theorem is a frequent occurrence among sequences $s(n, k)$ arising in combinatorial enumeration; the binomial coefficient case

* Much of the research for this paper was done as part of the author's Doctoral Dissertation at the University of California, San Diego, supported by NSF Grant GJ43333.

$s(n, k) = \binom{n}{k}$ is an instance of the classical central limit theorem; see [2] for further examples of combinatorial interest.

Harper [8] established (1.1) for the Stirling numbers of the second kind, that is, $s(n, k) =$ the number of partitions of a labeled n-set into k nonempty blocks. A crucial fact in his proof is that the polynomials $P_n(x) = \sum_k s(n, k)\, x^k$ have all roots real and nonpositive. Additional examples of (1.1) via the approach of real roots may be found in [3, 4], both including a local limit theorem with error estimate. (Local limit theorems are discussed in Section 3.) However, the polynomials whose coefficients are Stirling numbers of the second kind have another property of note, namely, they satisfy the identifies

$$P_n(x + y) = \sum_{k=0}^{n} \binom{n}{k} P_k(x)\, P_{n-k}(y). \tag{1.2}$$

A sequence of polynomials satisfying (1.2), and having $P_0(x) \equiv 1$, is said to be of *binomial type*. For a sequence of polynomials to be of binomial type it is both necessary and sufficient that there exists a (formal) power series $g(u)$ lacking constant term such that

$$\exp(xg(u)) = \sum_{n=0}^{\infty} P_n(x)(u^n/n!). \tag{1.3}$$

If $g(u)$ is expressed as an exponential generating function

$$g(u) = \sum_{n=1}^{\infty} g_n(u^n/n!), \tag{1.4}$$

then expansion of (1.3) reveals the following formula for $s(n, k)$, the coefficients of $P_n(x)$

$$s(n, k) = (1/k!) \sum \binom{n}{v_1, ..., v_k} g_{v_1} \cdots g_{v_k}, \tag{1.5}$$

in which the summation is over all k-tuples $(v_1, ..., v_k)$ with $v_i \geqslant 1$ and $\sum v_i = n$. Hence we have the following combinatorial interpretation of the coefficients $s(n, k)$: If g_n is the number of ways to construct an object of some sort on a labeled n-set, then $s(n, k)$ is the number of ways to partition a labeled n-set into k nonempty blocks, and construct such an object on each block. The Stirling numbers of the second kind are the case when $g_n \equiv 1$, that is, $g(u) = e^u - 1$. When $g(u) = -ln(1 - u)$, that is, $g_n = (n - 1)!$, then $s(n, k) =$ the number of permutations of an n-element set having k cycles, because $(n - 1)!$ is the number of cycles one may construct on an n-set. The theory of polynomials of binomial type is a rich one for combinatorial enumeration, and many properties have been developed. Concerning some of these properties, Eqs. (1.2–1.5), and the above combinatorial interpretation, see [6, 7, 12].

In view of Harper's result, we may pose a question: When do the coefficients $s(n, k)$ of a sequence of polynomials of binomial type $P_n(x)$ satisfy (1.1)? Since these coefficients are completely determined by $g(u)$, we seek suitable conditions on that function. There are unpublished results [3] revealing a wide class of such functions $g(u)$, recovering in particular Harper's original findings. Section 2 considers the case when $g(u)$ is a polynomial, for which it is possible to give a complete and affirmative answer to the above question. Section 3 presents known results and difficulties concerning the derivation of local limit theorems from central. Some familiar combinatorial distributions covered by Theorem I below include:

(a) Restricted occupancy problem: $s(n, k)$ = the number of partitions of an n-element set into k nonempty blocks of maximal size M. That is, $g(u) = \sum_{n=1}^{M} (u^n/n!)$. This includes the problem of distributing labeled balls into unlabeled boxes of limited capacity.

(b) Permutations with special cycle lengths: Included here is $s(n, k)$ = the number of permutations having k cycles of maximal size M, for which $g(u) = \sum_{n=1}^{M} (u^n/n)$; and also $s(n, k)$ = the number of permutations π having k cycles such that π^M is the identity, for which $g(u) = \sum_{d|M} (u^d/d)$.

(c) Graphs with components of bounded size: How many graphs on n labeled vertices have k components? As $n \to \infty$, almost all graphs are connected [5]; but when component size is bounded, we see by Theorem I that the distribution of k is asymptotically normal. A similar situation holds for forests of rooted trees: as $n \to \infty$, the average forest contains two trees (of necessity, large trees) and the standard deviation approaches 1. However when the size of the trees is bounded, the distribution of the number of trees in a forest is asymptotically normal, again according to Theorem I. Incidentally, concerning $s(n, k)$ = the number of forests on n vertices containing k rooted trees, the above results on the mean and standard deviation follow directly from the fact that $P_n(x) \equiv \sum_k s(n, k)\, x^k = x(x + n)^{n-1}$. This latter formula may be established, among other ways, by an application of the general theory of polynomials of binomial type (see [12]).

2. The Central Limit Theorem When $g(u)$ = Polynomial

Throughout this section we adhere to the following notation:

$$g(u) = \sum_{j=1}^{m} c_j u^j, \qquad c_j \geqslant 0, \qquad c_m \neq 0;$$

$$A(u) = ug'(u) = \sum_{j=1}^{m} jc_j u^j;$$

$$B(u) = uA'(u) = \sum_{j=1}^{m} j^2 c_j u^j;$$

$P_n(x)$ are generated by $g(u)$: (2.1)

$$\exp(xg(u)) = \sum_{n=0}^{\infty} P_n(x)(u^n/n!);$$

$r(y)$ is the inverse function of A:

$$A(r(y)) \equiv y;$$

$$\sigma(x) = (g(x) - (A(x))^2/B(x))^{1/2};$$

$$\sigma_n \equiv \sigma(r(n));$$

$$\mu_n \equiv g(r(n));$$

$$\alpha = e^{t/\sigma_n}, \qquad t \text{ a real number.}$$

Since all $c_j \geqslant 0$ (for combinatorial applications $c_j = g_j/j!$), $A(x)$ is monotone increasing for positive x, so $r(y)$ is a well-defined real, positive function for $y > 0$. We also note

$$g(x)\, B(x) - (A(x))^2 = \sum_i \left(\sum_{\substack{j+k=i \\ j>k}} (k-j)^2\, c_k c_j \right) x^i, \tag{2.2}$$

so that for $x > 0$ we have $B > 0$ and $gB - A^2 \geqslant 0$, meaning that $\sigma(x)$ is a real positive function.

THEOREM I. *Let $g(u)$ be a polynomial with real nonnegative coefficients: $g(u) = \sum_{j=1}^{m} c_j u^j$, $c_m \neq 0$. Assume that* GCD $\{j \mid c_j \neq 0\} = 1$. *Then the coefficients $s(n, k)$ of the polynomials $P_n(x)$ generated by $g(u)$ are asymptotically normal with mean μ_n and variance σ_n^2, defined as above.*

Remark 1. The assumption about the greatest common divisor is no loss. Indeed, when GCD$\{j \mid c_j \neq 0\} = d > 1$, then $s(n, k) \equiv 0$ for $d \nmid n$. In such a case define a polynomial $h(u) = g(u^{1/d})$. Then $h(u)$ satisfies the hypotheses of the theorem, and the $s_h(n, k)$ which it generates are precisely the nontrivial $s(nd, k)$ generated by the original $g(u)$.

Remark 2. When $m = 1$, $g(u) = c_1 u$ satisfies the hypotheses of the theorem. In this case, $\sigma = 0$ and $\mu_n = n$. While in some sense the resulting $s(n, k)$ are indeed "asymptotically normal with mean n and variance 0," basically this is a degenerate situation. Hence we shall make use in the proof that follows of the integer $J = \max\{j < m \mid c_j \neq 0\}$. From (2.2) we note

$$\sigma_n^2 \sim c \cdot r(n)^J, \qquad \text{some positive } c. \tag{2.3}$$

The rest of this section is devoted to the proof of Theorem I. If we let Y_n be the sequence of discrete random variables defined by: $\mathrm{Prob}\{Y_n = k\} = s(n, k)/\sum_k s(n, k)$; and let X_n be defined by: $X_n = (Y_n - \mu_n)/\sigma_n$; then, we may establish (1.1) by proving that the moment generating functions of X_n converge pointwise to the moment generating function of the normal distribution. Expressing these moment generating functions as functions of t, with the notation (2.1) for e^{t/σ_n}, we must show

$$\frac{e^{-\mu_n t/\sigma_n} P_n(\alpha)}{P_n(1)} \to e^{t^2/2}, \qquad \text{as} \quad n \to \infty. \tag{2.4}$$

The convergence in (2.4) will be shown to be uniform for $t \in K$, a compact set. The proof consists of two major parts, each involving much calculation. First we shall show

$$P_n(\alpha)/P_n(1) \sim \exp \left\{ \int_1^\alpha g(r(n/\beta))\, d\beta \right\}, \qquad \text{as } n \to \infty, \text{ uniformly for } t \in K. \tag{2.5}$$

Second we verify

$$(-\mu_n t/\sigma_n) + \int_1^\alpha g(r(n/\beta))\, d\beta \to t^2/2, \qquad \text{as } n \to \infty, \text{ uniformly for } t \in K. \tag{2.6}$$

Let $C = [a, b]$, $0 < a < 1 < b$, a and b fixed. Following an idea used in [9], we produce an asymptotic formula (2.8) for $P_n(x)/n!$ which is uniform for $x \in C$ as $n \to \infty$. Start by writing the Cauchy integral formula

$$P_n(x)/n! = (1/2\pi i) \oint_{|z|=r} \exp\{xg(z)\} dz/z^{n+1},$$

as a sum of two integrals, with r^n transposed:

$$\begin{aligned} (P_n(x)/n!)\, r^n &= (1/2\pi) \int_{-\delta}^{+\delta} \exp\{xg(re^{i\theta}) - in\theta\}\, d\theta \\ &\quad + (1/2\pi) \int_{+\delta}^{2\pi-\delta} \exp\{xg(re^{i\theta}) - in\theta\}\, d\theta \\ &= I_1 + I_2 . \end{aligned}$$

With δ taken as $r^{-m/2+1/8}$ (m is the degree of g), we can approximate I_1 and show that I_2 is negligible for larger r.

For some ζ between 0 and θ, by Taylor:

$$g(re^{i\theta}) = g(r) + i\theta A(r) - \tfrac{1}{2}\theta^2 B(r) - ire^{i\zeta}(\theta^3/6)\, B'(re^{i\zeta}).$$

Exponentiating, since x is bounded and δ is properly small,

$$\exp\{xg(re^{i\theta})\} = \exp\{x(g(r) + i\theta \cdot A(r) - \tfrac{1}{2}\theta^2 \cdot B(r))\}[1 + o(1)],$$

as $r \to \infty$, where $o(1) \to 0$ uniformly for $x \in C$ and $|\theta| \leqslant \delta$. Then,

$$I_1 = \frac{\exp\{xg(r)\}}{2\pi}\Big[\int_{-\delta}^{+\delta} \exp\{i\theta(xA - n) - \tfrac{1}{2}x\theta^2 B\}\, d\theta$$

$$+ o\Big\{\int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}x\theta^2 B\}\, d\theta\Big\}\Big]$$

$$= \frac{\exp\{xg(r)\}}{2\pi}\Big[\int_{-\delta(xB)^{1/2}}^{+\delta(xB)^{1/2}} \exp\{-\tfrac{1}{2}y^2 + icy\}(xB)^{-1/2}\, dy + o((xB)^{-1/2})\Big],$$

after the change of variables $\theta(xB)^{1/2} = y$ and $c = (xA - n)/(xB)^{1/2}$. Since $\delta^2 B \to \infty$,

$$\int_{-\delta(xB)^{1/2}}^{+\delta(xB)^{1/2}} \exp\{-\tfrac{1}{2}y^2 + icy\}\, dy = \int_{-\infty}^{+\infty} \exp\{-\tfrac{1}{2}y^2 + icy\}\, dy + o(1)$$

$$= (2\pi)^{1/2} \cdot \exp\Big[-\frac{(xA(r) - n)^2}{2xB(r)}\Big] + o(1).$$

By the corollary to Lemma 1 below, $I_2 = o(\exp\{xg(r)\})/(B(r))^{1/2}$, as $r \to \infty$, so that altogether

$$\frac{P_n(x)}{n!} r^n = \frac{\exp\{xg(r)\}}{(2\pi xB(r))^{1/2}}\Big\{\exp\Big[-\frac{(xA(r) - n)^2}{2xB(r)}\Big] + o(1)\Big\}, \quad \text{as } r \to \infty, \tag{2.7}$$

in which it is seen that $o(1)$ is independent of n and uniform for $x \in C$. Now, for each n, x we choose $r = r(n/x)$, where the *function* r is the inverse of the polynomial A, so that $xA(r(n/x)) = n$. Since $r(n/x) \to \infty$ as $n \to \infty$, from (2.7) we deduce the desired

$$\frac{P_n(x)}{n!} \sim \frac{\exp\{xg(r(n/x))\}}{(r(n/x))^n(2\pi xB(r(n/x)))^{1/2}}, \qquad \text{as } n \to \infty, \text{ uniformly for } x \in C. \tag{2.8}$$

From (2.3), $\sigma_n \to \infty$ so that $\alpha \to 1$ uniformly for $t \in K$. By (2.1), $r(y) \sim (y/mc_m)^{1/m}$ as $y \to \infty$, so it follows that $B(r(n/\alpha)) \sim B(r(n))$, as $n \to \infty$, uniformly for $t \in K$. Hence by division from (2.8)

$$\frac{P_n(\alpha)}{P_n(1)} \sim \frac{\exp\{\alpha g(r(n/\alpha))\}}{\exp\{g(r(n))\}} \cdot \Big(\frac{r(n)}{r(n/\alpha)}\Big)^n, \qquad \text{as } n \to \infty. \tag{2.9}$$

Since

$$(d/d\beta)\{\beta \cdot g(r(n/\beta)) - n \log r(n/\beta)\} = g(r(n/\beta)),$$

(2.5) follows from (2.9). The above differentiation is a routine check, the only trick being that

$$g'\left(r\left(\frac{n}{\beta}\right)\right) = \frac{A(r(n/\beta))}{r(n/\beta)} = \frac{n/\beta}{r(n/\beta)}.$$

To complete Theorem I by establishing (2.6), it is necessary to obtain explicit information about the higher order derivatives with respect to β of the function $\beta \to g(r(n/\beta))$ appearing in the integral (2.6). We denote this function as $F(\beta)$, with the dependence on n being understood.

Let T_j, $j = 1, 2, \ldots$, be a countable collection of variables. Define an operator Q on real polynomials in the variables T_j by these rules:

(i) $Q(T_j) = T_{j+1}$;

(ii) $Q(T_{j_1} \cdots T_{j_l}) = \sum_{i=1}^{l} T(j_1) \cdots Q(T_{j_i}) \cdots T_{j_l}$ (Leibnitz product rule);

(iii) Q is linear with respect to addition and scalar multiplication.

Then inductively we may define polynomials $R_k(T)$ in the variables T_j by

$$\begin{aligned} R_1(T) &= -T_1^2, \\ R_{k+1}(T) &= -T_1T_2Q(R_k) - \{kT_2^2 - (2k-1)\,T_1T_3\}\,R_k. \end{aligned} \tag{2.10}$$

We use $R_k(1)$ to denote the value of R_k resulting when all T_j are set equal to 1. From the above recursion we see that R_k is a homogeneous polynomial of degree $2k$; hence, $Q(R_k)\,|_{T=1} = 2k\,R_k(1)$. Therefore,

$$\begin{aligned} R_{k+1}(1) &= -2kR_k(1) - \{-k+1\}\,R_k(1) \\ &= (-k-1)\,R_k(1). \end{aligned}$$

It follows then that

$$R_k(1) = (-1)^k\,k! \tag{2.11}$$

It is also easy to conclude from the recursion that if $T_{j_1}T_{j_2} \cdots T_{j_{2k}}$ is a term appearing in R_k, then

$$j_1 + j_2 + \cdots + j_{2k} = 4k - 2. \tag{2.12}$$

The relation of the polynomials $R_k(T)$ to our function $F(\beta) = g(r(n/\beta))$ is given by the following.

PROPOSITION. *For $k \geqslant 1$*

$$F^{(k)}(\beta) = R_k(T)/(\beta^k T_2^{2k-1}), \tag{2.13}$$

in which each T_j is set equal to the polynomial $\sum_{i=1}^{m} i^j c_i r^i$, evaluated at $r = r(n/\beta)$. In particular, $T_1(r) = A(r)$ and $T_2(r) = B(r)$ in our earlier notation.

Proof. By definition, $T_1(r(y)) = y$. Hence,

$$\frac{r'(y)}{r(y)} = \frac{1}{T_2(r(y))},$$

and so,

$$\begin{aligned}\frac{d}{d\beta} T_j\left(r\left(\frac{n}{\beta}\right)\right) &= T_j'\left(r\left(\frac{n}{\beta}\right)\right) \cdot r'\left(\frac{n}{\beta}\right) \cdot \frac{-n}{\beta^2} \\ &= T_{j+1}\left(r\left(\frac{n}{\beta}\right)\right) \cdot \frac{r'(n/\beta)}{r(n/\beta)} \cdot \frac{-n}{\beta^2} \\ &= \frac{-T_1(r(n/\beta))}{\beta \cdot T_2(r(n/\beta))} \cdot T_{j+1}\left(r\left(\frac{n}{\beta}\right)\right),\end{aligned}$$

or, if all polynomials T_j are understood to be evaluated at $r(n/\beta)$,

$$(d/d\beta)\, T_j = (-T_1/\beta T_2) \cdot T_{j+1};$$

so that

$$(d/d\beta)\, R_k(T) = (-T_1/\beta T_2) \cdot Q(R_k(T)). \tag{2.14}$$

Using (2.14), the proof of (2.13) is a routine induction: When (2.13) is differentiated with respect to β, the result is Eq. (2.13) with k replaced by $k + 1$. This completes the proposition.

From (2.11) and (2.12) there follows

$$F^{(k)}(\beta) = \frac{(-1)^k\, k!\, c_m^{2k} m^{4k-2} (r(n/\beta))^{2km} + \text{lower powers of } r(n/\beta)}{\beta^k \{c_m^{2k-1} m^{4k-2} (r(n/\beta))^{(2k-1)m} + \text{lower powers of } (r(n/\beta)\}}, \tag{2.15}$$

since each term

$$(\text{constant}) \cdot T_{j_1} \cdots T_{j_{2k}}$$

appearing in $R_k(T)$ has as its highest power of $r(n/\beta)$

$$\begin{aligned}&(\text{constant}) \cdot c_m^{2k} \cdot m^{j_1+\cdots+j_{2k}} \cdot r(n/\beta)^{2km} \\ &= (\text{constant}) \cdot c_m^{2k} \cdot m^{4k-2} r(n/\beta)^{2km} \qquad \text{by (2.12)}\end{aligned}$$

and the sum of these constants is $R_k(1) = (-1)^k\, k!$ by (2.11). This accounts for the numerator in (2.15). For the denominator, it is a matter of finding the leading coefficient in T_2^{2k-1}. (Where T_2 is a polynomial in $r(n/\beta)$.)

We need one more result before turning to the proof of (2.6). Recall the expansion,

$$\frac{(e^u - 1)^k}{k!} = \sum_{n \leqslant K} \frac{S(n, k)\, u^n}{n!} + \sum_{n > K} \frac{S(n, k)\, u^n}{n!},$$

in which $S(n, k)$ denotes Stirling numbers of the second kind. Since $S(n, k) \leqslant n^n < 3^n \cdot n!$,

$$\sum_{n > K} \frac{S(n, k)\, u^n}{n!} \leqslant \frac{(3 \mid u \mid)^{K+1}}{1 - 3 \mid u \mid}, \qquad \text{provided} \quad \mid u \mid < 1/3.$$

Recalling that $\alpha = e^{t/\sigma_n}$, and $\sigma_n \to \infty$, we have

$$\frac{(\alpha - 1)^{j+1}}{(j + 1)!} = \sum_{l \leqslant K} \frac{S(l, j + 1)}{\sigma_n{}^l} \frac{t^l}{l!} + O\left(\frac{1}{\sigma_n^{K+1}}\right), \qquad \text{as} \quad n \to \infty. \quad (2.16)$$

We shall let K be a positive integer such that $(K + 1)J/2 > m$. J was defined in the first remark following the statement of Theorem I.

Returning finally to the proof of (2.6):

$$\frac{-\mu_n t}{\sigma_n} + \int_1^\alpha F(\beta)\, d\beta$$

$$= \frac{-\mu_n t}{\sigma_n} + \sum_{j=0}^{K-1} F^{(j)}(1) \frac{(\alpha - 1)^{j+1}}{(j + 1)!} + F^{(K)}(\beta) \cdot \frac{(\alpha - 1)^{K+1}}{(K + 1)!}, \quad (2.17)$$

with β between 1 and α, where we have expanded the integral about $\alpha = 1$ by Taylor's formula. This is the quantity which must be shown to converge to $t^2/2$ as $n \to \infty$. According to (2.15),

$$F^K(\beta) = O(r(n/\beta)^m)$$
$$= O(r(n)^m)$$

since $r(n) \sim r(n/\beta)$. By (2.3), our choice of K, and $\alpha - 1 = O(t/\sigma_n)$, the right most term in (2.17) goes to 0. Similarly, by (2.16), we can expand the other terms on the right of (2.17) and conclude

$$\frac{-\mu_n t}{\sigma_n} + \int_1^\alpha F(\beta)\, d\beta = \frac{-\mu_n t}{\sigma_n} + \sum_{j=0}^{K-1} F^{(j)}(1) \left[\sum_{l=j+1}^{K} \frac{S(l, j + 1)}{\sigma_n{}^l} \cdot \frac{t^l}{l!} \right] + o(1). \quad (2.18)$$

We must show that this approaches $t^2/2$ as $n \to \infty$.

Now, μ_n and σ_n were originally defined so that the coefficient of t in the above equation is 0, and the coefficient of $t^2/2$ is 1. The coefficient of $t^l/l!$ for $3 \leqslant l \leqslant K$ is

$$\frac{1}{\sigma_n{}^l} \sum_{j=0}^{l-1} F^{(j)}(1)\, S(l, j+1)$$
$$= \frac{1}{\sigma_n{}^l} \cdot \frac{g \cdot T_2^{2l-3} S(l, 1) + \sum_{k=1}^{l-1} R_k(T) \cdot T_2^{2l-2k-2} \cdot S(l, k+1)}{T_2^{2l-3}}, \tag{2.19}$$

using (2.13) with $\beta = 1$ and a common denominator of T_2^{2l-3}. All polynomials g and T_j are evaluated at $r(n)$. Each expression $R_k(T) \cdot T_2^{2l-2k-2}$ has as its leading term, by (2.15), for $k \geqslant 1$

$$(-1)^k \cdot k!\, c_m^{2l-2} m^{4l-6} r(n)^{m(2l-2)};$$

this formula is also correct when $k = 0$ for the term $g \cdot T_2^{2l-3}$. It follows that the total coefficient of $c_m^{2l-2} m^{4l-6} r(n)^{m(2l-2)}$ in the numerator of (2.19) is

$$\sum_{k=0}^{l-1} (-1)^k k!\, S(l, k+1).$$

However, a well-known combinatorial identity (see [10, p. 183]) says that the above expression is identically 0 for $l \geqslant 3$. Hence the numerator does not have the maximum possible $= m(2l-2)$ degree in $r(n)$, so its degree is $\leqslant m(2l-3) + J$. This means that the coefficient of $t^l/l!$ in (2.18) is $(1/\sigma_n{}^l) \cdot O(r(n)^J)$, which goes to 0 as $n \to \infty$, since $l \geqslant 3$. This completes the proof of the theorem, except for the following lemma and corollary.

LEMMA 1. *There are positive constants c and R such that* $g(r) - \operatorname{Re}(g(re^{i\theta})) \geqslant c \cdot r^{1/4}$ *when* $r^{-m/2+1/8} \leqslant |\theta| \leqslant \pi$, *and* $r \geqslant R$ (*m denotes the degree of the polynomial g, and* Re *is real part*).

Proof. Let $c = \frac{1}{8} \min\{c_k \mid c_k \neq 0\}$. Note that

$$g(r) - \operatorname{Re}(g(re^{i\theta})) = \sum_{k=1}^{m} c_k r^k (1 - \cos k\theta) \geqslant c_k r^k (1 - \cos k\theta).$$

If $1 - \cos k\theta \geqslant \frac{1}{8} r^{-3/4}$ for some k with $c_k \neq 0$, then the conclusion holds. Else it follows that $e^{ik\theta}$ is within a distance $r^{-3/4}$ of 1 (all distances and neighborhoods are measured as arc length on the unit circle in this lemma), so that $e^{i\theta}$ must be within a distance of $r^{-3/4}$ of a kth root of unity for all k with $c_k \neq 0$. Consider all kth roots of 1 with $c_k \neq 0$. Some of these may coincide, but for large r the $r^{-3/4}$ neighborhoods of the distinct ones are disjoint. Hence, $e^{i\theta}$ is within $r^{-3/4}$ of a point which is simultaneously a kth root of unity for all k with $c_k \neq 0$. Since $\mathrm{GCD}\{k \mid c_k \neq 0\} = 1$, this point can only be 1. Hence, $|\theta| \leqslant r^{-3/4}$. For large r, such a θ is so small that

$$(1 - \cos m\theta) \geqslant 1 - \cos\theta \geqslant \tfrac{1}{8}\theta^2 \geqslant \tfrac{1}{8} \cdot r^{-m+1/4};$$

then

$$g(r) - \operatorname{Re} g(re^{i\theta}) \geqslant c_m r^m \cdot \tfrac{1}{8} r^{-m+1/4} \geqslant c r^{1/4},$$

and the lemma is proved.

COROLLARY TO LEMMA 1.

$$\exp\{xg(re^{i\theta})\} = \frac{o(\exp\{xg(r)\})}{(xB(r))^{1/2}}, \qquad as \quad r \to \infty,$$

uniformly for $x \in C$, with θ as in Lemma 1.

Proof. In absolute value $\exp\{xg(re^{i\theta})\}/\exp\{xg(r)\}$ is $\exp\{x(\operatorname{Re}(g(re^{i\theta}) - g(r)))\}$ but x is bounded away from 0 and $B(r)$ has only polynomial growth. This establishes the corollary.

3. THE LOCAL LIMIT THEOREM

Although the central limit theorem (1.1) provides a certain qualitative feel for the numbers $s(n, k)$, greater information is provided by a local limit theorem. We say that a doubly indexed sequence of nonnegative numbers $s(n, k)$ satisfies a *local limit theorem on the set* S provided that for the probabilities $p_n(k) = s(n, k)/\sum_k s(n, k))$ there are numbers μ_n and σ_n such that

$$\lim_{n\to\infty} \sup_{x\in S} \left| \sigma_n p_n([\mu_n + x\sigma_n]) - (1/(2\pi)^{1/2})\, e^{-x^2/2} \right| = 0. \tag{3.1}$$

Generally it is not possible to conclude (3.1) from (1.1). Bender [2] has indicated that certain "smoothness conditions" on the $s(n, k)$ are sufficient; in particular, if the $s(n, k)$ are unimodal and $\sigma_n \to \infty$, then (3.1) follows from (1.1) for $S = \{x\colon |x| \geqslant \epsilon\}$, any positive ϵ. Moreover, he shows that in the presence of (1.1) the local theorem (3.1) is valid for $S =$ all reals provided the $s(n, k)$ are log concave:

$$(s(n, k))^2 \geqslant s(n, k - 1) \cdot s(n, k + 1). \tag{3.2}$$

Two consequences of log concavity are used in his proof:

$$\begin{gathered}(s(n, k + l))^2 \geqslant s(n, k) \cdot s(n, k + 2l),\\ s(n, k) \text{ are unimodal.}\end{gathered} \tag{3.3}$$

It is important to remark, however, that concluding (3.3) from (3.2) actually assumes certain $s(n, k)$ are nonzero. We illustrate with an

EXAMPLE. Returning to the context of polynomials of bionomial type, let $g(u) = u + u^4/24$ be the generating function for the polynomials $P_n(x)$. According to our combinatorial interpretation, the coefficient $s(n, k)$ of x^k in $P_n(x)$ is the number of ways to partition an n-element labeled set into k nonempty blocks, each block containing one or four elements. By Theorem I, the $s(n, k)$ satisfy (1.1); from elementary considerations it is clear that

$$(s(n, 1), s(n, 2),..., s(n, n)) = (0,..., 0, *, 0, 0, *, 0, 0,..., *, 0, 0, *),$$

where * indicates a nonzero number. More specifically, the first nonzero $s(n, k)$ is for $k = n - 3[n/4]$; additional nonzero $s(n, k)$ occur for $k = n - 3[n/4] + 3l$, $l = 1, 2,..., [n/4]$. Relation (3.2) is then trivially satisfied. Yet a local limit theorem (3.1) is obviously impossible for any reasonable set S. Notice that the second condition of (3.3) is not valid, even though $s(n, k)$ are log concave according to definition (3.2).

We wish to avoid behavior as in the previous example. Relations (3.3) *will* be valid provided

$$\begin{aligned} &\text{(i)} \quad s(n, k) \text{ are log concave, that is, satisfy (3.2),} \\ &\text{(ii)} \quad \{k: s(n, k) \neq 0\} = \{k: l_n \leq k \leq u_n\}, \text{ for some integers } l_n \text{ and } u_n. \end{aligned} \tag{3.4}$$

That is, the nonzero $s(n, k)$ must appear in an interval, not separated by $s(n, k)$ which are zero. We now state the only known general local limit theorem for polynomials of binomial type.

THEOREM II. *Let $s(n, k)$ be the coefficients of a sequence of polynomials of binomial type. (We are not assuming that $g(u)$ is a polynomial.) Assuming that $\sigma_n \to \infty$, that the $s(n, k)$ satisfy the central limit theorem* (1.1), *and that the $s(n, k)$ satisfy* (3.4), *then they also satisfy the local limit theorem* (3.1) *with S = all reals. Consequently we have the asymptotic formula*

$$s(n, k) \sim (P_n(1)\, e^{-x^2/2}/\sigma_n(2\pi)^{1/2}), \qquad \text{as} \quad n \to \infty, \tag{3.5}$$

where

$$k = \mu_n + x\sigma_n \qquad \text{and} \qquad x = O(1).$$

Proof. See [2] for a proof that the more carefully stated (3.4) and (1.1) imply (3.1); [3] provides some detail as to how the rate of convergence in (3.1) depends on the rate of convergence in (1.1) and the rate of convergence $\sigma_n \to \infty$. Formula (3.5) is of course an immediate consequence of the local limit theorem. Since we have developed an asymptotic formula for $P_n(1)$

(see (2.8)) (3.5) achieves the desirable goal of estimating $s(n, k)$ purely in terms of the generating function $g(u)$. Notice that (3.5) is valid only for k in a restricted range.

We next wish to demonstrate conditions on the generating function $g(u)$ which will assure (3.4) for the numbers $s(n, k)$, so that Theorem II is applicable. Such a result is of independent interest in its own right, as log concavity is a frequently studied property of combinatorial sequences, and Theorem III is the only such general result for the coefficients of polynomials of binomial type. The easiest way to assure the second condition of (3.4), that is, that the nonzero $s(n, k)$ contain no gaps, is to assume the same for the coefficients of $g(u)$. Interestingly enough, a way to assure the first condition (log concavity) is again to assume the same for the coefficients of $g(u)$. We now prove all this: as

THEOREM III. *Let* $g(u) = \sum_{j=1}^{\infty} c_j u^j$, *with* $c_j \geqslant 0$, *satisfy*:

$$\text{(a)} \quad \{j: c_j \neq 0\} = \{j: l \leqslant j \leqslant L \leqslant \infty\},$$

that is, the nonzero c_j *occur in an interval without gaps*;

$$\text{(b)} \quad c_j^2 \geqslant c_{j-1} c_{j+1} .$$

Then the coefficients $s(n, k)$ *of the polynomials* $P_n(x)$ *generated by* $g(u)$ *satisfy both conditions* (i) *and* (ii) *of* (3.4).

Proof. Notice that $g(u)$ is not assumed to be a polynomial in this theorem That (a) above implies (ii) of (3.4) is not difficult to see. If $s(n, k_1) \neq 0$ and $s(n, k_2) \neq 0$, there is clearly a nonzero term in the summation (1.5) for any $s(n, k)$ with $k_1 \leqslant k \leqslant k_2$. Consequently we concentrate on the log concavity part of the theorem. Notice that (a) and (b) *together* do indeed tell us that

$$c_i c_j \geqslant c_{i+l} c_{j-l}, \qquad \text{whenever} \qquad i \geqslant j. \tag{3.6}$$

We are going to show that

$$\left(\frac{k!\, s(n, k)}{n!}\right)^2 \geqslant \frac{(k-1)!\, s(n, k-1)}{n!} \cdot \frac{(k+1)!\, s(n, k+1)}{n!}, \tag{3.7}$$

so that the $s(n, k)$ are *strictly* log concave. First some notation:

$$A(n, k) = \left\{(v_1, v_2, \ldots, v_k) \mid v_i \geqslant 1 \text{ and } \sum v_i = n\right\};$$

$$c_v = c_{v_1} \cdot c_{v_2} \cdots c_{v_k}, \qquad \text{when} \quad v \in A(n, k);$$

$$A_\alpha(n, k) = \{v \in A(n, k) \mid v_k = \alpha\};$$

$$A_\alpha^\beta(n, k+1) = \{v \in A(n, k+1) \mid v_k = \alpha, v_{k+1} = \beta\}.$$

With this notation we note

$$\text{the coefficient of } u^n \text{ in } (g(u))^{k-1} = \sum_{v \in A(n,k-1)} c_v . \tag{3.8}$$

Referring to (1.5), (3.7) is equivalent to

$$\left(\sum_{v \in A(n,k)} c_v \right)^2 \geqslant \left(\sum_{v \in A(n,k+1)} c_v \right) \left(\sum_{v \in A(n,k-1)} c_v \right).$$

Since $A(n, k)$ is the disjoint union $\bigcup_\alpha A_\alpha(n, k)$, and $A(n, k+1)$ is the disjoint union $\bigcup_{\alpha,\beta} A_\alpha{}^\beta(n, k+1)$, it suffices to prove that for any ordered pair (α, β):

$$\left(\sum_{v \in A_\alpha(n,k)} c_v \right) \left(\sum_{v \in A_\beta(n,k)} c_v \right) \geqslant \left(\sum_{v \in A(n,k-1)} c_v \right) \left(\sum_{v \in A_\alpha{}^\beta(n,k+1)} c_v \right). \tag{3.9}$$

By a simple correspondence, the left side is the same as

$$c_\alpha c_\beta \left(\sum_{v \in A(n-\alpha,k-1)} c_v \right) \left(\sum_{v \in A(n-\beta,k-1)} c_v \right),$$

while the right side is

$$c_\alpha c_\beta \left(\sum_{v \in A(n,\, k-1)} c_v \right) \left(\sum_{v \in A(n-\alpha-\beta,k-1)} c_v \right).$$

In view of (3.8), the desired inequality (3.9) follows provided the coefficients of $(g(u))^{k-1}$ satisfy (3.6). Fortunately, the convolution of two sequences satisfying (3.6) itself satisfies (3.6) [11, Chap. 8] and so the whole proof is complete.

We return now to the estimate for $s(n, k)$ given by (3.5), which we noted permitted only a restricted range for k. It is possible to extend this range as follows:

THEOREM IV. *Let $g(u)$ be a polynomial satisfying the hypotheses of Theorems* I *and* III, *and let ρ be a positive real number. Then*

$$s(n, k) \sim P_n(\rho)/(\rho^k \sigma_n(\rho)(2\pi)^{1/2}), \qquad \text{as} \quad n \to \infty, \tag{3.10}$$

where

$$k = \rho \left[g\left(r\left(\frac{n}{\rho}\right)\right)\right], \qquad \text{and} \qquad \sigma_n{}^2(\rho) = \rho g\left(r\left(\frac{n}{\rho}\right)\right) - \frac{n^2}{\rho B(r(n/\rho))}.$$

Proof. If $g(u)$ generates the coefficients $s(n, k)$, then $\rho g(u)$ generates the coefficients $\rho^k s(n, k)$. Moreover, $\rho g(u)$ still satisfies the hypotheses of Theorems I and III. $\rho g(r(n/\rho))$ and $\sigma_n^2(\rho)$ are the mean and variance for $\rho^k s(n, k)$ according to Theorem I; (3.10) is the local limit theorem applied with $k =$ mean, that is, $x = 0$ in (3.5). The proof is complete.

It is desirable to know, and natural to conjecture, that Eq. (3.10) is uniform for $\rho \in [\epsilon, 1/\epsilon]$, any ϵ small and fixed. However, thus far this conjecture has not been verified.

The derivative of $\rho g(r(n/\rho))$ with respect to ρ is $\sigma_n^2(\rho)/\rho$. Hence $\rho \cdot g(r(n/\rho)) - \mu_n$ is $(\rho - 1)\, \sigma_n^2(\zeta)/\zeta$ for some ζ between 1 and ρ. We point this out to indicate that the range of k for which (3.10) estimates $s(n, k)$ does indeed extend the range covered by (3.5). On the other hand, for all k covered even by the conjectured uniform version of (3.10), it is easily checked that $k/n \to 1/m$ where m is the degree of $g(u)$. Notice that n/m is the smallest possible k for which $s(n, k) \neq 0$.

Let us now point out the following fact about the numbers $k!\, s(n, k)/n!$ used in the proof of Theorem III.

THEOREM V. *Let $g(u)$ be a polynomial satisfying the hypotheses of Theorems* I *and* III, *and generating the coefficients $s(n, k)$ of a sequence of polynomials of binomial type. Assume that the root of $g(u) = 1$ which is smallest in absolute value is simple. Define another sequence $a(n, k)$ by $a(n, k) = k!\, s(n, k)/n!$. Then the sequence $a(n, k)$ satisfies* (1.1) *and* (1.3), *the central and local limit theorems, with μ_n and σ_n given by $\mu_n = c_1 n$ and $\sigma_n = c_2 n^{1/2}$ for suitable constants c_1 and c_2.*

Proof. Note

$$\sum_{n,k} a(n, k)\, u^n x^k = 1/(1 - xg(u)).$$

The stated result for the central limit theorem is Bender's observation [2, 3.1]. For the local limit theorem our Theorem III applies. An examination of Bender's formula reveals that the range of k for which his technique estimates $s(n, k)$ is totally disjoint from the range of (3.10) for $\rho \in [\epsilon, 1/\epsilon]$ and n large.

In closing we mention that (1.1) is known to hold for the $s(n, k)$ generated by a wide class of nonpolynomial power series $g(u)$, with μ_n and σ_n still defined as in Theorem I [3]. This is the class of functions which Hayman calls *admissible*, and studies in [9]. We have not included any of these results here, but hope to publish them at a later time.

I would like to thank Professors Edward Bender and S. G. Williamson for many interesting conversations concerning the material of this paper.

REFERENCES

1. E. A. BENDER, Asymptotic methods in enumeration, *SIAM Rev.* **16** (1974), 485–515.
2. E. A. BENDER, Central and local limit theorem applied to asymptotic enumeration, *J. Combinatorial Theory Ser. A* **15** (1973), 91–111.
3. E. RODNEY CANFIELD, "Asymptotic Normality in Binomial Type Enumeration," Ph.D. Thesis, University of California, San Diego, 1975.
4. L. CARLITZ, D. C. KURTZ, R. SCOVILLE, AND O. P. STACKELBERG, Asymptotic properties of Eulerian numbers, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* (1972), 47–54.
5. P. ERDÖS AND A. RENYI, On random graphs I, *Publ. Math. Debrecen* **6** (1959), 290–297.
6. J. FILLMORE AND S. WILLIAMSON, A linear algebra setting for the Rota–Mullin theory of polynomials of binomial type, *J. Linear and Multilinear Algebra* **1** (1973), 67–80.
7. A. GARSIA, An expose of the Rota–Mullin theory of polynomials of binomial type, *J. Linear and Mültilinear Algebra* **1** (1973), 47–65.
8. L. H. HARPER, Stirling behavior is asymptotically normal, *Ann. Math. Statist.* **38** (1967), 410–414.
9. W. K. HAYMAN, A generalization of Stirling's formula, *J. Reine Angew. Math.* **196** (1956).
10. C. JORDAN, "Calculus of Finite Differences," Chelsea, New York, 1965.
11. S. KARLIN, "Total Positivity," Vol. I, Stanford Univ. Press, Stanford, Calif., 1968.
12. G.-C. ROTA AND R. MULLIN, On the foundations of combinatory theory, *in* "Graph Theory and Its Applications" (Benard Harris, Ed.), Academic Press, New York, 1970.