Calderón–Lozanovskiĭ construction on weighted Banach function lattices

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Abstract

We show that the Calderón–Lozanovskiĭ construction \( \psi(\cdot) \) commute with arbitrary weighted Banach function lattices, that is, \( \psi(E_0(w_0), E_1(w_1)) = \psi(E_0, E_1)(w) \) with \( w = 1/\psi(1/w_0, 1/w_1) \) if and only if \( \psi \) is equivalent to a power function. From this result we can also get the Stein–Weiss theorem on interpolation of weighted \( L^p \) spaces and its generalizations.

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0. Introduction

The first interpolation result in weighted spaces was proved by Stein and Weiss [16] in 1958. They showed that if a linear operator \( T \) is bounded from \( L^p(v_i) \) into \( L^q(w_i) \), for \( i = 0, 1 \), then \( T \) is bounded from \( L^p(v) \) into \( L^q(w) \), where \( 1/p = (1-\theta)/p_0 + \theta/p_1 \), \( 1/q = (1-\theta)/q_0 + \theta/q_1 \), \( v = v_0^{1-\theta} v_1^{\theta} \), \( w = w_0^{1-\theta} w_1^{\theta} \), \( 0 < \theta < 1 \), and

\[
\| T \|_{L^p(v) \to L^q(w)} \leq C \| T \|_{L^p(v_0) \to L^q(w_0)}^{1-\theta} \| T \|_{L^p(v_1) \to L^q(w_1)}^{\theta}
\]
for some $C \geq 1$. Peetre (1970), Gilbert (1972), Lizorkin (1976) and Freitag (1978) (cf. [1, Parts 5.4 and 5.5]) proved even more by describing the Lions–Peetre real interpolation space for weighted $L_p$-spaces. Namely, if $1 \leq p_0 < p_1 \leq \infty$, then

\[
(L_{p_0}(w_0), L_{p_1}(w_1))_{\theta,p} = L_p(w_0^{1-\theta}w_1^{\theta}),
\]

where $1/p = (1 - \theta)/p_0 + \theta/p_1$, $0 < \theta < 1$.

Similar result is true for the complex method (cf. [1])

\[
[L_{p_0}(w_0), L_{p_1}(w_1)]_{\theta} = L_p(w_1^{1-\theta}w_0^{\theta}),
\]

but its generalization to the Calderón–Lozanovskiǐ construction is not clear.

In this paper we investigate imbedding properties of this construction. Since, by Proposition 1, we have equality (equality means that the sets are the same and the norms are equivalent)

\[
\phi(E(w_0), E(w_1)) = E(\phi^*(w_0, w_1)), \quad \phi^*(w_0, w_1) = 1/\phi(1/w_0, 1/w_1),
\]

therefore it looks interesting to find conditions on $\phi$ for which we have imbeddings

\[
\phi(E_0(w_0), E_1(w_1)) \hookrightarrow \phi(E_0, E_1)(\phi^*(w_0, w_1))
\]

and

\[
\phi(E_0(w_0), E_1(w_1)) \hookrightarrow \phi(E_0, E_1)(\phi^*(w_0, w_1)).
\]

Our main results in Theorem 2 show that necessary and sufficient condition for (2) is a $C$-submultiplicativity of the function $\phi$

\[
\phi(1, s t) \leq C\phi(1, s)\phi(1, t),
\]

and for (3) a $C$-supermultiplicativity of the function $\phi$

\[
\phi(1, s)\phi(1, t) \leq C\phi(1, s t).
\]

These imbedding theorems have interesting corollaries. For example, the Calderón–Lozanovskiǐ construction $\phi(\cdot)$ commute with arbitrary weighted Banach function lattices, that is, the equality

\[
\phi(E_0(w_0), E_1(w_1)) = E_0(E_0, E_1)(\phi^*(w_0, w_1))
\]

holds for arbitrary Banach function lattices and arbitrary weights $w_0, w_1$ if and only if $\phi$ is equivalent to the power function $\phi(s, t) = s^{1-\theta}t^\theta$.

This description for power functions, that is, the equality

\[
L_{p_0}(w_0)^{1-\theta}L_{p_1}(w_1)^\theta = L_p(w_0^{1-\theta}w_1^{\theta})
\]

together with the interpolation property of the Calderón–Lozanovskiǐ construction on the maximal Banach function lattices gives the Stein–Weiss interpolation theorem and information about possible its generalizations.

The content of the paper is as follows. In Section 1 we define the Calderón–Lozanovskiǐ construction on the class of Banach function lattices.

Section 2 contains the main result of the paper. The imbeddings (2) and (3) are characterized in the terms of function $\phi$. The results obtained here are then applied to prove that the equality (6) holds if and only if $\phi$ is equivalent to the power function.
1. Calderón–Lozanovskiǐ construction

Recall some notions and definitions which we will need further. Let $(\Omega, \mu)$ be a complete $\sigma$-finite measure space, and let $L^0(\mu)$ denote, as usual, the space of all equivalence classes of measurable functions on $\Omega$ with the topology of convergence in measure on $\mu$-finite sets.

If a normed subspace $E$ of $L^0(\mu)$ is such that there exists $u \in L^0(\mu)$ with $u > 0$ a.e. and $|x| \leq |y|$ $\mu$-a.e. on $\Omega$ implies $\|x\| \leq \|y\|$, we say that $E$ is a normed function lattice (on $\Omega$).

If, in addition, the unit ball $B_E = \{x: \|x\|_E \leq 1\}$ is closed in $L^0(\mu)$, so that $E$ has the Fatou property, then $E$ is a Banach space which is called a maximal Banach function lattice. The Fatou property of $E$ means that if $0 \leq x_n \uparrow x$, $x_n \in E$ and $\sup_n \|x_n\|_E < \infty$, then $x \in X$ and $\|x\|_E \leq \|x\|_E$.

If $E$ is a Banach function lattice on $\Omega$ and $w \in L^0$ with $w > 0$ a.e. on $\Omega$, we define the weighted space $E(w)$ by $\|x\|_{E(w)} := \|xw\|_E$.

We define the Calderón–Lozanovskiǐ construction. Let $\mathcal{U}$ denotes the set of all concave, positively homogeneous of degree one, nondecreasing continuous in each variable functions $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\phi(0, 0) = 0$.

If $\overline{E} = (E_0, E_1)$ is a pair of Banach function lattices on $\Omega$ and $\phi \in \mathcal{U}$, then the Calderón–Lozanovskiǐ construction or Calderón–Lozanovskiǐ space $\phi(\overline{E}) = \phi(E_0, E_1)$ consists of all $x \in L^0(\mu)$ such that $|x| \leq \lambda \phi(|x_0|, |x_1|)$ $\mu$-a.e. on $\Omega$ for some $x_i \in E_i$, with $\|x_0\|_{E_0} \leq 1$, $i = 0, 1$. The space $\phi(\overline{E})$ is a Banach function lattice equipped with the norm

$$\|x\|_\phi = \inf \{\lambda > 0: |x| \leq \lambda \phi(|x_0|, |x_1|), \ \|x_0\|_{E_0} \leq 1, \ \|x_1\|_{E_1} \leq 1\}$$

(see [8]). In the case of the power function $\phi_\theta(s, t) = s^{1-\theta} t^\theta$ with $0 \leq \theta \leq 1$, $\phi_\theta(\overline{E})$ is well known Calderón space $E_0^{1-\theta} E_1^\theta$ (see [3]).

The properties of $\phi(\overline{E})$ were studied by Lozanovskiǐ in [8] and [9] (see also [11] and [2]), where among other facts it is proved the Köthe duality result

$$\phi(E_0, E_1)' = \phi(E_0', E_1')$$

with equivalent norms. Here, for $\phi \in \mathcal{U}$, the conjugate function $\hat{\phi}$ is defined by

$$\hat{\phi}(s, t) := \inf \left\{\frac{\alpha s + \beta t}{\phi(\alpha, \beta)}: \alpha, \beta > 0\right\}, \ s, t \geq 0.$$

We have $\hat{\phi} \in \mathcal{U}$ and $\hat{\phi} = \phi$ (see [9,11]).

Observe also that if $M : [0, \infty) \to [0, \infty]$ is nondecreasing convex and left-continuous function not identical $0$ or $\infty$ on $(0, \infty)$ with $M(0) = 0$, and $\phi \in \mathcal{U}$ is defined by $\phi(s, t) = t M^{-1}(s/t)$ if $t > 0$ and $0$ if $t = 0$, where $M^{-1}$ is the right continuous inverse of $M$. Then
for any Banach lattice $E$ on $\Omega$ the Calderón–Lozanovskiĭ space $\varphi(E, L_\infty(\mu))$ coincides isometrically with the Banach lattice

$$E_M = \{ x \in L^0(\mu) : M(|x|/\lambda) \in E \text{ for some } \lambda > 0 \}$$

equipped with the norm

$$\|x\|_{E_M} = \inf \{ \lambda > 0 : \|M(|x|/\lambda)\|_E \leq 1 \}.$$  

In particular, $\varphi(L_1, L_\infty)$ coincides isometrically with the Orlicz space $L_M$ (see [2,11,15]). For $\varphi \in \mathcal{U}$ let $\varphi^*(s,t) = 1/\varphi(1/s, 1/t)$ for $s, t > 0$.

2. Calderón–Lozanovskiĭ construction for weighted Banach function lattices

Let us start with the description of Calderón–Lozanovskiĭ construction for an easy couple of weighted spaces $(E(w_0), E(w_1))$, which is well-known for $E = L^p$ (see [15, p. 459]) but which will motivate us for the corresponding generalizations. We include here a proof for the sake of completeness.

**Proposition 1.** Let $\varphi \in \mathcal{U}$ and $E$ be any Banach function lattice. Then

$$\varphi(E(w_0), E(w_1)) = E(\varphi^*(w_0, w_1)), \quad \varphi^*(w_0, w_1) = 1/\varphi(1/w_0, 1/w_1). \quad (7)$$

**Proof.** Let $w := \varphi^*(w_0, w_1)$. If $x \in \varphi(E(w_0), E(w_1))$ with the norm $\|x\| < 1$, then we can find $x_i \in E(w_i)$ with $\|x_i\|_{E(w_i)} \leq 1$, $i = 0, 1$, such that $|x| \leq \varphi(|x_0|, |x_1|) \mu$-a.e. on $\Omega$. We have

$$|x|/w \leq \varphi(|x_0|, |x_1|)w \leq \max(|x_0|w_0, |x_1|w_1)\varphi\left(\frac{1}{w_0}, \frac{1}{w_1}\right)w$$

$$= \max(|x_0|w_0, |x_1|w_1) \leq |x_0|w_0 + |x_1|w_1, \quad \mu$-a.e. on $\Omega,$

and so

$$\|x\|_E \leq \|x_0w_0\|_E + \|x_1w_1\|_E \leq 2,$$

i.e., $x \in E(w)$ and $\|x\|_{E(w)} \leq 2\|x\|_{\varphi(E(w_0), E(w_1))}$.

On the other hand, if $x \in E(w)$ with the norm $\|x\| \leq 1$, then for

$$x_i = \frac{\varphi^*(w_0, w_1)}{w_i}|x|, \quad i = 0, 1,$$

we have

$$|x| = \varphi(|x_0|, |x_1|) \quad \text{and} \quad \|x_i\|_{E(w_i)} = \|x\|_E \leq 1.$$  

Thus $x \in \varphi(E(w_0), E(w_1))$ and $\|x\|_{\varphi(E(w_0), E(w_1))} \leq \|x\|_{E(w)}$.  

The above result we will generalize by considering the following imbeddings for the Calderón–Lozanovskiĭ construction.
Theorem 2. (i) The continuous imbedding
\[ \varphi(E_0(w_0), E_1(w_1)) \hookrightarrow \varphi(E_0, E_1)(\varphi^*(w_0, w_1)) \tag{8} \]
holds for any Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) if and only if \( \varphi \) is \( C \)-submultiplicative function for some \( C > 0 \), that is,
\[ \varphi(1, st) \leq C \varphi(1, s) \varphi(1, t) \tag{9} \]
for all \( s, t > 0 \).
(ii) The continuous imbedding
\[ \varphi(E_0, E_1)(\varphi^*(w_0, w_1)) \hookrightarrow \varphi(E_0(w_0), E_1(w_1)) \tag{10} \]
holds for any Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) if and only if \( \varphi \) is \( C \)-supermultiplicative function for some \( C > 0 \), that is,
\[ \varphi(1, s) \varphi(1, t) \leq C \varphi(1, st) \tag{11} \]
for all \( s, t > 0 \).

Proof. (i) Sufficiency. Assume that (9) is true. Then
\[ \varphi\left(\frac{s_0}{t_0}, \frac{s_1}{t_1}\right) = \frac{s_0}{t_0} \varphi\left(1, \frac{s_1 t_0}{s_0 t_1}\right) \leq C \frac{s_0}{t_0} \varphi\left(1, \frac{s_1}{s_0 t_1}\right) \varphi\left(1, \frac{t_0}{t_1}\right) = C \varphi(s_0, s_1) \varphi\left(\frac{1}{t_0}, \frac{1}{t_1}\right), \]
for all \( s_0, s_1, t_0, t_1 > 0 \)
Now, if \( x \in \varphi(E_0(w_0), E_1(w_1)) \) with \( \left\| x \right\|_\varphi < 1 \) we can find \( x_i \in E_i(w_i) \) with \( \left\| x_i \right\|_{E_i(w_i)} \leq 1, i = 0, 1 \), such that \( |x| \leq \varphi(|x_0|, |x_1|) \) \( \mu \)-a.e. on \( \Omega \).
Thus \( x \varphi^*(w_0, w_1) \in \varphi(E_0, E_1) \) and \( \left\| x \varphi^*(w_0, w_1) \right\|_{\varphi(E_0, E_1)} \leq C \) or \( x \in \varphi(E_0, E_1)(\varphi^*(w_0, w_1)) \) and \( \left\| x \right\|_{\varphi(E_0, E_1)(\varphi^*(w_0, w_1))} \leq C \). We proved in fact here that (9) gives a continuous imbedding
\[ \varphi(E_0(w_0), E_1(w_1)) \hookrightarrow \varphi(E_0, E_1)(\varphi^*(w_0, w_1)). \]

Necessity. Assume that (8) holds for any Banach function lattices \( E_0, E_1 \) on \( \Omega \) and arbitrary weights \( w_0, w_1 \) on \( \Omega \) but (9) does not hold, i.e.,
\[ \sup_{s, t > 0} \frac{\varphi(1, st)}{\varphi(1, s) \varphi(1, t)} = \infty. \]
Since ϕ is a positive concave function it follows that for $2^{k} \leq s < 2^{k+1}$ and $2^{l} \leq t < 2^{l+1}$ we have
\[
\frac{1}{4} \frac{\varphi(1, 2^{k+l})}{\varphi(1, 2^{k})\varphi(1, 2^{l})} \leq \frac{\varphi(1, st)}{\varphi(1, s)\varphi(1, t)} \leq \frac{4\varphi(1, 2^{k+l})}{\varphi(1, 2^{k})\varphi(1, 2^{l})}
\]
and this gives that
\[
\sup_{k, l \in \mathbb{N}} \frac{\varphi(1, 2^{k+l})}{\varphi(1, 2^{k})\varphi(1, 2^{l})} = \infty.
\]
Thus there are sequences $k_n, l_n$ such that
\[
\frac{\varphi(1, 2^{k_n+l_n})}{\varphi(1, 2^{k_n})\varphi(1, 2^{l_n})} \geq 2^n, \quad n = 1, 2, \ldots. \tag{12}
\]
Note that $|k_n| \to \infty$ and $|l_n| \to \infty$ as $n \to \infty$. In fact, if $|k_n| \leq M$ for all $n \in \mathbb{N}$ then
\[
\frac{\varphi(1, 2^{k_n+l_n})}{\varphi(1, 2^{k_n})\varphi(1, 2^{l_n})} \leq \frac{\varphi(1, 2^{M+l_n})}{\varphi(1, 2^{M})\varphi(1, 2^{l_n})} \leq \frac{2^{M} \varphi(1, 2^{l_n})}{\varphi(1, 2^{M})\varphi(1, 2^{l_n})} = \frac{2^{M}}{\varphi(1, 2^{M})}
\]
which is a contradiction with (12).

Taking a subsequence, if necessary, we may assume that either
(a) $0 > k_1 > k_2 > \cdots > k_n > k_{n+1} > \cdots$, or
(b) $0 < k_1 < k_2 < \cdots < k_n < k_{n+1} < \cdots$.

Consider the case (a). Let $E_0 = l_{\infty}$, $E_1 = l_1$, $w_0 = 1$ and $w_1$ be constructed in the following way:
\[
w_1 = \left( \frac{2^{-l_1}, \ldots, 2^{-l_1}, 2^{-l_2}, \ldots, 2^{-l_2}, \ldots}{2^{-l_1 \text{-terms}}, 2^{-l_2 \text{-terms}}} \right).
\]
We will show that the continuous imbedding
\[
\varphi(l_{\infty}, l_1(w_1)) \hookrightarrow \varphi(l_{\infty}, l_1)(w), \quad w = \varphi^*(1, w_1) = \frac{1}{\varphi(1, w_1)}, \tag{13}
\]
contradicts (12).

Take a sequence of functions
\[
x_1 = \left( \varphi(1, 2^{k_1+l_1}), \ldots, \varphi(1, 2^{k_1+l_1}), 0, 0, \ldots, \right), \tag{2^{-k_1 \text{-terms}}}
\]
\[
x_2 = \left( 0, \ldots, 0, \varphi(1, 2^{k_2+l_2}), \ldots, \varphi(1, 2^{k_2+l_2}), 0, 0, \ldots, \right), \tag{2^{-k_2 \text{-terms}}}
\]
\[
\vdots
\]
\[
x_n = \left( 0, \ldots, 0, \varphi(1, 2^{k_n+l_n}), \ldots, \varphi(1, 2^{k_n+l_n}), 0, 0, \ldots, \right), \tag{2^{-l_n \text{-terms}}}
\]
Since for
\[ y_n = (0, \ldots, 0, 2^{k_1} + l_n, \ldots, 2^{k_n} + l_n, 0, 0, \ldots) \]
we have
\[ x_n = \varphi(1, y_n) \chi_{\{k < n : y_n(k) > 0\}} \leq \varphi(1, y_n) \]
and
\[ \|y_n\|_{l_1(w_1)} = 2^{-k_n} 2^{k_n + l_n} 2^{-l_n} = 1 \]
it follows that \( \|x_n\|_{\varphi(l_\infty, l_1(w_1))} \leq 1 \).

Continuous imbedding (13) means that there exist \( C > 0 \) and sequences \( b_i^{(n)} \geq 0 \) such that
\[ \sum_{i \in \mathbb{Z}} b_i^{(n)} \leq 1 \] and
\[ x_n w = \frac{x_n}{\varphi(1, \frac{b_i^{(n)}}{W})} \leq C \varphi(1, b_i^{(n)}). \]

In particular, for \( \sum_{j=1}^{n-1} 2^{-k_j} < i \leq \sum_{j=1}^{n} 2^{-k_j} \) we have
\[ \frac{\varphi(1, 2^{k_n} + l_n)}{\varphi(1, 2^n)} \leq C \varphi(1, b_i^{(n)}), \]
and hence for such \( i \), by using (12), we get
\[ \frac{2^n}{C} \varphi(1, 2^{k_n}) \leq \varphi(1, b_i^{(n)}). \] (14)
We can choose \( n \) such that \( 2^n > C \) and then, from (14), follows that there are \( s_n \) such that
\[ s_n = \min \left\{ s > 0 : \varphi(1, s) = \frac{2^n}{C} \varphi(1, 2^{k_n}) \right\}. \]

Then, by (14),
\[ s_n \leq b_i^{(n)} \quad \text{for} \quad \sum_{j=1}^{n-1} 2^{-k_j} < i \leq \sum_{j=1}^{n} 2^{-k_j}. \] (15)
Summing both sides over all such \( i \) we obtain \( 2^{-k_n} s_n \leq 1 \). Hence
\[ \varphi(1, 2^{k_n}) = \frac{2^n}{C} \varphi(1, 2^{k_n}) = \varphi(1, s_n) \leq \varphi(1, 2^{k_n}), \]
which is a contradiction. This shows that the case (a) is impossible. The proof of impossibility in the case (b) we do by a change of \( \varphi \). Let us observe that if continuous imbedding
\[ \varphi(E_0(w_0), E_1(w_1)) \hookrightarrow \varphi(E_0, E_1)(\varphi^*(w_0, w_1)) \]
holds for \( \varphi \), then for \( \psi \) given by \( \psi(s, t) = \varphi(t, s) \) we have imbedding
\[ \psi(E_0(w_0), E_1(w_1)) = \psi(E_1(w_1), E_0(w_0)) \]
\[ \hookrightarrow \psi(E_1, E_0)(\varphi^*(w_1, w_0)) = \psi(E_0, E_1)(\varphi^*(w_0, w_1)). \]
But for $\psi$ inequality (12) has form
\[
\frac{\psi(1, 2^{-kn})}{\psi(1, 2^{-kn})} \geq 2^n, \quad n = 1, 2, \ldots,
\]
and the case (b) for $\varphi$ is going into the case (a) for $\psi$, which is impossible to hold as it was shown above. Therefore necessity of (9) is proved.

(ii) Sufficiency. Let (11) be satisfied. If $x \in \varphi(E_0, E_1)(\varphi^*(w_1, w_0))$ with $\|x\| < 1$, then we can find $x_i \in E_i$ with $\|x_i\|_{E_i} \leq 1$, $i = 0, 1$, such that $|x| \varphi^*(w_1, w_0) \leq \varphi(|x_0|, |x_1|)$ or
\[
|x| \leq \varphi(|x_0|, |x_1|)\varphi\left(\frac{1}{w_0}, \frac{1}{w_1}\right) \mu\text{-a.e. on } \Omega.
\]
Let $y_i = x_i/w_i$, $i = 0, 1$. Then $\|y_i\|_{E_i(w_i)} = \|x_i\|_{E_i} \leq 1$, $i = 0, 1$, and
\[
|x| \leq \varphi(|y_0|, |y_1|)\varphi\left(\frac{1}{w_0}, \frac{1}{w_1}\right) = \varphi(|y_0|w_0, |y_1|w_1)\varphi\left(\frac{1}{w_0}, \frac{1}{w_1}\right)
\]
\[
\leq C\varphi(|y_0|, |y_1|) \mu\text{-a.e. on } \Omega.
\]
Thus $x \in \varphi(E_0(w_0), E_1(w_1))$ and $\|x\|_\varphi \leq C$. We proved here that (11) gives a continuous imbedding
\[
\varphi(E_0, E_1)(\varphi^*(w_0, w_1)) \overset{C}{\hookrightarrow} \varphi(E_0(w_0), E_1(w_1)).
\]

Necessity. Assume now that (11) does not hold. Then, in the same way as in the case (i), we can find sequences $k_n, l_n$ such that
\[
\frac{\varphi(1, 2^{k_1})\varphi(1, 2^{k_2})}{\varphi(1, 2^{k_n+l_n})} \geq 2^n, \quad n = 1, 2, \ldots,
\]
and we can also assume that
\[
0 > k_1 > k_2 > \ldots > k_n > k_{n+1} > \ldots.
\]
Take $E_0 = l_\infty$, $E_1 = l_1$, $w_0 = 1$ and $w_1$ as in the case (i). Consider a sequence of functions
\[
x_1 = \left(\frac{\varphi(1, 2^{k_1})\varphi(1, 2^{l_1}), \ldots, \varphi(1, 2^{k_1})\varphi(1, 2^{l_1}), 0, 0, \ldots}\right),
\]
\[
x_2 = \left(0, \ldots, 0, \varphi(1, 2^{l_2})\varphi(1, 2^{l_2}), \ldots, \varphi(1, 2^{l_2})\varphi(1, 2^{l_2}), 0, 0, \ldots\right),
\]
...,
\[
x_n = \left(0, \ldots, 0, \varphi(1, 2^{l_n})\varphi(1, 2^{l_n}), \ldots, \varphi(1, 2^{l_n})\varphi(1, 2^{l_n}), 0, 0, \ldots\right).
\]
Since for
\[
y_n = \left(0, \ldots, 0, 2^{k_0}, 2^{k_0}, 2^{k_0}, 0, 0, \ldots\right)
\]

we have
\[ x_n = \left( \phi(1, y_n)/w \right) \chi_{\{ k \in \mathbb{N} : y_n(k) > 0 \}} \leq \phi(1, y_n)/w, \quad w = \phi^*(1, w_1), \]
and
\[ \|y_n\|_{l_1} = 2^{k_n} 2^{-k_n} = 1 \]
it follows that \[ \|x_n\|_{\psi(l_{\infty}, l_1)(w_1)} \leq \phi(1, y_n)/w. \]

Hence, by the imbedding
\[ \phi(l_{\infty}, l_1)(w) \hookrightarrow \phi(l_{\infty}, l_1(w_1)), \]
there exist \( C > 0 \) and sequences \( b_i^{(n)} \geq 0 \) such that \( \sum_{i \in \mathbb{N}} b_i^{(n)} \leq 1 \) and
\[ x_n \leq C \phi(1, b_i^{(n)}/w_1(i)). \]
In particular, for \( \sum_{j=1}^{n-1} 2^{-k_j} < i \leq \sum_{j=1}^{n} 2^{-k_j} \) we have
\[ \phi(1, 2^{k_n}) \phi(1, 2^{k_n}) \leq C \phi(1, b_i^{(n)} 2^{k_n}), \]
and so
\[ \frac{2^n}{C} \phi(2^{k_n + i}) \leq \phi(1, b_i^{(n)} 2^{k_n}). \]

We can again have that \( 2^n > C \) and so there are \( s_n \) such that
\[ s_n = \min \left\{ s > 0 : \phi(1, s) = \frac{2^n}{C} \phi(1, 2^{k_n + i}) \right\}. \]
Then, by (16),
\[ s_n \leq b_i^{(n)} 2^{k_n} \quad \text{for} \quad \sum_{j=1}^{n-1} 2^{-k_j} < i \leq \sum_{j=1}^{n} 2^{-k_j}. \]

Summing both sides over all such \( i \) we obtain
\[ 2^{-k_n} s_n \leq \sum b_i^{(n)} 2^{k_n} \leq 2^{k_n} \]
or \( s_n \leq 2^{k_n + i} \).

Hence
\[ \phi(1, 2^{k_n + i}) \leq \frac{2^n}{C} \phi(1, 2^{k_n + i}) = \phi(1, s_n) \leq \phi(1, 2^{k_n + i}), \]
which is a contradiction. \( \square \)

**Remark 1.** Theorem 2 can be generalized to three functions \( \varphi, \varphi_0, \varphi_1 \in \mathcal{U} \).

(i) The continuous imbedding
\[ \phi(E_0(w_0), E_1(w_1)) \hookrightarrow \phi_0(E_0, E_1)(\varphi_1(w_0, w_1)) \]
holds for any Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) if and only if there exists a constant \( C > 0 \) such that
\[ \varphi(1, s) \varphi_1(1, t) \leq C \varphi(1, st) \quad \text{for all} \ s, t > 0. \]
(ii) The continuous imbedding
\[ \varphi_0(E_0, E_1) \hookrightarrow \varphi(E_0(w_0), E_1(w_1)) \]
holds for any Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) if and only if there exists a constant \( C > 0 \) such that
\[ \varphi_0(1, st) \leq C \varphi_1(1, s) \varphi(1, t) \quad \text{for all } s, t > 0. \]
The above results are also true for the quasi-Banach function lattices \( E_0, E_1 \) and we should mention here that the Calderón–Lozanovskii spaces can be also defined in this case (cf. [6, 12, 14]).

**Remark 2.** Theorem 2 and Remark 1 in their necessity parts will be still true if instead of all Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) we take only \( L_{p_0}, L_{p_1} \) for all \( 1 \leq p_0, p_1 \leq \infty \) and arbitrary weights \( w_0, w_1 \). \n
**Remark 3.** Note that if \( 0 < p_0 < p_1 \leq \infty \), then
\[ \varphi(L_{p_0}(w_0), L_{p_1}(w_1)) = \varphi_0 \quad \text{(17)} \]
holds for any Banach function lattices \( E_0, E_1 \) and arbitrary weights \( w_0, w_1 \) if and only if \( \varphi \) is equivalent to a power function.

**Proof.** By Theorem 2 the equality (17) holds if and only if there exist positive constants \( c_0 \) and \( c_1 \) such that for all \( s, t > 0 \)
\[ c_0 \varphi_0(1, s) \varphi(1, t) \leq \varphi(1, st) \leq c_1 \varphi_1(1, s) \varphi(1, t). \]
Hence, the functions \( \varphi_i(t) := c_i \varphi(1, t) \) and \( \varphi_0(t) := 1/c_0 \varphi(1, t) \) are submultiplicative on \((0, \infty)\), that is,
\[ \varphi_i(st) \leq \varphi_i(s) \varphi_i(t) \quad \text{for all } s, t > 0 \ (i = 0, 1). \]
For submultiplicative functions \( \varphi_0 \) and \( \varphi_1 \), as it is known (see [5, pp. 241–250], [7, Theorem 1.3], and [11, Theorem 11.3]), we can find real numbers \( \alpha_i, \beta_i \) such that \(-\infty < \alpha_i \leq \beta_i < \infty, i = 0, 1, \) and
\[ \alpha_i = \sup_{0 < t < 1} \frac{\ln \varphi_i(t)}{\ln t} = \lim_{t \to 0^+} \frac{\ln \varphi_i(t)}{\ln t}, \quad \beta_i = \inf_{t > 1} \frac{\ln \varphi_i(t)}{\ln t} = \lim_{t \to \infty} \frac{\ln \varphi_i(t)}{\ln t}. \]
Note that
\[ \alpha_0 = \lim_{t \to 0^+} \frac{\ln \frac{1}{c_0 \varphi(1, t)}}{\ln t} = \lim_{t \to 0^+} \frac{-\ln c_0 - \ln \varphi(1, t)}{\ln t} = -\alpha_1, \]
and similarly \( \beta_0 = -\beta_1 \). Therefore for \( 0 < t < 1 \)
\[ \frac{1}{c_1} t^{\alpha_1} \leq \frac{\varphi_1(t)}{c_1} \leq \varphi(1, t) = \frac{1}{c_0 \varphi_0(t)} \leq \frac{1}{c_0} t^{\alpha_0} = \frac{1}{c_0} t^{\alpha_1}, \]
and similarly for \( t > 1 \)
\[ \frac{1}{c_1} t^{\beta_1} \leq \varphi(1, t) \leq \frac{1}{c_0} t^{\beta_1}. \]

We show that \( \alpha_1 = \beta_1 \). In fact, for any \( t > 1 \) let \( 0 < s < 1 \) be such that \( 0 < st < 1 \) (for example, \( s = t^{-2} \)); then
\[ \frac{1}{c_0} (st)^{\alpha_1} \geq \varphi(1, st) \geq c_0 \varphi(1, s) \varphi(1, t) \geq c_0 \frac{\varphi_1(s)}{c_1} \frac{\varphi_1(t)}{c_1} \]
or
\[ t^{\alpha_1} \geq \left( \frac{c_0}{c_1} \right)^2 t^{\beta_1}, \]
which as \( t \to \infty \) gives that \( \alpha_1 \geq \beta_1 \) (but from the definition we have \( \alpha_1 \leq \beta_1 \)) and so \( \alpha_1 = \beta_1 \). This means that
\[ \frac{1}{c_1} t^{\alpha_1} \leq \varphi(1, t) \leq \frac{1}{c_0} t^{\alpha_1} \]
for all \( t > 0 \). \( \square \)

**Remark 4.** The above proof shows that if measurable function \( \varphi : (0, \infty) \to (0, \infty) \) is \( c_0 \)-supermultiplicative and \( c_1 \)-submultiplicative, then \( \varphi \) is equivalent to a power function. By taking \( u(t) = \ln \varphi(e^t) \) we will have equivalently that if measurable function \( u : \mathbb{R} \to \mathbb{R} \) is \( c_0 \)-superadditive and \( c_1 \)-subadditive, then \( u \) is equivalent to a linear function.

### 3. Interpolation theorems of the Stein–Weiss type

Proposition 1 and Theorem 3 together with the well-known interpolation property of the Calderón–Lozanovskii construction on the maximal Banach function lattices (cf. [2,8,11,15]) give immediately the Stein–Weiss theorem on interpolation of \( L_p \)-spaces mentioned in the introduction and also, by Remark 1 on quasi-Banach function lattices, its improvements considered by Gustavsson in [4].

We may also apply Theorem 3 to establish another interpolation theorem of the Stein–Weiss type.
We say that operator \( T : (E_0, E_1) \to (F_0, F_1) \) is a bounded linear operator from the Banach couple \( (E_0, E_1) \) into the Banach couple \( (F_0, F_1) \) if \( T : E_0 + E_1 \to F_0 + F_1 \) is bounded and restrictions \( T|_{E_0} : E_0 \to F_0 \) and \( T|_{E_1} : E_1 \to F_1 \) are bounded.

**Theorem 4.** Let \( T : (l_\infty, l_\infty(2^{-n})) \to (L_{p_0}(w_0), L_{p_1}(w_1)) \) be a bounded linear operator. If \( \varphi \) is a \( C \)-submultiplicative function for some \( C > 0 \), then \( T \) maps \( l_\infty(\varphi^*(1, 2^{-n})) \) into \( \varphi(L_{p_0}, L_{p_1})(\varphi^*(w_0, w_1)) \). The result is sharp, in the sense that if \( \varphi \) is not \( C \)-submultiplicative function for any \( C > 0 \), then we can find \( p_0, p_1 \in [1, \infty] \), weights \( w_0, w_1 \) and a bounded linear operator \( T : (l_\infty, l_\infty(2^{-n})) \to (L_{p_0}(w_0), L_{p_1}(w_1)) \) such that \( T \) does not map \( l_\infty(\varphi^*(1, 2^{-n})) \) into \( \varphi(L_{p_0}, L_{p_1})(\varphi^*(w_0, w_1)) \).

**Proof.** We have \( \varphi(l_\infty, l_\infty(2^{-n})) = l_\infty(\varphi^*(1, 2^{-n})) \). This follows, for example, from our Proposition 1.

If \( \varphi \) is \( C \)-submultiplicative then by Theorem 2(i) we have continuous imbedding \( \varphi(L_{p_0}(w_0), L_{p_1}(w_1)) \rightleftharpoons \varphi(L_{p_0}, L_{p_1})(\varphi^*(w_0, w_1)) \), and by using the interpolation property of the Calderón–Lozanovskii construction on the maximal Banach function lattices we obtain the first part of the theorem.

Now, we show that result is sharp. We first note that for any couple of maximal Banach function lattices \( \vec{E} = (E_0, E_1) \) we have

\[
\varphi(E_0, E_1) = \text{Orb}_{l_\infty(\varphi^*(1, 2^{-n}))}(l_\infty, l_\infty(2^{-n})); (E_0, E_1))
\]

(cf. [15] and also [2], where the definition of the orbit space and some results are presented) and, in particular,

\[
\varphi(L_{p_0}(w_0), L_{p_1}(w_1)) = \text{Orb}_{l_\infty(\varphi^*(1, 2^{-n}))}(l_\infty, l_\infty(2^{-n})); (L_{p_0}(w_0), L_{p_1}(w_1))
\]

Thus if \( \varphi \) is not \( C \)-submultiplicative for any \( C > 0 \), then by Theorem 2(i) (in fact, by Remark 2) we can find \( p_0, p_1 \) and weights \( w_0, w_1 \) such that \( \varphi(L_{p_0}, L_{p_1})(\varphi^*(w_0, w_1)) \) does not contain \( \varphi(L_{p_0}(w_0), L_{p_1}(w_1)) \) and since orbit is minimal in inclusion interpolation method which contains \( l_\infty(\varphi^*(1, 2^{-n})) \) it follows the result. This also shows the importance of \( C \)-submultiplicativity of \( \varphi \). \( \square \)

The interpolation property of the Gagliardo relative completion \( \varphi^*(\cdot) \) of the Calderón–Lozanovskii construction \( \varphi(\cdot) \) with respect to the sum together with the Theorem 2(i) show the following interpolation result.

**Corollary 5.** Let \( \varphi \) be a \( C \)-submultiplicative function for some \( C > 0 \) and \( F_0, F_1 \) be two maximal Banach function lattices. If \( T : (E_0, E_1) \to (F_0(w_0), F_1(w_1)) \) is a bounded linear operator, then \( T : \varphi(E_0, E_1) \to \varphi(F_0, F_1)(\varphi^*(w_0, w_1)) \) is bounded.

The proof of Theorem 4 and Theorem 3 give another result.

**Theorem 6.** If \( \varphi \) is not equivalent to a power function, then there exist \( p_0, p_1, w_0, w_1 \) and \( q_0, q_1, v_0, v_1 \) and a bounded operator \( T : (L_{p_0}(w_0), L_{p_1}(w_1)) \to (L_{q_0}(v_0), L_{q_1}(v_1)) \) such that \( T \) does not map \( \varphi(L_{p_0}, L_{p_1})(\varphi^*(w_0, w_1)) \) into \( \varphi(L_{q_0}, L_{q_1})(\varphi^*(v_0, v_1)) \).
Theorem 4. If ϕ is not C-submultiplicative for any C > 0, then this follows directly from the

Proof. If ϕ is not C-submultiplicative for any C > 0, then this follows directly from the

Theorem 4. If ϕ is not C-submultiplicative for some C > 0, then we can consider couple for

which ϕ(Lp0(w0), Lp1(w1)) is an proper subspace of ϕ(Lp0, Lp1)(ϕ*(w0, w1)).

On the other hand, since Lp0(w0) and Lp1(w1) have the Fatou property it follows that

ϕ(Lp0(w0), Lp1(w1)) = Corb11(ϕ⋆([1, 2^n]))((Lp0(w0), Lp1(w1); (l1, l1(2^n))))

(cf. [15] and also [2], where the definition of the coorbit space and also some results are

included). But coorbit is a maximal interpolation method in a sense of inclusion and is

 contained in l1(ϕ*(1, 2^n)).

Thus, for x ∈ ϕ(Lp0, Lp1)(ϕ*(w0, w1)) \ ϕ(Lp0(w0), Lp1(w1)) we can find a bounded operator

T : (Lp0(w0), Lp1(w1)) → (l1, l1(2^n)) such that Tx \neq ϕ(l1, l1(2^n)) = l1(ϕ*(1, 2^n)). □

The results we just proved allow us to characterize the complex interpolation method on couples of weighted Lp-spaces. The necessary definitions we can find in [2].

Theorem 7. Let F be an interpolation functor on the category of (Lp0(w0), Lp1(w1))
couples such that

F(Lp0(w0), Lp1(w1)) = F(Lp0, Lp1)(ϕ*(w0, w1)).

(18)

If F is relatively complete, then F is an upper complex method Cθ for some 0 ≤ θ ≤ 1.

Proof. Let F be a relatively complete. Since F(l∞, l∞(2−n)) = l∞(ϕ*(1, 2^n)) and

F(l1, l1(2−n)) = l1(ϕ*(1, 2^n)) we can find on any couple (E0, E1) of Banach function lattices

with the Fatou property that F(E0, E1) contains Orb1(ϕ*(1, 2^n))((l∞, l∞(2−n)); (E0, E1))

and is contained in Corb11(ϕ*(1, 2^n))((E0, E1); (l1, l1(2^n))).

By the Ovchinnikov theorem (see [13] or [15]) the functor F coincide with the

Calderón–Lozanovskii spaces ϕ(E0, E1). The assumption (18) gives that

ϕ(Lp0(w0), Lp1(w1)) = ϕ(Lp0, Lp1)(ϕ*(w0, w1))

and, by the Theorem 3, ϕ must be equivalent to the power function ϕθ = s^{1-θ}t^θ. Thus the

functor F is the Calderón–Lozanovskii construction E0^{1-θ}E1^{θ}.

Now, by using the Lozanovskii theorem from [10, Theorem 3.3] we have that on the

Banach function lattices with the Fatou property spaces E0^{1-θ}E1^{θ} are equal to the upper

complex method [E0, E1]^θ. Finally, we obtain that F(E0, E1) = [E0, E1]^θ and the proof is

finished. □

References


