Some generalized Ostrowski–Grüss type integral inequalities

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A R T I C L E   I N F O

Article history:
Received 9 June 2011
Received in revised form 16 September 2011
Accepted 10 November 2011

Keywords:
Ostrowski–Grüss type inequality
Numerical integration
Error bound
Sharp bound

A B S T R A C T

In this paper, we establish some new Ostrowski–Grüss type integral inequalities involving \((k - 1)\) interior points in 1D case, which are generalizations of some known results in the literature, and one of which is sharp. Then we deduce an Ostrowski–Grüss type integral inequality in 2D case involving \((k - 1)^2\) interior points for the first time. We also present one application on the estimate of error bound for numerical integration formula, in which a sharp error bound for a new numerical integration formula is provided by the results established.

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1. Introduction

In recent years, the research for the Ostrowski type inequalities has been a hot topic in the literature. The Ostrowski type inequality, which can be used to estimate the absolute deviation of a function from its integral mean, was originally presented by Ostrowski in [1] as follows (see also in [2, pp. 468]).

Theorem 1.1. Let \( f : I \rightarrow R \) be a differentiable mapping in the interior \( \text{Int} \) of \( I \), where \( I \subset R \) is an interval, and let \( a, b \in \text{Int} I \). \( a < b \). If \( |f'(t)| \leq M \), \( \forall t \in [a, b] \), then we have

\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(t)dt \right| \leq \left[ \frac{1}{4} + \frac{(x - a + b)^2}{(b - a)^2} \right] (b - a)M, \quad \text{for } x \in [a, b].
\]

Since then, various generalizations of the Ostrowski inequality have been established (for example, see [3–15] and the references therein), one of which is the inequalities of Ostrowski–Grüss type (for example, see [16–23]). Such inequalities can be used to provide explicit error bounds for some known and some new numerical quadrature formulas. The first inequality of the Ostrowski–Grüss type was presented by Dragomir and Wang in [16], which reads as in the following theorem.

Theorem 1.2 ([16, Theorem 2.1]). Let \( I \subset R \) be an open interval, \( a, b \in I, \quad a < b \). If \( f : I \rightarrow R \) is a differential function such that there exist constants \( \gamma, \Gamma \in R \) with \( \gamma \leq f'(x) \leq \Gamma \), \( x \in [a, b] \), then we have

\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(t)dt - \frac{f(b) - f(a)}{b - a} \left( x - \frac{a + b}{2} \right) \right| \leq \frac{1}{4}(b - a)(\Gamma - \gamma) \quad \text{for all } x \in [a, b].
\]

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doi:10.1016/j.camwa.2011.11.017
In [17], Cheng presented the sharp version for (1), which is shown in the following theorem.

**Theorem 1.3** ([17, Theorem 1.5]). Let the assumptions of Theorem 1.1 hold. Then for all \( x \in [a, b] \), we have

\[
|f(x) - \frac{1}{b - a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b - a} (x - \frac{a + b}{2})| \leq \frac{1}{8} (b - a)(\Gamma - \gamma).
\]

The inequality (2) is sharp in the sense that the constant \( \frac{1}{8} \) cannot be replaced by a smaller one.

Motivated by the above work, in this paper, we will present some new Ostrowski–Grüss type integral inequalities, which are generalizations of Theorems 1.2 and 1.3 to the case involving \( \gamma \), \( \Gamma \in \mathbb{R} \) with \( \gamma \leq f(x) \leq \gamma \), \( x \in [a, b] \). Furthermore, suppose that \( x_k \in [a, b] \), \( i = 0, 1, \ldots, k, a = x_0 < x_1 < \cdots < x_k < b \) is a division of the interval \( [a, b] \), and \( m_i \in [x_{i-1}, x_i], i = 1, 2, \ldots, k, m_0 = a, m_{k+1} = b \). Then we have

\[
\int_{x_{i+1}}^{x_i} (m_{i+1} - m_i) f(x_i) = \int_a^b f(t)dt + \int_a^b k(t, l_i) f'(t)dt,
\]

where

\[
k(t, l_i) = \begin{cases}
    t - m_1, & t \in [x_0, x_1), \\
    t - m_2, & t \in [x_1, x_2), \\
    \vdots & \\
    t - m_{k-1}, & t \in [x_{k-2}, x_{k-1}), \\
    t - m_k, & t \in [x_{k-1}, x_k].
\end{cases}
\]

**Lemma 2.1** (Montgomery Identity). Let \( I \subset \mathbb{R} \) be an open interval, \( a, b \in I, a < b \). \( f : I \to \mathbb{R} \) is a differential function such that there exist constants \( \gamma, \Gamma \in \mathbb{R} \) with \( \gamma \leq f(x) \leq \gamma \), \( x \in [a, b] \). Furthermore, suppose that \( x_k \in [a, b], i = 0, 1, \ldots, k, a = x_0 < x_1 < \cdots < x_k < b \) is a division of the interval \( [a, b] \), and \( m_i \in [x_{i-1}, x_i], i = 1, 2, \ldots, k, m_0 = a, m_{k+1} = b \). Then we have

\[
\int_{x_{i+1}}^{x_i} (m_{i+1} - m_i) f(x_i) = \int_a^b f(t)dt + \int_a^b k(t, l_i) f'(t)dt.
\]

2. Main results

**Lemma 2.2** ([25, pp. 295, Grüss’ Inequality]). Let \( f, g : [a, b] \to \mathbb{R} \) be two integrable functions such that \( \Phi \leq f(x) \leq \Phi \) and \( \gamma \leq g(x) \leq \gamma \) for all \( x \in [a, b] \), where \( \Phi, \Phi, \gamma, \Gamma \) are constants. Then we have

\[
\left| \frac{1}{b - a} \int_a^b f(t)g(t)dt - \frac{1}{b - a} \int_a^b f(t)dt \cdot \frac{1}{b - a} \int_a^b g(t)dt \right| \leq \frac{1}{4} (\Phi - \Phi)(\Gamma - \gamma).
\]

**Theorem 2.3.** Under the conditions of Lemma 2.1, we have the following inequality

\[
\frac{1}{b - a} \left( \sum_{i=0}^{k} (m_{i+1} - m_i) f(x_i) - \frac{1}{b - a} \int_a^b f(t)dt \cdot \frac{f(b) - f(a)}{b - a} \right)^2 \leq \frac{1}{4} (b - a)(\Gamma - \gamma).
\]

**Proof.** From (4), we can obtain \( t - b \leq k(t, l_i) \leq t - a \). So a combination of Lemmas 2.1 and 2.2 yields

\[
\frac{1}{b - a} \left( \sum_{i=0}^{k} (m_{i+1} - m_i) f(x_i) - \frac{1}{b - a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b - a} \right)^2 \leq \frac{1}{4} (t - a)(t - b)(\Gamma - \gamma) = \frac{1}{4} (b - a)(\Gamma - \gamma).
\]

On the other hand,

\[
\int_a^b k(t, l_i)dt = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} (t - m_{i+1})dt = \sum_{i=0}^{k-1} \left[ \frac{(x_{i+1} - m_{i+1})^2}{2} - \frac{(x_i - m_{i+1})^2}{2} \right]
\]
\[
\text{Proof.}
\]
Under the conditions of Theorem 2.3, furthermore, let \( l^+_{[a, b]}, \ l^-_{[a, b]} \) be two subsets of the interval \([a, b]\), and \( p(t, l_k) \)
\[
\begin{cases}
  \geq 0, & t \in l^+_{[a, b]} \\
  \leq 0, & t \in l^-_{[a, b]}
\end{cases}
\]
where
\[
p(t, l_k) = \begin{cases}
  t - m_1 - C, & t \in [x_0, x_1), \\
  t - m_2 - C, & t \in [x_1, x_2), \\
  \ldots \\
  t - m_{k-1} - C, & t \in [x_{k-2}, x_{k-1}), \\
  t - m_k - C, & t \in [x_{k-1}, x_k].
\end{cases}
\]
and \( C = \frac{1}{2}(b + a) - \frac{1}{b-a} \sum_{i=0}^{k-1} m_{i+1}(x_{i+1} - x_i) \). Then we have the following inequality
\[
\left| \frac{1}{b-a} \sum_{i=0}^{k} (m_{i+1} - m_i) f(x_i) - \frac{1}{b-a} \int_{a}^{b} f(t)dt - \frac{f(b) - f(a)}{(b-a)^2} \left[ b^2 - a^2 - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1} - x_i) \right] \right|
\leq \frac{1}{8} (b-a)(\gamma^2 - \gamma)
\]
provided that one of the following two conditions holds:

(i) there exist \([x_{j-1}, x_j), j = 1, 2, \ldots, l_1\) such that \([x_{j-1}, x_j) \cap l^+_{[a, b]} \neq \emptyset, [x_{j-1}, x_j) \cap l^-_{[a, b]} \neq \emptyset\) for \(j = 1, 2, \ldots, l_1, l^+_{[a, b]} \subset \bigcup_{j=1}^{l_1} [x_{j-1}, x_j),\) and \(m(l^+_{[a, b]}) \leq \frac{b-a}{2}\).

(ii) there exist \([x_{j-1}, x_j), j = 1, 2, \ldots, l_2\) such that \([x_{j-1}, x_j) \cap l^+_{[a, b]} \neq \emptyset, [x_{j-1}, x_j) \cap l^-_{[a, b]} \neq \emptyset\) for \(j = 1, 2, \ldots, l_2, l^-_{[a, b]} \subset \bigcup_{j=1}^{l_2} [x_{j-1}, x_j),\) and \(m(l^-_{[a, b]}) \leq \frac{b-a}{2}\).

where \(m(.)\) denotes the measure of a Lebesgue measurable set.

The inequality (10) is sharp in the sense that the constant \(\frac{1}{8}\) cannot be replaced by a smaller one.

**Proof.** First, we have the following observations:

\[
\int_{a}^{b} p(t, l_k) f'(t)dt = \sum_{i=0}^{k-1} \int_{x_{i+1}}^{x_{i}} (t - m_{i+1} - C) f'(t)dt
\]
\[
= \sum_{i=0}^{k-1} [(x_{i+1} - m_{i+1} - C) f(x_{i+1}) - (x_{i} - m_{i+1} - C) f(x_{i})] - \sum_{i=0}^{k-1} \int_{x_{i}}^{x_{i+1}} f(t)dt
\]
\[
= \sum_{i=0}^{k} (m_{i+1} - m_i) f(x_i) - \int_{a}^{b} f(t)dt - \frac{f(b) - f(a)}{b-a} \left[ b^2 - a^2 - \sum_{i=0}^{k-1} m_{i+1}(x_{i+1} - x_i) \right].
\]

\[
\int_{a}^{b} p(t, l_k)dt = \sum_{i=0}^{k-1} \left[ \frac{(x_{i+1} - m_{i+1} - C)^2}{2} - \frac{(x_{i} - m_{i+1} - C)^2}{2} \right]
\]
\[
= \frac{1}{2} \sum_{i=0}^{k-1} [(x_{i+1} - x_i)(x_{i+1} + x_i - 2m_{i+1} - 2C)]
\]
\[
= \frac{1}{2} \sum_{i=0}^{k-1} (x_{i+1} - x_i)^2 - 2 \sum_{i=0}^{k-1} (x_{i+1} - x_i) m_{i+1} - 2C \sum_{i=0}^{k-1} (x_{i+1} - x_i)
\]
\[
= \frac{1}{2} (b^2 - a^2) - \sum_{i=0}^{k} (x_{i+1} - x_i)m_{i+1} - C(b-a) = 0.
\]

From the definition of \( l^+_{[a, b]} \) and \( l^-_{[a, b]} \), we can see \(\int_{a}^{b} p(t, l_k)dt = \int_{l^+_{[a, b]}} p(t, l_k)dt + \int_{l^-_{[a, b]}} p(t, l_k)dt = 0.\)
Now we suppose that (i) holds. Then, furthermore we have \( l^+_{[a,b]} = \bigcup_{j=1}^{l^+_{[a,b]}} [m_j + C, x_j] \). So
\[
\int_{l^+_{[a,b]}} p(t, l_k) dt = \sum_{j=1}^{l^+_{[a,b]}} \int_{m_j + C}^{x_j} (t - m_j - C) dt = \frac{1}{2} \sum_{j=1}^{l^+_{[a,b]}} (x_j - m_j - C)^2 \\
\leq \frac{1}{2} \left( \sum_{j=1}^{l^+_{[a,b]}} (x_j - m_j - C) \right)^2 = \frac{1}{2} \left( m(l^+_{[a,b]}) \right)^2 \leq \frac{(b - a)^2}{8},
\]
which implies
\[
\int_{l^+_{[a,b]}} p(t, l_k) dt = -\frac{1}{2} \sum_{j=1}^{l^+_{[a,b]}} (x_j - m_j - C)^2 \geq -\frac{(b - a)^2}{8}.
\]
Then we have
\[
\int_a^b p(t, l_k) f'(t) dt \leq (\Gamma - \gamma) \frac{(b - a)^2}{8}
\]
and
\[-\int_a^b p(t, l_k) f'(t) dt \leq (\Gamma - \gamma) \frac{(b - a)^2}{8},
\]
which implies
\[
\left| \int_a^b p(t, l_k) f'(t) dt \right| \leq (\Gamma - \gamma) \frac{(b - a)^2}{8}.
\]
Combining (11) and (15), we get the desired inequality (10).

If we suppose that (ii) holds, then following in the same manner as (13)–(15), we also get the desired inequality (10).

To prove the sharpness of (10), we take \( k = 2, m_0 = m_1 = a, m_2 = m_3 = b, x_1 = \frac{3b + a}{4} \), and
\[
f(t) = \begin{cases} 
\gamma t - \frac{b + a}{2} \gamma - \frac{b - a}{2} \Gamma, & a \leq t < \frac{b + 3a}{4}, \\
\Gamma t + \frac{a - b}{4} \gamma - \frac{3b + a}{4} \Gamma, & \frac{b + 3a}{4} \leq t < \frac{3b + a}{4}, \\
\gamma t - \frac{3b + a}{4} \gamma, & \frac{3b + a}{4} \leq t \leq b.
\end{cases}
\]

Then
\[
p(t, l_k) = \begin{cases} 
\frac{b + 3a}{4}, & t \in \left[ a, \frac{3b + a}{4} \right), \\
\frac{5b - a}{4}, & t \in \left[ \frac{3b + a}{4}, b \right].
\end{cases}
\]
From (17), we can see that \( l^+_{[a,b]} = \left[ \frac{b + 3a}{4}, \frac{3b + a}{4} \right) \subset [x_0, x_1] \), which gives \( l^+_{[a,b]} \cap [x_0, x_1] \neq \emptyset \), \( l^-_{[a,b]} \cap [x_0, x_1] \neq \emptyset \), and \( m(l^+_{[a,b]}) = \frac{b - a}{2} \). Then condition (i) holds. On the other hand, from the expression of \( f(t) \) in (16), we can see \( f'(t) = \begin{cases} 
\gamma, & t \in \left[ l^+_{[a,b]} \right), \\
\gamma, & t \in \left[ l^-_{[a,b]} \right].
\end{cases} \)
So (13)–(15) all hold in the equality form, which confirms the proof. □

**Remark 2.5.** If we set \( k = 2 \), then Theorems 2.3 and 2.4 reduce to [16, Theorem 2.1] and [17, Theorem 1.5] respectively.

In the following, we will extend the result in Theorem 2.3 to 2D case.

**Lemma 2.6** (2D Grüss’ Inequality). Let \( f, g : [a, b] \times [c, d] \rightarrow \mathbb{R} \) be two integrable functions such that \( \Phi \leq f(x, y) \leq \Phi \) and \( \gamma \leq g(x, y) \leq \Gamma \) for all \( x \in [a, b], \ y \in [c, d] \), where \( \Phi, \ \Gamma, \ \gamma \) are constants. Then we have
\[
\left| \frac{1}{(d - c)(b - a)} \int_a^b \int_c^d f(s, t) g(s, t) dt ds - \frac{1}{(d - c)(b - a)} \int_a^b \int_c^d f(s, t) dt ds \frac{1}{(d - c)(b - a)} \int_a^b \int_c^d g(s, t) dt ds \right| \\
\leq \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma).
\]

The proof for Lemma 2.6 is similar to [25, pp. 295–296], and we omit it here.
Theorem 2.7. Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be an absolutely continuous function such that the partial derivative of order 2 exists and there exist constants $K_1$, $K_2$ with $K_1 \leq \frac{\partial^2 f(x, t)}{\partial x^2} \leq K_2$. Suppose that $x_i \in [a, b]$, $y_i \in [c, d]$, $i = 0, 1, \ldots, k$. Let $l_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$ is a division of the interval $[a, b]$, while $l_k : a = y_0 < y_1 < \cdots < y_{k-1} < y_k = d$ is a division of the interval $[c, d]$. Let $\alpha_i \in [x_{i-1}, x_i]$, $\beta_i \in [y_{i-1}, y_i]$, $i = 1, 2, \ldots, k$, $\alpha_0 = a$, $\alpha_{k+1} = b$, $\beta_0 = c$, $\beta_{k+1} = d$. Then we have

\[
\begin{align*}
\sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (\alpha_{i+1} - \alpha_i)(\beta_{j+1} - \beta_j) f(x_i, y_j) + \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i)(y_k - \beta_k) f(x_i, y_k) - \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i)(y_0 - \beta_1) f(x_i, y_0) \\
+ \sum_{j=1}^{k-1} (x_k - \alpha_k)(\beta_{j+1} - \beta_j) f(x_k, y_j) + (x_k - \alpha_k)(y_k - \beta_k) f(x_k, y_k) - (x_k - \alpha_k)(y_0 - \beta_1) f(x_k, y_0) \\
- \sum_{j=1}^{k-1} (x_0 - \alpha_1)(\beta_{j+1} - \beta_j) f(x_0, y_j) - (x_0 - \alpha_1)(y_k - \beta_k) f(x_0, y_k) + (x_0 - \alpha_1)(y_0 - \beta_1) f(x_0, y_0) \\
- \sum_{j=1}^{k-1} \int_a^b (\beta_{j+1} - \beta_j) f(s, y_j) ds - \int_a^b [y_k - \beta_k] f(s, y_k) - (y_0 - \beta_1) f(s, y_0) ds \\
- \sum_{j=1}^{k-1} \int_c^d (\alpha_{i+1} - \alpha_i) f(x_i, t) dt - \int_c^d [x_k - \alpha_k] f(x_k, t) - (x_0 - \alpha_1) f(x_0, t) dt + \int_a^b \int_c^d f(s, t) dt ds \\
- \frac{\partial f(b, d) - f(b, c) - f(a, d) + f(a, c)}{(b - a)(d - c)} \left[ \frac{1}{2} (b^2 - a^2) - \sum_{i=0}^{k-1} \alpha_{i+1} (x_{i+1} - x_i) \right] \\
\times \left[ \frac{1}{2} (d^2 - c^2) - \sum_{j=0}^{k-1} \beta_{j+1} (y_{j+1} - y_j) \right] \right| \leq \frac{(b - a)(d - c)^2}{4} (K_2 - K_1).
\end{align*}
\]  

(19)

Proof. Define

\[
H(s, t, l_k, l_k) = (s - \alpha_{i+1})(t - \beta_{j+1}), \quad (s, t) \in [x_i, x_{i+1}] \times [y_j, y_{j+1}], \quad i, j = 0, 1, \ldots, k - 1.
\]

Then from (20), we obtain

\[
\begin{align*}
\int_a^b \int_c^d H(s, t, l_k, l_k) \frac{\partial^2 f(s, t)}{\partial s \partial t} dt ds \\
= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (s - \alpha_{i+1})(t - \beta_{j+1}) \frac{\partial^2 f(s, t)}{\partial s \partial t} dt ds \\
= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} (s - \alpha_{i+1}) \left[ \frac{\partial f(s, y_{j+1})}{\partial s} - (y_{j+1} - \beta_{j+1}) \frac{\partial f(s, y_j)}{\partial s} \right] ds \\
- \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \left[ (x_{i+1} - \alpha_{i+1}) f(x_{i+1}, y_{j+1}) - (x_i - \alpha_{i+1}) f(x_i, y_{j+1}) \right] (y_{j+1} - \beta_{j+1}) \\
- \int_{x_i}^{x_{i+1}} (y_{j+1} - \beta_{j+1}) f(s, y_{j+1}) - (y_j - \beta_{j+1}) f(s, y_j) ds \\
- \sum_{j=0}^{y_{j+1}} \left[ (x_{i+1} - \alpha_{i+1}) f(x_{i+1}, t) - (x_i - \alpha_{i+1}) f(x_i, t) \right] dt + \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} f(s, t) dt ds \\
= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} (\alpha_{i+1} - \alpha_i)(y_{j+1} - \beta_{j+1}) f(x_i, y_{j+1}) + \sum_{j=0}^{y_{j+1}} (x_{i+1} - \alpha_{i+1})(y_{j+1} - \beta_{j+1}) f(x_{i+1}, y_{j+1})
\end{align*}
\]
On the other hand, similar to (8), we can deduce
\[
\begin{align*}
\int_{a}^{b} \int_{c}^{d} H(s, t, l_k, J_k) dt ds &= \sum_{i=1}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_i+1} \int_{y_i}^{y_i+1} (s - \alpha_{i+1}(t - \beta_{j+1})) dt ds \\
&= \left[ \frac{1}{2} (b^2 - a^2) - \sum_{i=0}^{k-1} \alpha_{i+1} (x_{i+1} - x_i) \right] \left[ \frac{1}{2} (d^2 - c^2) - \sum_{j=0}^{k-1} \beta_{j+1} (y_{j+1} - y_j) \right],
\end{align*}
\]
and by the definition of $H(s, t, l_k, J_k)$, we observe that $\max(H(s, t, l_k, J_k)) - \min(H(s, t, l_k, J_k)) \leq (b-a)(d-c)$. Furthermore, by Lemma 2.6, we have
\[
\left| \frac{1}{(b-a)(d-c)} \int_{a}^{b} \int_{c}^{d} H(s, t, l_k, J_k) \frac{\partial^2 f(s, t)}{\partial s \partial t} dt ds \right|
\]
\[\leq \left( \frac{\max(H(s, t, l_k, J_k)) - \min(H(s, t, l_k, J_k))}{4} \right) (K_2 - K_1) \leq \frac{(b-a)(d-c)}{4} (K_2 - K_1).
\]
Then combining (21)–(23), we get the desired inequality (19). \qed

3. Applications in numerical integration

In this section, we will present one application for the results established above, and derive a sharp error bound for a new numerical integration formula.

Theorem 3.1. Suppose $f(t)$ satisfies the conditions of Theorem 2.4, $l_k : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ is a division of $[a, b]$, and $h_i = t_{i+1} - t_i$, $i = 0, 1, \ldots, n-1$. If $l_k : t_i = x_i, 0 < x_i, 1 < \cdots < x_i, k-1 < x_i, k = t_{i+1}$ is a division of the interval $[t_i, t_{i+1}]$, $m_{ij} \in [x_i, j, x_i, j]$; $j = 1, 2, \ldots, k, i = 0, 1, \ldots, n-1$ and $m_{ii} = t_i$, $m_{i, k+1} = t_{i+1}$. Let $l_{\{t_i, t_{i+1}\}}, l_{\{t_i, t_{i+1}\}^+}$ be two subsets of the interval $[t_i, t_{i+1}]$, and $p(t, l_i) \begin{cases} \geq 0, & t \in l_{\{t_i, t_{i+1}\}^+} \\ \leq 0, & t \in l_{\{t_i, t_{i+1}\}} \end{cases}$
and \( C_i = \frac{1}{2}(t_{i+1} + t_i) - \frac{1}{h_i} \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j}) \). Then for any matrices \( X = (x_{i,j})_{n \times (k+1)}, M = (m_{i,j})_{n \times (k+2)}, \) and

\[
H(f, X, M) = \sum_{i=0}^{n-1} \left( \frac{f(t_{i+1}) - f(t_i)}{h_i} \left[ \frac{(t_{i+1}^2 - t_i^2)}{2} - \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j}) \right] - \sum_{j=0}^{k-1} (m_{i,j+1} - m_{i,j}) f(x_{i,j}) \right),
\]

we have the following estimate

\[
\left| \int_a^b f(t) \, dt - H(f, X, M) \right| \leq \frac{(\Gamma - \gamma)}{8} \sum_{i=0}^{n-1} h_i^2
\]

provided that for each \([t_i, t_{i+1}], \ i = 0, 1, \ldots, n - 1, \) one of the following two conditions holds:

(i) there exist \([x_{i,p_{j-1}}, x_{i,p_j}], \ j = 1, 2, \ldots, l_1 \) such that \([x_{i,p_{j-1}}, x_{i,p_j}] \cap I_{[t_i, t_{i+1}]} \neq \emptyset, [x_{i,p_{j-1}}, x_{i,p_j}] \cap I_{[t_i, t_{i+1}]} \neq \emptyset \) for \( j = 1, 2, \ldots, l_1, I_{[t_i, t_{i+1}]} \subseteq \bigcup_{j=1}^{l_1} [x_{i,p_{j-1}}, x_{i,p_j}], \) and \( m(I_{[t_i, t_{i+1}]}) \leq \frac{h_i}{2} \),

(ii) there exist \([x_{i,n_{j-1}}, x_{i,n_j}], \ j = 1, 2, \ldots, l_2 \) such that \([x_{i,n_{j-1}}, x_{i,n_j}] \cap I_{[t_i, t_{i+1}]} \neq \emptyset, [x_{i,n_{j-1}}, x_{i,n_j}] \cap I_{[t_i, t_{i+1}]} \neq \emptyset \) for \( j = 1, 2, \ldots, l_2, I_{[t_i, t_{i+1}]} \subseteq \bigcup_{j=1}^{l_2} [x_{i,n_{j-1}}, x_{i,n_j}], \) and \( m(I_{[t_i, t_{i+1}]}) \leq \frac{h_i}{2} \).

**Proof.** Applying Theorem 2.4 with \([a, b] \) replaced by \([t_i, t_{i+1}] \), we obtain

\[
\left| \sum_{j=0}^{k} (m_{i,j+1} - m_{i,j}) f(x_{i,j}) - \int_{t_i}^{t_{i+1}} f(t) \, dt - \frac{f(t_i) - f(t_{i+1})}{h_i} \left[ \frac{(t_{i+1}^2 - t_i^2)}{2} - \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j}) \right] \right| \leq \frac{(\Gamma - \gamma)}{8} h_i^2,
\]

which implies

\[
\left| \sum_{j=0}^{n-1} \left\{ \sum_{j=0}^{k} (m_{i,j+1} - m_{i,j}) f(x_{i,j}) - \int_{t_i}^{t_{i+1}} f(t) \, dt - \frac{f(t_i) - f(t_{i+1})}{h_i} \left[ \frac{(t_{i+1}^2 - t_i^2)}{2} - \sum_{j=0}^{k-1} m_{i,j+1}(x_{i,j+1} - x_{i,j}) \right] \right\} \right| \leq \frac{(\Gamma - \gamma)}{8} \sum_{i=0}^{n-1} h_i^2.
\]

Then the proof is complete. \( \square \)

**Remark 3.2.** The estimate for error bound in (25) is sharp in the sense that the constant \( \frac{1}{8} \) cannot be replaced by a smaller one.

4. Conclusions

In this paper, we generalize some known Ostrowski–Grüss type inequalities to the case involving \((k - 1)\) interior points, and get the sharp version in 1D case. Furthermore, we derive an Ostrowski–Grüss type inequality in 2D case involving \((k - 1)^2\) interior points for the first time. Finally, we note that the inequality presented in 2D case is not sharp, and the sharp version of it is supposed to undergo further research.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (11171178), the Natural Science Foundation of Shandong Province (ZR2009AM011) (China) and the Specialized Research Fund for the Doctoral Program of Higher Education (20103705110003) (China).

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