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The covariant quantum superstring and superparticle from their classical actions

P.A. Grassi [a], G. Policastro [b,c], P. van Nieuwenhuizen [a]

[a] *C.N. Yang Institute for Theoretical Physics, State University of New York at Stony Brook, Stony Brook, NY 11794-3840, USA*
[b] *Scuola Normale Superiore, Piazza dei Cavalieri 7, Pisa 56126, Italy*
[c] *New York University, Department of Physics, 4 Washington Place, New York, NY 10003, USA*

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Abstract

We develop an approach based on the Noether method to construct nilpotent BRST charges and BRST-invariant actions. We apply this approach first to the holomorphic part of the flat-space covariant superstring, and we find that the ghosts b, c_z, which we introduced by hand in our earlier work, are needed to fix gauge symmetries of the ghost action. Then we apply this technique to the superparticle and determine its cohomology. Finally, we extend our results to the combined left- and right-moving sectors of the superstring.

1. Introduction and summary

Recently, a new approach to the completely super-Poincaré covariant quantization of the superstring with spacetime supersymmetry was developed in [1–3], based on earlier work by Berkovits [4–7]. A free quantum action invariant under BRST transformations and a nilpotent BRST generator Q were constructed [1]. The correct massless and massive spectrum for the open and closed string was obtained [2]. The definition of physical states in terms of equivariant cohomology was established [3]. In [1] a ghost pair (c_z, b) was introduced by hand to make the BRST charge nilpotent, and another BRST-inert ghost system (namely, η^m, ω_z^m in [1], replaced by η_z^m, ω^m in [2]) was introduced by hand to cancel the central charge. In this Letter we shall construct the quantum action and the BRST charge using the Noether method, and we obtain in this way a derivation of the ghost pair b, c_z.

We start from the classical Green–Schwarz action, but we take a flat worldsheet metric,[1] and we replace the κ transformation $\delta_\kappa \theta^\alpha = \gamma_m^{\alpha\beta} \Pi_z^m \kappa_\beta^z$ by the more general expression $\delta_\lambda \theta^\alpha = \lambda^\alpha$ where λ^α is a real commuting 16-component $D = (9, 1)$ spinor. Using the Noether method applied to BRST symmetry, new ghosts are added to the action. A preliminary

E-mail addresses: pgrassi@insti.physics.sunysb.edu (P.A. Grassi), g.policastro@sns.it (G. Policastro), vannieu@insti.physics.sunysb.edu (P. van Nieuwenhuizen).

[1] At the tree level the choice of a flat worldsheet metric is sufficient, but clearly at one loop or for higher genus surfaces (with or without punctures) it is inadequate.

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ghost action will turn out to have a rigid symmetry but is not BRST-invariant. Making this symmetry local leads to the ghost system b, c_z leads and a BRST-invariant action. We apply this general method to several cases: (i) the heterotic superstring, (ii) the superparticle and, (iii) the flat space superstring with combined left- and right-moving sectors. In all the cases we do arrive at an invariant action and a nilpotent BRST charge.

There exists now a derivation of the b, c_z system from first principles. For the η_z^m, ω^m ghost system a similar derivation is still lacking.

A different approach, starting from a twisted version of the complexified $N = 2$ superembedding formulation of the superstring, has been studied in [8].

2. Heterotic superstring and superparticle

The basis for our work is a remarkable identity between the free classical (i.e., without ghosts) superstring $S_{\text{free}}^{\text{class}}$, the full nonlinear classical Green–Schwarz (GS) superstring S_{GS}, and antihermitian composite objects $d_{L\alpha}$ and $d_{R\alpha}$ [9]. In the conformal gauge, $h^{\mu\nu} = \eta^{\mu\nu}$, one has in Minkowski space

$$S_{\text{free}}^{\text{class}} = S_{\text{GS}} - \int d^2 z \left(d_{L\mu\alpha}(\eta^{\mu\nu} - \epsilon^{\mu\nu})\partial_\nu \theta_L^\alpha + d_{R\mu\alpha}(\eta^{\mu\nu} + \epsilon^{\mu\nu})\partial_\nu \theta_R^\alpha \right),$$

$$\mathcal{L}_{\text{free}}^{\text{class}} = -\frac{1}{2}\partial_\mu x^m \partial^\mu x_m - p_{L\mu\alpha} P^{\mu\nu} \partial_\nu \theta_L^\alpha - p_{R\mu\alpha} \bar{P}^{\mu\nu} \partial_\nu \theta_R^\alpha, \tag{2.1}$$

where $P^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu}$ and $\bar{P}^{\mu\nu} = \eta^{\mu\nu} + \epsilon^{\mu\nu}$. Furthermore, $S_{\text{GS}} = S_{\text{kin}} + S_{\text{WZ}}$ with

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\Pi_\mu^m \Pi_m^\mu,$$

$$\mathcal{L}_{\text{WZ}} = -\epsilon^{\mu\nu}\left[i\partial_\mu x^m (\theta_L \gamma_m \partial_\nu \theta_L - \theta_R \gamma_m \partial_\nu \theta_R) - (\theta_L \gamma^m \partial_\mu \theta_L)(\theta_R \gamma_m \partial_\nu \theta_R) \right] \tag{2.2}$$

and

$$d_{L\mu\alpha} = p_{L\mu\alpha} + \left(i\partial_\mu x^m + \frac{1}{2}\theta_L \gamma^m \partial_\mu \theta_L + \frac{1}{2}\theta_R \gamma^m \partial_\mu \theta_R \right)(\gamma_m \theta_L)_\alpha,$$

$$d_{R\mu\alpha} = p_{R\mu\alpha} + \left(i\partial_\mu x^m + \frac{1}{2}\theta_L \gamma^m \partial_\mu \theta_L + \frac{1}{2}\theta_R \gamma^m \partial_\mu \theta_R \right)(\gamma_m \theta_R)_\alpha,$$

$$\Pi_\mu^m = \partial_\mu x^m - i\theta_L^\alpha \gamma_{\alpha\beta}^m \partial_\mu \theta_L^\beta - i\theta_R^\alpha \gamma_{\alpha\beta}^m \partial_\mu \theta_R^\beta. \tag{2.3}$$

In chiral notation one has

$$\mathcal{L}_{\text{free}}^{\text{class}} = -\frac{1}{2}\partial x^m \bar{\partial} x_m - p_{L\alpha} \bar{\partial} \theta_L^\alpha - p_{R\alpha} \bar{\partial} \theta_R^\alpha$$

with $\partial = \partial_\sigma - \partial_t$ and $\bar{\partial} = \partial_\sigma + \partial_t$. Further,

$$d_{L\alpha} = p_{L\alpha} + \left(i\partial x^m + \frac{1}{2}\theta_L \gamma^m \partial \theta_L + \frac{1}{2}\theta_R \gamma^m \partial \theta_R \right)(\gamma_m \theta_L)_\alpha,$$

$$d_{R\alpha} = p_{R\alpha} + \left(i\bar{\partial} x^m + \frac{1}{2}\theta_L \gamma^m \bar{\partial} \theta_L + \frac{1}{2}\theta_R \gamma^m \bar{\partial} \theta_R \right)(\gamma_m \theta_R)_\alpha.$$

For us the identity in (2.1) is useful because it defines objects $d_{L\mu\alpha}$ and $d_{R\mu\alpha}$ which play a crucial role in what follows. They become constraints in the quantum theory and form the starting point for the BRST charge. We denote the left-moving spinor in the Green–Schwarz action by θ_L, while θ_R is the right-moving spinor. Chiral θ's have spinorial superscript θ_L^α and θ_R^α and antichiral θ's are denoted by θ_α. Thus for the IIA case, we use the notation $\theta_{\alpha R}$.

There also exists a relation in Berkovits' approach between the free quantum action, the GS action and a BRST exact term. It reads (we use the notation w_α for the conjugate momentum of λ^α instead of β_α of our earlier work to facilitate the comparison with [4–7])

$$S_{\text{free}}^{\text{qu}} = S_{\text{GS}} + Q_B \int d^2 z \left(w_{L\mu\alpha} P^{\mu\nu} \partial_\nu \theta_L^\alpha + w_{R\mu\alpha} \bar{P}^{\mu\nu} \partial_\nu \theta_R^\alpha \right), \tag{2.4}$$

where

$$\mathcal{L}_{\text{free}}^{\text{qu}} = \mathcal{L}_{\text{free}}^{\text{class}} - w_{L\mu\alpha} P^{\mu\nu} \partial_\nu \lambda_L^\alpha - w_{R\mu\alpha} \bar{P}^{\mu\nu} \partial_\nu \lambda_R^\alpha.$$

Further, $Q_B = (Q_{B,L} + Q_{B,R})$ with

$$Q_{B,L} = \int d\sigma\, dt \left(i\lambda_L^\alpha \frac{\delta}{\delta\theta_L^\alpha} + \lambda_L\gamma^m\theta_L \frac{\delta}{\delta x^m} + d_{L\mu}\frac{\delta}{\delta w_{L\alpha}} - \Pi^m(\lambda_L\gamma_m)_\alpha \frac{\delta}{\delta d_{L\alpha}} \right), \tag{2.5}$$

and similarly $Q_{B,R}$, which satisfy

$$Q_{B,L}^2 = \int d\sigma\, dt\, \left(-i\lambda_L\gamma^m\lambda_L\right) \times \left(\frac{\delta}{\delta x^m} + \left(\partial_m\theta\gamma^m\right)_\alpha \frac{\delta}{\delta d_{L\alpha}} \right) - \Pi^m(\lambda_L\gamma_m)_\alpha \frac{\delta}{\delta w_{L\alpha}}. \tag{2.6}$$

In Berkovits approach the BRST operator Q_B is not hermitian or antihermitian, because his λ^α is complex, but in our approach the BRST operator, denoted by Q, is antihermitian. For pure spinors λ satisfying $\lambda\gamma^m\lambda = 0$, Q_B is clearly nilpotent on x^m, θ^α, λ^α and $d_{z\alpha}$, but does not vanish on w_α. The free quantum action (2.4) is invariant under the gauge transformation $\delta w_\alpha^\mu = \Lambda_m^\mu(\gamma^m\lambda)_\alpha$ if the λ's are pure spinors, and the BRST operators are nilpotent up to a gauge transformation. The Q_B variation of S_{GS} does not vanish either, but $S_{\mathrm{free}}^{\mathrm{qu}}$ is Q_B-invariant. The relation in (2.4) was discovered by Oda and Tonin [10], and has been used by Berkovits to construct the pure spinor action in a curved background [11]. In our derivation below this relation plays no role. We shall use the Noether method, applied to BRST symmetry.

In this section we restrict ourselves to one (left-moving) sector (the heterotic string). In Section 4 we discuss the combined left- and right-moving sector. We start from the GS action which we decompose into a kinetic term and a Wess–Zumino (WZ) term, $S_{\mathrm{GS}} = S_{\mathrm{kin}} + S_{\mathrm{WZ}}$. We shall not need S_{WZ} but only its exterior derivative which is given by the following 3-form both for the IIB and the IIA cases

$$d\mathcal{L}_{\mathrm{WZ}} = -i\, d\theta_L \not{\Pi}\, d\theta_L + i\, d\theta_R \not{\Pi}\, d\theta_R. \tag{2.7}$$

The action is invariant under local κ (Siegel) gauge transformations if one does not fix the conformal gauge. We consider the GS action in the conformal gauge. In this gauge the κ symmetry transformations acquire extra compensating terms and are quite complicated. We follow, therefore, a different approach. We choose the conformal gauge and replace the composite parameters $\not{\Pi}\kappa$ of κ symmetry by a new local classical gauge parameter λ. The GS action (from now on in the conformal gauge) is of course not invariant under the λ transformations of x^m and θ^α, but we shall use the Noether method to obtain a BRST-invariant free quantum action. The new local gauge transformations of x and θ follow straightforwardly by replacing $\Pi_{zm}\gamma^{\alpha\beta}\kappa_\beta^z$ by λ^α

$$\delta_\lambda x^m = -i\lambda\gamma^m\theta, \qquad \delta_\lambda\theta^\alpha = \lambda^\alpha. \tag{2.8}$$

The matrices $\gamma_{\alpha\beta}^m$ are real and symmetric, hence the reality of $\delta_\lambda x^m$ and of $\delta_\lambda\theta^\alpha$ is preserved.

The geometrical meaning is at this point unclear. However, Eq. (2.8) has the same form as the BRST transformations generated by the BRST charge Q_B in Berkovits' formalism. Therefore, we interpret λ from this point on as a real ghost which changes its statistics: λ becomes commuting. The BRST transformations with constant anticommuting antihermitian parameter Λ read $\delta_B\theta^\alpha = i\Lambda\lambda^\alpha$ and $\delta_B x^m = i\Lambda\delta_\lambda x^m$. Denoting the BRST transformation of x^m and θ^α without Λ by s, we obtain $s\theta^\alpha = i\lambda^\alpha$ and $sx^m = \lambda\gamma^m\theta$. The BRST transformations close (they are nilpotent) if the λ's are pure spinors. In our approach [1] we do not impose any constraints on the spinors λ, and therefore, to still regain nilpotency of the λ transformation, we modify the λ transformation rules of x and θ by adding further fields such that they become nilpotent. Nilpotency of s is achieved by defining $s\lambda^\alpha = 0$, but since s is not nilpotent on x, we introduce a new ghost ξ_m in sx^m

$$sx^m = \lambda\gamma^m\theta + \xi^m, \qquad s\xi^m = -i\lambda\gamma^m\lambda, \tag{2.9}$$

where ξ_m is anticommuting and real. We have obtained $s^2 = 0$ on x. For the variation of the action we need the variation of Π_μ^m which is given by

$$s\Pi_\mu^m = \partial_\mu\xi^m + 2\lambda\gamma^m\partial_\mu\theta. \tag{2.10}$$

The variation of S_{kin} contains a term with a derivative of a ghost which we can handle with the Noether approach, and a term with $\partial_\mu\theta$ which poses a problem as far as the Noether method is concerned and which therefore should be removed

$$s\left(\frac{1}{2}\Pi_\mu^m\Pi_{\nu m}\right) = \Pi_{(\mu}^m\left(\partial_{\nu)}\xi_m + 2\lambda\gamma_m\partial_{\nu)}\theta\right). \tag{2.11}$$

To remove the term with $\partial_\nu\theta$ we modify the induced metric $G_{\mu\nu} = \Pi^m_{(\mu}\Pi_{\nu)m}$ by adding a suitable term to it

$$G^{\mathrm{mod}}_{\mu\nu} = \Pi^m_{(\mu}\Pi_{\nu)m} + 2d_{(\mu\alpha}\partial_{\nu)}\theta^\alpha, \tag{2.12}$$

where $d_{\mu\alpha}$ is a new antihermitian anticommuting field. The extra term $-d_{\mu\alpha}P^{\mu\nu}\partial_\nu\theta^\alpha$ in the action should be interpreted as a gauge fixing term which breaks the κ-symmetry. The gauge fixed kinetic term varies as follows

$$\begin{aligned} sG^{\mathrm{mod}}_{\mu\nu} &= 2\Pi^m_{(\mu}\partial_{\nu)}\xi_m + \big[4\big(\lambda\not{\Pi}_{(\mu}\big)_\alpha + 2sd_{(\mu\alpha}\big]\partial_{\nu)}\theta^\alpha \\ &\quad - 2id_{\mu\alpha}\partial_\nu\lambda^\alpha. \end{aligned} \tag{2.13}$$

The most general expression for $sd_{\mu\alpha}$ which leaves only terms with derivatives of ghosts is given by

$$sd_{\mu\alpha} = -2\big(\not{\Pi}_\mu\lambda\big)_\alpha + \partial_\mu\chi + A_m\big(\gamma^m\partial_\mu\theta\big)_\alpha, \tag{2.14}$$

where A_m is an antihermitian anticommuting vector to be fixed. We used that $\partial_{(\mu}\gamma^m\partial_{\nu)}\theta$ vanishes, made a Fierz rearangement and introduced a new real commuting ghost field χ_α, which can be interpreted as the antichiral counterpart of the chiral λ^α. We fix these free objects by requiring that $sd_{\mu\alpha}$ be s inert (nilpotency of s on $d_{\mu\alpha}$). This yields

$$\begin{aligned} sd_\mu &= \partial_\mu\chi - 2\not{\Pi}_\mu\lambda - 2i\xi^m\gamma_m\partial_\mu\theta, \\ s\chi &= 2\xi^m\gamma_m\lambda. \end{aligned} \tag{2.15}$$

So far we have achieved that the s variation of

$$\mathcal{L}^{\mathrm{mod}}_{\mathrm{kin}} = -\frac{1}{2}\Pi^\mu_m\Pi^m_\mu - d_{\mu\alpha}\partial^\mu\theta^\alpha \tag{2.16}$$

contains only terms with derivatives of the ghosts λ^α, χ_α, and ξ^m, namely

$$s\mathcal{L}^{\mathrm{mod}}_{\mathrm{kin}} = -\Pi^\mu_m\partial_\mu\xi^m - \partial^\mu\theta\partial_\mu\chi + id_{\mu\alpha}\partial^\mu\lambda^\alpha. \tag{2.17}$$

We now repeat this program for the WZ term. It is a good consistency check that this is possible at all. We define a modified WZ term as follows

$$\mathcal{L}^{\mathrm{mod}}_{\mathrm{WZ}} = \mathcal{L}_{\mathrm{WZ}} + \epsilon^{\mu\nu}d_{\mu\alpha}\partial_\nu\theta^\alpha. \tag{2.18}$$

One finds that also $s\mathcal{L}^{\mathrm{mod}}_{\mathrm{WZ}}$ only contains terms with derivatives of ghosts

$$s\mathcal{L}^{\mathrm{mod}}_{\mathrm{WZ}} = \epsilon^{\mu\nu}\big[\Pi^m_\mu\partial_\nu\xi_m + \partial_\mu\theta\partial_\nu\chi - id_{\mu\alpha}\partial_\nu\lambda^\alpha\big]. \tag{2.19}$$

The sum of all variations is given by

$$\begin{aligned} s\big(\mathcal{L}^{\mathrm{mod}}_{\mathrm{kin}} + \mathcal{L}^{\mathrm{mod}}_{\mathrm{WZ}}\big) &= -\Pi^m_\mu P^{\mu\nu}\partial_\nu\xi_m + id_{\mu\alpha}P^{\mu\nu}\partial_\nu\lambda^\alpha \\ &\quad - \partial_\mu\theta^\alpha P^{\mu\nu}\partial_\nu\chi_\alpha. \end{aligned} \tag{2.20}$$

The next step is to cancel these variations by adding free ghost actions and defining suitable transformation laws for the antighost fields

$$\begin{aligned} \mathcal{L}_{\mathrm{gh}} &= -\beta_{\mu m}P^{\mu\nu}\partial_\nu\xi^m - w_{\mu\alpha}P^{\mu\nu}\partial_\nu\lambda^\alpha \\ &\quad - \kappa^\alpha_\mu P^{\mu\nu}\partial_\nu\chi_\alpha. \end{aligned} \tag{2.21}$$

The antighost β^μ_m is anticommuting and antihermitian, while $w_{\mu\alpha}$ and κ^α_μ are commuting and real. Because the variation of $\mathcal{L}_{\mathrm{kin}} + \mathcal{L}_{\mathrm{WZ}}$ contain the operator $P^{\mu\nu} = \eta^{\mu\nu} - \epsilon^{\mu\nu}$, the antighosts are holomorphic (chiral on the worldsheet: they have the index structure β^m_z, $\beta_{\alpha z}$ and κ^α_z). One finds easily a particular solution for the variation of the antighosts, but the most general solution contains a free constant b and a target-space bispinor $\eta^{\mu,\alpha\beta}$

$$\begin{aligned} s\beta^\mu_m &= \big(-\Pi^\mu_m - 2\kappa^\mu\gamma_m\lambda\big) + \left(b\partial^\mu\xi_m + \frac{1}{2}\partial^\mu b\xi_m\right) \\ &\quad + \big(\chi\eta^\mu\gamma_m\lambda\big), \\ sw^\mu_\alpha &= \big(id^\mu - 2i\beta^\mu_m\gamma^m\lambda - 2\xi_m\gamma^m\kappa^\mu\big)_\alpha \\ &\quad - i\left(b\partial^\mu\chi_\alpha + \frac{3}{4}\partial^\mu b\chi_\alpha\right) + \big(\not{\xi}\eta^\mu\chi\big), \\ s\kappa^{\alpha\mu} &= \big(-\partial^\mu\theta^\alpha\big) + i\left(b\partial^\mu\lambda^\alpha + \frac{1}{4}\partial^\mu b\lambda^\alpha\right) \\ &\quad + \big(\eta^\mu\not{\xi}\lambda\big). \end{aligned} \tag{2.22}$$

The transformations with b map β into its own ghost ξ and w and κ into the other commuting ghosts while the transformations with $\eta^{\mu,\alpha\beta}$ map each antighost into the two noncorresponding ghosts.

Setting the anticommuting and antihermitian b and the real commuting $\eta^{\mu,\alpha\beta}$ to zero yields a solution of the inhomogeneous equation for the transformation laws of the antighosts, but the terms with *constant* b and $\eta^{\mu,\alpha\beta}$ yield further homogeneous solutions. In other words, we are encountering a system with constant ghosts-for-ghosts. We have already added the terms with a derivative of b for reasons to be explained now.

The terms in the transformation rules with constant b and $\eta^{\mu,\alpha\beta}$ yield new rigid symmetries of the ghost action. Although we have obtained an s-invariant action, the transformation rules for the antighosts are not nilpotent. We now let b become a field and add the terms with $\partial_\mu b$ in (2.22). The action then ceases to be invariant, but the transformation laws of the antighosts can be made nilpotent by defining suitable transformation laws for b and η, namely,

$$sb = 1, \qquad s\eta^{\mu,\alpha\beta} = 0. \tag{2.23}$$

In fact the terms in (2.22) with $\eta^{\mu,\alpha\beta}$ can be removed by redefining $\kappa^{\alpha\mu} \to \kappa^{\alpha\mu} + (1/2)(\eta^{\mu}\chi)^{\alpha}$ and for this reason we omit them from now on. This redefinition leads to a new term in the action of the form $\chi_\alpha \eta^{\mu,\alpha\beta}\partial_\mu \chi_\beta$; however, this extra term is a total derivative which we also omit.

Returning to the problem of making the action BRST-invariant, we need a kinetic term for b. Hence, we introduce also a new real anticommuting ghost c_μ and add the following term to the ghost action: $\mathcal{L}_{\text{gh}}^{\text{extra}} = -bP^{\mu\nu}\partial_\mu c_\nu$. We determine the transformation rule of c_μ such that the action becomes s-invariant. One finds

$$sc_\mu = -\frac{1}{2}\left(\xi^m \partial_\mu \xi_m - \frac{3i}{2}\chi_\alpha \partial_\mu \lambda^\alpha + \frac{i}{2}\partial_\mu \chi_\alpha \lambda^\alpha\right). \tag{2.24}$$

Also this transformation law is nilpotent.

In this way we have reobtained the free BRST-invariant action and the nilpotent BRST transformation rules of [1]. In particular, we have given a derivation of the need for the b, c_μ system which follows from the Noether procedure applied to symmetries of the ghost action. However, the problem of giving a similar fundamental derivation of the η, ω system remains. For the string the η, ω system was needed to cancel the central charge. For the superparticle, to which we now turn, the b, c system is needed, but the η, ω system is not needed because for the superparticle there is no central charge and hence we do not need to cancel it.

3. The superparticle

In this section we apply the procedure presented in the previous section to the point particle. The operator formalism of [1] cannot directly be applied in this case because $\dot\theta$ vanishes on-shell. The off-shell BRST approach is successful. We consider the open string, hence rigid $N = 1$ spacetime susy with one θ. We shall show that the correct spectrum, namely the field equations of $d = (9, 1)$ $N = 1$ super-Yang–Mills theory, is obtained.

We start from the $N = 1$ supersymmetric action [12]

$$S = \int d\tau\, \frac{1}{2e}\left(\dot x^m - i\theta^\alpha \gamma^m_{\alpha\beta}\dot\theta^\beta\right)^2, \quad \alpha = 1, \ldots, 16, \tag{3.1}$$

which is invariant under κ-symmetry:

$$\delta_\kappa \theta^\alpha = \Pi_m\left(\gamma^m\kappa\right)^\alpha, \qquad \delta_\kappa x^m = i\theta\gamma^m\delta_\kappa\theta,$$
$$\delta_\kappa e = 4ie\dot\theta^\alpha\kappa_\alpha, \tag{3.2}$$

where $\Pi_m = \dot x^m - i\theta^\alpha\gamma^m_{\alpha\beta}\dot\theta^\beta$. The quantization of (3.1) is nontrivial because of the fermionic constraint $\delta S/\delta\dot\theta^\alpha = p_\alpha = iP^m(\gamma_m\theta)_\alpha$ with P_m and p_α the conjugate momenta to the x and θ coordinates. The anticommutator

$$\left\{p_\alpha - iP^m(\gamma_m\theta)_\alpha,\ p_\beta - iP^m(\gamma_m\theta)_\beta\right\} = -2\gamma^m_{\alpha\beta}P_m$$

shows that the fermionic constraints are both first and second class: only half of them anticommute with each other.[2] However, it is difficult to disentangle these two classes and construct a covariant set of independent basis vectors for these constraints.[3] The theory is invariant under reparametrization of the worldline; however, we will set $e = 1$ from the beginning and construct a consistent model with local transformation rules. In the original superparticle, one could choose the gauge $e = 1$, but then κ transformations acquire extra nonlocal compensating terms with $\xi(t) = \int^t dt'\,(4i\dot\theta k)(t')$.[4]

[2] Decomposing $d_\alpha = p_\alpha - iP^m(\gamma_m\theta)_\alpha$ into $\not{P} d_\alpha + (1 - \not{P})d_\alpha$, the $\not{P} d_\alpha$ are first class and the $(1 - \not{P})d_\alpha$ are second class.

[3] Recently, two of the authors [13] presented a solution of the quantization of the superparticle using a "twistor"-like redefinition of variables $P^m\gamma_m^{\alpha\beta} = \lambda^\alpha_a(\sigma^+ + P^2\sigma^-)^a{}_b\lambda^{\beta b}$ where λ^α_a are the twistor-like variables and $\sigma^\pm$ the Pauli matrices. One way to disentangle the two types of constraints is an infinite number of ghosts. Using Batalin–Vilkovisky techniques the ghosts of level greater than three do not interact with the ghost of lower levels and with the other fields of the theory.

[4] There should be a better way to do this: first go to the light-cone gauge for the superparticle action (3.1) and reparameterize the fermions by $\zeta^a = \sqrt{p^+}(\gamma^-\theta)^\alpha$ where $\gamma^\pm = \frac{1}{2}(\gamma^0 \pm \gamma^9)$. The BRST operator for the quantized model is only $Q = cP^2$ and the states are representations of the Clifford algebra $\{\zeta^a, \zeta^b\} = 2\delta^{a,b}$. Berkovits [14] finds an interpolating BRST operator $\hat Q$ in an enlarged functional space with the unconstrained spinors $\hat\lambda^\alpha$ and their conjugate momenta $\hat w_\alpha$, and the composite field d_α. One can show that the cohomology can be constructed in two equivalent ways: the first reproduces the light-cone massless states of the

We compute the variation of (3.1) under the BRST transformations

$$s x^m = \xi^m + \theta\gamma^m\lambda, \qquad s\theta^\alpha = i\lambda^\alpha,$$
$$s\xi^m = -i\lambda\gamma^m\lambda, \qquad s\lambda^\alpha = 0. \tag{3.3}$$

In order that the variation of (3.1) be proportional to the equations of motion of the ghost fields, we add the term $\int d\tau\, d_\alpha\dot{\theta}^\alpha$ where d_α and its BRST variation are given by

$$d_\alpha = p_\alpha + i\dot{x}_m(\gamma^m\theta)_\alpha + \frac{1}{2}(\gamma^m\theta)_\alpha(\theta\gamma_m\dot{\theta}),$$
$$s d_\alpha = \dot{\chi}_\alpha - 2\Pi_m\gamma^m\lambda + \Lambda_m(\gamma^m\dot{\theta})_\alpha, \tag{3.4}$$

where Λ_m and χ_α are two arbitrary fields. Notice that we can freely add the ghost χ_α since on-shell this term vanishes. The BRST transformation of d_α is nilpotent if

$$\Lambda_m = -2i\xi_m, \qquad s\chi_\alpha = 2\xi^m(\gamma_m\lambda)_\alpha.$$

Then, following the procedure already discussed, we add ghost terms to the action

$$S_{\mathrm{gh},1} = \int d\tau\,\left(\beta_m\dot{\xi}^m + w_\alpha\dot{\lambda}^\alpha + \kappa^\alpha\dot{\chi}_\alpha\right) \tag{3.5}$$

whose variation cancels against the variation of $S + \int d\tau\, d_\alpha\dot{\theta}^\alpha$ if the antighosts transform in the following way

$$s\beta_m = -\Pi_m - 2\kappa\gamma_m\lambda + b\dot{\xi}_m + \frac{1}{2}\dot{b}\xi_m,$$
$$s w_\alpha = i d_\alpha - 2i\beta_m(\gamma^m\lambda)_\alpha - 2\xi_m(\gamma^m\kappa)_\alpha$$
$$\qquad - ib\dot{\chi}_\alpha - \frac{3i}{4}\dot{b}\chi_\alpha,$$
$$s\kappa^\alpha = -\dot{\theta}^\alpha + ib\dot{\lambda}^\alpha + \frac{i}{4}\dot{b}\lambda^\alpha. \tag{3.6}$$

The contributions with ghosts–antighosts in the transformation rules are needed to compensate the nonlinear variations of the ghost fields ξ^m and χ_α in the action (3.5). Further, the terms proportional to b or $\dot{b}$ are needed to obtain a nilpotent BRST symmetry. As we learned from the previous section, a suitable redefinition of κ^α removes the η^m terms from the symmetry, therefore, we have already chosen the basis without η^m. The nilpotency of the BRST symmetry is achieved by defining $sb = 1$.

The last step is to add a $b - c$ term to the action and derive the BRST transformation for the ghost c

$$S_{\mathrm{gh},2} = \int d\tau\, b\dot{c},$$
$$sc = -\frac{1}{2}\left(\xi^m\dot{\xi}_m - \frac{3i}{2}\chi_\alpha\dot{\lambda}^\alpha + \frac{i}{2}\dot{\chi}_\alpha\lambda^\alpha\right). \tag{3.7}$$

The sum $S + S_{\mathrm{gh},1} + S_{\mathrm{gh},2}$ is now invariant under BRST symmetry. At this point, we can rewrite the terms of the action which contain the field x^m in a first-order formalism. Namely, $\int d\tau\,\frac{1}{2}\Pi^2 = \int d\tau\,\times (P_m\Pi^m - \frac{1}{2}P^2)$. Canonical quantization implies that $[P^m, x^n] = -i\eta^{mn}$. This will be used in the next section.

We now turn to the determination of the massless cohomology for the superparticle. The physical states of the superparticle should be found at ghost number 1. Without further restriction, the cohomology is, however, trivial, but following [2] we assign a grading to the ghost fields

$$gr(\lambda^\alpha) = 1, \qquad gr(\xi^m) = 2,$$
$$gr(\chi_\alpha) = 3, \qquad gr(c) = 4, \tag{3.8}$$

and the corresponding opposite numbers for antighosts. We cannot use the affine Lie algebra to determine the grading of χ and c as in [2], because $\dot{\theta} = 0$ is a here a field equation and there is no central charge for a point particle. However, observing that the part Q_0 of the BRST operator which only contains ghost and antighost fields is nilpotent by itself, one can introduce a grading which explains this. Namely, Q_0 has vanishing grading and this yields $gr(\chi) = 3$ and $gr(b) = -4$. The relevant cohomology is selected in the functional space of non-negatively graded polynomials denoted in the following by $\mathcal{H}_+$.[5]

superparticle, the other reproduces the BRST cohomology with pure spinor constraints. It would be interesting to repeat this approach for our formulation.

[5] Notice that in the pure spinor formulation, λ^α should be complex and its complex conjugate $\bar{\lambda}_\alpha$ should transform under the conjugated representation of $Spin(9,1)$. This implies that one can construct a homotopy operator $\mathcal{K}$ for the BRST charge $Q_B = \lambda^\alpha d_\alpha$. It is easy to show that $\mathcal{K} = \bar{\lambda}_\alpha\theta^\alpha/(\bar{\lambda}\lambda)$ with $(\bar{\lambda}\lambda) = \bar{\lambda}_\alpha\lambda^\alpha$ satisfies $\{Q, \mathcal{K}\} = 1$. This obviously renders the cohomology in [15] trivial since every Q-closed expression is also Q-exact. In order to obtain a nontrivial cohomology one may use the grading in (3.8) and observe that the homotopy operator $\mathcal{K}$ has negative grading.

The most general scalar expression in $\mathcal{H}_+$ with ghost number one is

$$\mathcal{U}^{(1)}(z) = i\lambda^\alpha A_\alpha + \xi^m A_m + \chi_\alpha W^\alpha + b\big(\xi^m \xi^n F_{mn} + i\lambda^\alpha \chi_\beta F_\alpha{}^\beta + \chi_\alpha \xi^m F^\alpha{}_m + \chi_\alpha \chi_\beta F^{\alpha\beta}\big), \tag{3.9}$$

where $A_\alpha, \ldots, F^{\alpha\beta}$ are arbitrary superfields depending on x_m, θ^α. The requirement of positive grading has ruled out $b\lambda^\alpha\lambda^\beta$ and $b\lambda^\alpha\xi^m$.

The condition $\{Q, \mathcal{U}^{(1)}(z)\} = 0$ implies the following equations

$$\begin{aligned}
&D_{(\alpha} A_{\beta)} + i\gamma^m_{\alpha\beta} A_m = 0,\\
&\partial_m A_\alpha - D_\alpha A_m - 2i\gamma_{m\alpha\beta} W^\beta = 0,\\
&\partial_{[m} A_{n]} + F_{mn} = 0, \qquad D_\beta W^\alpha + F_\beta{}^\alpha = 0,\\
&\partial_m W^\alpha + F^\alpha{}_m = 0, \qquad F^{\alpha\beta} = 0,
\end{aligned} \tag{3.10}$$

where $D_\alpha \equiv \partial/\partial\theta^\alpha - i\theta^\beta \gamma^m_{\alpha\beta}\partial/\partial x^m$.[6] The terms in $\{Q, \mathcal{U}^{(1)}(z)\}$ which contain the field b yield equations which are the Bianchi identities [1]. From the first two equations of (3.10) one gets the field equations for $N = 1$, $d = (9, 1)$ super-Maxwell theory

$$\gamma^{\alpha\beta}_{[mnpqr]} D_\alpha A_\beta = 0, \tag{3.11}$$

as well as the definition of the vector potential A_m and the spinorial field strength W^α in terms of A_α

$$\begin{aligned}
A_m &= \frac{1}{16}\gamma^{\alpha\beta}_m D_\alpha A_\beta,\\
W^\alpha &= \frac{1}{20}\gamma^{\alpha\beta}_m (D_\beta A_m - \partial_m A_\beta).
\end{aligned} \tag{3.12}$$

Moreover, the remaining equations in (3.10) imply that the curvatures F_{mn}, $F^\alpha{}_m$, and $F_\beta{}^\alpha$ are expressed in terms of the spinor potential A_α.

The gauge transformations of the vertex $\mathcal{U}^{(1)}(z)$ are generated by the BRST variation of a spin-zero ghost-number-zero field $\Omega^{(0)}(z) \in \mathcal{H}_+$, whose most general expression is given by $\Omega^{(0)}(z) = C$, with C arbitrary superfield. The BRST variation of $\Omega^{(0)}$ is $\delta\mathcal{U}^{(1)}(z) = [Q, \Omega^{(0)}(z)] = i\lambda^\alpha D_\alpha C + \xi^m \partial_m C$. One can easily check that C is the usual parameter of the gauge transformations on the super-Maxwell potentials: $\delta A_\alpha = D_\alpha C$, $\delta A_m = \partial_m C$. Thus, the only independent superfield is A_α, and it satisfies (3.11) which is gauge-invariant. For further discussion of these field equations we refer to [1].

[6] Notice that D_α is hermitian. We define $D_{(\alpha} A_{\beta)} = \frac{1}{2}(D_\alpha A_\beta + D_\beta A_\alpha)$ and $\partial_{[m} A_{n]} = \frac{1}{2}(\partial_m A_n - \partial_n A_m)$.

4. Closed superstrings

In this section we again apply the procedure of Section 2, but now to the combined left-moving and right-moving sector of the Green–Schwarz superstring simultaneously.

We start from the GS action in (2.2). The transformation rules are now given by

$$\begin{aligned}
&s x^m = \big(\theta_L \gamma^m \lambda_L + \xi_L\big) + \big(\theta_R \gamma^m \lambda_R + \xi_R\big),\\
&s\theta^\alpha_L = i\lambda^\alpha_L, \qquad s\theta^{\hat\alpha}_R = i\lambda^{\hat\alpha}_R,\\
&s\lambda^\alpha_L = s\lambda^{\hat\alpha}_R = 0,\\
&s\xi^m_L = -i\lambda_L\gamma^m\lambda_L, \qquad s\xi^m_R = -i\lambda_R\gamma^m\lambda_R.
\end{aligned} \tag{4.1}$$

One clearly has nilpotency on these fields.

Next we add to $\mathcal{L}_{\mathrm{GS}}$ the terms with $d_{Lz\alpha} \equiv d_{L,1\alpha} - d_{L,0\alpha}$ and $d_{R\bar z\alpha} \equiv d_{R,1\alpha} + d_{R,0\alpha}$

$$\mathcal{L}_d = -d_{Lz\alpha}\bar\partial\theta^\alpha_L - d_{R\bar z\alpha}\partial\theta^\alpha_R. \tag{4.2}$$

We recall that $d_{Lz\alpha}$ and $d_{R\bar z\alpha}$, given below (2.3), are such that in $\mathcal{L}_{\mathrm{GS}} + \mathcal{L}_d$ only the free kinetic terms for x, $\theta_{L/R}$ and $p_{L/R}$ remain. As before we determine the variations of $d_{Lz\alpha}$ and $d_{R\bar z\alpha}$ (hence of $p_{Lz\alpha}$ and $p_{R\bar z\alpha}$) by requiring that in the s-variation of $\mathcal{L}_{\mathrm{GS}} + \mathcal{L}_d$ the terms without derivatives of ghosts cancel. However, we also require nilpotency on $d_{Lz\alpha}$ and $d_{R\bar z\alpha}$; since there are cross-terms, this is less trivial. We find it convenient to introduce an auxiliary field for Π^m_0, so we replace $(1/2)(\Pi^m_0)^2$ by $-(1/2)P^m_0 P_{0m} + P^m_0 \Pi_{0m}$. There are now two ways to proceed.

(i) We take the rules of the heterotic string in each sector, but the cross-terms in $s d_{Lz\alpha}$ are determined by requiring nilpotency on P^m_0 and $d_{Lz\alpha}$. One can achieve this, but one has then only nilpotency on $d_{Rz\alpha}$ modulo the free field equations of $\theta_{L/R}$ and $\xi_{L/R}$.

(ii) We write all transformation rules with only ∂_1 derivatives, but not with any ∂_0 derivatives. This can be achieved by using the free field equations. This changes the rules of the heterotic string, but we obtain nilpotency on all fields.

Since one either works with the heterotic string or with the Green–Schwarz string, we adopt the second procedure. We obtain then

$$\begin{aligned}
sd_{Lz\alpha} &= 2\partial_1\chi_{L\alpha} - 2(\Pi_{1m} - P_{0m})\gamma^m_{\alpha\beta}\lambda^\alpha_L \\
&\quad - 4i\xi_{Lm}\gamma^m_{\alpha\beta}\partial_1\theta^\beta_L, \\
sd_{R\bar{z}\alpha} &= 2\partial_1\chi_{R\alpha} - 2(\Pi_{1m} + P_{0m})\gamma^m_{\alpha\beta}\lambda^\alpha_R \\
&\quad - 4i\xi_{Rm}\gamma^m_{\alpha\beta}\partial_1\theta^\beta_R, \\
sP^m_0 &= -2\big(\lambda_L\gamma^m\partial_1\theta_L - \lambda_R\gamma^m\partial_1\theta_R\big) \\
&\quad - \partial_1\xi^m_L + \partial_1\xi^m_R, \\
s\Pi^m_1 &= 2\lambda_L\gamma^m\partial_1\theta_L + 2\lambda_R\gamma^m\partial_1\theta_R \\
&\quad + \partial_1\xi^m_L + \partial_1\xi^m_R, \\
s\chi_{L\alpha} &= 2\xi^m_L(\gamma_m\lambda_L)_\alpha, \\
s\chi_{R\alpha} &= 2\xi^m_R(\gamma_m\lambda_R)_\alpha.
\end{aligned} \tag{4.3}$$

It is clear that nilpotency of s holds on Π^m_1, Π^m_0 and P^m_0 in each sector separately. We have written $s\Pi^m_1$ below sP^m_0 so that the difference becomes clear: in sP^m_0 we have used the field equations

$$(\partial_1 + \partial_0)\theta^\alpha_L = 0, \qquad (\partial_1 - \partial_0)\theta^\alpha_R = 0,$$
$$(\partial_1 + \partial_0)\xi^m_L = 0, \qquad (\partial_1 - \partial_0)\xi^m_R = 0.$$

Because there are only ∂_1 derivatives in Π^m_1 and P^m_0, nilpotency of $sd_{Lz\alpha}$ and $sd_{R\bar{z}\alpha}$ is relatively easy to prove.

Using these transformation rules, one finds

$$\begin{aligned}
sS = \int d^2z\,\big[&\big(P^m_0 - \Pi^m_1\big)\bar\partial\xi_{Lm} - \big(P^m_0 + \Pi^m_1\big)\partial\xi_{Rm} \\
&- 2\partial_1\chi_{L\alpha}\bar\partial\theta^\alpha_L - 2\partial_1\chi_{R\alpha}\partial\theta^\alpha_R \\
&+ id_{Lz\alpha}\bar\partial\lambda^\alpha_L + id_{R\bar{z}\alpha}\partial\lambda^\alpha_R\big].
\end{aligned} \tag{4.4}$$

To prove this simple result requires multiple partial integrations and Fierz identities. To cancel these variations we add the ghost action

$$\begin{aligned}
S_{\mathrm{gh},1} = \int d^2z\,\big(&w_{Lz\alpha}\bar\partial\lambda^\alpha_L + w_{R\bar{z}\alpha}\partial\lambda^\alpha_R \\
&+ \beta_{Lzm}\bar\partial\xi^m_L + \beta_{R\bar{z}m}\partial\xi^m_R \\
&+ \kappa^\alpha_{Lz}\bar\partial\chi_{L\alpha} + \kappa^\alpha_{R\bar{z}}\partial\chi_{R\alpha}\big)
\end{aligned} \tag{4.5}$$

and choose the appropriate transformation laws for the antighosts

$$\begin{aligned}
sw_{L\alpha} &= -id_{L\alpha} - 2i\beta_{Lm}\big(\gamma^m\lambda_L\big)_\alpha - 2\xi_{Lm}\big(\gamma^m\kappa_L\big)_\alpha \\
&\quad + 2ib_L\partial_1\chi_{L\alpha} + \frac{3i}{2}\partial_1 b_L\chi_{L\alpha}, \\
s\beta_{Lm} &= -P_{0m} + \Pi_{1m} - 2\kappa_L\gamma^m\lambda_L \\
&\quad - 2b_L\partial_1\xi_{Lm} - \partial_1 b_L\xi_{Lm}, \\
s\kappa^\alpha_L &= 2\partial_1\theta^\alpha_L - 2ib_L\partial_1\lambda^\alpha_L - \frac{i}{2}\partial_1 b_L\lambda^\alpha_L.
\end{aligned} \tag{4.6}$$

The rules for the right-moving antighosts $w_{R\alpha}$, β^m_R and κ^α_R are obtained by replacing $-P^m_0$ by P^m_0 (and L by R of course). These rules are nilpotent if $sb_L = sb_R = 1$, but the action is not yet invariant. Since it varies into term with b we add the ghost action

$$S_{\mathrm{gh},2} = \int d^2z\,\big[b_L\bar\partial c_L + b_R\partial c_R\big], \tag{4.7}$$

and find the transformation rules for c_L and c_R from the BRST invariance of the action

$$sc_L = -\xi_L\partial_1\xi_L + \frac{3i}{2}\chi_{L\alpha}\partial_1\lambda^\alpha_L - \frac{i}{2}\partial_1\chi_{L\alpha}\lambda^\alpha_L \tag{4.8}$$

and, analogously, for c_R. Nilpotency only fixes the terms with $\partial_1 b_L$ in (4.6) up to an overall constant, but invariance of the action fixes this constant. All transformation rules for the combined sectors are now nilpotent; this has been achieved by introducing only one auxiliary field, namely P^m_0.

Needless to say, we can again define the grading current and we define the BRST cohomology on the space of non-negatively graded vertices.

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References

[1] P.A. Grassi, G. Policastro, M. Porrati, P. van Nieuwenhuizen, hep-th/0112162.
[2] P.A. Grassi, G. Policastro, P. van Nieuwenhuizen, hep-th/0202123.
[3] P.A. Grassi, G. Policastro, P. van Nieuwenhuizen, hep-th/0206216.

[4] N. Berkovits, JHEP 0004 (2000) 018, hep-th/0001035.
[5] N. Berkovits, B.C. Vallilo, JHEP 0007 (2000) 015, hep-th/0004171.
[6] N. Berkovits, JHEP 0009 (2000) 046, hep-th/0006003.
[7] N. Berkovits, Int. J. Mod. Phys. A 16 (2001) 801, hep-th/0008145.
[8] M. Matone, L. Mazzucato, I. Oda, D. Sorokin, M. Tonin, Nucl. Phys. B 639 (2002) 182, hep-th/0206104.
[9] W. Siegel, Nucl. Phys. B 263 (1986) 93;
W. Siegel, Phys. Rev. D 50 (1994) 2799.
[10] I. Oda, M. Tonin, Phys. Lett. B 520 (2001) 398, hep-th/0109051.
[11] N. Berkovits, hep-th/0201151.
[12] L. Brink, J.H. Schwarz, Phys. Lett. B 100 (1981) 310.
[13] P.A. Grassi, G. Policastro, M. Porrati, Nucl. Phys. B 606 (2001) 380, hep-th/0009239.
[14] N. Berkovits, Lectures on Covariant Quantization of Superstrings and Supermembranes, ICTP Miramare, Trieste, 18–26 March, 2002.
[15] N. Berkovits, hep-th/0105050.