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Between compactness and completeness

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This paper is dedicated to Som Naimpally

Abstract

Call a sequence in a metric space cofinally Cauchy if for each positive ε there exists a cofinal (rather than residual) set of indices whose corresponding terms are ε-close. We give a number of new characterizations of metric spaces for which each cofinally Cauchy sequence has a cluster point. For example, a space has such a metric if and only each continuous function defined on it is uniformly locally bounded. A number of results exploit a measure of local compactness functional that we introduce. We conclude with a short proof of Romaguera's Theorem: a metrizable space admits such a metric if and only if its set of points having a compact neighborhood has compact complement.

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1. Introduction

All mathematicians are familiar with compact metric spaces and complete metric spaces and their standard properties. Between these lies the class of *boundedly compact metric spaces*—spaces in which closed and bounded sets are compact, to which Euclidean spaces belong. One invariably learns the following facts about a compact metric space $\langle X, d\rangle$: (1) each continuous function defined on X with values in an arbitrary metric space $\langle Y, \rho\rangle$ is uniformly continuous; (2) each pair of disjoint closed nonempty subsets of X lie a positive distance apart; and (3) each open cover of X has a Lebesgue number. While none of these properties are characteristic properties of compact spaces, they are each characteristic properties of a larger class of spaces most frequently called UC spaces in the literature. It is obvious that the UC spaces neither contain nor are contained in the spaces in which closed and bounded sets are compact: an infinite set equipped with the zero–one metric belongs to the former but not the latter, whereas n-dimensional Euclidean space in which closed and bounded sets are compact is not UC.

The UC spaces, first systematically studied by Atsuji [2], have been the subject of a number of articles over the years, most recently the survey article [17], where they are called *Atsuji spaces*, following [4,5]. Occasionally, they have been called *normal metric spaces* [19] and *Lebesgue metric spaces* [20,24] in the literature. One sequential characterization that was discovered early on is this: if $\langle x_n\rangle$ is a sequence in X with $\lim d(x_n, \{x_n\}^c) = 0$, then $\langle x_n\rangle$

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has a cluster point [2,15]. This implies immediately that the set of limit points of a UC space is compact. But there is a second sequential characterization of UC spaces discovered by Toader [26] that clarifies their relation to complete spaces as distinct from compact ones. A sequence $\langle x_n \rangle$ is of course Cauchy if $\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall \{n, j\} \subseteq \mathbb{N}$, $n > n_0$, and $j > n_0 \Rightarrow d(x_n, x_j) < \varepsilon$. If we permute the inner two quantifiers we call the result a *pseudo-Cauchy sequence* [5]:

$$\forall \varepsilon > 0\ \forall n_0 \in \mathbb{N}\ \exists \{n, j\} \subseteq \mathbb{N} \quad \text{with } n > n_0 \text{ and } j > n_0 \text{ and } d(x_n, x_j) < \varepsilon.$$

A metric space is evidently complete if and only if each Cauchy sequence with distinct terms has a cluster point. Toader proved that $\langle X, d \rangle$ is a UC space if and only if each pseudo-Cauchy sequence with distinct terms has a cluster point.

Toader's pseudo-Cauchy sequences are those for which pairs of terms are arbitrarily close frequently. But there is a second natural way to generalize the definition of Cauchy sequence [13,14]. A sequence is Cauchy if for each $\varepsilon > 0$, there exists a residual set of indices $\mathbb{N}_\varepsilon$ such that each pair of terms whose indices come from $\mathbb{N}_\varepsilon$ are within ε of each other. If we replace *residual* by *cofinal* then we obtain sequences that we here call cofinally Cauchy.

Definition 1.1. A sequence $\langle x_n \rangle$ in a metric space $\langle X, d \rangle$ is called *cofinally Cauchy* if $\forall \varepsilon > 0$ there exists an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that for each $n, j \in \mathbb{N}_\varepsilon$ we have $d(x_n, x_j) < \varepsilon$.

It is the purpose of this paper to cast new light on those metric spaces in which each cofinally Cauchy sequence has a cluster point, a collection of spaces that we call here the cofinally complete metric spaces. Our results reveal that such spaces position themselves relative to uniformly locally compact spaces in the same way that UC spaces sit relative to the uniformly discrete spaces, and we exhibit more generally a striking parallelism between the two classes of spaces. Central in our analysis is a measure of local compactness functional that parallels the index of isolation so important in the study of UC spaces and that leads to Cantor-type theorems. We also characterize such spaces in terms of a uniform property that continuous functions defined on them must have, and produce a short proof of Romaguera's Theorem [23, Thm. 2]: a metrizable space has a compatible cofinally complete metric if and only if its set of points having no compact neighborhood is compact.

Cofinal completeness can of course be formulated in terms of nets and entourages and it is in this more general form that it was considered first implicitly by Corson [8] and then by Howes [13] who showed that a completely regular Hausdorff space is paracompact if and only if it admits a compatible cofinally complete uniformity. A few years later, Rice [22] introduced the notion of uniform paracompactness for a Hausdorff uniform space X: for each open cover $\{V_i\colon i \in I\}$ of X there exists an open refinement and an entourage U such that for each $x \in X$, $U(x)$ meets only finitely many members of the refinement. Subsequently, the reviewer of Rice's paper [25] observed that uniform paracompactness is equivalent to net cofinal completeness for a Hausdorff uniform space (see also [14, Thm. 4.6]).

2. Preliminaries

First we list some notational conventions. Let x_0 be a point in a metric space $\langle X, d \rangle$ and let $\varepsilon > 0$. We write $S_\varepsilon(x_0)$ (resp., $S_\varepsilon[x_0]$) for the open (resp., closed) ε-ball with center x_0. If A is a nonempty subset of X, we write $d(x_0, A)$ for the distance from x_0 to A, and if $A = \emptyset$ we agree that $d(x_0, A) = \infty$. We denote the open *ε-enlargement* of A by A^ε, i.e.,

$$A^\varepsilon = \{x\colon d(x, A) < \varepsilon\} = \bigcup_{x \in A} S_\varepsilon(x).$$

If A, B are subsets of X, we define the *Hausdorff distance* [16,6] between them by

$$\begin{aligned} H_d(A, B) &= \max\{\sup\{d(a, B)\colon a \in A\}, \sup\{d(b, A)\colon b \in B\}\} \\ &= \inf\{\varepsilon > 0\colon B \subseteq A^\varepsilon \text{ and } A \subseteq B^\varepsilon\}. \end{aligned}$$

Restricted to the nonempty closed subsets of X, Hausdorff distance so defined is an extended real-valued metric which is finite valued when restricted to the nonempty closed and bounded sets. Of course, $x \to \{x\}$ is an isometry.

If A is a subset of X we write $\operatorname{diam}(A)$, $\operatorname{int}(A)$, $\operatorname{cl}(A)$, $\operatorname{bd}(A)$ and A' for the diameter, interior, closure, boundary and set of limit points of A. Perhaps the most visual characterization of UC spaces involves the set of limit points of X itself [2,15,6]: X' is compact and $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\{x, w\} \cap (X')^\varepsilon = \emptyset \Rightarrow d(x, w) \geqslant \delta$. Atsuji introduced the notation $I(x)$ for $d(x, \{x\}^c)$ to connote the degree of isolation of a point $x \in X$. Clearly, $I(x) = 0$ if and only if $x \in X'$. As mentioned in the introduction the functional I arises in a basic sequential characterization of UC spaces. In our work here, another geometric functional comes into play that measures the local compactness of the space at each point. If $x \in X$ has a compact neighborhood, set $\nu(x) = \sup\{\varepsilon > 0\colon\ S_\varepsilon[x] \text{ is compact}\}$; otherwise, set $\nu(x) = 0$. The set $\{x\colon\ \nu(x) = 0\}$ is the set of points of non-local compactness of X, which we denote by $\mathrm{nlc}(X)$ in the sequel. Notice that if $\nu(x_0) = \infty$ for some x_0, then $\nu(x) = \infty$ for all x and X is a boundedly compact space. Also notice that uniform local compactness of X amounts to $\inf\{\nu(x)\colon\ x \in X\} > 0$. We collect some facts in an elementary lemma whose proof is left to the reader.

Lemma 2.1. *Let $\langle X, d\rangle$ be a metric space.*

(1) $\forall x \in X,\ I(x) \leqslant \nu(x)$;
(2) $\forall x \in X,\ \nu(x) = \inf\{\liminf_{n\to\infty} d(x, x_n)\colon\ \langle x_n\rangle$ *has no cluster point*$\}$;
(3) *If X is not boundedly compact, then $\nu : X \to [0, \infty)$ is uniformly continuous.*

From condition (3) of Lemma 2.1 we obtain the well-known fact $\mathrm{nlc}(X)$ is closed. We also note that condition (3) is a special case of condition (5) of Lemma 3.9 *infra* which we do prove. The spaces that are the focus of this paper are next properly defined.

Definition 2.2. Let $\langle X, d\rangle$ be a metric space. We call X *cofinally complete* provided each cofinally Cauchy sequence in X has a cluster point.

Evidently, a uniformly locally compact metric space is cofinally complete. A cofinally complete space need not be locally compact: as a metric subspace of the Hilbert space ℓ_2 of square summable sequences with origin θ and standard o.n. base $\{e_n\colon\ n \in \mathbb{N}\}$, let $X = \{\theta\} \cup \{\frac{1}{j} e_n\colon\ j \in \mathbb{N},\ n \in \mathbb{N}\}$. Since Cauchy sequences are cofinally Cauchy, each cofinally complete metric space is complete. The reverse inclusion fails. To clarify this, we include the next functional analytic result (see less generally [14, p. 34]).

Proposition 2.3. *Let X be a Banach space. Then X is cofinally complete if and only if X is finite dimensional.*

Proof. As is well known, the closed unit ball B of a Banach space X is compact if and only if X is finite dimensional [11]. Now there exists a compact closed ball in X if and only if each closed ball in X is compact, so a finite dimensional normed linear space is boundedly compact and is thus cofinally complete. If X is not finite dimensional, take $x_0 \in X$ with $\|x_0\| = 17$. For each $j \in \mathbb{N}$ let $\langle x_{nj}\rangle$ be a sequence in $jx_0 + j^{-1}B$ without a cluster point. Partition $\mathbb{N}$ into a countable family of infinite subsets $\{\mathbb{M}_j\colon\ j \in \mathbb{N}\}$ where for each j $\mathbb{M}_j = \{k_n^j\colon\ n \in \mathbb{N}\}$. Then the assignment $a : \mathbb{N} \to X$ defined by $a(k_n^j) = x_{nj}$ yields a cofinally Cauchy sequence in the Banach space without a cluster point. □

In the Toader sequential characterization of UC spaces given in the Introduction, it is required that, the sequence have distinct terms, for in the UC space of a integers equipped with the Euclidean metric, the sequence $1, 1, 2, 2, 3, 3, 4, 4, \ldots$ is a pseudo-Cauchy sequence without a cluster point. The unpleasant reality is that a pseudo-Cauchy sequence need not have a pseudo-Cauchy subsequence with distinct terms. This unpleasantness does not occur with cofinally Cauchy sequences, provided there is no constant subsequence. Further, if $\langle\varepsilon_n\rangle$ is a positive real sequence convergent to zero, we can take the infinite sets of integers $\mathbb{N}_{\varepsilon_n}$ in the definition of cofinally Cauchy sequence to be pairwise disjoint. This observation greatly simplifies several proofs in the sequel.

Proposition 2.4. *Let $\langle x_n\rangle$ be a cofinally Cauchy sequence without a constant subsequence. Then there is a pairwise disjoint family $\{\mathbb{M}_j\colon\ j \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ such that*

(1) *if $\{i, l\} \subseteq \bigcup\{\mathbb{M}_j\colon\ j \in \mathbb{N}\}$ then $x_i \neq x_l$; and*
(2) *if $i \in \mathbb{M}_j$ and $l \in \mathbb{M}_j$ then $d(x_i, x_l) < \frac{1}{j}$.*

Proof. Suppose $\langle x_n\rangle$ has a Cauchy subsequence. Since $\langle x_n\rangle$ has no constant subsequence, we can assume that this Cauchy subsequence based on $\mathbb{N}_0 \subseteq \mathbb{N}$ has distinct terms. For each $j \in \mathbb{N}$ choose $N_j \in \mathbb{N}$ such that $i > l \geqslant N_j$ and $\{i, l\} \subseteq \mathbb{N}_0 \Rightarrow d(x_i, x_l) < \frac{1}{j}$. Partition $\mathbb{N}_0$ into countably many infinite subsets $\{\mathbb{K}_j\colon j \in \mathbb{N}\}$. Then $\{\mathbb{M}_j\colon j \in \mathbb{N}\}$ does the job where for each j

$$\mathbb{M}_j = \{n \in \mathbb{K}_j\colon n \geqslant N_j\}.$$

Otherwise choose an infinite $\mathbb{M}_1 \subseteq \mathbb{N}$ such that $\{i, l\} \subseteq \mathbb{M}_1 \Rightarrow 0 < d(x_i, x_l) < 1$. Since we are now assuming $\langle x_n\rangle$ has no Cauchy subsequence, $\{x_i\colon i \in \mathbb{M}_1\}$ cannot be totally bounded [9, p. 312]. By passing to an infinite subset of $\mathbb{M}_1$ we can find $\varepsilon_1 < \frac{1}{2}$ such that $\{i, l\} \subseteq \mathbb{M}_1 \Rightarrow \varepsilon_1 < d(x_i, x_l) < 1$. Now choose an infinite $\mathbb{M}_2 \subseteq \mathbb{N}$ such that $\{i, l\} \subseteq \mathbb{M}_2 \Rightarrow 0 < d(x_i, x_l) < \varepsilon_1$. By construction $\{x_i\colon i \in \mathbb{M}_1\} \cap \{x_i\colon i \in \mathbb{M}_2\}$ consists of at most one point. Also, $\{x_i\colon i \in \mathbb{M}_2\}$ is not totally bounded, so by passing to an infinite subset of $\{x_i\colon i \in \mathbb{M}_2\}$ we can assume the two sets are disjoint and further that there exists $\varepsilon_2 < \frac{1}{3}$ such that

$$\{i, l\} \subseteq \mathbb{M}_2 \quad \Rightarrow \quad \varepsilon_2 < d(x_i, x_l) < \frac{1}{2}.$$

Choosing an infinite $\mathbb{M}_3 \subseteq \mathbb{N}$ such that $\{i, l\} \subseteq \mathbb{M}_3 \Rightarrow 0 < d(x_i, x_l) < \varepsilon_2$, by deleting at most two indices from $\mathbb{M}_3$ we can assume $\{\{x_i\colon i \in \mathbb{M}_j\}\colon j = 1, 2, 3\}$ is a pairwise disjoint family. Continuing in this way we produce $\{\mathbb{M}_j\colon j \in \mathbb{N}\}$ with the asserted properties. □

A cofinally Cauchy sequence that has a constant subsequence obviously has a cluster point. Otherwise, by Proposition 2.4, the cofinally Cauchy sequence has a pseudo-Cauchy subsequence with distinct terms. From this, it follows that each UC space is cofinally complete. On the other hand, $\mathbb{R}$ with the usual metric is a cofinally complete space that is not UC.

3. Characterizations of cofinally complete metric spaces

For one of our first characterizations of cofinally complete metric spaces, we require a standard result regarding convergence of a decreasing sequence of closed sets in Hausdorff distance [6, p. 90].

Proposition 3.1. *Let $\langle A_n\rangle$ be a decreasing sequence of nonempty closed sets in a metric space $\langle X, d\rangle$ with intersection A. The following conditions are equivalent*:

(1) *Whenever $\langle a_n\rangle$ is a sequence in X with $a_n \in A_n$ for each n, then $\langle a_n\rangle$ has a cluster point*;
(2) *A is a nonempty compact set and for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $A_n \subseteq A^\varepsilon$.*

Theorem 3.2. *Let $\langle X, d\rangle$ be a metric space. The following are equivalent*:

(1) *X is cofinally complete*;
(2) *Whenever $\langle x_n\rangle$ is a sequence in X with $\lim_{n\to\infty} \nu(x_n) = 0$, then $\langle x_n\rangle$ has a cluster point*;
(3) *Either X is uniformly locally compact or $\mathrm{nlc}(X)$ is nonempty and compact and $\langle\{x\colon \nu(x) \leqslant \frac{1}{n}\}\rangle$ converges to $\mathrm{nlc}(X)$ in Hausdorff distance*;
(4) *$\mathrm{nlc}(X)$ is compact and $\forall\varepsilon > 0$ $(\mathrm{nlc}(X)^\varepsilon)^c$ is uniformly locally compact in its relative topology.*

Proof. (1) $\Rightarrow$ (2) Suppose condition (2) fails, and let $\langle x_n\rangle$ be a sequence in X with $\lim_{n\to\infty} \nu(x_n) = 0$ that has no cluster point. Without loss of generality, we can assume by passing to a subsequence that $\nu(x_n) < \frac{1}{n}$ and so we can find a sequence $\langle w_j^n\rangle$ in $S_{1/n}[x_n]$ with no cluster point. Partition $\mathbb{N}$ into an countable family of infinite subsets $\{\mathbb{K}_n\colon n \in \mathbb{N}\}$. Then the assignment $y_j = w_n^j$ for $j \in \mathbb{K}_n$ defines a cofinally Cauchy sequence $\langle y_j\rangle$ in X without a cluster point. Thus (1) fails.

(2) $\Rightarrow$ (3) For each $n \in \mathbb{N}$, let $F_n = \{x\colon \nu(x) \leqslant 1/n\}$, a closed set by the continuity of ν. If for some n, F_n is empty, then X is uniformly locally compact. Otherwise pick $x_n \in F_n$ for $n = 1, 2, 3, \ldots$. By (2), $\langle x_n\rangle$ has a cluster point which by the continuity of ν must lie in $\mathrm{nlc}(X)$. By the general result on Hausdorff metric convergence of Proposition 3.1, (3) follows.

(3) ⇒ (4) Assume (3) holds. If $\mathrm{nlc}(X) = \emptyset$, then $\mathrm{nlc}(X)$ is compact and X is uniformly locally compact. Thus, $\forall \varepsilon > 0$ $(\mathrm{nlc}(X)^{\varepsilon})^c = X$ is (relatively) uniformly locally compact. Otherwise $\mathrm{nlc}(X)$ is nonempty and compact. Let $\varepsilon > 0$ and choose by (3) n such that $\{x\colon \nu(x) \leqslant 1/n\} \subseteq \mathrm{nlc}(X)^{\varepsilon}$. As a result, if $x \in (\mathrm{nlc}(X)^{\varepsilon})^c$, then $S_{1/n}[x]$ is compact, and since complements of ε-enlargements are closed, $S_{1/n}[x] \cap (\mathrm{nlc}(X)^{\varepsilon})^c$ is compact. Condition (4) now follows.

(4) ⇒ (1) If $\mathrm{nlc}(X) = \emptyset$, then by (4) X is uniformly locally compact and hence is cofinally complete. Otherwise $\mathrm{nlc}(X)$ is nonempty and compact. Let $\langle x_n \rangle$ be a cofinally Cauchy sequence without a constant subsequence, and let $\{\mathbb{M}_j\colon j \in \mathbb{N}\}$ be the family of infinite subsets of $\mathbb{N}$ for $\langle x_n \rangle$ described in Proposition 2.4. If for some $\varepsilon > 0$ and for infinitely many $j \in \mathbb{N}$ $\{x_n\colon n \in \mathbb{M}_j\} \cap (\mathrm{nlc}(X)^{\varepsilon})^c$ is infinite, then by (4), $\langle x_n \rangle$ has a cluster point by (relative) uniform local compactness of $(\mathrm{nlc}(X)^{\varepsilon})^c$. Otherwise, for each $\varepsilon > 0$ $\exists j_0 \in \mathbb{N}$ such that $j \geqslant j_0 \Rightarrow \{x_n\colon n \in \mathbb{M}_j\} \cap (\mathrm{nlc}(X)^{\varepsilon})^c$ is a finite set. In particular, $\forall \varepsilon > 0$ $(\mathrm{nlc}(X))^{\varepsilon}$ contains infinitely many terms of $\langle x_n \rangle$, and by compactness $\langle x_n \rangle$ has a cluster point in $\mathrm{nlc}(X)$. This proves that X is cofinally complete. □

Conditions (2) and (4) are analogs of characterizations of UC spaces mentioned earlier. On the other hand, the condition parallel to (3) for UC spaces, namely, either X is uniformly discrete, or X' is nonempty and compact and $\langle \{x\colon I(x) \leqslant \frac{1}{n}\} \rangle$ converges to X' in Hausdorff distance seems not to be explicit in the literature. Noncompactness of $\mathrm{nlc}(X)$ is the first thing one looks for to spot a space that is not cofinally complete; this works in the case that X is an infinite dimensional normed linear space. In the same way the completely metrizable space of irrationals has no admissible cofinally complete metric (see Theorem 4.1 *infra*).

Condition (4) is a variant of a condition presented by Hohti [12, Thm. 2.1.1] in his thesis that characterizes uniform paracompactness in metric space as defined by Rice [22]. Since uniform paracompactness is equivalent to net cofinal completeness in a general uniform space [25,14], it follows that sequential cofinal completeness and the formally stronger net cofinal completeness coincide in the context of metric spaces. This is not always the case in uniform spaces. In fact, Burdick [7] has given an example showing that cofinal completeness for all transfinite sequences may be properly weaker than net cofinal completeness.

Our next goal is to characterize cofinally complete spaces in terms of an additional uniform property that continuous functions defined on them must enjoy. For complete clarity we include the following definition.

Definition 3.3. Let f be an arbitrary function defined on a metric space $\langle X, d \rangle$ with values in a metric space $\langle Y, \rho \rangle$. We say f is *uniformly locally bounded* if $\exists \delta > 0$ such that $\forall x \in X$ $f(S_{\delta}(x))$ is a bounded subset of Y.

Continuous functions are locally bounded but need not be uniformly locally bounded, e.g., $f(x) = 1/x$ restricted to $(0, \infty)$. Continuous functions defined on uniformly locally compact spaces as well as uniformly continuous functions are uniformly locally bounded.

Theorem 3.4. *Let $\langle X, d \rangle$ be a metric space. The following conditions are equivalent*:

(1) *X is cofinally complete*;
(2) *Each continuous function on X with values in a metric space $\langle Y, \rho \rangle$ is uniformly locally bounded*;
(3) *Each real-valued continuous function on X is uniformly locally bounded.*

Proof. (1) ⇒ (2) Suppose X is cofinally complete yet some continuous function $f : \langle X, d \rangle \to \langle Y, \rho \rangle$ fails to be uniformly locally bounded. Obviously, X cannot be uniformly locally compact, so by Theorem 3.2, $\mathrm{nlc}(X)$ is nonempty and compact. Choose a sequence $\langle x_n \rangle$ in X such that for each n, $f(S_{1/n}(x_n))$ is an unbounded subset of Y. Again by Theorem 3.2. if $\varepsilon > 0$ is arbitrary, $\exists \delta > 0$ such that $d(x, \mathrm{nlc}(X)) \geqslant \varepsilon \Rightarrow S_{\delta}[x]$ is compact. As a result, if $\frac{1}{n} < \delta$, then $d(x_n, \mathrm{nlc}(X)) < \varepsilon$. By the compactness of $\mathrm{nlc}(X)$, $\langle x_n \rangle$ has a cluster point $p \in \mathrm{nlc}(X)$ at which f fails to be continuous.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) If $\langle X, d\rangle$ is not cofinally complete, there exists a cofinally Cauchy sequence $\langle x_n\rangle \in X$ without a cluster point. Since this sequence has no constant subsequence, we once again make use of the family $\{\mathbb{M}_j\colon j \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ for $\langle x_n\rangle$ described in Proposition 2.4. Clearly,

$$\left\{x\colon x = x_i \text{ for some } i \in \bigcup_{j=1}^{\infty} \mathbb{M}_j\right\}$$

is a closed discrete set. Since different indices in $\bigcup_{j=1}^{\infty} \mathbb{M}_j$ give distinct terms of the sequence, we can by the Tietze Extension Theorem [9, p. 149] define a continuous real-valued function on X whose restriction to each set of the form $\{x_i\colon i \in \mathbb{M}_j\}$ is unbounded as we can assign values to the points of each such set of terms as we choose. Such an f, while continuous, is not uniformly locally bounded. □

Each of the classes of metric spaces we have introduced can be characterized by boundedness properties of continuous functions defined on them. In the context of metrizable spaces, pseudo-compactness reduces to compactness [10]: $\langle X, d\rangle$ is compact iff each continuous function on X is bounded. A metric space is boundedly compact (resp., complete) iff each continuous function defined on it is bounded on bounded (resp., totally bounded) subsets of X. Finally, Atsuji [2, Theorem 1] has shown that $\langle X, d\rangle$ is UC iff each continuous function defined on it is bounded on $\{x\colon I(x) < \delta\}$ for some positive δ.

If X is a UC space, then each continuous function defined on X is uniformly continuous. Since cofinally complete spaces properly contain the UC spaces, it is natural to try to characterize the cofinally complete spaces as those for which each function in some proper subclass of the continuous functions is uniformly continuous. In this direction, we offer

Theorem 3.5. *Let $\langle X, d\rangle$ be a metric space. The following are equivalent*:

(1) *X is cofinally complete*;
(2) *Each continuous function on X with values in a metric space $\langle Y, \rho\rangle$ that for each $\varepsilon > 0$ is uniformly continuous on $\{x\colon \nu(x) > \varepsilon\}$ is globally uniformly continuous*;
(3) *Each bounded continuous real valued function on X that for each $\varepsilon > 0$ is uniformly continuous on $\{x\colon \nu(x) > \varepsilon\}$ is globally uniformly continuous.*

Proof. (1) ⇒ (2) Assume X is cofinally complete and f is uniformly continuous on $\{x\colon \nu(x) > \varepsilon\}$ for each positive ε, yet f is not globally uniformly continuous. There exists $\lambda > 0$ and for each $n \in \mathbb{N}$ $x_n, w_n \in X$ with $d(x_n, w_n) < \frac{1}{n}$ but $\rho(f(x_n), f(w_n)) > \lambda$. Then for each $\varepsilon > 0$ eventually $\{x_n, w_n\} \cap \{x\colon \nu(x) \leqslant \varepsilon\} \neq \emptyset$. By Lemma 2.1 actually $\lim_{n\to\infty} \nu(x_n) = \lim_{n\to\infty} \nu(w_n) = 0$, so that by condition (2) of Theorem 3.2, a common cluster point p exists at which continuity of f fails. This contradiction establishes the implication.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) Suppose X is not cofinally complete. Let $\langle x_n\rangle$ be a sequence with distinct terms in X without a cluster point such that $\lim_{n\to\infty} \nu(x_n) = 0$. By passing to a subsequence, we can guarantee that one of two scenarios occurs:

(i) $\forall n$ x_n is a limit point of X, or
(ii) $\forall n$ x_n is an isolated point of X.

In case (i), for each $n \in \mathbb{N}$ choose $\delta_n \in (0, 1/n)$ such that $\{S_{\delta_n}[x_n]\colon n \in \mathbb{N}\}$ is a discrete family of closed balls. Since $\forall n$ $x_n \in X'$ we can pick $w_n \neq x_n$ in $S_{\delta_n}[x_n]$. For each n set $\varepsilon_n = d(x_n, w_n)$ and define a Lipschitz function f_n on X by

$$f_n(x) = \begin{cases} 1 - \frac{d(x, x_n)}{\varepsilon_n} & \text{if } x \in S_{\varepsilon_n}[x_n], \\ 0 & \text{otherwise.} \end{cases}$$

By the discreteness of the family of closed balls, $f := \sum_{n=1}^{\infty} f_n$ is continuous with values in $[0, 1]$. Now let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \nu(x_n) = 0$ and $\forall n\ \delta_n < \frac{1}{n}$, the uniform continuity of ν guarantees that for some $k \in \mathbb{N}$

$$\{x\colon\ \nu(x) > \varepsilon\} \cap \bigcup_{n=k+1}^{\infty} S_{\delta_n}[x_n] = \emptyset$$

so that f restricted to $\{x\colon\ \nu(x) > \varepsilon\}$ is represented by $\sum_{n=1}^{k} f_n$. Thus f restricted to each set of the form $\{x\colon\ \nu(x) > \varepsilon\}$ is actually Lipschitz continuous. But f is not globally uniformly continuous as $\forall n\ d(x_n, w_n) < \frac{1}{n}$ yet $f(x_n) - f(w_n) = 1$.

In case (ii), where all terms x_n are isolated, we consider two subcases: either $\langle x_n \rangle$ is a pseudo-Cauchy sequence, or not. In the first subcase, by passing to a subsequence, we may assume $\forall n\ d(x_{2n-1}, x_{2n}) < \frac{1}{n}$. Then $g : X \to [0, 1]$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = x_{2n} \text{ for some } n, \\ 0 & \text{otherwise,} \end{cases}$$

obviously fails to be globally uniformly continuous. Yet g is uniformly continuous restricted to $\{x\colon\ \nu(x) > \varepsilon\}$ for each positive ε, because each such set contains only finitely many points of $\{x_{2n}\colon\ n \in \mathbb{N}\}$. In fact g so restricted is uniformly locally constant. In the second subcase, after passing to a subsequence, we may assume for some $\delta > 0$ and for all $n, j \in \mathbb{N}$ that $d(x_n, x_j) > \delta$. Then it is easy to verify that h defined below does the job:

$$h(x) = \begin{cases} 1 & \text{if } x = x_n \text{ for some } n, \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof of the theorem. □

In a UC space, disjoint nonempty closed sets lie a positive distance apart which means that disjoint closed sets cannot be asymptotic.

Definition 3.6. Let A and B be disjoint subsets of $\langle X, d \rangle$. We call A and B *asymptotic* if $\forall \varepsilon > 0\ \exists a \in A, b \in B$ with $d(a, b) < \varepsilon$.

Our next result characterizes cofinally complete spaces in terms of the asymptotics of disjoint closed sets.

Theorem 3.7. *A metric space $\langle X, d \rangle$ is cofinally complete if and only if whenever F_1 and F_2 are disjoint asymptotic closed sets, there exists $\delta > 0$ such that $F_1 \cap \{x\colon\ \nu(x) > \delta\}$ and $F_2 \cap \{x\colon\ \nu(x) > \delta\}$ are asymptotic.*

Proof. Suppose no such δ exists. Then for each $n \in \mathbb{N}$ there exists $x_n \in F_1$ and $w_n \in F_2$ with $d(x_n, w_n) < \frac{1}{n}$ and $\{x_n, w_n\} \cap \{x\colon\ \nu(x) \leqslant \frac{1}{n}\} \neq \emptyset$. Since the functional ν is uniformly continuous, we have $\lim_{n\to\infty} \nu(x_n) = \lim_{n\to\infty} \nu(w_n) = 0$. Clearly, $\langle x_n \rangle$ can have no cluster point, else it would lie in $F_1 \cap F_2$. By condition (2) of Theorem 3.2, we see that X is not cofinally complete. Conversely, suppose X is not cofinally complete. Again, we call on Proposition 2.4. Let $\langle x_n \rangle$ be a cofinally Cauchy sequence without a cluster point, and let $\{\mathbb{M}_j\colon\ j \in \mathbb{N}\}$ be our infinite family of infinite subsets of $\mathbb{N}$ for $\langle x_n \rangle$ as described therein. Since each pair of terms in $\{x_i\colon\ i \in \mathbb{M}_j\}$ lie within $\frac{1}{j}$ of each of other, we see that $\forall i \in \mathbb{M}_j\ \nu(x_i) < \frac{1}{j}$. For each $j \in \mathbb{N}$ choose $\{i_{j1}, i_{j2}\} \subseteq \mathbb{M}_j$ arbitrarily. Then

$$F_1 = \{x_{i_{j1}}\colon\ j \in \mathbb{N}\} \quad \text{and} \quad F_2 = \{x_{i_{j2}}\colon\ j \in \mathbb{N}\}$$

are asymptotic disjoint closed sets such that for each $\delta > 0$ both $F_1 \cap \{x\colon\ \nu(x) > \delta\}$ and $F_2 \cap \{x\colon\ \nu(x) > \delta\}$ are finite and are *a fortiori* not asymptotic. □

We note that complete metric spaces can be characterized in a similar way in terms of asymptotics: $\langle X, d \rangle$ is complete if and only if whenever F_1, F_2 are asymptotic closed sets and T is totally bounded, then for some $\delta > 0$ $F_1 \cap (T^\delta)^c$ and $F_2 \cap (T^\delta)^c$ are asymptotic. For the amusement of the reader, we now give as an application of Theorem 3.4 a second asymptotics result for cofinally complete spaces that unfortunately does not characterize this class.

Proposition 3.8. *Let F_1 and F_2 be disjoint closed sets in a cofinally complete metric space $\langle X, d\rangle$. Then there exists $\delta > 0$ such that whenever $E \subseteq X$ and $\operatorname{diam}(E) < \delta$ then $E \cap F_1$ and $E \cap F_2$ are not asymptotic.*

Proof. Define $f : X \to (0, \infty)$ by

$$f(x) = \max\bigl\{d(x, F_1), d(x, F_2)\bigr\}.$$

Evidently, f is 1-Lipschitz, so $g(x) = \frac{1}{f(x)}$ is globally continuous. By Theorem 3.4 $\exists\delta > 0$ such that g restricted to each ball of radius δ is bounded above and f so restricted is thus bounded away from zero. As a result, if $\operatorname{diam}(E) < \delta$, then $E \cap F_1$ and $E \cap F_2$ are not asymptotic. □

One of course wonders what property of open covers for cofinally complete metric spaces corresponds to the Lebesgue number property for open covers of UC spaces. Rice [22] showed that uniform paracompactness of a Hausdorff uniform space is equivalent to this formally weaker property: for each open cover $\{V_i\colon i \in I\}$ of X, $\exists$ an entourage U such that $\forall x \in X$, $U(x)$ has a finite subcover from $\{V_i\colon i \in I\}$. Given the equivalence of uniform paracompactness and net cofinal completeness, in the context of metric spaces, Rice's result says this: $\langle X, d\rangle$ is cofinally complete if and only if for each open cover $\{V_i\colon i \in I\}$, there exists $\mu > 0$ such that $\operatorname{diam}(A) < \mu \Rightarrow A$ has a finite subcover from $\{V_i\colon i \in I\}$. This of course is the sought-after parallel property.

Complete metric spaces are characterized by the nonempty intersection of decreasing sequences of nonempty closed sets $\langle F_n\rangle$ having either of the following properties [18]: (1) $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$, or (2) $\lim_{n\to\infty} \alpha(F_n) = 0$, where for each $E \subseteq X$

$$\alpha(E) = \inf\{\varepsilon > 0\colon\ E \text{ is a contained in a finite union of sets of diameter} < \varepsilon\}.$$

Notice that $\alpha(E) = 0$ if and only if E is totally bounded. In the literature, the functional α is called the *Kuratowski measure of noncompactness functional* [3]. In fact, α is really a measure of non-total boundedness. In [5], the author easily characterized UC spaces in a similar manner, replacing the two set functionals above by either of two set functionals defined in terms of Atsuji's measure of isolation defined for points of X:

$$\bar{I}(E) = \sup\bigl\{I(x)\colon\ x \in E\bigr\}$$

or

$$\underline{I}(E) = \inf\bigl\{I(x)\colon\ x \in E\bigr\}.$$

The Cantor-type characterizations obtained for UC spaces motivate us to look at two parallel set functionals for cofinally complete spaces, namely

$$\bar{\nu}(E) = \sup\bigl\{\nu(x)\colon\ x \in E\bigr\}$$

and

$$\underline{\nu}(E) = \inf\bigl\{\nu(x)\colon\ x \in E\bigr\}.$$

We next collect some facts about these two new set functionals.

Lemma 3.9. *Let $\langle X, d\rangle$ be a metric space. Then the extended real-valued set functional $\bar{\nu}(\cdot)$ defined for subsets of X has these properties*:

(1) $A \subseteq B \Rightarrow \bar{\nu}(A) \leqslant \bar{\nu}(B)$;
(2) $\bar{\nu}(A \cup B) = \max\{\bar{\nu}(A), \bar{\nu}(B)\}$;
(3) $\bar{\nu}(A) = \bar{\nu}(\operatorname{cl}(A))$;
(4) *If $H_d(A, B)$ is finite, then $\bar{\nu}(A) = \infty$ if and only if $\bar{\nu}(B) = \infty$*;
(5) *$\bar{\nu}(\cdot)$ is H_d-uniformly continuous restricted to $\{A\colon\ \bar{\nu}(A) < \infty\}$.*

Proof. Properties (1)–(3) are obvious. We establish property (5); the arguments required to establish (4) are similar. Suppose $H_d(A, B) < \varepsilon$; we claim that $|\bar{\nu}(A) - \bar{\nu}(B)| \leqslant \varepsilon$. We just show $\bar{\nu}(A) \leqslant \bar{\nu}(B) + \varepsilon$. Suppose to the contrary that $\bar{\nu}(A) > \bar{\nu}(B) + \varepsilon$. Pick $a_0 \in A$ with $\nu(a_0) > \bar{\nu}(B) + \varepsilon$. Then pick $\lambda > \bar{\nu}(B) + \varepsilon$ such that $S_\lambda[a_0]$ is compact and $b_0 \in B$ such that $d(a_0, b_0) < \varepsilon$. If $d(x, b_0) \leqslant \lambda - \varepsilon$, then $d(x, a_0) < \lambda$. As a result $S_{\lambda-\varepsilon}[b_0]$ is compact, and as

$$\lambda - \varepsilon > \bar{\nu}(B) \geqslant \nu(b_0),$$

we obtain a contradiction. □

Lemma 3.10. *Let $\langle X, d\rangle$ be a metric space. Then the real-valued set functional $\underline{\nu}(\cdot)$ defined for subsets of X has these properties*:

(1) $A \subseteq B \Rightarrow \underline{\nu}(A) \geqslant \underline{\nu}(B)$;
(2) $\underline{\nu}(A \cup B) = \min\{\underline{\nu}(A), \underline{\nu}(B)\}$;
(3) $\underline{\nu}(A) = \underline{\nu}(\mathrm{cl}(A))$;
(4) $\underline{\nu}(\cdot)$ *is H_d-uniformly continuous.*

Theorem 3.11. *Let $\langle X, d\rangle$ be a metric space. The following are equivalent*:

(1) *X is cofinally complete*;
(2) *Whenever $\langle F_n\rangle$ is a decreasing sequence of nonempty closed subsets of X with $\lim_{n\to\infty} \underline{\nu}(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$*;
(3) *Whenever $\langle F_n\rangle$ is a decreasing sequence of nonempty closed subsets of X with $\lim_{n\to\infty} \bar{\nu}(F_n) = 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof. (1) ⇒ (2) For each $n \in \mathbb{N}$ pick $x_n \in F_n$ with $\nu(x_n) < \max\{2\underline{\nu}(F_n), \frac{1}{n}\}$. Then by condition (2) of Theorem 3.2, $\langle x_n\rangle$ has a cluster which must lie in each F_n as each F_n is closed and $\forall n\ F_{n+1} \subseteq F_n$.

(2) ⇒ (3) This is trivial.

(3) ⇒ (1) We show that condition (2) of Theorem 3.2 is met. Let $\langle x_n\rangle$ be a sequence in X with $\lim_{n\to\infty} \nu(x_n) = 0$. Without loss of generality, we may assume for each n, $\nu(x_n) \geqslant \nu(x_{n+1})$. For each n set $F_n = \mathrm{cl}(\{x_k\colon k \geqslant n\})$. By Lemma 3.9

$$\bar{\nu}(F_n) = \nu(\{x_k\colon k \geqslant n\}) = \nu(x_n).$$

By condition (3) the set of cluster points of $\langle x_n\rangle$ which is just $\bigcap_{n=1}^{\infty} \mathrm{cl}(\{x_k\colon k \geqslant n\})$ is nonempty. This verifies that X is cofinally complete. □

The following corollary is anticipated by [5, Theorem 2].

Corollary 3.12. *Let $\langle X, d\rangle$ be a complete metric space. Then X is cofinally complete if and only if $\forall \varepsilon > 0\ \exists \delta > 0$ such that $\forall A \subseteq X$, $\bar{\nu}(A) < \delta \Rightarrow \alpha(A) < \varepsilon$.*

Proof. In light of Kuratowski's Theorem [18, p. 412] and condition (3) of Theorem 3.11, the continuity condition is sufficient. For necessity, if $\mathrm{nlc}(X) = \emptyset$, by Theorem 3.2 $\exists \delta > 0$ such that $\bar{\nu}(A) < \delta \Rightarrow A = \emptyset$ and so the condition holds vacuously. On the other hand, if $\mathrm{nlc}(X)$ is nonempty and compact, let $\varepsilon > 0$. By total boundedness of $\mathrm{nlc}(X)$ we have $\alpha(\mathrm{nlc}(X)^{\varepsilon/3}) < \varepsilon$. Now choose by Theorem 3.2 $\delta > 0$ such that $\{x\colon \nu(x) < \delta\} \subseteq \mathrm{nlc}(X)^{\varepsilon/3}$; clearly this choice of δ is adequate for the continuity condition to hold. □

Total boundedness is usually coupled with completeness as a criterion for compactness. This seems quite natural in that a metric space is totally bounded if and only if each sequence in the space has a Cauchy subsequence [9, p. 312]. Obviously, total boundedness can equally well be paired with cofinal completeness or UCness to yield compactness. Our final simple result shows that such pairings are equally natural.

Proposition 3.13. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent*:

(1) *d is a totally bounded metric*;
(2) *Each sequence in X is cofinally Cauchy*;
(3) *Each sequence in X is pseudo-Cauchy.*

Proof. (1) $\Rightarrow$ (2) Let $\langle x_n\rangle$ be a sequence in X and let $\varepsilon > 0$. Choose $\{w_1, w_2, w_3, \ldots, w_k\}$ in X with $X = \{w_1, w_2, w_3, \ldots, w_k\}^{\varepsilon/2}$. Then for some infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ and some $i \leqslant k$ we have $\{x_n\colon n \in \mathbb{N}_\varepsilon\} \subseteq S_{\varepsilon/2}(x_i)$ so that $\operatorname{diam}\{x_n\colon n \in \mathbb{N}_\varepsilon\} \leqslant \varepsilon$.

(2) $\Rightarrow$ (3) This is trivial.

(3) $\Rightarrow$ (1) If X is not totally bounded, then X has a uniformly discrete infinite subset $\{x_n\colon n \in \mathbb{N}\}$ and the associated sequence $\langle x_n\rangle$ is not pseudo-Cauchy. □

4. Which metrizable spaces admit a cofinally complete metric?

According to the celebrated theorem of Alexandrov, a metrizable topological space X is completely metrizable if and only if it is homeomorphic to a G_δ subset of some complete metric space; this is true if and only if X is Čech complete, i.e., X sits as a G_δ set in its Stone–Čech compactification [10]. By a theorem of Vaughan [27], X admits a boundedly compact metric if and only if X is locally compact and separable, or equivalently, there exists a countable cover of X by compacta $\{K_n\colon n \in \mathbb{N}\}$ such that for each n $K_n \subseteq \operatorname{int}(K_{n+1})$ [10, p. 250]. By a theorem of Arens [1], a metrizable space X admits a boundedly compact metric if and only the compact-open topology on the real-valued continuous functions on X is metrizable. The UC-metrizability of X is equivalent to any of the following conditions [21]: (1) X' is compact; (2) for any subset E of X, $\operatorname{bd}(E)$ is compact; (3) each closed subset F of X has a countable neighborhood base (see also [24]).

Ten years ago, Romaguera [23] introduced a condition for completely regular Hausdorff spaces aimed to parallel Čech completness that he called cofinal Čech completeness. He showed that a metrizable space admits a cofinally complete metric if and only if it is cofinally Čech complete, and then obtained as an application that the space has a cofinally complete metric if and only if $\operatorname{nlc}(X)$ is compact (necessity was first observed by Rice [22, Thm. 5]).

We give a short direct proof of Romaguera's Theorem on compactness of $\operatorname{nlc}(X)$ based on a well-know result regarding refinements of a sequence of open covers of a metrizable space [9, p. 196]: if X is metrizable and $\{\Omega_k\colon k \in \mathbb{N}\}$ is a family of open covers of X, then there exists an admissible metric d for X such that $\forall k \in \mathbb{N}$ $\{S_{1/k}(x)\colon x \in X\}$ refines Ω_k.

Theorem 4.1. *Let X be a metrizable topological space. The following conditions are equivalent*:

(1) *X has a cofinally complete admissible metric d*;
(2) $\operatorname{nlc}(X)$ *is compact*;
(3) *Whenever F is a closed subset of* $\operatorname{nlc}(X)$, *F has a countable base for its neighborhood system.*

Proof. By Theorem 3.2 condition (1) is sufficient for (2) and evidently condition (2) is sufficient for (3). We intend to prove (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

For (3) $\Rightarrow$ (2), suppose $\operatorname{nlc}(X)$ is noncompact. Let $\langle x_n\rangle$ be a sequence in $\operatorname{nlc}(X)$ with distinct terms but without a cluster point. Then $F = \{x_n\colon n \in \mathbb{N}\}$ is a closed discrete subset of $\operatorname{nlc}(X)$. Let $\{S_{\varepsilon_n}(x_n)\colon n \in \mathbb{N}\}$ be a pairwise disjoint family of open balls. Let $\{V_n\colon n \in \mathbb{N}\}$ be an arbitrary countable family of neighborhoods of F. We produce a neighborhood V of F such that $\forall n$ $V_n \subsetneq V$. For each n, x_n is a limit point of X so $\exists \delta_n > 0$ such that $S_{\delta_n}(x_n)$ is a proper subset of $V_n \cap S_{\varepsilon_n}(x_n)$. Then $V = \bigcup_{n=1}^\infty S_{\delta_n}(x_n)$ does the job.

For (2) $\Rightarrow$ (1), first suppose $\operatorname{nlc}(x) = \emptyset$; then each $x \in X$ has an open neighborhood V_x such that $\operatorname{cl}(V_x)$ is compact. By the refinement result there exists an admissible metric d such that $\{S_1(x)\colon x \in X\}$ refines $\{V_x\colon x \in X\}$. As a result $\forall x$ $S_{1/2}[x]$ is compact, and so d is cofinally complete. Otherwise, $\operatorname{nlc}(X)$ is nonempty and compact and thus has a countable base $\{W_k\colon k \in \mathbb{N}\}$ for its system of neighborhoods. Again for each $x \notin \operatorname{nlc}(X)$ let V_x be an open neighborhood of x with compact closure. For each $k \in \mathbb{N}$, define an open cover Ω_k of X as follows:

$$\Omega_k = \{V_x\colon x \notin W_k\} \cup \{W_k\}.$$

Choose an admissible metric d such that for each k, $\{S_{1/k}[x]\colon x \in X\}$ refines Ω_k. Now let $\langle x_n\rangle$ satisfy $\lim_{n\to\infty} \nu(x_n) = 0$ with respect to the metric d. Then for each k, W_k contains a tail of $\langle x_n\rangle$, specifically $x_n \in W_k$ when $\nu(x_n) < \frac{1}{k}$. But $\forall \varepsilon > 0$ $\exists k$ with $W_k \subseteq \operatorname{nlc}(X)^\varepsilon$. Since $\operatorname{nlc}(X)$ is nonempty and compact, $\langle x_n\rangle$ has a cluster point and d is cofinally complete in this second case. □

Corollary 4.2. *Suppose X is a metrizable space that has an admissible cofinally complete metric and Y is a metrizable space. If there exists an open continuous surjection f from X to Y, then Y has an admissible cofinally complete metric.*

Proof. If $x \in X$ has a compact neighborhood, then so does $f(x)$. As a result, $\mathrm{nlc}(Y) \subseteq f(\mathrm{nlc}(X))$. By continuity, $f(\mathrm{nlc}(X))$ is compact and so $\mathrm{nlc}(Y)$ is a closed subset of a compact set. □

We note that if each admissible metric for X were cofinally complete, then X itself must be compact, as this is true when each admissible metric is just complete [10, p. 347]. Thus, cofinally completeness of a metric is not preserved under bicontinuous maps between metric spaces. In fact, if $\langle X, d\rangle$ is cofinally complete and $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ is bicontinuous and Lipschitz, $\langle Y, \rho\rangle$ need not be cofinally complete. As a simple example, the homeomorphism $f : \mathbb{R} \to (-1, 1)$ defined by

$$f(x) = \frac{x}{1 + |x|}$$

is 1-Lipschitz, as f is continuously differentiable with $\max |f'(x)| = |f'(0)| = 1$.

We close with a positive result.

Theorem 4.3. *Let $\langle X, d\rangle$ be a cofinally complete metric space and let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ be a continuous bijection such that f^{-1} is uniformly continuous. Then $\langle Y, \rho\rangle$ is cofinally complete.*

Proof. Let $\langle y_n\rangle$ be a sequence in Y with $\lim_{n\to\infty} \nu(y_n) = 0$. For each n let $x_n = f^{-1}(y_n)$. The assumptions of the theorem ensure that $\lim_{n\to\infty} \nu(x_n) = 0$, so $\langle x_n\rangle$ has a cluster point p which by continuity of f is mapped to a cluster point of $\langle y_n\rangle$. □

In particular if $\langle X, d\rangle$ is cofinally complete, any admissible metric for X whose metric uniformity is finer than the d-uniformity also gives rise to a cofinally complete metric space.

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