A fundamental solution method for three-dimensional viscous flow problems with obstacles in a periodic array

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Abstract

In this paper, we propose a fundamental solution method for three-dimensional viscous flow problems with obstacles in a periodic array. Our problem is mathematically a boundary value problem of the Stokes equation with periodic boundary conditions, to which it is difficult to give a good approximation by the ordinary fundamental solution method. Our method gives an approximate solution by a linear combination of the periodic fundamental solutions. In addition, we can compute the drag forces on the obstacles by using the data obtained in our method. Numerical examples for the problems of flows past spheres show the effectiveness of our method.

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1. Introduction

The fundamental solution method (or the charge simulation method) is a fast solver for problems of partial differential equations [12,11,17] and is widely used in science and engineering for the reasons that (i) it is easy to program, (ii) its computational cost is low and (iii) it achieves high accuracy under

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some conditions. The method approximates the solution by a linear combination of the fundamental solutions.

In this paper, we are concerned with three-dimensional viscous flow problems with obstacles arranged infinitely in a three-dimensional periodic array (see Fig. 1) and propose a fundamental solution method for these problems. The problems of viscous flow past obstacles in a periodic array are important from theoretical and practical viewpoints, and have been studied by many authors. Hasimoto presented the periodic fundamental solution of the Stokes equation and applied it to the analysis of the viscous flow past an infinite array of spheres [4], whose results were improved by Sangani and Acrivos [16]. Ishii discussed three-dimensional Stokes flow problems with multiple planar arrays of small spheres in his paper [6], where he presented the periodic fundamental solution of the Stokes equation with a planar array of singularity points. Tamada and Fujikawa investigated the motion of steady two-dimensional viscous flow past an infinite row of circular cylinders based on the Oseen equation [18]. Liron investigated the Stokes flow due to an infinite array of Stokeslets parallel to flat planes or in a pipe with application to the analysis of fluid transport by ciliated organisms [7–9].

We assume that the flow is steady, incompressible and slow, that is, the Reynolds number is low and that a uniform force is acting on the whole fluid. Our problem is a boundary value problem of the Stokes equation, a linearization of the Navier–Stokes equation,

\[ \mu \Delta \mathbf{v} - \nabla p + \mathbf{K} = 0 \]  

and the continuity equation

\[ \nabla \cdot \mathbf{v} = 0, \]

where \( \mathbf{v}(v_1, v_2, v_3) \) is the velocity of the flow, \( p \) is the pressure, \( \mathbf{K}(K_1, K_2, K_3) \) is the uniform force per unit volume and \( \mu \) is the viscosity. If we apply the fundamental solution method to this problem, it is
natural to use the Stokeslet (see [5], Section 69), which is the solution of the equations
\[ \mu \Delta v - \nabla p = 4 \pi \mu Q \delta(x), \]
\[ \nabla \cdot v = 0, \]

\( Q(Q_1, Q_2, Q_3) \) is a constant vector and is explicitly given by
\[ v_i(x) = -\frac{1}{2} \left( \frac{Q_i}{|x|} + \sum_{j=1}^{3} \frac{Q_j x_j}{|x|^3} \right) \quad (i = 1, 2, 3), \]
\[ p(x) = -\mu \sum_{j=1}^{3} \frac{Q_j x_j}{|x|^3}. \]

Introducing the \( 3 \times 3 \) matrix
\[ T(x) = \begin{pmatrix} T_{11}(x) & T_{12}(x) & T_{13}(x) \\ T_{21}(x) & T_{22}(x) & T_{23}(x) \\ T_{31}(x) & T_{32}(x) & T_{33}(x) \end{pmatrix}, \]
\[ T_{ij}(x) = -\frac{1}{2} \left( \frac{x_i x_j}{|x|^3} + \delta_{ij} \frac{|x|}{|x|^3} \right) \quad (i, j = 1, 2, 3), \]

where the symbol \( \delta_{ij} \) is the Kronecker delta:
\[ \delta_{ij} = \begin{cases} 1 & (i = j), \\ 0 & (i \neq j). \end{cases} \]

and the vector
\[ \tau(x) = -\frac{\mu x}{|x|^3}, \]

the solutions (3) and (4) are simply expressed as
\[ v(x) = T(x)Q, \]
\[ p(x) = \tau(x) \cdot Q. \]

Physically the Stokeslet (3), (4) or (5), (6) illustrates the flow induced by a point force \(-4\pi \mu Q\) acting on the point \(x = 0\). The fundamental solution method with the Stokeslets approximates the solution by a linear combination of the Stokeslets, that is,
\[ v(x) \approx v_N(x) = v_0 + \sum_{j=1}^{N} T(x - \xi_j)Q_j, \]
\[ p(x) \approx p_N(x) = \sum_{j=1}^{N} \tau(x - \xi_j) \cdot Q_j, \]

where \(v_0, Q_j \ (j = 1, 2, \ldots, N)\) are constant vectors to be suitably chosen and \(\xi_j \ (j = 1, 2, \ldots, N)\) are given points outside the problem domain. Physically the approximation (7) and (8) is the flow \(v_0\) plus
the superposition of the flows induced by the point forces $-4\pi\mu Q_j$ acting on the points $\xi_j$. Remark that the approximate solution $v_N(x),\ p_N(x)$ exactly satisfies the Eqs. (1) and (2) in the problem domain. The boundary condition is approximately satisfied by choosing the vectors $Q_j$ appropriately for the given points $\xi_j$.

The above approximation by the Stokeslet is applicable to a problem of flow with a finite number of obstacles. But it is difficult to approximate the solution of our problem, i.e., a problem with infinite number of obstacles in a periodic array, by the above method because the solution obviously includes periodic functions and is difficult to approximate by the expression (7) and (8). In order to overcome this difficulty, we introduce the periodic fundamental solutions, the solution of the equations

$$\mu \Delta v - \nabla p = 4\pi\mu Q \sum_a \delta(x - a),$$

$$\nabla \cdot v = 0,$$

(9)

where $Q$ is a constant vector and the summation in (9) is taken over all the lattice points $a$. The periodic fundamental solution was first presented by Hasimoto and was applied in his paper [4] to a study of viscous flows past spheres or cylinders of small radius in a periodic array. We use this periodic fundamental solution to construct a fundamental solution method for our problem, namely, we approximate the solution of our problem by a linear combination of the periodic fundamental solutions. This linear combination is a periodic function as the exact solution is, therefore is expected to approximate the solution with high accuracy. The method presented in this paper is an extension to three-dimensional cases of our method presented for two-dimensional cases [14]. We also remark that the presented method is similar to our method for problems of numerical conformal mapping of periodic structure domains [15] in the sense that the solution is approximated by using fundamental solutions with sources in a periodic array.

In Section 2, we prepare some notations and formulate mathematically our problem. In Section 3, we make a brief review of the periodic fundamental solution and, using this solution, we construct a fundamental solution method for our problem. In addition, we present an approximation formula of the drag forces on the obstacles exerted by the fluid, which includes the data obtained in our method. In Section 4, we show some numerical examples for the problems of flows past spheres in the typical lattices such as the simple cubic lattice, the body-centered lattice and the face-centered lattice. In Section 5, we make concluding remarks and refer to future problems related to this paper.

2. Formulation of our problem

We here introduce some notations. Throughout this paper, we denote by $\mathbb{R}$ the set of all the real numbers and denote by $\mathbb{Z}$ the set of all the integers.

In order to describe the geometrical structure of the array (or lattice) of the obstacles, we adopt the terminology used in the theory of crystal structures in solid-state physics (see the textbook of Ashcroft and Mermin [2]). Let $a_1, a_2, a_3$ be linearly independent three-dimensional constant vectors which span the obstacles, that is, every obstacle is given by

$$D_a = \{x + a \mid x \in D_0\},$$
where \(a(a_1, a_2, a_3)\) is a so-called lattice vector

\[
a = n_1 a_1 + n_2 a_2 + n_3 a_3 \quad (n_1, n_2, n_3 \in \mathbb{Z})
\]

and \(D_0\) is a specified obstacle. Let \(\mathcal{L}\) be the set of all the lattice vectors:

\[
\mathcal{L} = \{ n_1 a_1 + n_2 a_2 + n_3 a_3 \mid n_1, n_2, n_3 \in \mathbb{Z} \}.
\]

The problem domain is the exterior of the obstacles

\[
\mathcal{D} = \mathbb{R}^3 \setminus \bigcup_{a \in \mathcal{L}} \overline{D_a}.
\]

It is also convenient to introduce so-called reciprocal lattice vectors \(k(k_1, k_2, k_3)\) of the form

\[
k = m_1 b_1 + m_2 b_2 + m_3 b_3 \quad (m_1, m_2, m_3 \in \mathbb{Z}),
\]

where \(b_1, b_2, b_3\) are the bi-orthogonal basis with respect to \(a_1, a_2, a_3\), i.e., the vectors satisfying the relations

\[
a_i \cdot b_j = \delta_{ij}.
\]

We denote the set of all the reciprocal lattice vectors by \(\mathcal{L}^*\), that is,

\[
\mathcal{L}^* = \{ m_1 b_1 + m_2 b_2 + m_3 b_3 \mid m_1, m_2, m_3 \in \mathbb{Z} \}.
\]

Our problem is a boundary value problem of the Stokes equation and the continuity one coupled with the no-slip boundary condition

\[
\mu \Delta v - \nabla p + K = 0 \quad \text{in} \ \mathcal{D},
\]
\[
\nabla \cdot v = 0 \quad \text{in} \ \mathcal{D},
\]
\[
v = 0 \quad \text{on} \ \partial \mathcal{D}.
\]

3. Fundamental solution method by the periodic fundamental solution

The solution \(v(x)\) of our problem is obviously a periodic solution of the periods \(a_1, a_2, a_3\), i.e., a function satisfying

\[
v(x + a) = v(x) \quad (x \in \mathcal{D}, \ a \in \mathcal{L})
\]

and is difficult to approximate by the ordinary fundamental solution method. In order to approximate this periodic solution, we propose a new fundamental solution method by using the periodic fundamental solution presented by Hasimoto [4].
The periodic fundamental solution of the Stokes equation for the lattice \( \mathcal{L} \) is the solution of the equations

\[
\mu \dot{\mathbf{v}} - \nabla p = 4\pi \mu \mathbf{Q} \sum_{\mathbf{a} \in \mathcal{L}} \delta(\mathbf{x} - \mathbf{a}),
\]
\[
\nabla \cdot \mathbf{v} = 0,
\]

where \( \mathbf{Q}(Q_1, Q_2, Q_3) \) is a constant vector. The solution is explicitly given by

\[
v_i(x) = \sum_{j=1}^{3} Q_j \frac{\partial^2 S_2(x)}{\partial x_i \partial x_j} - Q_i S_1(x) \quad (i = 1, 2, 3),
\]

\[
p(x) = -\frac{4\pi \mu}{\tau_0} \sum_{j=1}^{3} Q_j x_j + \mu \sum_{j=1}^{3} Q_j \frac{\partial S_1(x)}{\partial x_j},
\]

where

\[
S_1(x) = \frac{1}{\pi \tau_0} \sum_{k \in \mathcal{L}^* \setminus \{\mathbf{0}\}} \frac{e^{i2\pi k \cdot x}}{|k|^2}, \quad S_2(x) = -\frac{1}{4\pi \tau_0} \sum_{k \in \mathcal{L}^* \setminus \{\mathbf{0}\}} \frac{e^{i2\pi k \cdot x}}{|k|^4}
\]

and \( \tau_0 \) is the volume of a unit cell, i.e.,

\[\tau_0 = a_1 \cdot (a_2 \times a_3).\]

Introducing the \( 3 \times 3 \) matrix

\[
\mathbf{T}(x) = \begin{pmatrix}
T_{11}(x) & T_{12}(x) & T_{13}(x) \\
T_{21}(x) & T_{22}(x) & T_{23}(x) \\
T_{31}(x) & T_{32}(x) & T_{33}(x)
\end{pmatrix},
\]

\[
T_{ij}(x) = \frac{\partial^2 S_2(x)}{\partial x_i \partial x_j} - \delta_{ij} S_1(x) \quad (i, j = 1, 2, 3)
\]

and the vector

\[
\mathbf{\tau}(x) = -\frac{4\pi \mu}{\tau_0} x + \mu \nabla S_1(x),
\]

the solutions (14), (15) are simply expressed as

\[
\mathbf{v}(x) = \mathbf{T}(x) \mathbf{Q},
\]

\[
p(x) = \mathbf{\tau}(x) \cdot \mathbf{Q}.
\]

1 In the case of the periodic fundamental solution with a planar array of singularity points discussed in [6], the pressure of Eq. (2.5) in [6] has a term of discontinuity on the plane of singularity array. However, in our case, that is, in the case of the periodic fundamental solution with three-dimensional periodic array of singularity points, we are free of the above problem of the pressure discontinuity.
Physically the periodic fundamental solutions (14), (15) or (19), (20) illustrate the flow induced by the force $-4\pi\mu Q$ acting uniformly on each lattice point $a(\in \mathcal{L})$. The Fourier series on the right-hand sides of (17), (18) converge slowly but can be computed by Ewald’s technique [3], which will be presented in Appendix A in detail.

We here return to our original problem. We approximate the solution of the problem (10), (11), (12) by a linear combination of the periodic fundamental solutions, that is,

\[
v(x) \approx v_N(x) = v_0 + \sum_{j=1}^{N} T(x - \xi_j)Q_j,
\]

\[
p(x) \approx p_N(x) = K \cdot x + \sum_{j=1}^{N} \tau(x - \xi_j) \cdot Q_j,
\]

where $v_0(v_{01}, v_{02}, v_{03})$, $Q_j(Q_{j1}, Q_{j2}, Q_{j3})$ $(j = 1, 2, \ldots, N)$ are unknown constant vectors and $\xi_j$ $(j = 1, 2, \ldots, N)$ are given points inside an obstacle $D_0$. The constant vector $v_0$ can be related to the mean velocity of the flow as shown in the numerical examples in Section 4. Physically the approximation (21), (22) is the flow $v_0$ plus the superposition of the flows induced by the point force $-4\pi\mu Q_j$ acting uniformly on the lattice points $\xi_j + a$. Remark that the approximate solution (21), (22) exactly satisfies the Eqs. (10) and (11) in $\mathcal{D}$ and, in addition, that the approximate velocity $v_N(x)$ has the property of periodicity (13). The boundary condition (12) is approximately satisfied by choosing the unknown vectors $Q_j$ appropriately. More exactly, we pose the collocation condition

\[
v_N(x_i) = 0 \quad (i = 1, 2, \ldots, N)
\]

on the approximate solution, where $x_i$ $(i = 1, 2, \ldots, N)$ are given points on the boundary $\partial D_0$, instead of the boundary condition (12). The collocation condition (23) is rewritten as

\[
v_0 + \sum_{j=1}^{N} T(x_i - \xi_j)Q_j = 0 \quad (i = 1, 2, \ldots, N).
\]

Besides, the drag force $D$, which is approximated by

\[
D \approx D_N = 4\pi\mu \sum_{j=1}^{N} Q_j,
\]

(see Appendix B) and the force $-(\text{vol } D_0)K$ ($(\text{vol } D_0)$ is the volume of the obstacle $D_0$) due to $K$, which is the buoyancy if $K$ is the gravity, are balanced, i.e., $D - (\text{vol } D_0)K = 0$. Thus, taking into account (25), we have

\[
4\pi\mu \sum_{j=1}^{N} Q_j = (\text{vol } D_0)K.
\]

The equalities (24), (25) form $(3N + 3)$ simultaneous linear equations with respect to the unknowns $v_0$, $Q_j$. We determine the unknowns $v_0$, $Q_j$ by solving these equations and obtain the approximate solutions $v_N$, $p_N$. 

4. Numerical examples

In this section, we show numerical examples for some typical cases. All the computations in this section were carried out on a Sun Blade 150 workstation using programs coded in C with double precision working.

The problem domain is the exterior of spheres of radius \( r > 0 \) in the typical lattices \( \mathcal{L} \) such as

(1) simple cubic lattice (S.C.L.)

\[
\begin{align*}
    a_1 &= (1, 0, 0), & b_1 &= (1, 0, 0), \\
    a_2 &= (0, 1, 0), & b_2 &= (0, 1, 0), \\
    a_3 &= (0, 0, 1), & b_3 &= (0, 0, 1) \quad (a > 2r),
\end{align*}
\]

(2) body-centered lattice (B.C.L.)

\[
\begin{align*}
    a_1 &= \frac{a}{2} \left\{ (1, 1, -1), \right. & b_1 &= \frac{a}{2} \left\{ (1, 1, 0), \\
    a_2 &= \left. (0, 1, 1), \right. & b_2 &= (0, 1, 1), \\
    a_3 &= (1, 0, 1), & b_3 &= (1, 0, 1) \quad (a > \frac{4}{\sqrt{3}}r),
\end{align*}
\]

(3) face-centered lattice (F.C.L.)

\[
\begin{align*}
    a_1 &= \frac{a}{2} \left\{ (1, 1, 0), \right. & b_1 &= \frac{a}{2} \left\{ (1, 1, -1), \\
    a_2 &= \left. (0, 1, 1), \right. & b_2 &= (-1, 1, 1), \\
    a_3 &= (1, 0, 1), & b_3 &= (-1, 1, 1) \quad (a > 2\sqrt{2}r).
\end{align*}
\]

The uniform force \( \mathbf{K} \) is taken in the \( x_1 \)-direction in all the above cases.

In these cases, the mean velocity

\[
\bar{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3) = \frac{1}{a^2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} v(x) \, dx_2 \, dx_3,
\]

where the \( x_1 \)-coordinate in the integral is taken so that the surface \(-a/2 \leq x_2, x_3 \leq a/2\) is outside every sphere, is obviously in the \( x_1 \)-direction, i.e., \( \bar{v}_2 = \bar{v}_3 = 0 \). Besides, it is shown that, in these cases, the \( x_1 \)-element of the approximate mean velocity

\[
\bar{v}_{N1} = \frac{1}{a^2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} v_{N1}(x) \, dx_2 \, dx_3,
\]

where \( v_{N1}(x) \) is the \( x_1 \)-element of the approximate velocity \( v_N(x) \), is equal to the \( x_1 \)-element of \( v_0 \), i.e., \( \bar{v}_{N1} = v_{01} \). In fact, the \( x_1 \)-element \( \bar{v}_{N1} \) of the mean approximate velocity is given by

\[
\bar{v}_{N1} = v_{01} + \frac{1}{a^2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} \sum_{j=1}^{N} \sum_{i=1}^{3} \frac{Q_{ji}}{i^2 \xi_i} \left[ \frac{\partial S_2(x - \xi_j)}{\partial x_1 \partial x_i} - \frac{\partial S_1(x - \xi_j)}{\partial x_i} \right] \, dx_2 \, dx_3
\]

\[
= v_{01} + \frac{1}{a^2} \sum_{j=1}^{N} \sum_{k \in \mathbb{Z}^+ \setminus \{0\}} \frac{e^{-i2\pi k \cdot \xi_j}}{|k|^4} \left[ -(k_2^2 + k_3^2) Q_{j1} + k_1 k_2 Q_{j2} + k_1 k_3 Q_{j3} \right] \times \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} e^{i2\pi k \cdot x} \, dx_2 \, dx_3
\]
Fig. 2. The velocity field in the $z = 0$ plane of the Stokes flow past spheres in (a) the simple cubic lattice, (b) the body-centered lattice and (c) the face-centered lattice. The volume concentrations of the lattice are $C = (0.3)^3$.

and we have

$$
\frac{1}{a^2} \int_{-a/2}^{a/2} \int_{-a/2}^{a/2} e^{i2\pi k \cdot x} \, dx_2 \, dx_3 = 0 \quad \text{if } k_2 \neq 0 \, \text{or} \, k_3 \neq 0
$$

for the S.C.L., the B.C.L. and the F.C.L. Therefore the mean velocity is approximated by

$$
\mathbf{v} = (\mathbf{v}_1, 0, 0) \approx (v_{01}, 0, 0), \quad (28)
$$

The collocation points $\mathbf{x}_i$ and the points $\xi_j$ are positioned by the slice point method proposed by Nishida [13] as follows. For a given integer $n > 3$, we put

$$
x_{11} = r(0, 0, 1), \quad x_{n1} = r(0, 0, -1),
$$

$$
x_{kl} = r(\sin \theta_k \cos \varphi_{kl}, \sin \theta_k \sin \varphi_{kl}, \cos \theta_k) \quad (2 \leq k \leq n - 1, 1 \leq l \leq n_k),
$$

where

$$
\theta_k = \frac{\pi(k - 1)}{n - 1}, \quad n_k = [2(n - 1) \sin \theta_k + 0.5], \quad \varphi_{kl} = \frac{2\pi(l + (-1)^k/4)}{n_k}.
$$

The collocation points $\mathbf{x}_i$ are given by renumbering $x_{kl}$ and the points $\xi_j$ are given by $\xi_j = q x_j$ for a given parameter $q$ ($0 < q < 1$). The parameter $q$ is taken to be $q = 0.1$ in the examples presented here.

Fig. 2 shows the velocity field of the flows obtained by our method, where the volume concentration

$$
C = \frac{4\pi r^3}{\tau_0}
$$

is taken to be $C = (0.3)^3$ for the three lattices.

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2 The symbol $[\cdot]$ denotes the floor function, that is, $[x]$ is the largest integer less than or equal to $x$. 
Table 1
Error estimates $\varepsilon$ and the ratios $r/a$ for volume concentrations $C = (0.1)^3$, $(0.5)^3$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$C = (0.1)^3$</th>
<th>$C = (0.5)^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>S.C.L.</td>
<td>B.C.L.</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon$</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>9.5E-2</td>
<td>9.1E-2</td>
</tr>
<tr>
<td>34</td>
<td>9.4E-5</td>
<td>2.2E-4</td>
</tr>
<tr>
<td>64</td>
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<td>3.3E-8</td>
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<td>3.6E-10</td>
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<td>1.9E-12</td>
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<td>288</td>
<td>3.5E-13</td>
<td>7.9E-13</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$r/a$</td>
<td>$r/a$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.06203...</td>
<td>0.04924...</td>
<td>0.03909...</td>
</tr>
</tbody>
</table>

In order to estimate the error of the approximation, we computed the value

$$
\varepsilon = \max_{x \in \partial D_0} \left\{ \frac{|v_N(x)|}{v_{\text{mean}}} \right\},
$$

(29)

where $v_{\text{mean}}$ is the magnitude of the mean velocity and computed by $v_{\text{mean}} \approx v_{01}$. The value $\varepsilon$ shows how accurately the boundary condition (12) is satisfied. In the numerical examples, the maximum over $\partial D_0$ is approximated by the maximum over 1000 points on $\partial D_0$ distributed by random numbers. Table 1 shows the value $\varepsilon$ for volume concentrations $C = (0.1)^3$, $(0.5)^3$ as a function of the number of the points $N$. From this table, we find that the error estimates $\varepsilon$ decrease as the number of the points $N$ increases and that the error estimates $\varepsilon$ are small if the spheres are small. In addition, we find that there is little difference in the accuracy of our method among the three types of the lattices if the volume concentration $C$ and the number of the points $N$ are the same.

We also investigated the relation between the drag force $D$ on the spheres and the mean velocity. Fig. 3 shows the non-dimensional drag force $|D|/(6\pi \mu v_{\text{mean}})$ on the spheres as a function of $C^{1/3}$ and Table 2 shows the force as a function of $(C/C_{\text{max}})^{1/3}$, where $C_{\text{max}}$ is the volume concentration of the spheres in the touching configuration, i.e.,

$$
C_{\text{max}} = \begin{cases} 
\pi/6 & \text{(S.C.L.)}, \\
\sqrt{3}\pi/8 & \text{(B.C.L.)}, \\
\sqrt{2}\pi/6 & \text{(F.C.L.)}.
\end{cases}
$$

Part of our results listed in Table 2 are compared with the results of Sangani and Acrivos [16]. From Fig. 3, we find that there is little difference in the values of $|D|/(6\pi \mu v_{\text{mean}})$ among the three types of lattices if the volume concentrations $C$ are the same and small. From Table 2 our results make good agreement with the results of Sangani and Acrivos and, in addition, we have

$$
|D| \sim 6\pi \mu r v_{\text{mean}} \quad \text{as} \quad r \to 0
$$

which corresponds to the Stokes formula for isolated spheres.
Fig. 3. The non-dimensional drag force $|D|/(6\pi \mu v_{\text{mean}})$ on the spheres as a function of $C^{1/3}$, where $C$ is the volume concentration $C = (4\pi r^3/3)/\tau_0$. The number of the points $N$ is taken to be $N = 106$.

Table 2
The non-dimensional drag force $|D|/(6\pi \mu v_{\text{mean}})$ on the spheres as a function of $(C/C_{\text{max}})^{1/3}$ for the three types of lattices

<table>
<thead>
<tr>
<th>$(C/C_{\text{max}})^{1/3}$</th>
<th>Our method</th>
<th>Sangani and Acrivos</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.001</td>
<td>1.0014</td>
<td>1.0016</td>
</tr>
<tr>
<td>0.01</td>
<td>1.0144</td>
<td>1.0160</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1646</td>
<td>1.1861</td>
</tr>
<tr>
<td>0.2</td>
<td>1.3881</td>
<td>1.4487</td>
</tr>
<tr>
<td>0.3</td>
<td>1.6999</td>
<td>1.8331</td>
</tr>
<tr>
<td>0.4</td>
<td>2.1518</td>
<td>2.4234</td>
</tr>
<tr>
<td>0.5</td>
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<td>3.3886</td>
</tr>
<tr>
<td>0.6</td>
<td>3.9737</td>
<td>5.1082</td>
</tr>
<tr>
<td>0.7</td>
<td>6.0037</td>
<td>8.5648</td>
</tr>
</tbody>
</table>

Results by our method are compared with the results of Sangani and Acrivos [16]. The blanks in the column “Sangani and Acrivos” mean that there are no data for $(C/C_{\text{max}})^{1/3} = 0.001, 0.01$ in their results. The number of points $N$ in our method is taken to be $N = 106$.

5. Concluding remarks

In this paper, we proposed a fundamental solution method for three-dimensional Stokes flow problems with obstacles in a periodic array. In our method, the solution, which includes periodic functions,
is approximated by a linear combination of the periodic fundamental solutions, i.e., the fundamental solutions with sources in a periodic array. We also presented an approximation formula of the drag force on the obstacles which includes the data obtained in our method. Numerical examples for the problems of flows past spheres show that our method gives good approximations for the cases of small radius of spheres and that the results by our method make good agreement with the results of the previous authors. However, approximations by our method are not good for the cases of large radius of spheres. It is a future problem to improve our method for the problems of large volume concentration. In addition, the following problems are considered to be future problems related to this paper:

(1) constructing a fundamental solution method for Oseen flow problems with obstacles in a periodic array, and
(2) applying our method to practical problems such as the analysis of flow through porous media.

Acknowledgements

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Appendix A. Ewald’s technique

Ewald’s technique is a method for computing the elements of the matrix $T(x)$ given by (17), which include slowly convergent sums with respect to reciprocal lattice vectors $k$. The technique is based on the relation

$$
\sum_{k \in \mathcal{L}^* \setminus \{0\}} e^{i2\pi k \cdot x} \frac{1}{|k|^{2m}} = \pi^m \zeta^m \left( \tau_0 \zeta^{3/2} \sum_{a \in \mathcal{L}} \phi_{-m+1/2} \left( \frac{\pi}{\zeta} |x - a| \right) - \frac{1}{m} \right) + \sum_{k \in \mathcal{L}^* \setminus \{0\}} e^{i2\pi k \cdot x} \phi_{m-1} (\pi \zeta |k|^2) \right) \quad (m = 1, 2, \ldots),
$$

where $\zeta$ is an arbitrary positive number and $\phi_v(x) \ (x > 0)$ is the incomplete gamma function

$$
\phi_v(x) = \int_1^\infty \xi^v e^{-\xi x} d\xi.
$$

The function $\phi_v(x)$ is evaluated by the recurrence relations

$$
\phi'_v = -\phi_{v+1}, \quad x \phi_v = e^{-x} + v \phi_{v-1}
$$

with

$$
\phi_0(x) = \frac{e^{-x}}{x}, \quad \phi_{-1/2}(x) = \frac{\sqrt{\pi}}{\sqrt{x}} \text{erfc} \left( \sqrt{x} \right).
$$
Substituting (A.1) into (16) and differentiating it, we have

\[
\frac{\partial^2 S_2(x)}{\partial x_i \partial x_j} \approx U^{(1)}_{ijn}(x) + U^{(2)}_{ijm}(x) \quad (i, j = 1, 2, 3),
\]

where

\[
U^{(1)}_{ijn}(x) = -\frac{\pi}{\alpha^{3/2}} \sum_{a=a_n \in V} \left\{(x_i - a_i)(x_j - a_j) - \delta_{ij} |x - a|^2 \phi_{1/2} \left(\frac{\pi}{\alpha} |x - a|^2\right)\right\} + \delta_{ij} \frac{\pi}{\alpha} \exp \left(-\frac{\pi}{\alpha} |x - a|^2\right) \quad (i, j = 1, 2, 3)
\]

\[
(a_n = n_1a_1 + n_2a_2 + n_3a_3, \ n = (n_1, n_2, n_3) \in \mathbb{Z}^3), \quad (A.2)
\]

\[
U^{(2)}_{ijm}(x) = \frac{\pi \alpha^2}{\tau_0} \sum_{k=k_m \in V \setminus \emptyset} k_i k_j \phi_{1} (\pi \alpha |k|^2) \cos (2\pi k \cdot x) \quad (i, j = 1, 2, 3)
\]

\[
(k_m = m_1b_1 + m_2b_2 + m_3b_3, \ m = (m_1, m_2, m_3) \in \mathbb{Z}^3). \quad (A.3)
\]

Then, using the relation \(\Delta S_2(x) = S_1(x)\), we have

\[
T(x) \approx T_{n}^{(1)}(x) + T_{m}^{(2)}(x), \quad (A.4)
\]

where

\[
T_{n}^{(1)}(x) = \begin{bmatrix}
-U_{22n}^{(1)}(x) - U_{33n}^{(1)}(x) & U_{12n}^{(1)}(x) & U_{13n}^{(1)}(x) \\
U_{22n}^{(1)}(x) & -U_{33n}^{(1)}(x) - U_{11n}^{(1)}(x) & U_{13n}^{(1)}(x) \\
U_{32n}^{(1)}(x) & U_{12n}^{(1)}(x) & -U_{11n}^{(1)}(x) - U_{22n}^{(1)}(x)
\end{bmatrix},
\]

\[
T_{m}^{(2)}(x) = \begin{bmatrix}
-U_{22m}^{(2)}(x) - U_{33m}^{(2)}(x) & U_{12m}^{(2)}(x) & U_{13m}^{(2)}(x) \\
U_{22m}^{(2)}(x) & -U_{33m}^{(2)}(x) - U_{11m}^{(2)}(x) & U_{13m}^{(2)}(x) \\
U_{32m}^{(2)}(x) & U_{12m}^{(2)}(x) & -U_{11m}^{(2)}(x) - U_{22m}^{(2)}(x)
\end{bmatrix}.
\]

The infinite sums on the right-hand sides of (A.2) and (A.3) converge rapidly since we have the asymptotic expansion for the incomplete gamma function

\[
\phi_{n}(x) \sim e^{-x} \left[1 + \frac{v}{x} + \frac{v(v - 1)}{x^2} + \cdots\right] \quad (x \to +\infty)
\]

(see the formula 6.5.32 in [1]). Thus we can compute \(T_{ij}(x)\) accurately with small integers \(n, m\).

We investigated how many terms in (A.2) and (A.3) are needed for obtaining the elements of \(T(x)\) with sufficient accuracy. Namely, we investigated the values

\[
n^* = \max\{n^*(x) \mid x \in A\}, \quad m^* = \max\{m^*(x) \mid x \in A\},
\]
Table 3

The values of $n^*$, $\max_{x \in A} \| T^{(1)}_{n^*(x)}(x) \|$, $m^*$, $\max_{x \in A} \| T^{(2)}_{m^*(x)}(x) \|

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n^*$</th>
<th>$\max_{x \in A} | T^{(1)}_{n^*(x)}(x) |$</th>
<th>$m^*$</th>
<th>$\max_{x \in A} | T^{(2)}_{m^*(x)}(x) |$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>S.C.L.</td>
<td>4</td>
<td>8.82630</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>B.C.L.</td>
<td>5</td>
<td>1.10816 \times 10^{-1}</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>F.C.L.</td>
<td>6</td>
<td>1.43641 \times 10^{-1}</td>
<td>2</td>
</tr>
<tr>
<td>0.5</td>
<td>S.C.L.</td>
<td>3</td>
<td>7.84829</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>B.C.L.</td>
<td>4</td>
<td>1.05055 \times 10^{-1}</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>F.C.L.</td>
<td>5</td>
<td>1.31224 \times 10^{-1}</td>
<td>3</td>
</tr>
<tr>
<td>0.25</td>
<td>S.C.L.</td>
<td>2</td>
<td>5.46733</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>B.C.L.</td>
<td>3</td>
<td>9.36923</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>F.C.L.</td>
<td>3</td>
<td>1.31167 \times 10^{-1}</td>
<td>4</td>
</tr>
</tbody>
</table>

where

$$n^*(x) = \min \{ n | \| T^{(1)}_{n+1}(x) - T^{(1)}_{n}(x) \| \leq \varepsilon \| T^{(1)}_{n+1}(x) \| \},$$

$$m^*(x) = \min \{ m | \| T^{(2)}_{m+1}(x) - T^{(2)}_{m}(x) \| \leq \varepsilon \| T^{(2)}_{m+1}(x) \| \},$$

$$A = \left\{ x = \xi_1 a_1 + \xi_2 a_2 + \xi_3 a_3 \middle| 0 \leq \xi_1, \xi_2, \xi_3 \leq 1, \| x - a \| > r \ (\forall a \in \mathcal{L}) \right\},$$

$\varepsilon$ is a positive number of the order of machine epsilon, $r$ is a positive number and, for a $3 \times 3$ matrix $T = [T_{ij}]$, the matrix norm $\| T \|$ is determined by

$$\| T \| = \left\{ \sum_{i,j=1}^{3} T_{ij}^2 \right\}^{1/2}.$$

Table 3 shows the values of $n^*$, $\max_{x \in A} \| T^{(1)}_{n^*(x)}(x) \|$, $m^*$, $\max_{x \in A} \| T^{(2)}_{m^*(x)}(x) \|$ for the simple cubic lattice (S.C.L.), the body-centered lattice (B.C.L.) and the face-centered lattice (F.C.L.), where $\varepsilon$, $r$ are respectively taken to be

$$\varepsilon = 10^{-15}, \quad r = 0.1 r_0^{1/3}$$

and the maximums over $A$ are approximated by the maximums over 1000 points in $A$ distributed by random numbers. Taking into account the results in Table 3, we take $\alpha = 0.5$, $n = 3$, $m = 4$ for the simple-cubic lattice, $\alpha = 0.5$, $n = 4$, $m = 4$ for the body-centered lattice and $\alpha = 0.25$, $n = 3$, $m = 4$ for the face-centered lattice in computing $T(x)$ in the numerical examples in Section 4 so that we can compute $T(x)$ by truncating the infinite sums with small numbers of terms $n$, $m$.
Appendix B. Forces on an obstacle

In this section, we discuss the relations of the forces acting on an obstacle and obtain an approximation formula of the drag force on the obstacle.

The force \( F_S(F_{S1}, F_{S2}, F_{S3}) \) acting on the inside of a closed surface \( S \) is equal to the momentum flow per unit time \( G \) out of the surface \( S \), which is obtained as follows. The momentum flow \( dG \) through an infinitesimal element of surface \( dS \) (see Fig. 4) is given by

\[
dG = \rho v (v \cdot n) dS - Pn dS \approx -Pn dS,
\]

where \( \rho \) is the mass density, \( n \) is the unit normal vector and \( P \) is the stress tensor

\[
P = \begin{bmatrix}
p_{11} & p_{12} & p_{13} \\
p_{21} & p_{22} & p_{23} \\
p_{31} & p_{32} & p_{33}
\end{bmatrix},
\]

\[
p_{ij} = -p \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\]

(see Section 21.14 in [10]). Then the force \( F_S \) is obtained by integrating (B.1) over the surface \( S \), that is,

\[
F_S = -\int_S dG \approx \int_S Pn dS
\]

and the \( x_i \)-element of the force \( F_S \) is given by

\[
F_{Si} = -\int_S p n_i dS + \mu \sum_{j=1}^{3} \int_S \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) n_j dS.
\]

Here, we divide the pressure \( p \) as

\[
p = p^* + K \cdot x,
\]
where $K \cdot x$ is the term due to the existence of the uniform force $K$. The first term on the right-hand side of (B.3) is computed as

$$- \int_S p n_i \, dS = - \int_S p^* n_i \, dS - \int_S (K \cdot x) n_i \, dS$$

$$= - \int_S p^* n_i \, dS - \int_V \frac{\partial}{\partial x_i} (K \cdot x) \, dV$$

$$= - \int_S p^* n_i \, dS - (\text{vol} \, V) K_i,$$

where $V$ is the inside of $S$ and we used the Gauss formula on the second equality. Then we have

$$F_S = -(\text{vol} \, V) K + \int_S p^* n \, dS, \quad (B.4)$$

where

$$p^* = \begin{bmatrix} p_{11}^* & p_{12}^* & p_{13}^* \\ p_{21}^* & p_{22}^* & p_{23}^* \\ p_{31}^* & p_{32}^* & p_{33}^* \end{bmatrix}, \quad (B.5)$$

$$p_{ij}^* = - p^* \delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right). \quad (B.6)$$

The first term on the right-hand side of (B.4) gives the “buoyancy” and the second term gives the drag force.

We here suppose that the surface $S$ surrounds an obstacle $D_a (a \in L)$ and take the limit $V \to D_a$. Then the force $\lim_{V \to D_a} F_S$ gives the total force acting on the obstacle $D_a$ and is equal to zero since the obstacle does not move. It means that the drag force $D(D_1, D_2, D_3)$ on the obstacle $D_a$ and the force $(\text{vol} \, D_a) \, K$ are balanced, i.e.,

$$D = \int_{D_a} p^* n \, dS = (\text{vol} \, D_a) K. \quad (B.7)$$

Next we obtain an approximation of the drag force $D$ by using the approximate solution by the fundamental solution method. We put $p_N = p_N^* + K \cdot x$, i.e.,

$$p_N^* = \sum_{j=1}^N \tau(x - \xi_j) \cdot Q_j \quad (B.8)$$

and substitute $v \approx v_N, p^* \approx p_N^*$ into

$$D_1 = \int_{\partial D_a} \sum_{j=1}^3 p_{ij}^* n_j \, dS = - \int_{\partial D_a} p^* n_i \, dS + \mu \int_{\partial D_a} \sum_{j=1}^3 \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) n_j \, dS.$$

Using the Gauss formula, we have

$$D_1 \approx \int_{D_a} \left( \mu \Delta v_N - \frac{\partial p_N^*}{\partial x_i} + \mu \frac{\partial}{\partial x_i} (\nabla \cdot v_N) \right) \, dV,$$
i.e.,

\[ D \approx \int_{D_a} (\mu \Delta v_N - \nabla p_N^* + \mu \nabla (\nabla \cdot v_N)) \, dV \]

and, using the relations

\[ \mu \Delta v_N - \nabla p_N^* = 4\pi \mu \sum_{j=1}^{N} \sum_{a \in \mathcal{L}} Q_j \delta (x - \xi_j - a), \]

\[ \nabla \cdot v_N = 0, \]

we obtain the approximation

\[ D \approx D_N = 4\pi \mu \sum_{j=1}^{N} Q_j. \quad (B.9) \]

References