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Asymptotic Behaviour of Solutions to a Class of Semilinear Hyperbolic Systems in Arbitrary Domains

Frank Jochmann

Institut für angewandte Mathematik, Humboldt Universität Berlin, Unter den Linden 6, 10099 Berlin; and Weierstrass-Institut für angewandte Analysis und Stochastik, im Forschungsverbund Berlin e.v., Mohrenstr. 39, 10117 Berlin

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The subject of this paper is the long time asymptotic behavior of solutions of semilinear hyperbolic systems of the form

$$\partial_t \mathbf{E} = E^{(1)} \cdot \left[\left(\sum_{k=1}^{3} H_k^* \partial_k \mathbf{F} \right) - \mathbf{S}(t, x, \mathbf{E}, \mathbf{F}) \right] + \mathbf{G}^{(1)}, \tag{1.1}$$

$$\partial_t \mathbf{F} = E^{(2)} \cdot \sum_{k=1}^{3} H_k \partial_k \mathbf{E} + \mathbf{G}^{(2)}, \tag{1.2}$$

with the initial-condition

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \qquad \mathbf{F}(0, x) = \mathbf{F}_0(x). \tag{1.3}$$

Here $\mathbf{E} \in C([0, \infty), L^2(\Omega, \mathbb{R}^M))$ and $\mathbf{F} \in C([0, \infty), L^2(\Omega, \mathbb{R}^N))$ are the unknown functions depending on the time $t \geqslant 0$ and the space-variable $x \in \Omega$. $\mathbf{G}^{(1)} \in L^1((0, \infty), L^2(\Omega, \mathbb{R}^M))$ and $\mathbf{G}^{(2)} \in L^1((0, \infty), L^2(\Omega, \mathbb{R}^N))$ are prescribed functions.

The domain $\Omega \subset \mathbb{R}^3$ is arbitrary. $H_k \in \mathbb{R}^{N \times M}$ are constant matrices, $E^{(1)} \in L^\infty(\Omega, \mathbb{R}^{M \times M})$ and $E^{(2)} \in L^\infty(\Omega, \mathbb{R}^{N \times N})$ are positive symmetric variable matrices, which depend on the space-variable $x \in \Omega$ and satisfy $E^{(1)} = 1$ and $E^{(2)} = 1$ on $\Omega_0 =^{\text{def}} \Omega \backslash G$ with some subset $G \subset \Omega$.

The generally nonlinear function $\mathbf{S}\colon [0, \infty) \times \Omega \times \mathbb{R}^{M+N} \to \mathbb{R}^M$ satisfies

$$\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) = 0 \quad \text{for all} \quad x \in \Omega_0 = \Omega \backslash G \quad \text{and}$$
$$\mathbf{S}(t, x, 0) = 0 \quad \text{for all} \quad x \in \Omega,\ t \in (0, \infty).$$

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In particular the damping-term $\mathbf{S}(t, x, \mathbf{E}, \mathbf{F})$ is only present on a certain subset $G \subset \Omega$. The following dissipativity-assumption is imposed.

$$\mathbf{y}\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) \geqslant \gamma(x) \min\{|\mathbf{y}|^p, |\mathbf{y}|\} \qquad \text{for all} \quad t \geqslant 0,$$
$$\mathbf{y} \in \mathbb{R}^M, \quad \mathbf{z} \in \mathbb{R}^N, \quad x \in G.$$

Here $p \in [2, \infty)$ and $\gamma \in L^\infty(G)$ is a positive function on G, which does not necessarily have a uniform positive lower bound on G.

This means that $\mathbf{S}(t, x, \mathbf{y}, \mathbf{z})$ is allowed to be bounded as $|y| \to \infty$ and $|\mathbf{S}(t, x, \mathbf{y}, \mathbf{z})|$ behaves like $|y|^{p-1}$ for small $|y|$. In particular a linear damping-term $\mathbf{S}(t, x, \mathbf{E}, \mathbf{F}) = \sigma(t, x)\,\mathbf{E}$ with $\sigma \in L^\infty([0, \infty) \times G)$, $\sigma \geqslant 0$ is possible.

A domain $D(B) \subset L^2(\Omega, \mathbb{R}^{M+N})$ containing $C_0^\infty(\Omega, \mathbb{R}^{M+N})$ is chosen, such that the operartor

$$B(\mathbf{E}, \mathbf{F}) \stackrel{\text{def}}{=} \left(E^{(1)} \left[\sum_{k=1}^{3} H_k^* \partial_k \mathbf{F} \right], E^{(2)} \left[\sum_{k=1}^{3} H_k \partial_k \mathbf{E} \right] \right)$$

is skew-adjoint on $D(B)$, i.e., $B^* = -B$ with respect to a weighted scalar-product. The choice of $D(B)$ depends on the boundary conditions on $\partial\Omega$ supplementing (1.1)–(1.2).

A physically important example for this system are Maxwell's equations describing the propagation of the electromagnetic field

$$\varepsilon \partial_t \mathbf{E} = \operatorname{curl} \mathbf{H} - \mathbf{S}(t, x, \mathbf{E}, \mathbf{H}) - \mathbf{j} \qquad \text{and} \qquad \mu \partial_t \mathbf{H} = -\operatorname{curl} \mathbf{E}, \tag{1.4}$$

supplemented by the initial-boundary conditions

$$\vec{n} \wedge \mathbf{E} = 0 \quad \text{on} \quad (0, \infty) \times \Gamma_1, \qquad \vec{n} \wedge \mathbf{H} = 0 \quad \text{on} \quad (0, \infty) \times \Gamma_2, \tag{1.5}$$

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \qquad \mathbf{H}(0, x) = \mathbf{H}_0(x). \tag{1.6}$$

In (1.5) $\Gamma_1 \subset \partial\Omega$ and $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \backslash \Gamma_1$. $\mathbf{E}, \mathbf{H}$ denote the electric and magnetic field respectively which depend on the time $t \geqslant 0$ and the space-variable $x \in \Omega$, whereas $\mathbf{j} \in L^1((0, \infty), L^2(\Omega, \mathbb{R}^3))$ is a prescribed external current. The term $\mathbf{S}(t, x, \mathbf{E}, \mathbf{H})$ describes a possibly nonlinear resistor. The dielectric and magnetic susceptibilities $\varepsilon, \mu \in L^\infty(\Omega)$ are assumed to be uniformly positive.

For (1.4), (1.5) the operator B is defined in the space $X =^{\text{def}} L^2(\Omega, \mathbb{C}^6)$ by

$$B(\mathbf{E}, \mathbf{F}) \stackrel{\text{def}}{=} (\varepsilon^{-1} \operatorname{curl} \mathbf{F}, -\mu^{-1} \operatorname{curl} \mathbf{E}) \qquad \text{for} \quad (\mathbf{E}, \mathbf{F}) \in D(B) \stackrel{\text{def}}{=} W_E \times W_H.$$

Here W_H is the closure of $C_0^\infty(\mathbb{R}^3 \backslash \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{curl}(\Omega)$, where $H_{curl}(\Omega)$, is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with $\operatorname{curl} \mathbf{E} \in L^2(\Omega)$.

W_E denotes the set of all $\mathbf{E} \in H_{curl}(\Omega)$, such that

$$\int_\Omega \mathbf{E} \operatorname{curl} \mathbf{F} - \mathbf{F} \operatorname{curl} \mathbf{E} \, dx = 0 \qquad \text{for all} \quad \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1, see [8] and [9].

Another example for (1.1)–(1.2) is the first-order system corresponding to the initial-boundary-value-problem of the scalar wave-equation with nonlinear damping, for which the long-time behaviour in the case of a bounded domain has been investigated in [3, 4–6, 10, 14, and 17].

$$\partial_t^2 \varphi = \operatorname{div}(E \nabla \varphi) - S(x, \partial_t \varphi) \tag{1.7}$$

supplemented by the initial-boundary-onditions

$$\varphi = 0 \qquad \text{on} \quad (0, \infty) \times \partial\Omega \tag{1.8}$$

$$\varphi(0, x) = f_0(x) \qquad \text{and} \qquad \partial_t \varphi(0, x) = f_1(x) \tag{1.9}$$

for initial-data $f_0 \in \overset{0}{H}{}^1(\Omega)$ and $f_1 \in L^2(\Omega)$. Here $E \in L^\infty(\Omega, \mathbb{R}^{3 \times 3})$ is a symmetric matrix-valued function satisfying $E = 1$ on $\Omega_0 = \Omega \backslash G$.

Note that $\mathbf{u} \overset{\text{def}}{=} (\partial_t \varphi, E \nabla \varphi) \in C([0, \infty), L^2(\Omega, \mathbb{R}^4))$ solves the system

$$\partial_t \mathbf{u} = (\operatorname{div}(\mathbf{u}_2, ..., \mathbf{u}_4) - S(t, x, \mathbf{u}_1), E \nabla \mathbf{u}_1) \tag{1.10}$$

which is of the form (1.1)–(1.3).

The aim of this paper is to show that the solution $(\mathbf{E}, \mathbf{F})$ of (1.1)–(1.3) satisfies

$$(\mathbf{E}(t), \mathbf{F}(t)) \xrightarrow{t \to \infty} 0 \qquad \text{in } L^2(\Omega) \text{ weakly} \tag{1.11}$$

if and only if the initial-data $(\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)$ obey

$$\int_\Omega (E^{(1)-1} \tilde{\mathbf{E}}_0 \mathbf{e} + E^{(2)-1} \tilde{\mathbf{F}}_0 \mathbf{f}) \, dx = 0 \qquad \text{for all} \quad (\mathbf{e}, \mathbf{f}) \in \mathcal{N}. \tag{1.12}$$

Here

$$\tilde{\mathbf{E}}_0 \overset{\text{def}}{=} \mathbf{E}_0 + \int_0^\infty \mathbf{G}^{(1)} \, dt \qquad \text{and} \qquad \tilde{\mathbf{F}}_0 \overset{\text{def}}{=} \mathbf{F}_0 + \int_0^\infty \mathbf{G}^{(2)} \, dt$$

and $\mathcal{N} \subset L^2(\Omega, \mathbb{R}^{M+N})$ denotes the set of all $(\mathbf{E}, \mathbf{F}) \in \ker B$ with $\mathbf{E} = 0$ on G.

Furthermore it is shown that for arbitrary initial-states $(\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)$ the solution $(\mathbf{E}, \mathbf{F})$ of (1.1)–(1.3) converges weakly in $L^2(\Omega)$ to some element of $\mathcal{N}$ as $t \to \infty$.

It follows easily from the assumptions on $\mathbf{S}$ that $\mathcal{N}$ is the set of stationary states of the system (1.1)–(1.3) provided that $\mathbf{G}=0$.

In the case of Maxwell's equations (1.4)–(1.6) the condition (1.12) on $(\mathbf{E}_0, \mathbf{F}_0)$ implies

$$\operatorname{div}\left(\varepsilon \mathbf{E}_0 - \int_0^\infty \mathbf{j}\,dt\right)=0 \quad \text{on } \Omega_0 \qquad \text{and} \qquad \operatorname{div}(\mu \mathbf{H}_0)=0 \quad \text{on } \Omega \tag{1.13}$$

since $\mathcal{N}$ contains all elements of the form $(\nabla\varphi, \nabla\psi)$ with $\varphi \in C_0^\infty(\Omega_0)$ and $\psi \in C_0^\infty(\Omega)$.

If $\mathbf{S}$ is independent of t and monotone with respect to $\mathbf{E}$ strong L^r-convergence is shown, i.e.,

$$\|\mathbf{E}(t)\|_{L^r(K)} + \|\mathbf{F}(t)\|_{L^2(K)} \xrightarrow{t\to\infty} 0 \qquad \text{for all} \quad 1 \leqslant r < 2, \quad \text{and compact sets } K \subset \Omega \tag{1.14}$$

if the initial-data $(\mathbf{E}_0, \mathbf{F}_0) \in L^2(\Omega)$ obey condition (1.12).

Finally (1.11) is used to prove that the solution the wave-equation (1.7)–(1.8) in an arbitrary domain $\Omega \subset \mathbb{R}^3$ decays with respect to the energy-norm on each bounded subdomain of Ω. For all $R \in (0, \infty)$, $f_0 \in \overset{0}{H}{}^1(\Omega)$ and $f_1 \in L^2(\Omega)$ it is shown that

$$(\|\nabla\varphi(t)\|_{L^2(\Omega \cap B_R)} + \|\partial_t \varphi(t)\|_{L^2(\Omega \cap B_R)}) \xrightarrow{t\to\infty} 0.$$

The proof of (1.11) is based on a suitable modification of the approach in [4] for the case that the operator B does not necessarily have purely discrete spectrum. The basic idea is to show that for each $f \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and $\mathbf{g} \in \omega_0(\mathbf{E}_0, \mathbf{F}_0)$ the function $f(iB)\,\mathbf{g}$ is real-analytic and vanishes on G, where $\omega_0(\mathbf{E}_0, \mathbf{F}_0)$ denotes the ω-limit-set with respect to the weak topology of the orbit belonging to the initial-state $(\mathbf{E}_0, \mathbf{F}_0)$. This implies $f(iB)\,\mathbf{g}=0$ for all $f \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and hence $\mathbf{g} \in \ker B$. (Here the operator $f(iB)$ can be defined by the spectral-theorem, since iB is self-adjoint in $L^2(\Omega, \mathbb{C}^{M+N})$.)

In [14] it is shown that the solution of the scalar wave-equation in a bounded domain tends to zero weakly in the energy-space if $S(x, y) = a(x)\,g(y)$ obeys $\ker g \subset (-\infty, 0]$ or $\ker g \subset [0, \infty)$. The assumptions on the nonlinear damping-term have been further weakened in [5] where strong convergence is obtained in the case that Ω is a bounded one-dimensional interval. In [17] also decay-rates for the energy-norm are obtained, which depend on the behaviour of the damping term for y near zero.

In [4, 6, 14] the following unique-continuation-principle is used. Let $\Omega \subset \mathbb{R}^N$ be bounded and $u \in C([0, \infty), \overset{0}{H}{}^1(\Omega)) \cap C^1([0, \infty), L^2(\Omega))$ be a solution of the wave-equation $\partial_t^2 u = \Delta u$ on $[0, \infty) \times \Omega$ with the property that $u(t, x) = 0$ on $[0, \infty) \times E$ for some subset $E \subset \Omega$ with positive measure. Then $u = 0$ on all of $[0, \infty) \times \Omega$.

In this paper the following modification for not necessarily bounded domains is proved, see Theorem 1. Let $(\mathbf{e}, \mathbf{f}) \in C(\mathbb{R}, L^2(\Omega, \mathbb{R}^{M+N}))$ solve $\partial_t(\mathbf{e}, \mathbf{f}) = B(\mathbf{e}, \mathbf{f})$ with the property that $\mathbf{e}(t, x) = 0$ for all $t \in \mathbb{R}$ and $x \in G$. Then $(\mathbf{e}(0), \mathbf{f}(0)) \in \ker B$.

2. NOTATION, ASSUMPTIONS

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$.

Let $\Omega \subset \mathbb{R}^3$ be a (connected) domain and let $\Omega_0 \subset \Omega$ be an open subset, such that $G \overset{\text{def}}{=} \Omega \backslash \Omega_0$ has nonempty interior. The variable matrices $E^{(1)} \in L^\infty(\Omega, \mathbb{R}^{(M \times M)})$ and $E^{(2)} \in L^\infty(\Omega, \mathbb{R}^{(N \times N)})$ assumed to be symmetric and uniformly positive in the sense that

$$y^\perp \cdot E^{(1)}(x)\, y \geqslant c_0\, |y|^2 \qquad \text{and} \qquad z^\perp \cdot E^{(2)}(x)\, z \geqslant c_0\, |z|^2 \tag{2.15}$$

for all $x \in \Omega$, $y \in \mathbb{R}^M$ and $z \in \mathbb{R}^N$ with some $c_0 \in (0, \infty)$ independent of x, y, z.

Next,

$$E^{(1)}(x) = 1 \qquad \text{and} \qquad E^{(2)}(x) = 1 \quad \text{for all} \quad x \in \Omega_0. \tag{2.16}$$

The assumptions on $\mathbf{S}\colon [0, \infty) \times \Omega \times \mathbb{R}^{M+N} \to \mathbb{R}^M$ are the following.

$$\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) = 0 \qquad \text{if} \quad x \in \Omega_0 = \Omega \backslash G, \tag{2.17}$$

$$\mathbf{S}(\cdot, \cdot, \mathbf{y}, \mathbf{z}) \qquad \text{measurable for fixed} \quad \mathbf{y} \in \mathbb{R}^M, \quad \mathbf{z} \in \mathbb{R}^N \tag{2.18}$$

and Lipschitz-continuous, i.e., there exists $L \in (0, \infty)$, such that

$$|\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) - \mathbf{S}(t, x, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})| \leqslant L(|\mathbf{y} - \tilde{\mathbf{y}}| + |\mathbf{z} - \tilde{\mathbf{z}}|) \tag{2.19}$$

for all $\mathbf{y}, \tilde{\mathbf{y}} \in \mathbb{R}^M$, $\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{R}^N$ and $x \in \Omega$.

$$|\mathbf{S}(t, x, \mathbf{y}, \mathbf{z})|^2 \leqslant C_0 \mathbf{y} \cdot \mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) \tag{2.20}$$

for all $t \geqslant 0$, $x \in G$, $\mathbf{y} \in \mathbb{R}^M$, $\mathbf{z} \in \mathbb{R}^N$, with some $C_0 \in (0, \infty)$. Moreover,

$$\mathbf{y}\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) \geqslant \gamma(x) \min\{|\mathbf{y}|^p, |\mathbf{y}|\} \tag{2.21}$$

for all $t \geqslant 0$, $\mathbf{y} \in \mathbb{R}^M$, $\mathbf{z} \in \mathbb{R}^N$, $x \in G$.

Here $\gamma \in L^\infty(G)$ with $\gamma > 0$ and $p \in [2, \infty)$. The function γ does not necessarily have a uniform positive lower bound on G. It follows from the two latter assumptions that $\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) = 0$ if and only if $\mathbf{y} = 0$ for all $x \in G$.

In the sequel $L^q_\gamma(K)$ denotes for a measurable subset $K \subset G$ the weighted L^q-space endowed with the norm

$$\|u\|_{L^q_\gamma(K)} \overset{\text{def}}{=} \left(\int_K |u|^q \, \gamma \, dx \right)^{1/q}$$

where $q \in [1, \infty)$ and γ as in (2.21).

The matrices $H_j \in \mathbb{R}^{N \times M}$ obey the following algebraic condition, which is fulfilled in the examples (1.4)–(1.6) and (1.7)–(1.9).

$$\left(\sum_{k=1}^{3} \xi_k H_k \right) \left(\sum_{k=1}^{3} \xi_k H_k^* \right) \left(\sum_{k=1}^{3} \xi_k H_k \right) = |\xi|^2 \left(\sum_{k=1}^{3} \xi_k H_k \right) \qquad \text{for all} \quad \xi \in \mathbb{R}^3 \tag{2.22}$$

Let $W_0 \subset L^2(\Omega, \mathbb{C}^M)$ be the space of all $\mathbf{e} \in L^2(\Omega, \mathbb{C}^M)$ with $\sum_{k=1}^3 \partial_k(H_k \mathbf{e}) \in L^2(\Omega)$ in the sense of distributions endowed with the norm

$$\|\mathbf{e}\|^2_{W_0} \overset{\text{def}}{=} \|\mathbf{e}\|^2_{L^2} + \left\| \sum_{k=1}^{3} \partial_k(H_k \mathbf{e}) \right\|^2_{L^2}.$$

Furthermore, let $D(A)$ with $C_0^\infty(\Omega, \mathbb{C}^M) \subset D(A)$ be closed subspace of W_0 with respect to the above norm and

$$A\mathbf{e} \overset{\text{def}}{=} \sum_{k=1}^{3} \partial_k(H_k \mathbf{e}) \qquad \text{for} \quad \mathbf{e} \in D(A). \tag{2.23}$$

Then the adjoint operator A^* obeys $C_0^\infty(\Omega, \mathbb{C}^N) \subset D(A^*)$ and

$$A^* \mathbf{F} = -\sum_{k=1}^{3} \partial_k(H_k^* \mathbf{F}) \qquad \text{for all} \quad \mathbf{F} \in D(A^*). \tag{2.24}$$

For a vector $\mathbf{w} \in \mathbb{C}^{M+N}$ we denote by $\underline{\mathbf{w}}_1$ the first M and by $\underline{\mathbf{w}}_2$ the last N components of $\mathbf{w}$.

Now, the following operators are defined.

Let $D(B_0) \overset{\text{def}}{=} D(A) \times D(A^*)$ and

$$B_0 \mathbf{w} \overset{\text{def}}{=} (-A^* \underline{\mathbf{w}}_2, A \underline{\mathbf{w}}_1) \qquad \text{for} \quad \mathbf{w} \in D(B_0) = D(A) \times D(A^*).$$

Next, $B \overset{\text{def}}{=} EB_0$ with $E \overset{\text{def}}{=} \operatorname{diag}(E^{(1)}, E^{(2)})$, i.e., $D(B) \overset{\text{def}}{=} D(B_0)$ and

$$B\mathbf{w} \overset{\text{def}}{=} EB_0 \mathbf{w} = (-E^{(1)} A^* \underline{\mathbf{w}}_2, E^{(2)} A \underline{\mathbf{w}}_1) \tag{2.25}$$

for $\mathbf{w} \in D(B)$. It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X \overset{\text{def}}{=} L^2(\Omega, \mathbb{C}^{M+N})$ endowed with the scalar-product

$$\langle \mathbf{u}, \mathbf{v} \rangle_X \overset{\text{def}}{=} \int_\Omega E^{-1} \mathbf{u} \bar{\mathbf{v}} \, dx$$

This follows from the closedness of A, which implies that $A^{**} = \bar{A} = A$. (It is advantageous for following considerations to consider a complex space X. But whenever the term $\mathbf{S}(t, x, \mathbf{E}, \mathbf{F})$ occurs in an equation, the functions $\mathbf{E}$ and $\mathbf{F}$ are of course assumed to be real-valued.)

Now, let $\mathcal{N}$ be the set of all $\mathbf{a} \in \ker B$ with $\underline{\mathbf{a}}_1(x) = 0$ for all $x \in G$.

Moreover, let $X^0 \overset{\text{def}}{=} \mathcal{N}^\perp$ be the space of all $\mathbf{w} \in X$ with $\langle \mathbf{u}, \mathbf{w} \rangle_X = 0$ for all $\mathbf{u} \in \mathcal{N}$.

For $\mathbf{G} = (\mathbf{G}^{(1)}, \mathbf{G}^{(2)}) \in L^1((0, \infty), L^2(\Omega, \mathbb{R}^{M+N}))$ and $\mathbf{w} \in L^2(\Omega, \mathbb{R}^{M+N})$ a function $\mathbf{u} \in C([0, \infty), X)$ is called a weak soution to the problem (1.1)–(1.3), if

$$\frac{d}{dt} \langle \mathbf{u}(t), \mathbf{a} \rangle_X = -\langle \mathbf{u}(t), B\mathbf{a} \rangle_X + \langle \mathbf{G}(t) - F(t, \mathbf{u}(t)), \mathbf{a} \rangle_X \quad \text{for all} \quad \mathbf{a} \in D(B) \tag{2.26}$$

and $\mathbf{u}$ fulfilles the initial-condition.

Here $F\colon (0, \infty) \times X \to X$ is defined by

$$F(t, \mathbf{u}) \overset{\text{def}}{=} (E^{(1)} \mathbf{S}(t, \cdot, \mathbf{u}(\cdot)), 0).$$

(2.26) is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp(tB)\, \mathbf{w} + \int_0^t \exp((t-s)\, B)[\mathbf{G}(s) - F(s, \mathbf{u}(s))]\, ds \tag{2.27}$$

where $(\exp(tB))_{t \in \mathbb{R}}$ is the unitary group generated by B. Since $F(t, \cdot)$ is assumed to be Lipschitz-continuous in X by assumption (2.19), it follows from a standard result that this integal-equation has a unique solution $\mathbf{u} \in C([0, \infty), X)$, (see [11, chap. 7]).

(2.27) yields the energy-estimate

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{u}(t)\|_X^2 &= \langle \mathbf{G}(t) - F(t, \mathbf{u}(t)), \mathbf{u}(t) \rangle_X \\ &= \langle \mathbf{G}(t), \mathbf{u}(t) \rangle_X - \int_G \mathbf{S}(t, x, \mathbf{u}(t)) \cdot \underline{\mathbf{u}(t)}_1 \, dx \\ &\leqslant \langle \mathbf{G}(t), \mathbf{u}(t) \rangle_X. \end{aligned} \tag{2.28}$$

In the sequel $T(\cdot)\,\mathbf{w} \in C([0, \infty), X)$ denotes the unique solution to (1.1)–(1.3) in the sense of (2.26).

3. WEAK CONVERGENCE FOR $T \to \infty$

In the following lemma it is shown in particular that $T(\cdot)\,\mathbf{w} \in L^\infty((0, \infty), X)$, i.e., $\|T(t)\,\mathbf{w}\|_X$ is bounded as $t \to \infty$.

LEMMA 1. *Suppose* $\mathbf{w} \in X$ *and* $\mathbf{u}(t) \overset{\text{def}}{=} T(t)\,\mathbf{w}$. *Then*

$$\begin{aligned} \|\mathbf{u}(t)\|_X &\leqslant \|\mathbf{w}\|_X + \|\mathbf{G}\|_{L^1((0,\infty),X)}, \\ \int_0^\infty \langle \mathbf{u}(t), F(t, \mathbf{u}(t))\rangle_X\,dt &\leqslant (\|\mathbf{w}\|_X + \|\mathbf{G}\|_{L^1((0,\infty),X)})^2 \end{aligned} \tag{3.29}$$

and

$$\int_0^\infty \|F(t, \mathbf{u}(t))\|_X^2\,dt \leqslant C_0(\|\mathbf{w}\|_X + \|\mathbf{G}\|_{L^1((0,\infty),X)})^2$$

with some $C_0 \in (0, \infty)$ *independent of* $\mathbf{w}$. *Moreover,*

$$\underline{\mathbf{u}}_1 \in L^p((0, \infty), L_\gamma{}^1(K)) \qquad \text{for all bounded measurable subsets} \quad K \subset G. \tag{3.30}$$

Proof. Let $\mathbf{u}(t) = (\mathbf{E}(t), \mathbf{F}(t)) \overset{\text{def}}{=} T(t)\,\mathbf{w}$. By the assumptions (2.20) on $\mathbf{S}$ one has

$$\|F(t, \mathbf{f})\|_X^2 \leqslant C_0 \langle F(t, \mathbf{f}), \mathbf{f}\rangle_X \qquad \text{for all} \quad \mathbf{f} \in X$$

with some $C_0 > 0$ independent of $\mathbf{f}$. Therefore, the energy-estimate (2.28) yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\mathbf{u}(t)\|_X^2 &\leqslant \langle \mathbf{G}(t) - F(t, \mathbf{u}(t)), \mathbf{u}(t)\rangle_X \\ &\leqslant \|\mathbf{G}(t)\|_X \|\mathbf{u}(t)\|_X - \langle F(t, \mathbf{u}(t)), \mathbf{u}(t)\rangle_X \\ &\leqslant \|\mathbf{G}(t)\|_X \|\mathbf{u}(t)\|_X - C_0^{-1} \|F(t, \mathbf{u}(t))\|_X^2. \end{aligned}$$

This implies (3.29) by Gronwall's lemma.

To prove (3.30) let $\mathbf{f} \in X$ and define $\mathbf{a}, \mathbf{b} \in L^2(G, \mathbb{R}^M)$ by $\mathbf{a}(x) \overset{\text{def}}{=} \underline{\mathbf{f}}_1(x)$ if $|\underline{\mathbf{f}}_1(x)| \leqslant 1$ and $\mathbf{a}(x) \overset{\text{def}}{=} 0$ if $|\underline{\mathbf{f}}_1(x)| > 1$. Moreover, $\mathbf{b}(x) \overset{\text{def}}{=} \underline{\mathbf{f}}_1(x)$ if $|\underline{\mathbf{f}}_1(x)| > 1$ and $\mathbf{b}(x) \overset{\text{def}}{=} 0$ if $|\underline{\mathbf{f}}_1(x)| \leqslant 1$.

Then it follows from assumption (2.21) that

$$\mathbf{a}(x)\,\mathbf{S}(t, x, \mathbf{a}(x), \underline{\mathbf{f}}_2(x)) \geqslant \gamma(x)\,|\mathbf{a}(x)|^p \qquad \text{and}$$
$$\mathbf{b}(x)\,\mathbf{S}(t, x, \mathbf{b}(x), \underline{\mathbf{f}}_2(x)) \geqslant \gamma(x)\,|\mathbf{b}(x)|$$

for all $x \in G$. Hölder's inequality yields

$$\begin{aligned}
\|\underline{\mathbf{f}}_1\|_{L^1_\gamma(K)} &\leqslant \|\mathbf{a}\|_{L^1_\gamma(K)} + \|\mathbf{b}\|_{L^1_\gamma(K)}\\
&\leqslant C_{K,1}\,\|\mathbf{a}\|_{L^p_\gamma(K)} + \|\mathbf{b}\|_{L^1_\gamma(K)}\\
&= C_{K,1}\left(\int_G |\mathbf{a}(x)|^p\,\gamma\,dx\right)^{1/p} + \int_G |\mathbf{b}(x)|\,\gamma\,dx\\
&\leqslant C_{K,1}\left(\int_G \mathbf{a}(x)\,\mathbf{S}(t, x, \mathbf{a}(x), \underline{\mathbf{f}}_2(x))\,dx\right)^{1/p}\\
&\quad + \int_G \mathbf{b}(x)\,\mathbf{S}(t, x, \mathbf{b}(x), \underline{\mathbf{f}}_2(x))\,dx\\
&\leqslant C_{K,1}\left(\int_G \mathbf{f}(x)\,\mathbf{S}(t, x, \mathbf{f}(x))\,dx\right)^{1/p}\\
&\quad + \int_G \mathbf{f}(x)\,\mathbf{S}(t, x, \mathbf{f}(x))\,dx\\
&= C_{K,1}(\langle \mathbf{f}, F(t, \mathbf{f})\rangle_X)^{1/p} + \langle \mathbf{f}, F(t, \mathbf{f})\rangle_X\\
&\leqslant C_{K,2}(1 + \|\mathbf{f}\|_X^{2-2/p})(\langle \mathbf{f}, F(t, \mathbf{f})\rangle_X)^{1/p}
\end{aligned} \tag{3.31}$$

Finally, the assertion (3.30) follows from (3.29) and (3.31). ∎

Next some lemmata concerning the operator B are given.

Lemma 2. (i) *$\Delta\mathbf{w} = B_0^2\mathbf{w}$ on Ω for all $\mathbf{w} \in (\text{rang}\,B_0) \cap D(B_0^2)$, in particular $-\Delta\mathbf{e} = A^*A\mathbf{e}$ and $-\Delta\mathbf{f} = AA^*\mathbf{f}$ on Ω for all $\mathbf{e} \in (\text{rang}\,A^*) \cap D(A)$ and $\mathbf{f} \in (\text{rang}\,A) \cap D(A^*)$ with $A\mathbf{e} \in D(A^*)$ and $A^*\mathbf{f} \in D(A)$.*

(ii) *$\Delta\mathbf{w} = B^2\mathbf{w}$ on $\Omega_0 = \Omega \backslash G$ for all $\mathbf{w} \in X^0 \cap D(B^2)$.*

Proof. Let $\mathbf{u} \in C_0^\infty(\Omega, \mathbb{C}^{M+N}) \subset D(B_0^n)$ for all $n \in \mathbb{N}$. Then it follows from the algebraic condition (2.22) using Fourier-transform that

$$\begin{aligned}
\mathscr{F}\underline{(B_0^3\mathbf{u})}_1(\xi) &= -i\left(\sum_{j=1}^3 \xi_j H_j^*\right)\left(\sum_{k=1}^3 \xi_k H_k\right)\left(\sum_{l=1}^3 \xi_l H_l^*\right)\mathscr{F}(\underline{\mathbf{u}}_2)(\xi)\\
&= -i\,|\xi|^2\left(\sum_{l=1}^3 \xi_l H_l^*\right)\mathscr{F}(\underline{\mathbf{u}}_2)(\xi)
\end{aligned}$$

Analogously,

$$\underline{\mathcal{F}(B_0^3\mathbf{u})_2}(\xi) = -i\,|\xi|^2\left(\sum_{l=1}^{3}\xi_l H_l\right)\mathcal{F}(\underline{\mathbf{u}}_1)(\xi)$$

and hence

$$B_0^3\mathbf{u} = B_0\,\Delta\,\mathbf{u} \qquad \text{for all} \quad \mathbf{u}\in C_0^\infty(\Omega, \mathbb{C}^{M+N}). \tag{3.32}$$

Now, assume $\mathbf{w}\in(\text{rang } B_0)\cap D(B_0^2)$, i.e., $\mathbf{w}=B_0\mathbf{v}$ with some $\mathbf{v}\in D(B_0^3)$. Then

$$\int_\Omega (B_0^2\mathbf{w})\,\mathbf{u}\,dx = \langle B_0^3\mathbf{v}, \bar{\mathbf{u}}\rangle_{L^2} = -\langle \mathbf{v}, B_0^3\bar{\mathbf{u}}\rangle_{L^2}$$

$$= -\langle \mathbf{v}, B_0\,\Delta\,\bar{\mathbf{u}}\rangle_{L^2} = \langle \mathbf{w}, \Delta\,\bar{\mathbf{u}}\rangle_{L^2} = \int_\Omega \mathbf{w}\,\Delta\,\mathbf{u}\,dx$$

for all $\mathbf{u}\in C_0^\infty(\Omega)$, which means $B_0^2\mathbf{w}=\Delta\mathbf{w}$ in the sense of distributions.

To prove (ii) let $\mathbf{w}\in X^0\cap D(B^2)$. Suppose $\mathbf{u}\in C_0^\infty(\Omega_0, \mathbb{C}^{M+N})$, and define $\tilde{\mathbf{u}} \overset{\text{def}}{=} (B_0^2-\Delta)\,\mathbf{u}\in C_0^\infty(\Omega_0, \mathbb{C}^{M+N})\subset D(B_0^n)$. Then (3.32) yields $B_0\tilde{\mathbf{u}}=0$ and hence $\tilde{\mathbf{u}}\in\mathcal{N}$. In particular $0=\langle \mathbf{w}, \tilde{\mathbf{u}}\rangle_X$, because $\mathbf{w}\in X^0$. Since $E=1$ on Ω_0, it follows $B\mathbf{u}=B_0\mathbf{u}\in D(B)$ and $\tilde{\mathbf{u}}=(B^2-\Delta)\,\mathbf{u}$. Now,

$$0=\langle \mathbf{w}, \tilde{\mathbf{u}}\rangle_X = \langle \mathbf{w}, B^2\mathbf{u}\rangle_X - \langle \mathbf{w}, \Delta\,\mathbf{u}\rangle_X = \langle B^2\mathbf{w}, \mathbf{u}\rangle_X - \langle \mathbf{w}, \Delta\,\mathbf{u}\rangle_X$$

$$= \int_\Omega ([B^2\mathbf{w}]\,\bar{\mathbf{u}} - \mathbf{w}\,\Delta\,\bar{\mathbf{u}})\,dx$$

Since $\mathbf{u}\in C_0^\infty(\Omega_0, \mathbb{C}^{M+N})$ is arbitrary, the assertion follows. ∎

Remark 1. Due to the facts that generally $E^{(j)}\neq 1$ and $\underline{\mathbf{a}}_1=0$ on G for all $\mathbf{a}\in\mathcal{N}$ we have $\Delta\,\underline{\mathbf{w}}_1\neq\underline{(B^2\mathbf{w})}_1$ on G for all $\mathbf{w}\in X^0\cap D(B^2)$ in general.

For example is the case of Maxwell's Eqs. (1.4)–(1.6) all $\mathbf{w}\in X^0\cap D(B^2)$ obey $\underline{(B^2\mathbf{w})}_1 = -\varepsilon^{-1}\,\text{curl}(\mu^{-1}\,\text{curl}\,\underline{\mathbf{w}}_1)$. The condition $\mathbf{w}\in X^0$ implies $\text{div}(\varepsilon\underline{\mathbf{w}}_1)=0$ on Ω_0 and $\text{div}(\mu\underline{\mathbf{w}}_2)=0$ on Ω, as mentioned in the introduction, but it does not provide any information on the divergence of $\underline{\mathbf{w}}_1$ on the set G, since $\underline{\mathbf{a}}_1=0$ on G for all $\mathbf{a}\in\mathcal{N}$.

The next theorem is the generalization of the unique-continuation-principle in [4] and [6] as mentioned in the introduction.

THEOREM 1. *Let* $\mathbf{g}\in X$ *with the property*

$$\underline{(\exp(tB)\,\mathbf{g})}_1 = 0 \qquad \text{on } G \text{ for all} \quad t\in\mathbb{R}. \tag{3.33}$$

Then $\mathbf{g}\in\mathcal{N}\subset\ker B$.

Proof. Since iB is self-adjoint in X, $f(iB)=\int_{\mathbb{R}} f(\lambda)\, dE_\lambda$ can be defined by the spectral-theorem for a Borel-measurable function $f\colon \mathbb{R}\to\mathbb{C}$. Here $(E_\lambda)_{\lambda\in\mathbb{R}}$ denotes the family of spectral-projectors of iB. If $f\in C_0^\infty(\mathbb{R})$, then bounded operator $f(iB)$ has the representation

$$f(iB)\,\mathbf{u}=(2\pi)^{-1/2}\int_{\mathbb{R}} \hat{f}(t)\exp(-tB)\,\mathbf{u}\,dt \qquad \text{for all} \quad \mathbf{u}\in X. \tag{3.34}$$

Here $\hat{f}$ denotes the Fourier-transform of f. To see this let $\mathbf{u},\mathbf{v}\in X$. Then

$$\begin{aligned}\langle f(iB)\,\mathbf{u},\mathbf{v}\rangle_X &= \int_{\mathbb{R}} f(\lambda)\, d\langle E_\lambda \mathbf{u},\mathbf{v}\rangle_X \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(t)\exp(it\lambda)\,dt\,d\langle E_\lambda\mathbf{u},\mathbf{v}\rangle_X \\ &= (2\pi)^{-1/2}\int_{\mathbb{R}} \hat{f}(t)\langle \exp(-tB)\,\mathbf{u},\mathbf{v}\rangle_X\,dt\end{aligned}$$

Suppose $f\in C_0^\infty(\mathbb{R}\backslash\{0\})$. Then (3.33) and (3.34) yield

$$\underline{(f(iB)\,\mathbf{g})}_1=0 \qquad \text{on } G. \tag{3.35}$$

Moreover,

$$\tilde{f}(iB)\,\mathbf{g}=iBf(iB)\,\mathbf{g}=i(-E^{(1)}A^*\,\underline{(f(iB)\,\mathbf{g})}_2,\, E^{(2)}A\,\underline{(f(iB)\,\mathbf{g})}_1) \qquad \text{on } \Omega, \tag{3.36}$$

where $\tilde{f}(\lambda)=\lambda f(\lambda)$. In particular (3.35) and (3.36) yield by replacing f by $g(\lambda)\overset{\text{def}}{=}\lambda^{-1}f(\lambda)\in C_0^\infty(\mathbb{R}\backslash\{0\})$ that

$$\underline{(f(iB)\,\mathbf{g})}_2=iE^{(2)}A\,\underline{(g(iB)\,\mathbf{g})}_1=0 \qquad \text{on } G$$

and hence by (3.35)

$$f(iB)\,\mathbf{g}=0 \qquad \text{on } G \tag{3.37}$$

Since $E(x)=1$ on $\Omega\backslash G$, (3.35)–(3.37) yield

$$B_0 f(iB)\,\mathbf{g}=B(f(iB)\,\mathbf{g})=-i\tilde{f}(iB)\,\mathbf{g} \qquad \text{for all} \quad f\in C_0^\infty(\mathbb{R}\backslash\{0\}) \tag{3.38}$$

with $\tilde{f}(\lambda)=\lambda f(\lambda)$.

In particular it follows by induction

$$f(iB)\,\mathbf{g}\in(\operatorname{rang} B_0)\cap D(B_0^n) \qquad \text{with} \quad B_0^n f(iB)\,\mathbf{g}=B^n(f(iB)\,\mathbf{g}) \tag{3.39}$$

for all $f\in C_0^\infty(\mathbb{R}\backslash\{0\})$ and $n\in\mathbb{N}$.

The aim of the following considerations is to show that $f(iB)\,\mathbf{g}$ is real analytic on Ω. This will be achieved by means of a local integral representation.

Let $f \in C_0^\infty(\mathbb{R}\backslash\{0\})$ and choose $\chi \in C_0^\infty(\mathbb{R}\backslash\{0\})$ with $\chi(\lambda) = 1$ on supp f. Define

$$\mathbf{F}(t) \stackrel{\text{def}}{=} \exp(-tB)\,\chi(iB)\,\mathbf{g} = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\chi}(\xi) \exp((-t-\xi)\,B)\,\mathbf{g}\,d\xi.$$

Then (3.39) and Lemma 2(i) yield

$$\partial_t^2 \mathbf{F}(t) = B^2 \mathbf{F}(t) = B_0^2 \mathbf{F}(t) = \Delta \mathbf{F}, \tag{3.40}$$

in particular

$$\partial_t^j \Delta^k \mathbf{F} = (-1)^j B^{j+2k} \mathbf{F}(\cdot) \in L^\infty(\mathbb{R}, L^2(\Omega)) \quad \text{for all} \quad j \in \mathbb{N} \quad \text{and} \quad k \in \mathbb{N},$$

which implies $\mathbf{F} \in C^\infty(\mathbb{R} \times \Omega)$ and

$$\partial_t^j \partial^\alpha \mathbf{F} \in L^\infty(\mathbb{R} \times \mathscr{K}) \qquad \text{for all compact} \quad \mathscr{K} \subset \Omega, \quad j \in \mathbb{N}_0 \quad \text{and} \quad \alpha \in \mathbb{N}_0^3. \tag{3.41}$$

Suppose $x_0 \in \Omega$ and choose $R > 0$ with $B_{2R}(x_0) \subset \Omega$. Let

$$K(x, \xi) \stackrel{\text{def}}{=} (4\pi\,|x|)^{-1} \hat{f}(\xi - |x|) \qquad \text{for} \quad \xi \in \mathbb{R} \quad \text{and} \quad x \in \mathbb{R}^3$$

Then (3.41) yields for all $x \in B_{R/2}(x_0)$

$$\begin{aligned}
&\lim_{r \to 0} \int_{\mathbb{R}} \int_{\partial B_r(x)} \vec{n}(y)[K(x-y, \xi)\,\nabla_y \mathbf{F}_j(\xi, y) \\
&\qquad - \mathbf{F}_j(\xi, y)\,\nabla_y K(x-y, \xi)]\,dS(y)\,d\xi \\
&\quad = (4\pi)^{-1} \lim_{r \to 0} \left(r^{-3} \int_{\mathbb{R}} \hat{f}(\xi - r) \int_{\partial B_r(x)} [\vec{n}(y)(x-y)]\,\mathbf{F}_j(\xi, y)\,dS(y)\,d\xi \right) \\
&\quad = \int_{\mathbb{R}} \hat{f}(\xi)\,\mathbf{F}_j(\xi, x)\,d\xi \\
&\quad = \int_{\mathbb{R}} \hat{f}(\xi)(\exp(-\xi B)\,\chi(iB)\,\mathbf{g})_j\,(x)\,d\xi \\
&\quad = (2\pi)^{1/2}\,(f(iB)\,\chi(iB)\,\mathbf{g})_j(x) \\
&\quad = (2\pi)^{1/2}\,(f(iB)\,\mathbf{g})_j\,(x).
\end{aligned} \tag{3.42}$$

For all $x \in B_{R/2}(x_0)$ and all $y \in B_{2R}(x_0)$ with $y \neq x$ one has by (3.40)

$$\begin{aligned}
&\operatorname{div}_y[K(x-y,\xi)\,\nabla_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\,\nabla_y K(x-y,\xi)] \\
&\quad = K(x-y,\xi)\,\Delta_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\,\Delta_y K(x-y,\xi) \\
&\quad = K(x-y,\xi)\,\partial_\xi{}^2 \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\,\partial_\xi{}^2 K(x-y,\xi) \\
&\quad = \partial_\xi[K(x-y,\xi)\,\partial_\xi \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\,\partial_\xi K(x-y,\xi)]
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\partial B_R(x_0)} \vec{n}(y)[K(x-y,\xi)\,\nabla_y \mathbf{F}_j(\xi,y) - \mathbf{F}_j(\xi,y)\,\nabla_y K(x-y,\xi)]\,dS(y)\,d\xi \\
&\qquad - \int_{\mathbb{R}} \int_{\partial B_r(x)} \vec{n}(y)[K(x-y,\xi)\,\nabla_y \mathbf{F}_j(\xi,y) \\
&\qquad - \mathbf{F}_j(\xi,y)\,\nabla_y K(x-y,\xi)]\,dS(y)\,d\xi \\
&\quad = \int_{\mathbb{R}} \int_{B_R(x_0)\backslash B_r(x)} \operatorname{div}_y[K(x-y,\xi)\,\nabla_y \mathbf{F}_j(\xi,y) \\
&\qquad - \mathbf{F}_j(\xi,y)\,\nabla_y K(x-y,\xi)]\,dy\,d\xi \\
&\quad = \int_{B_R(x_0)\backslash B_r(x)} \int_{\mathbb{R}} \partial_\xi[K(x-y,\xi)\,\partial_\xi \mathbf{F}_j(\xi,y) \\
&\qquad - \mathbf{F}_j(\xi,y)\,\partial_\xi K(x-y,\xi)]\,d\xi\,dy = 0,
\end{aligned} \tag{3.43}$$

since $K(x-y,\xi) \xrightarrow{|\xi| \to \infty} 0$ and $\partial_\xi K(x-y,\xi) \xrightarrow{|\xi| \to \infty} 0$, whereas $\mathbf{F}$ and $\partial_\xi \mathbf{F}$ remain bounded as $|\xi| \to \infty$ by (3.41) for fixed $y \neq x$.

Now, (3.42) and (3.43) yield for all $x \in B_{R/2}(x_0)$

$$\begin{aligned}
(2\pi)^{1/2}\,(f(iB)\,\mathbf{g})_j(x) = \int_{\mathbb{R}} \int_{\partial B_R(x_0)} &\vec{n}(y)[K(x-y,\xi)\,\nabla_y \mathbf{F}_j(\xi,y) \\
&- \mathbf{F}_j(\xi,y)\,\nabla_y K(x-y,\xi)]\,dS(y)\,d\xi
\end{aligned} \tag{3.44}$$

Since $f \in C_0^\infty(\mathbb{R})$, there exists a constant $C_1 \in (0,\infty)$ with

$$(1+\xi^2)\,|\hat{f}^{(k)}(\xi)| \leqslant C_1^k \qquad \text{for all} \quad \xi \in \mathbb{R} \quad \text{and} \quad k \in \mathbb{N}.$$

Hence there exists a constant $C_2 \in (0, \infty)$ with

$$\int_{\mathbb{R}} \int_{\partial B_R(x_0)} \left(\left| \frac{d^k}{d\tau^k} K(x_0 + \tau\eta - y, \xi) \right| + \left| \frac{d^k}{d\tau^k} (\vec{n}(y) \nabla_y K(x_0 + \tau\eta - y, \xi)) \right| \right) dS(y)\, d\xi \leqslant C_2^k k!\, |\eta|^k$$

for all $\eta \in \mathbb{R}^3$ with $|\eta| \leqslant R/2$, $\tau \in (-1, 1)$ and $k \in \mathbb{N}$. Now it follows from (3.41) and (3.44) and the previous estimate that there exists a constant $C_3 \in (0, \infty)$ with

$$\left| \frac{d^k}{d\tau^k} (f(iB)\, \mathbf{g})(x_0 + \tau\eta) \right| \leqslant (C_3\, |\eta|)^k\, k!$$

for all $\eta \in \mathbb{R}^3$ with $|\eta| \leqslant R/2$, $\tau \in (-1, 1)$ and $k \in \mathbb{N}$, which yields the analycity of $f(iB)\, \mathbf{g}$.

Next this analycity yields by (3.37) and the assumptions that G has nonempty interior and Ω is connected that

$$f(iB)\, \mathbf{g} = 0 \qquad \text{for all} \quad f \in C_0^\infty(\mathbb{R} \backslash \{0\}). \tag{3.45}$$

Choose a sequence $f_n \in C_0^\infty(\mathbb{R} \backslash \{0\})$, $n \in \mathbb{N}$ with $|f_n(\lambda)| \leqslant 1$ and $f_n(\lambda) \xrightarrow{n \to \infty} 1$ for all $\lambda \in \mathbb{R} \backslash \{0\}$.

By the spectral-theorem (3.45) implies

$$0 = \langle f_n(iB)\, \mathbf{g}, \mathbf{g} \rangle_X \xrightarrow{n \to \infty} \langle (1 - P_{ker\, B})\, \mathbf{g}, \mathbf{g} \rangle_X$$

and hence $\mathbf{g} = P_{ker\, B} \mathbf{g} \in \ker B$. Together with (3.33) this yields $\mathbf{g} \in \mathcal{N}$, which completes the proof. ∎

Remark 2. In [7], Chap. VIII the following result can be found (Theorem 8.6.8), which is a consequence of Holmgren's uniqueness-theorem:

Let $X_1, X_2 \subset \mathbb{R}^N$ open and convex with $X_1 \subset X_2$. Let L be a differential operator with constant coefficients. Then the following conditions are equivalent:

(i) All $u \in \mathcal{D}'(X_2)$ with $Lu = 0$ on X_2 and $u = 0$ on X_1 are identically zero on all of X_2.

(ii) Every hyperplane which is characteristic with respect to L and intersects X_2 also intersects X_1.

This can be used in the proof of the previous theorem as follows. Let $\chi \in C_0^\infty(\mathbb{R}\setminus\{0\})$ and define

$$\mathbf{F}(t) \stackrel{\text{def}}{=} \exp(-tB)\,\chi(iB)\,\mathbf{g} = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{\chi}(\xi) \exp((-t-\xi)\,B)\,\mathbf{g}\,d\xi.$$

As above it follows from (3.37), (3.39) and Lemma 2(i) that $\mathbf{F} \in C^\infty(\mathbb{R}\times\Omega)$ solves the scalar wave-equation (3.40) and vanishes on the subset $\mathbb{R}\times G$. In order to apply Theorem 8.6.8 in [7] define U as the set of all $x \in \Omega$, such that there exists a neigbourhood $\mathscr{B}$ of x with $\mathbf{F}=0$ on $\mathbb{R}\times\mathscr{B}$. The aim of the following considerations is to show $U=\Omega$, in particular $\mathbf{F}$ is identically zero.

By (3.37) and the assumption that G has nonempty interior there exists some $x_0 \in G$ with this property, in particular $U \neq \emptyset$. Since U is open and Ω is connected, it suffices to show that U is relatively closed in Ω. Suppose $x_1 \in \Omega \cap \bar{U}$ and choose $R>0$ with $B_R(x_1) \subset \Omega$. Then one can find $y \in B_R(x_1) \cap U$ and $r>0$ with $B_r(y) \subset B_R(x_1)$ and $\mathbf{F}=0$ on $X_1 \stackrel{\text{def}}{=} \mathbb{R}\times B_r(y)$. Now every hyperplane, which is characteristic with respect to the wave-operator intersects X_1. Therefore Theorem 8.6.8 in [7] asserts that $\mathbf{F}=0$ on $X_2 \stackrel{\text{def}}{=} \mathbb{R}\times B_R(x_1)$, in particular $x_1 \in U$, which completes the proof of Theorem 1 with the aid of Theorem 8.6.8 in [7].

However the proof of Theorem 1 given in this paper is independent of Holmgren's theorem.

Remark 3. The proof of Theorem 1 can be simplyfied further under the additional assumption that

$$\overline{\Omega_0} \subset \Omega \tag{3.46}$$

Suppose that $\mathbf{g} \in X$ satifies the assumption in Theorem 1. As above one has for all $f \in C_0^\infty(\mathbb{R}\setminus\{0\})$

$$f(iB)\,\mathbf{g} = 0 \qquad \text{on } G \tag{3.47}$$

and $f(iB)\,\mathbf{g}$ satisfies (3.39).

Next it is shown that $f(iB)\,\mathbf{g}$ is real analytic on Ω. Lemma 2(i) and (3.39) yield

$$B^2 f(iB)\,\mathbf{g} = B_0^2 f(iB)\,\mathbf{g}(t) = \Delta f(iB)\,\mathbf{g}, \tag{3.48}$$

By induction it follows

$$(1-\Delta)^n\, f(iB)\,\mathbf{g} = (1-B^2)^n\, f(iB)\,\mathbf{g} = \int_{\mathbb{R}} (1+\lambda^2)^k\, f(\lambda)\, dE_\lambda \mathbf{g} \in L^2(\Omega) \tag{3.49}$$

and hence

$$\|(1-\Delta)^n f(iB)\,\mathbf{g}\|_X = \|(1-B^2)^n f(iB)\|_X \leqslant \sup_{\lambda\in\mathbb{R}} ((1+\lambda^2)^n\,|f(\lambda)|)\,\|\mathbf{g}\|_X \leqslant C_1^n \tag{3.50}$$

for all $n\in\mathbb{N}$ with some constant $C_1\in(0,\infty)$ independent of n.

Let $\mathbf{F}\in L^2(\mathbb{R}^3)$ be the extension of $f(iB)\,\mathbf{g}$ by zero defined by $\mathbf{F}(x)\overset{\text{def}}{=}(f(iB)\,\mathbf{g})(x)$ if $x\in\Omega$ and $\mathbf{F}(x)=0$ if $x\in\mathbb{R}^3\backslash\Omega$. Since $\mathbf{F}(x)=0$ for all $x\in G=\Omega\backslash\Omega_0$ by (3.47) the support of $\mathbf{F}$ is contained in the closed subset $\overline{\Omega_0}\subset\Omega$ by assumption (3.46). Now, it follows easily from (3.48)–(3.50) that $(1-\Delta)^n\,\mathbf{F}\in L^2(\mathbb{R}^3)$ and

$$\|(1-\Delta)^n\,\mathbf{F}\|_{L^2(\mathbb{R}^3)} \leqslant \|(1-\Delta)^n f(iB)\,\mathbf{g}\|_X \leqslant C_1^n \quad \text{for all} \quad n\in\mathbb{N}. \tag{3.51}$$

This yields by Sobolev's embedding-theorem $\mathbf{F}\in C^\infty(\mathbb{R}^3)$ and

$$\begin{aligned}\|\partial^\alpha\mathbf{F}\|_{L^\infty} &\leqslant C\,\|\partial^\alpha\mathbf{F}\|_{H^2(\mathbb{R}^3)} = C\,\|(1+\xi^2)\,\xi^\alpha\,\hat{\mathbf{F}}\|_{L^2(\mathbb{R}^3)}\\ &\leqslant C\,\|(1+\xi^2)^{n+1}\,\hat{\mathbf{F}}\|_{L^2(\mathbb{R}^3)} = C\,\|(1-\Delta)^{n+1}\,\mathbf{F}\|_{L^2(\mathbb{R}^3)}\\ &\leqslant C_1^{n+1}\end{aligned} \tag{3.52}$$

for all $n\in\mathbb{N}$ and $|\alpha|\leqslant 2n$ with $C_1\in(0,\infty)$ as in (3.51), which yields the analycity of $\mathbf{F}$. Since $\mathbf{F}(x)=0$ for all $x\in G$, this analycity implies $\mathbf{F}=0$ on all of $\mathbb{R}^3$ and hence (3.45)

In the sequel let $\omega_0(\mathbf{w})$ denote the ω-limit-set of the solution $T(\cdot)\,\mathbf{w}$ with respect to the weak topology of X, i.e., the set of all $\mathbf{g}\in X$, such that there exists a sequence $t_n \xrightarrow{n\to\infty} \infty$ with $T(t_n)\,\mathbf{w} \xrightarrow{n\to\infty} \mathbf{g}$ in X weakly, that means with $\langle T(t_n)\,\mathbf{w},\mathbf{f}\rangle_X \xrightarrow{n\to\infty} \langle\mathbf{g},\mathbf{f}\rangle_X$ for all $\mathbf{f}\in X$.

Since the $T(\cdot)\,\mathbf{w}\in L^\infty((0,\infty),X)$ by Lemma 1 the weak ω-limit-set $\omega_0(\mathbf{w})$ in nonempty for all $\mathbf{w}\in X$.

THEOREM 2. *Let* $\mathbf{w}\in X$. *Then* $\omega_0(\mathbf{w})\subset\mathcal{N}$.

Proof. Let $\mathbf{u}(t)\overset{\text{def}}{=}T(t)\,\mathbf{w}$ for $t\in\mathbb{R}$. Suppose $\mathbf{g}\in X$ and $t_n \xrightarrow{n\to\infty} \infty$ with $T(t_n)\,\mathbf{w} \xrightarrow{n\to\infty} \mathbf{g}$ in X weakly. Let $t\in\mathbb{R}$. By (2.27) one has

$$\mathbf{u}(t_n+t)=\exp(tB)\,\mathbf{u}(t_n)+\int_{t_n}^{t_n+t}\exp((t_n+t-\tau)\,B)[\mathbf{G}(\tau)-F(\tau,\mathbf{u}(\tau))]\,d\tau$$

for all sufficiently large $n\in\mathbb{N}$, such that $t_n+t\geqslant 0$. (In order to apply Theorem 1 it is necessary also to consider $t\leqslant 0$.) With $\mathbf{G}\in L^1((0,\infty),X)$ it follows from Lemma 1, (3.29) that

$$\begin{aligned}
&\|\mathbf{u}(t_n+t)-\exp(tB)\,\mathbf{u}(t_n)\|_X \\
&\qquad \leqslant \int_{[t_n,\,t_n+t]} (\|\mathbf{G}(\tau)\|_X + \|F(\tau,\mathbf{u}(\tau))\|_X)\,d\tau \\
&\qquad \leqslant \int_{[t_n,\,t_n+t]} \|\mathbf{G}(\tau)\|_X\,d\tau + |t|^{1/2} \\
&\qquad\quad \times \left(\int_{[t_n,\,t_n+t]} \|F(\tau,\mathbf{u}(\tau))\|_X^2\,d\tau\right)^{1/2} \xrightarrow{\mathrm{n}\to\infty} 0
\end{aligned}$$

and hence

$$\mathbf{u}(t_n+t) \xrightarrow{\mathrm{n}\to\infty} \exp(tB)\,\mathbf{g} \qquad \text{in } X \text{ weakly for all} \quad t\in\mathbb{R}. \tag{3.53}$$

Suppose $a, b\in\mathbb{R}$ with $a<b$ and define $\mathbf{f} \overset{\text{def}}{=} \int_a^b \exp(tB)\,\mathbf{g}\,dt$ and $\mathbf{f}^{(n)} \overset{\text{def}}{=} \int_a^b \mathbf{u}(t_n+t)\,dt$ for $n\in\mathbb{N}$ sufficiently large, such that $t_n+a\geqslant 0$. Then (3.53) yields by the dominated convergence-theorem

$$\begin{aligned}
\langle \mathbf{f}^{(n)}, \mathbf{h}\rangle_X &= \int_a^b \langle \mathbf{u}(t_n+t), \mathbf{h}\rangle_X\,dt \\
&\xrightarrow{\mathrm{n}\to\infty} \int_a^b \langle \exp(tB)\,\mathbf{g}, \mathbf{h}\rangle_X\,dt \\
&= \langle \mathbf{f}, \mathbf{h}\rangle_X
\end{aligned}$$

for all $\mathbf{h}\in X$, i.e., $\mathbf{f}^{(n)} \xrightarrow{\mathrm{n}\to\infty} \mathbf{f}$ weakly. In particular

$$\underline{\mathbf{f}^{(n)}}_1 \xrightarrow{\mathrm{n}\to\infty} \underline{\mathbf{f}}_1 \qquad \text{in} \quad L^2(G)\subset L^1_\gamma(K) \quad \text{weakly for all bounded} \quad K\subset G. \tag{3.54}$$

On the other hand it follows from Lemma 1(iii) that

$$\|\underline{\mathbf{f}^{(n)}}_1\|_{L^1_\gamma(K)} \leqslant (b-a)^{1/p^*} \left(\int_{a+t_n}^{b+t_n} \|\underline{\mathbf{u}(t)}_1\|^p_{L^1_\gamma(K)}\,dt\right)^{1/p} \xrightarrow{\mathrm{n}\to\infty} 0 \tag{3.55}$$

for all $t\in\mathbb{R}$. Now (3.54) and (3.55) yield

$$\int_a^b \underline{(\exp(tB)\,\mathbf{g})}_1\,dt = 0 \qquad \text{on } K \text{ for all bounded} \quad K\subset G$$
$$\text{and all} \quad a, b\in\mathbb{R}, \quad a<b.$$

This implies that $\mathbf{g}$ obeys condition (3.33) of Theorem 1. Hence $\mathbf{g}\in\mathcal{N}$. ▮

Let P be the orthogonal-projector on $\mathcal{N}$ in X.

LEMMA 3. *Suppose* $\mathbf{w} \in X$. *Then* $\|PT(t)\,\mathbf{w} - P(\mathbf{w} + \int_0^\infty \mathbf{G}(t)\,dt)\|_X \xrightarrow{t \to \infty} 0$.

Proof. Suppose $\mathbf{w} \in X$ and $\mathbf{a} \in \mathcal{N}$, that means $\mathbf{a} \in \ker B$ and $\underline{\mathbf{a}}_1 = 0$ on G. Then (2.27) yields

$$\begin{aligned}
\langle PT(t)\,\mathbf{w}, \mathbf{a}\rangle_X &= \langle T(t)\,\mathbf{w}, \mathbf{a}\rangle_X \\
&= \left\langle \exp(tB)\,\mathbf{w} + \int_0^t \exp((t-s)\,B)(\mathbf{G}(s)\right. \\
&\qquad \left. - F(s, T(s)\,\mathbf{w}))\,ds, \mathbf{a}\right\rangle_X \\
&= \langle \mathbf{w}, \exp(-tB)\,\mathbf{a}\rangle_X \\
&\qquad + \int_0^t \langle \mathbf{G}(s) - F(s, T(s)\,\mathbf{w}), \exp((s-t)\,B)\,\mathbf{a}\rangle_X\,ds \\
&= \langle \mathbf{w}, \mathbf{a}\rangle_X + \int_0^t \langle \mathbf{G}(s) - F(s, T(s)\,\mathbf{w}), \mathbf{a}\rangle_X\,ds \\
&= \langle \mathbf{w}, \mathbf{a}\rangle_X + \int_0^t \langle \mathbf{G}(s), \mathbf{a}\rangle_X\,ds \\
&= \left\langle P(\mathbf{w} + \int_0^t \mathbf{G}(s)\,ds), \mathbf{a}\right\rangle_X.
\end{aligned}$$

Hence

$$PT(t)\,\mathbf{w} = P\left(\mathbf{w} + \int_0^t \mathbf{G}(s)\,ds\right). \tag{3.56}$$

With $\mathbf{G} \in L^1(0, \infty, X)$ the assertion follows. ∎

Now, the main theorem concerning weak convergence can be proved.

THEOREM 3. *Suppose* $\mathbf{w} \in X$.
Then $T(t)\,\mathbf{w} \xrightarrow{t \to \infty} P(\mathbf{w} + \int_0^\infty \mathbf{G}(t)\,dt)$ *in* X *weakly.*

Proof. By Lemma 3 one has for all $\mathbf{g} \in \omega_0(\mathbf{w})$

$$P\mathbf{g} = P\left(\mathbf{w} + \int_0^\infty \mathbf{G}(s)\,ds\right).$$

On the other hand Theorem 2 yields $\mathbf{g} \in \mathcal{N}$ and hence

$$\mathbf{g} = P\mathbf{g} = P\left(\mathbf{w} + \int_0^\infty \mathbf{G}(s)\, ds\right) \qquad \text{for all} \quad \mathbf{g} \in \omega_0(\mathbf{w}). \tag{3.57}$$

Now it follows from (3.57) that

$$\omega_0(\mathbf{w}) \subset \left\{P\left(\mathbf{w} + \int_0^\infty \mathbf{G}(s)\, ds\right)\right\}. \tag{3.58}$$

Since the orbit $\{T(t)\,\mathbf{w}: t \geqslant 0\}$ is precompact in the weak topology by Lemma 1(i), this completes the proof. ▮

In particular it follows from the previous theorem that $T(t)\,\mathbf{w} \xrightarrow{t \to \infty} 0$ in X weakly if and only if $\mathbf{w} + \int_0^\infty \mathbf{G}(t)\, dt \in X^0 = \mathcal{N}^\perp$, which is condition 1.12.

4. STRONG L^Q-CONVERGENCE OF SOLUTIONS

The aim of the following considerations is find sufficient conditions for strong convergence. Assume that in addition $\mathbf{S}(t, x, \mathbf{y}, \mathbf{z})$ is independent of t, i.e., $\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) = \mathbf{S}_0(x, \mathbf{y}, \mathbf{z})$ and

$$(\mathbf{S}_0(x, \mathbf{y}, \mathbf{z}) - \mathbf{S}_0(x, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}))(\mathbf{y} - \tilde{\mathbf{y}}) \geqslant 0 \tag{4.59}$$

for all $t \geqslant 0$, $\mathbf{y} \in \mathbb{R}^M$, $\mathbf{z} \in \mathbb{R}^N$ and $x \in G$ with some function $\mathbf{S}_0: \Omega \times \mathbb{R}^{M+N} \to \mathbb{R}^M$.

The main purpose of this assumption is to ensure that $T(t)\,\mathbf{w} \in D(B)$, $\partial_t(T(t)\,\mathbf{w}) \in L^2(\Omega)$ and $BT(\cdot)\,\mathbf{w} \in L^\infty((0, \infty), X)$, i.e., $\|BT(t)\,\mathbf{w}\|_X$ is bounded as $t \to \infty$ if $\mathbf{w} \in D(B)$ as shown in the following lemma. (For example in the linear case $\mathbf{S}(t, x, \mathbf{y}, \mathbf{z}) = \sigma(t, x)\,\mathbf{y}$ the condition that $\mathbf{S}$ is independent of t can be replaced by the weaker assumption

$$\partial_t \sigma \in L^\infty((0, \infty) \times G) \qquad \text{and} \qquad |\partial_t \sigma(t, x)| \leqslant C_1 \sigma(t, x)$$

for all $t \geqslant 0$ and $x \in G$ with some constant C_1 independent of t, x.)

LEMMA 4. *Suppose in addition that* $\mathbf{G} \in W^{1,1}((0, \infty), X)$ *and* $\mathbf{w} \in D(B)$. *Then one has*

$$T(\cdot)\,\mathbf{w} \in W^{1,\infty}((0, \infty), X) \cap L^\infty((0, \infty), D(B)) \tag{4.60}$$

Proof. It follows from the assumption that there is a nonlinear operator $F_0: X \to X$ with $F(t, \mathbf{w}) = F_0(\mathbf{w})$ and

$$\langle F_0(\mathbf{w}) - F_0(\tilde{\mathbf{w}}), \mathbf{w} - \tilde{\mathbf{w}} \rangle_X \geqslant 0 \qquad \text{for all} \quad \mathbf{w}, \tilde{\mathbf{w}} \in X$$

Suppose $\mathbf{w} \in D(B)$ and set $\mathbf{u}(t) \overset{\text{def}}{=} T(t)\,\mathbf{w}$. It follows from a standard regularity-result that $\mathbf{u} \in C^1([0,\infty), X) \cap L^\infty_{loc}((0,\infty), D(B))$ is a strong solution of

$$\mathbf{u}'(t) = B\mathbf{u}(t) + \mathbf{G}(t) - F_0(\mathbf{u}(t)). \tag{4.61}$$

In analogy to Lemma 1 an energy-estimate for $\mathbf{u}'$ can be obtained using the monotonicity of F_0:

$$1/2 \frac{d}{dt} \|\partial_t \mathbf{u}(t)\|_X^2 \leqslant \langle \partial_t \mathbf{G}(t), \partial_t \mathbf{u}(t) \rangle_X \leqslant \|\partial_t \mathbf{G}(t)\|_X \, \|\partial_t \mathbf{u}(t)\|_X$$

With $\partial_t \mathbf{G} \in L^1((0,\infty), X)$ this yields $\mathbf{u} \in W^{1,\infty}((0,\infty), X)$.

By (4.61) one obtains also $\mathbf{u}(t) \in D(B^*) = D(B)$ and $B\mathbf{u}(\cdot) \in L^\infty((0,\infty), X)$. ∎

LEMMA 5. *$X^0 \cap D(B^n)$ is dense in $X^0 \cap D(B^m)$ for all $m, n \in \mathbb{N}$ with $m < n$.*

Proof. Let $\mathbf{w} \in X^0 \cap D(B^m)$ and define $\mathbf{w}_\tau \overset{\text{def}}{=} \tau^n(\tau - B)^{-n}\,\mathbf{w} \in D(B^n)$ for $\tau > 0$. Then

$$\|B^k(\mathbf{w}_\tau - \mathbf{w})\|_X = \|B^k\mathbf{w} - [\tau(\tau - B)^{-1}]^n\, B^k\mathbf{w}\|_X \xrightarrow{\tau \to \infty} 0$$
$$\text{for all} \quad k \in \{0, 1, .., m\}. \tag{4.62}$$

Suppose $\mathbf{a} \in \mathcal{N}$. Then

$$\langle \mathbf{w}_\tau, \mathbf{a} \rangle_X = \langle \mathbf{w}, \tau^n(\tau + B)^{-n}\,\mathbf{a} \rangle_X = \langle \mathbf{w}, \mathbf{a} \rangle_X = 0.$$

Hence $\mathbf{w}_\tau \in X^0$. By (4.62) the proof is complete. ∎

The next lemma concerns regularity-properties of elements of $X^0 \cap D(B)$.

LEMMA 6. (i) *Let $K \subset \Omega_0$ be a bounded open set with $\bar{K} \subset \Omega_0$. Then $\mathbf{w} \in H^1(K)$ and*

$$\|\mathbf{w}\|_{H^1(K)} \leqslant C_K \, \|\mathbf{w}\|_{D(B)} \qquad \textit{for all} \quad \mathbf{w} \in X^0 \cap D(B).$$

with some constant $C_K \in (0,\infty)$ depending only on K.

(ii) *Suppose in addition $E^{(2)} = 1$ on all of Ω.*

Let $U \subset \Omega$ be a bounded open set with $\bar{U} \subset \Omega$. Then $\mathbf{F} \in H^1(U)$ and

$$\|\mathbf{F}\|_{H^1(U)} \leqslant C_U \, \|\mathbf{w}\|_{D(B)} \qquad \textit{for all} \quad \mathbf{w} = (\mathbf{E}, \mathbf{F}) \in X^0 \cap D(B).$$

with some constant $C_U \in (0,\infty)$ depending only on U.

Proof. (i) Let $K \subset \Omega_0$ be a bounded open set with $\bar{K} \subset \Omega_0$. Choose $\chi \in C_0^\infty(\Omega_0)$ with $\chi = 1$ on K. Suppose $\mathbf{w} \in X^0 \cap D(B^2)$. Then Lemma 2(ii) yields $\mathbf{w} \in H^2_{loc}(\Omega_0)$ and

$$\sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, |\nabla \mathbf{w}_k|^2 \, dx$$

$$= \sum_{k=1}^{M+N} \int_{\Omega_0} \operatorname{div}(\chi^2 \, \nabla \mathbf{w}_k) \, \bar{\mathbf{w}}_k \, dx$$

$$\leqslant C_{K,1} \sum_{k=1}^{M+N} \int_{\Omega_0} |\chi \, \nabla \mathbf{w}_k| \, |\mathbf{w}_k| \, dx + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, \Delta \mathbf{w}_k \bar{\mathbf{w}}_k \, dx$$

$$\leqslant C_{K,2} \, \|\mathbf{w}\|^2_X + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, |\nabla \mathbf{w}_k|^2 \, dx + \langle \chi^2(B^2 \mathbf{w}), \mathbf{w} \rangle_X$$

$$\leqslant C_{K,3} \, \|\mathbf{w}\|^2_{D(B)} + 1/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, |\nabla \mathbf{w}_k|^2 \, dx + \langle \chi^2(B\mathbf{w}), B\mathbf{w} \rangle_X$$

$$\leqslant C_{K,4}(\|B\mathbf{w}\|^2_X + \|\mathbf{w}\|^2_X) + 2/3 \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, |\nabla \mathbf{w}_k|^2 \, dx$$

by assumption (2.16). Hence

$$\|\mathbf{w}\|^2_{H^1(K)} \leqslant \|\mathbf{w}\|^2_X + \sum_{k=1}^{M+N} \int_{\Omega_0} \chi^2 \, |\nabla \mathbf{w}_k|^2 \, dx \leqslant 3C_{K,4}(\|B\mathbf{w}\|^2_X + \|\mathbf{w}\|^2_X)$$

By Lemma 5 the estimate holds for all $\mathbf{w} \in X^0 \cap D(B)$.

To prove (ii) consider first $\mathbf{f} \in D(A^*) \cap (\ker A^*)^\perp$ with $A^*\mathbf{f} \in D(A)$.

Since $(\ker A^*)^\perp = \overline{\operatorname{rang} A}$ Lemma 2(i) yields $\Delta \mathbf{f} = -AA^*\mathbf{f}$. From a similar cut-off argument as in the proof of the first part it follows that

$$\|\mathbf{f}\|^2_{H^1(U)} \leqslant C_{U,4}(\|A^*\mathbf{f}\|^2_{L^2} + \|\mathbf{f}\|^2_{L^2}) \tag{4.63}$$

Since the set of all $\mathbf{f} \in D(A^*) \cap (\ker A^*)^\perp$ with $A^*\mathbf{f} \in D(A)$ is dense in $D(A^*) \cap (\ker A^*)^\perp$, (4.63) holds for all $\mathbf{f} \in D(A^*) \cap (\ker A^*)^\perp$.

Now let $(\mathbf{E}, \mathbf{F}) \in X^0 \cap D(B)$.

Since $(0, \mathbf{g}) \in \mathcal{N}$ for all $\mathbf{g} \in (\ker A)^*$, it follows from the assumption $E^{(2)} = 1$ on Ω that

$$\langle \mathbf{F}, \mathbf{g} \rangle_{L^2(\Omega)} = \langle (\mathbf{E}, \mathbf{F}), (0, \mathbf{g}) \rangle_X = 0 \qquad \text{for all} \quad \mathbf{g} \in (\ker A)^*,$$

in particular $\mathbf{F} \in D(A^*) \cap (\ker A^*)^\perp$. Finally, the assertion follows from (4.63). ∎

Remark 4. As described in Remark 1 the H^1_{loc}-regularty of $\underline{\mathbf{w}}_1$ for $\mathbf{w} \in X^0 \cap D(B)$ does generally not hold on the set $G = \Omega \backslash \Omega_0$ even if $E^{(j)} = 1$ on Ω.

LEMMA 7. *Suppose $E^{(2)} = 1$ on Ω.*
Then $(\mathbf{e}(t), \mathbf{f}(t)) \overset{\text{def}}{=} T(t)\,\mathbf{w} - P(\mathbf{w} + \int_0^\infty \mathbf{G}(t)\,dt)$ obeys

$$(\|\mathbf{e}(t)\|_{L^2(K)} + \|\mathbf{f}(t)\|_{L^2(U)}) \xrightarrow{t \to \infty} 0.$$

for all compact sets $K \subset \Omega_0$ and $U \subset \Omega$ and $\mathbf{w} \in X$.

Proof. First suppose in addition that $\mathbf{w} \in D(B)$ and $\mathbf{G} \in W^{1,1}((0, \infty), X)$. Define $(\tilde{\mathbf{e}}(t), \tilde{\mathbf{f}}(t)) \overset{\text{def}}{=} (1 - P)\,T(t)\,\mathbf{w} \in \mathcal{N}^\perp = X^0$. Since $PT(t)\,\mathbf{w} \in \mathcal{N} \subset D(B)$, Lemma 4 yields

$$(\tilde{\mathbf{e}}, \tilde{\mathbf{f}}) \in L^\infty((0, \infty), D(B) \cap X^0) \tag{4.64}$$

Hence, it follows from Lemma 6 and Sobolev's imbedding theorem that

$$\{\tilde{\mathbf{e}}(t) : t \geqslant 0\} \text{ is precompact in } L^2(K)$$
$$\text{and } \{\tilde{\mathbf{f}}(t) : t \geqslant 0\} \text{ is precompact in } L^2(U).$$

Therefore, Lemma 3 and Theorem 3 yield

$$\|\tilde{\mathbf{e}}(t)\|_{L^2(K)} + \|\tilde{\mathbf{f}}(t)\|_{L^2(U)} \xrightarrow{t \to \infty} 0. \tag{4.65}$$

Next it follows from Lemma 3 that

$$\|\tilde{\mathbf{e}}(t) - \mathbf{e}(t)\|_{L^2(K)} + \|\tilde{\mathbf{f}}(t) - \mathbf{e}(t)\|_{L^2(U)}$$
$$\leqslant \left\| PT(t)\,\mathbf{w} - P\left(\mathbf{w} + \int_0^\infty \mathbf{G}(t)\,dt\right)\right\|_X \xrightarrow{t \to \infty} 0. \tag{4.66}$$

Now, the assertion follows from (4.65) and (4.66) under the additional hypothesis $\mathbf{w} \in D(B)$ and $\mathbf{G} \in W^{1,1}((0, \infty), X)$.

In order to prove the theorem in the general case assume that $\mathbf{w}, \tilde{\mathbf{w}} \in X$ and $\mathbf{G}, \tilde{\mathbf{G}} \in L^1((0, \infty), X)$. Let $\tilde{\mathbf{u}}$ be the corresponding solution to (1.1)–(1.3) with $\mathbf{w}$, $\mathbf{G}$ replaced by $\tilde{\mathbf{w}}$ and $\tilde{\mathbf{G}}$ respectively. Then one obtains from (4.59) and a similar estimate as in (2.28)

$$\frac{d}{dt}\|T(t)\,\mathbf{w} - \tilde{\mathbf{u}}(t)\|_X^2 = 2\langle \mathbf{G}(t) - \tilde{\mathbf{G}}(t) - F_0(T(t)\,\mathbf{w})$$
$$+ F_0(\tilde{\mathbf{u}}(t)), T(t)\,\mathbf{w} - \tilde{\mathbf{u}}(t)\rangle_X$$
$$\leqslant \|\mathbf{G}(t) - \tilde{\mathbf{G}}(t)\|_X\,\|T(t)\,\mathbf{w} - \tilde{\mathbf{u}}(t)\|_X$$

and therefore

$$\|T(t)\,\mathbf{w}-\tilde{\mathbf{u}}(t)\|_X \leqslant \|\mathbf{w}-\tilde{\mathbf{w}}\|_X + \|\mathbf{G}-\tilde{\mathbf{G}}\|_{L^1((0,\infty),X)}.$$

Since $W^{1,1}((0,\infty),X)$ is dense in $L^1((0,\infty),X)$, it follows from the latter estimate that the assertion holds for all $\mathbf{w}\in X$ and $\mathbf{G}\in L^1((0,\infty),X)$. ∎

In the next lemma the strong L^r_{loc}-convergence of $\underline{\mathbf{u}}_1$ on the set G is proved, which in general does not follow from Lemma 6, see Remark 4.

LEMMA 8. *Suppose* $\mathbf{w}\in X$, $R>0$ *and* $r\in[1,2)$. *Then* $(\mathbf{e}(t),\mathbf{f}(t)) \stackrel{\text{def}}{=} T(t)\,\mathbf{w}-P(\mathbf{w}+\int_0^\infty \mathbf{G}(t)\,dt)$ *obeys*

$$\|\mathbf{e}(t)\|_{L^r(G\cap B_R)} \xrightarrow{t\to\infty} 0.$$

Proof. By the same density-argument as in the proof of the previous lemma it suffices to consider $\mathbf{w}\in D(B)$ and $\mathbf{G}\in W^{1,1}((0,\infty),X)$.

Let $G^{(R)} \stackrel{\text{def}}{=} G\cap B_R$ and $M \stackrel{\text{def}}{=} \|(\mathbf{e},\mathbf{f})\|_{L^\infty((0,\infty),L^2(\Omega))}$.

Suppose $\delta>0$. With $\gamma>0$ as in (2.21) one has $G=\bigcup_{n\in\mathbb{N}}\{x\in G\colon \gamma(x)>1/n\}$. Therefore there exists a subset $G^{(R)}_\delta\subset G^{(R)}$, such that

$$M\,|G^{(R)}\backslash G^{(R)}_\delta|^{(1/r-1/2)}\leqslant \delta/2, \tag{4.67}$$

and

$$\gamma(x)\geqslant c_\delta \qquad \text{for all} \quad x\in G^{(R)}_\delta \tag{4.68}$$

with some positive constant $c_\delta>0$. In (4.67) $|G^{(R)}\backslash G^{(R)}_\delta|$ denotes the Lebesgue-measure of this set.

Since $\underline{(P(\mathbf{w}+\int_0^\infty \mathbf{G}(t)\,dt))}_1=0$ on G, one obtains from (4.68) and Lemma 1 that

$$\mathbf{e}\in L^p((0,\infty),L^1_\gamma({G_\delta}^{(R)}))\subset L^p((0,\infty),L^1({G_\delta}^{(R)})). \tag{4.69}$$

Lemma 4 yields

$$\mathbf{e}\in W^{1,\infty}((0,\infty),L^2(\Omega))\subset W^{1,\infty}((0,\infty),L^1({G_\delta}^{(R)})). \tag{4.70}$$

By (4.69) and (4.70) the function $t\to\|\mathbf{e}(t)\|^p_{L^1(G^{(R)}_\delta)}$ is uniformly continuous and integrable over $(0,\infty)$ and hence

$$\|\mathbf{e}(t)\|_{L^1(G^{(R)}_\delta)} \xrightarrow{t\to\infty} 0.$$

Since $r\in(1,2)$, this yields

$$\begin{aligned}\|\mathbf{e}(t)\|_{L^r(G^{(R)}_\delta)} &\leqslant \|\mathbf{e}(t)\|^{\theta}_{L^2(G^{(R)}_\delta)}\,\|\mathbf{e}(t)\|^{1-\theta}_{L^1(G^{(R)}_\delta)}\\ &\leqslant M^\theta\,\|\mathbf{e}(t)\|^{1-\theta}_{L^1(G^{(R)}_\delta)} \xrightarrow{t\to\infty} 0.\end{aligned} \tag{4.71}$$

where $1/r = \theta/2 + 1 - \theta$. Next it follows from (4.67) that

$$\|\mathbf{e}(t)\|_{L^r(G^{(R)}\setminus G_\delta^{(R)})} \leqslant \|\mathbf{e}(t)\|_{L^2(\Omega)} \, |G^{(R)}\setminus G_\delta{}^{(R)}|^{(1/r-1/2)}$$
$$\leqslant M \, |G^{(R)}\setminus G_\delta{}^{(R)}|^{(1/r-1-2)} \leqslant \delta/2. \tag{4.72}$$

Finally, the assertion follows from (4.71) and (4.72), since $\delta > 0$ is arbitrary. ▮

Now the main theorem concerning strong L^q-convergence can be proved.

THEOREM 4. *Suppose $E^{(2)} = 1$ on Ω. Then it follows for all $q \in [1, 2)$, $\mathbf{w} = (\mathbf{E}_0, \mathbf{F}_0) \in X$ and all compact $U \subset \Omega$ that*

$$(\|\mathbf{e}(t)\|_{L^q(U)} + \|\mathbf{f}(t)\|_{L^2(U)}) \xrightarrow{t \to \infty} 0.$$

where $(\mathbf{e}(t), \mathbf{f}(t)) \stackrel{\text{def}}{=} T(t)\,\mathbf{w} - P(\mathbf{w} + \int_0^\infty \mathbf{G}(t)\,dt)$.

Proof. Define $M \stackrel{\text{def}}{=} \|(\mathbf{e}, \mathbf{f})\|_{L^\infty((0,\infty), L^2(\Omega))}$.

Suppose $\delta > 0$. Choose a compact set $K \subset U \cap \Omega_0$ with $M\,|(U \cap \Omega_0)\setminus K|^{(1/q-1/2)} \leqslant \delta$. Then Hölder's inequality yields

$$\|\mathbf{e}(t)\|_{L^q(U)} \leqslant \|\mathbf{e}(t)\|_{L^q(U\cap G)} + \|\mathbf{e}(t)\|_{L^q(K)}$$
$$+ \|\mathbf{e}(t)\|_{L^2(U)} \, |(U \cap \Omega_0)\setminus K|^{(1/q-1/2)}$$
$$\leqslant \|\mathbf{e}(t)\|_{L^q(U\cap G)} + \|\mathbf{e}(t)\|_{L^q(K)} + \delta.$$

Now, Lemma 7 and Lemma 8 yield $\limsup_{t\to\infty} \|\mathbf{w}(t)\|_{L^q(U)} \leqslant \delta$, which completes the proof. ▮

In the case of Maxwell's Eqs. (1.4)–(1.6) the assumption $E^{(2)} = 1$ on Ω can be omitted using the compactness-result in [8, 12, 15].

Under the general assumptions considered so far it cannot be expected that the assertion of the previons theorem holds for $q = 2$ or sets U which may overlap the boundary $\partial\Omega$. However, for the system corresponding to the scalar wave-equation the result can be improved in this direction. Consider

$$\partial_t^2 \varphi = \operatorname{div}(E\nabla\varphi) - S(x, \partial_t \varphi) \tag{4.73}$$

supplemented by the initial-boundary-onditions

$$\varphi = 0 \qquad \text{on} \quad (0,\infty) \times \partial\Omega \tag{4.74}$$

$$\varphi(0, x) = f_0(x) \qquad \text{and} \qquad \partial_t \varphi(0, x) = f_1(x). \tag{4.75}$$

Here the nonlinear function $S: \Omega \times \mathbb{R} \to \mathbb{R}$ obeys the assumptions (2.1)–(2.7). According to (4.59) it is assumed that S is independent of t and monotone with respect to $y \in \mathbb{R}^3$. For a domain $\Omega_1 \subset \Omega$ let $H^1(\Omega_1)$ be the usual first order Sobolev space and $\overset{0}{H}{}^1(\Omega_1)$ denotes the closure of $C_0^\infty(\Omega_1)$ in $H^1(\Omega_1)$.

Next, $D(\mathscr{A}) \subset \overset{0}{H}{}^1(\Omega)$ is defined as the set of all $f \in \overset{0}{H}{}^1(\Omega)$, such that

$$\mathscr{A}f \overset{\text{def}}{=} -\operatorname{div}(E\nabla f) \in L^2(\Omega).$$

It is well known that for $f_0 \in \overset{0}{H}{}^1(\Omega))$ and $f_1 \in L^2(\Omega))$ problem (4.73)–(4.75) admits a unique solution $\varphi \in C([0, \infty), \overset{0}{H}{}^1(\Omega))$ with $\partial_t \varphi \in C([0, \infty), L^2(\Omega))$. The usual energy-estimate yields

$$\partial_t \varphi \in L^\infty((0, \infty)\ L^2(\Omega)), \nabla\varphi \in L^\infty((0, \infty), L^2(\Omega)). \tag{4.76}$$

If in addition $f_1 \in \overset{0}{H}{}^1(\Omega))$ and $f_0 \in D(\mathscr{A})$ then $\varphi \in C([0, \infty), D(\mathscr{A}))$ and $\partial_t \varphi \in C([0, \infty), \overset{0}{H}{}^1(\Omega))$ with

$$\begin{aligned} &\partial_t \nabla\varphi, \partial_t^2 \varphi \in L^\infty((0, \infty)\ L^2(\Omega)), \\ &\operatorname{div}(E\nabla\varphi) = \mathscr{A}\varphi(\cdot) \in L^\infty((0, \infty), L^2(\Omega)). \end{aligned} \tag{4.77}$$

In order to consider problem (4.73)–(4.75) is the setting of Section 2 the following operators are introduced. Let $D(A) \overset{\text{def}}{=} \overset{0}{H}{}^1(\Omega, \mathbb{C})$, $A\varphi \overset{\text{def}}{=} \nabla\varphi$. $D(A^*)$ is the space of all vector-fields $\mathbf{a} \in L^2(\Omega, \mathbb{C}^3)$ with $A^*\mathbf{a} = -\operatorname{div} \mathbf{a} \in L^2(\Omega)$. Next, $D(B) \overset{\text{def}}{=} D(A) \times D(A^*)$ and

$$B(\mathbf{w}_1, ..., \mathbf{w}_4) \overset{\text{def}}{=} (-A^*(\mathbf{w}_2, ..., \mathbf{w}_4), EA\mathbf{w}_1) = (\operatorname{div}(\mathbf{w}_2, ..., \mathbf{w}_4), E\nabla\mathbf{w}_1)$$

for $\mathbf{w} \in D(B)$.

Suppose $\varphi \in C([0, \infty), \overset{0}{H}{}^1(\Omega))$ is for $f_0 \in \overset{0}{H}{}^1(\Omega))$ and $f_1 \in L^2(\Omega))$ a solution of problem (4.73)–(4.75). Then $\mathbf{u} \overset{\text{def}}{=} (\partial_t \varphi, E\nabla\varphi) \in C([0, \infty), L^2(\Omega, \mathbb{R}^4))$ is a weak solution of (2.26), i.e.,

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X - \langle F_0(\mathbf{u}(t)), \mathbf{a}\rangle_X \qquad \text{for all} \quad \mathbf{a} \in D(B)$$

where $F_0: L^2(\Omega, \mathbb{R}^4) \to L^2(\Omega, \mathbb{R}^4)$ is defined by

$$F_0(\mathbf{u}) \overset{\text{def}}{=} (S(\cdot, \mathbf{u}_1(\cdot)), 0).$$

If $f_0 \in D(\mathscr{A})$ and $f_1 \in \overset{0}{H}{}^1(\Omega))$ then $\mathbf{u}(0) \in D(B)$ and hence by Lemma 4 $\mathbf{u} \in L^\infty((0, \infty), D(B))$, whence again (4.77).

Next it is shown that

$$\nabla\varphi(t) \xrightarrow{t\to\infty} 0 \qquad \text{and} \qquad \partial_t\varphi(t) \xrightarrow{t\to\infty} 0 \qquad \text{in } L^2(\Omega) \text{ weakly.} \tag{4.78}$$

for all $f_0 \in \overset{0}{H}{}^1(\Omega))$ and $f_1 \in L^2(\Omega)$. For this purpose let $\mathbf{w} \overset{\text{def}}{=} (f_1, E\nabla f_0) \in L^2(\Omega, \mathbb{R}^4)$. Then $(\partial_t\varphi(t), E\nabla\varphi(t)) = \mathbf{u}(t) = T(t)\,\mathbf{w}$ solves (2.26). In order to apply Theorem 3 it suffices to show

$$\mathbf{w} \in X^0 \tag{4.79}$$

Suppose $\mathbf{a} \in \mathcal{N}$. Then $\mathbf{a}_1 \in \overset{0}{H}{}^1(\Omega)$, with $\nabla\mathbf{a}_1 = 0$, which implies $\mathbf{a}_1 = 0$. Moreover, $\operatorname{div}(\mathbf{a}_2, ..., \mathbf{a}_4) = 0$ by the definition of A, B. Hence

$$\langle \mathbf{w}, \mathbf{a}\rangle_X = \int_\Omega [E^{-1}(\mathbf{w}_2, ..., \mathbf{w}_4)](\mathbf{a}_2, ..., \mathbf{a}_4)\,dx = \int_\Omega (\mathbf{a}_2, ..., \mathbf{a}_4)\,\nabla f_0\,dx = 0$$

since $f_0 \in \overset{0}{H}{}^1(\Omega)$. Thus, (4.79) and (4.78) are proved. In the following theorem local strong convergence in the energy-norm is shown.

THEOREM 5. *For all $R \in (0, \infty)$, $f_0 \in \overset{0}{H}{}^1(\Omega))$ and $f_1 \in L^2(\Omega))$ one has*

$$(\|\nabla\varphi(t)\|_{L^2(\Omega\cap B_R)} + \|\partial_t\varphi(t)\|_{L^2(\Omega\cap B_R)}) \xrightarrow{t\to\infty} 0.$$

Proof. By a density-argument it suffices to consider $f_0 \in D(\mathscr{A})$ and $f_1 \in \overset{0}{H}{}^1(\Omega)$.

Choose $\chi \in C_0^\infty(B_{2R})$ with $\chi(x) = 1$ on B_R and define $\Omega_R \overset{\text{def}}{=} \Omega \cap B_{2R}$ and $\varphi_R(t, x) \overset{\text{def}}{=} \chi(x)\,\varphi(t, x)$. It follows easily from (4.77) using Poincare's inequality that $\varphi_R \in L^\infty((0, \infty), \overset{0}{H}{}^1(\Omega \cap B_{2R}))$ and $\partial_t\varphi_R \in L^\infty((0, \infty), \overset{0}{H}{}^1(\Omega \cap B_{2R}))$. Since $\Omega \cap B_{2R}$, is bounded, the imbedding $\overset{0}{H}{}^1(\Omega \cap B_{2R}) \hookrightarrow L^2(\Omega \cap B_{2R})$ is compact. Hence

$$\{\varphi(t) \colon t \geqslant 0\} \text{ is precompact in } L^2(\Omega \cap B_R) \tag{4.80}$$

$$\text{and } \{\partial_t\varphi(t) \colon t \geqslant 0\} \text{ is precompact in } L^2(\Omega \cap B_R). \tag{4.81}$$

for all $R \in (0, \infty)$. Next, one obtains by (2.25) and the definition of $\mathscr{A}$ that

$$c_0\,\|\nabla(\varphi(t_1) - \varphi(t_2))\|^2_{L^2(B_R)}$$

$$\leqslant \int_\Omega \chi E\nabla(\varphi(t_1) - \varphi(t_2))\,\nabla(\varphi(t_1) - \varphi(t_2))\,dx$$

$$= -\int_\Omega (\varphi(t_1) - \varphi(t_2)) \operatorname{div}(\chi E \nabla[\varphi(t_1) - \varphi(t_2)])\, dx$$

$$\leqslant \|\varphi(t_1) - \varphi(t_2)\|_{L^2(B_{2R})} (\|\mathscr{A}(\varphi(t_1) - \varphi(t_2))\|_{L^2(\Omega)}$$

$$+ K_R \|\nabla(\varphi(t_1) - \varphi(t_2))\|_{L^2(\Omega)}) \qquad \text{for all} \quad t_1, t_2 \geqslant 0.$$

which implies by (4.76), (4.77), and (4.80) also

$$\{\nabla\varphi(t) : t \geqslant 0\} \text{ is precompact in } L^2(\Omega \cap B_R) \tag{4.82}$$

Finally, the result follows from (4.78), (4.81), and (4.82). ∎

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