Discrete Applied Mathematics 156 (2008) 3325–3328

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

The connectivity of a graph and its complement

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ARTICLE INFO

Article history:
Received 23 September 2005
Received in revised form 7 November 2006
Accepted 21 May 2008
Available online 27 June 2008

Keywords:
Edge-connectivity
Vertex-connectivity
Complementary graph
Nordhaus–Gaddum type results

ABSTRACT

Let G be a graph with minimum degree $\delta(G)$, edge-connectivity $\lambda(G)$, vertex-connectivity $\kappa(G)$, and let $\bar{G}$ be the complement of G.

In this article we prove that either $\lambda(G)=\delta(G)$ or $\lambda(\bar{G})=\delta(\bar{G})$. In addition, we present the Nordhaus–Gaddum type result $\kappa(G)+\kappa(\bar{G})\geq\min\{\delta(G),\delta(\bar{G})\}+1$. A family of examples will show that this inequality is best possible.

1. Terminology and preliminary results

For graph-theoretical terminology and notation not defined here we follow Bondy and Murty [1]. We consider finite, undirected and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For each vertex $v\in V(G)$, the *neighborhood* $N(v)=N_G(v)$ of v is defined as the set of all vertices adjacent to v, and $d(v)=|N(v)|$ is the *degree* of v. We denote by $\delta(G)$ the *minimum degree*, by $\Delta(G)$ the *maximum degree* and by $n(G)=|V(G)|$ the *order* of G.

For a connected graph G, we define the *distance* $d_G(u,v)$ between two vertices u and v as the length of a shortest path from u to v in G. The *diameter* of G is the number $\mathrm{dm}(G)=\max\{d_G(u,v):u,v\in V(G)\}$. If a graph G is not connected, then we define $\mathrm{dm}(G)=\infty$. Furthermore, let $d_G(X,Y)=\min\{d_G(x,y)|x\in X,y\in Y\}$ for two vertex sets X and Y in the graph G, The *complement* $\bar{G}$ of a graph G is the graph with vertex set $V(G)$ and two vertices are adjacent in $\bar{G}$ if they are not adjacent in G. A graph G is called *self-complementary*if $\bar{G}$ is isomorphic to G.

An *edge-cut* or *vertex-cut* of a connected graph G is a set of edges or vertices whose removal disconnects G. The *edge-connectivity* $\lambda(G)$ is defined as the minimum cardinality of an edge-cut over all edge-cuts of G, and if G is non-complete, then the *vertex-connectivity* $\kappa(G)$ is defined as the minimum cardinality of a vertex-cut over all vertex-cuts of G. For the complete graph K_n of order n, we define $\kappa(K_n)=n-1$. In 1932, Whitney [6] proved the classical inequality chain $\kappa(G)\leq\lambda(G)\leq\delta(G)$ for every graph G.

Each edge-cut or vertex-cut S satisfying $|S|=\lambda(G)$ or $|S|=\kappa(G)$ is called a *minimum edge-cut* or a *minimum vertex-cut*.

The following known results play an important role in our investigations. We start with a nice result which can be found in the book by Bondy and Murty [1] on p. 14 as an exercise (for a proof cf. Volkmann [5], p. 19).

Theorem 1.1 (*Bondy, Murty [1] 1976*). *If G is a graph of diameter* $\mathrm{dm}(G)\geq 4$, *then* $\mathrm{dm}(\bar{G})\leq 2$.

Theorem 1.2 (*Jolivet [2] 1972, Plesník [3] 1975*). *If G is a graph with* $\mathrm{dm}(G)\leq 2$, *then* $\lambda(G)=\delta(G)$.

Theorem 1.3 (*Plesník, Znám [4] 1989*). *If G is a bipartite graph with* $\mathrm{dm}(G)\leq 3$, *then* $\lambda(G)=\delta(G)$.

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0166-218X/$ – see front matter
doi:10.1016/j.dam.2008.05.012

2. Main results

Theorem 2.1. *Let G be a bipartite graph. If* $\mathrm{dm}(G) \le 3$*, then* $\lambda(G) = \delta(G)$*. If* $\mathrm{dm}(G) \ge 4$*, then* $\lambda(\bar{G}) = \delta(\bar{G})$.

Proof. If $\mathrm{dm}(G) \le 3$, then we deduce from Theorem 1.3 that $\lambda(G) = \delta(G)$. If $\mathrm{dm}(G) \ge 4$, then Theorem 1.1 implies $\mathrm{dm}(\bar{G}) \le 2$ and thus Theorem 1.2 leads to $\lambda(\bar{G}) = \delta(\bar{G})$. □

In particular, it follows from Theorem 2.1 that if G is a bipartite graph, then $\lambda(G) = \delta(G)$ or $\lambda(\bar{G}) = \delta(\bar{G})$. The next result will show that this is valid for all graphs.

Theorem 2.2. *If G is an arbitrary graph, then*

$$\lambda(G) = \delta(G) \quad \textit{or} \quad \lambda(\bar{G}) = \delta(\bar{G}).$$

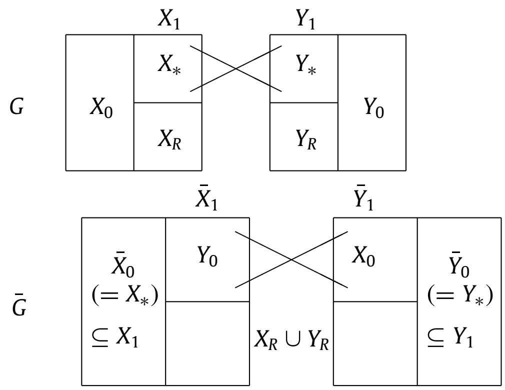
Proof. If $\mathrm{dm}(G) \le 2$ or $\mathrm{dm}(\bar{G}) \le 2$, then the desired result follows immediately from Theorem 1.2. Hence there remain the cases where $\mathrm{dm}(G) \ge 3$ and $\mathrm{dm}(\bar{G}) \ge 3$. If $\lambda(G) = \delta(G)$ or $\lambda(\bar{G}) = \delta(\bar{G})$, then we are done.

Therefore we suppose to the contrary that $\lambda(G) \le \delta(G) - 1$ and $\lambda(\bar{G}) \le \delta(\bar{G}) - 1$. If S is an arbitrary minimum edge-cut, then we denote the vertex sets of the two components of $G - S$ by X and Y. The vertex set $X_1 \subseteq X$ consists of the vertices with at least one neighbor in Y and the vertex set $Y_1 \subseteq Y$ consists of the vertices with at least one neighbor in X. In addition, let $X_0 = X \setminus X_1$ and $Y_0 = Y \setminus Y_1$. Using the assumption $\lambda(G) \le \delta(G) - 1$, we observe that

$$\delta(G)|X| \le \sum_{x \in X} d_G(x) \le |X|(|X| - 1) + \delta(G) - 1$$

and thus $|X| \ge \delta(G) + 1$. Combining this with the inequality $|X_1| \le \lambda(G) \le \delta(G) - 1$, we find that $|X_0| = |X| - |X_1| \ge 2$. Similar analysis shows that $|Y_0| \ge 2$. We note that $d_G(X_0, Y_0) \ge 3$.

Since $Y_0 \ne \emptyset$, we deduce that $d_{\bar{G}}(x, a) \le 2$, for $x \in X_0$ and $a \in V(G)$. Using the fact that $X_0 \ne \emptyset$, we obtain analogously $d_{\bar{G}}(y, b) \le 2$, where $y \in Y_0$ and $b \in V(G)$. Furthermore, it follows that $d_{\bar{G}}(a, b) \le 2$ for $a, b \in X_1$ or $a, b \in Y_1$.



Let $\bar{S}$ be an arbitrary minimum edge-cut in $\bar{G}$ and let $\bar{X}, \bar{Y}, \bar{X}_0, \bar{Y}_0, \bar{X}_1, \bar{Y}_1$ be defined as before. Analogously, we obtain $|\bar{X}_0|, |\bar{Y}_0| \ge 2$ and thus $d_{\bar{G}}(\bar{X}_0, \bar{Y}_0) \ge 3$, since $\lambda(\bar{G}) < \delta(\bar{G})$. Using our distance observations above, we conclude first that $\bar{X}_0, \bar{Y}_0 \subseteq X_1 \cup Y_1$ and then that $\bar{X}_0 \subseteq X_1, \bar{Y}_0 \subseteq Y_1$ or $\bar{X}_0 \subseteq Y_1, \bar{Y}_0 \subseteq X_1$, say $\bar{X}_0 \subseteq X_1$ and $\bar{Y}_0 \subseteq Y_1$. In G we denote the vertex set $\bar{X}_0$ by X_* and $\bar{Y}_0$ by Y_*, and we define $X_R = X_1 \setminus X_*$ and $Y_R = Y_1 \setminus Y_*$.

In $\bar{G}$ each vertex in X_0 is adjacent to every vertex in Y and each vertex in Y_0 is adjacent to every vertex in X. Thus $Y_0 \subseteq \bar{X}_1, X_0 \subseteq \bar{Y}_1$ and as $X_R \cap \bar{X}_0 = Y_R \cap \bar{Y}_0 = \emptyset$, we deduce that $X_R \cup Y_R \subseteq \bar{X}_1 \cup \bar{Y}_1$.

We collect some of the derived properties:

(1) If $x \in X_*$ and $y \in Y_*$, then $xy \in E(G)$.
(2) If $x \in X_0$ and $y \in Y_0$, then $xy \in E(\bar{G})$.
(3) It follows from (1) that $\lambda(G) \ge |X_*||Y_*| + \max\{|X_R|, |Y_R|\} \ge |X_*||Y_*| + \frac{|X_R|+|Y_R|}{2}$.
(4) It follows from (2) that $\lambda(\bar{G}) \ge |X_0||Y_0| + \frac{|X_R|+|Y_R|}{2}$.
(5) $\delta(G) \le |X| - 1 = |X_0| + |X_R| + |X_*| - 1$
(6) $\delta(G) \le |Y| - 1 = |Y_0| + |Y_R| + |Y_*| - 1$
(7) $\delta(\bar{G}) \le |X_*| + |Y_0| + \frac{|X_R|+|Y_R|}{2} - 1$ or $\delta(\bar{G}) \le |Y_*| + |X_0| + \frac{|X_R|+|Y_R|}{2} - 1$.

Case 1. Assume that $\delta(\bar{G}) \le |X_*| + |Y_0| + \frac{|X_R|+|Y_R|}{2} - 1$. Since $\lambda(G) \le \delta(G) - 1$; the inequalities (3) and (5) imply

$$|X_*||Y_*| + |X_R| \le \lambda(G) \le \delta(G) - 1 \le |X_0| + |X_R| + |X_*| - 2$$

and thus

$$|X_*||Y_*| \leq |X_0| + |X_*| - 2$$

and so

$$|X_0| - 1 \geq |X_*||Y_*| - |X_*| + 1.$$

Using (4), (7), $|Y_*| \geq 2$ and $|Y_0| \geq 2$, we arrive at the following contradiction:

$$\begin{aligned}
\lambda(\bar{G}) &\geq |X_0||Y_0| + \frac{|X_R| + |Y_R|}{2} \\
&= |Y_0| + |Y_0|(|X_0| - 1) + \frac{|X_R| + |Y_R|}{2} - 1 + 1 + |X_*| - |X_*| \\
&\geq \delta(\bar{G}) + |Y_0|(|X_0| - 1) + 1 - |X_*| \\
&\geq \delta(\bar{G}) + |Y_0|(|X_*||Y_*| - |X_*| + 1) + 1 - |X_*| \\
&= \delta(\bar{G}) + |X_*|(|Y_0||Y_*| - |Y_0| - 1) + 1 + |Y_0| \\
&\geq \delta(\bar{G}) + |X_*|(2|Y_0| - |Y_0| - 1) + 1 + |Y_0| \\
&= \delta(\bar{G}) + |X_*|(|Y_0| - 1) + 1 + |Y_0| \\
&\geq \delta(\bar{G}) + |X_*|(2 - 1) + 1 + |Y_0| \\
&> \delta(\bar{G}).
\end{aligned}$$

Case 2. Assume that $\delta(\bar{G}) \leq |Y_*| + |X_0| + \frac{|X_R|+|Y_R|}{2} - 1$. Combining $\lambda(G) \leq \delta(G) - 1$, (3) and (6) we find that

$$|X_*||Y_*| + |Y_R| \leq \lambda(G) \leq \delta(G) - 1 \leq |Y_0| + |Y_R| + |Y_*| - 2$$

and thus

$$|X_*||Y_*| \leq |Y_0| + |Y_*| - 2$$

and so

$$|Y_0| - 1 \geq |X_*||Y_*| - |Y_*| + 1.$$

Using (3), (7), $|X_*| \geq 2$ and $|X_0| \geq 2$, we obtain analogously to Case 1 the final contradiction

$$\begin{aligned}
\lambda(\bar{G}) &\geq |X_0||Y_0| + \frac{|X_R| + |Y_R|}{2} \\
&= |X_0| + |Y_*| + \frac{|X_R| + |Y_R|}{2} - 1 - |Y_*| + |X_0|(|Y_0| - 1) + 1 \\
&\geq \delta(\bar{G}) + |X_0|(|Y_0| - 1) + 1 - |Y_*| \\
&\geq \delta(\bar{G}) + |Y_*|(|X_0||X_*| - |X_0| - 1) + 1 + |X_0| \\
&> \delta(\bar{G}). \quad \square
\end{aligned}$$

The following family of examples shows that Theorem 2.2 is not valid in general for the vertex-connectivity of a graph and its complement.

Example 2.3. Let H_1 and H_2 be two copies of the complete graph K_p of order $p \geq 3$. We consider the disjoint union of the graphs H_1, H_2 and the empty graph H with p vertices together with the edges with one end in $V(H)$ and the other one in $V(H_1) \cup V(H_2)$. By the removal of one edge between $V(H)$ and $V(H_1) \cup V(H_2)$, we obtain the graph G. We note that $\delta(G) = 2p - 2$, $\kappa(G) \leq |V(H)| = p < \delta(G)$, $\delta(\bar{G}) = p - 1$ and $\kappa(\bar{G}) = 1 < \delta(\bar{G})$.

Corollary 2.4. *If G is a self-complementary graph, then* $\lambda(G) = \delta(G)$.

Corollary 2.5. *If G and* $\bar{G}$ *are connected graphs, then*

$$\lambda(G) + \lambda(\bar{G}) \geq \min\{\delta(G), \delta(\bar{G})\} + 1.$$

The next theorem shows that Corollary 2.5 is also valid for the sum of the vertex-connectivities of a graph and its complement.

Theorem 2.6. *If G and* $\bar{G}$ *are connected graphs, then*

$$\kappa(G) + \kappa(\bar{G}) \geq \min\{\delta(G), \delta(\bar{G})\} + 1.$$

Proof. If $\kappa(G) = \delta(G)$ or $\kappa(\bar{G}) = \delta(\bar{G})$, then we are done. Now we assume that $\kappa(G) < \delta(G)$ and $\kappa(\bar{G}) < \delta(\bar{G})$. Let S be an arbitrary minimum vertex-cut and let X denote the vertex set of an arbitrary component of $G - S$. Furthermore, let $Y = V(G) \setminus (X \cup S)$. Our assumption implies

$$|Y|, |X| \geq \delta(G) - \kappa(G) + 1 \geq 2. \quad (1)$$

Analogously, let $\bar{S}$ be an arbitrary minimum vertex-cut of $\bar{G}$. Furthermore, let $\bar{X}$ be a vertex set of a component of $\bar{G} - \bar{S}$ and $\bar{Y} = V(G) \setminus (\bar{X} \cup \bar{S})$. Since $\kappa(\bar{G}) < \delta(\bar{G})$, we obtain

$$|\bar{Y}|, |\bar{X}| \geq \delta(\bar{G}) - \kappa(\bar{G}) + 1 \geq 2. \quad (2)$$

Case 1. If $(\bar{X} \cup \bar{Y}) \subseteq S$, then $(X \cup Y) \subseteq \bar{S}$ and we arrive at the contradiction

$$\begin{aligned} \kappa(\bar{G}) &= |\bar{S}| \geq |X| + |Y| \\ &= |V(G) \setminus S| = n(G) - \kappa(G) \\ &\geq n(G) - (\delta(G) - 1) = \Delta(\bar{G}) + 2 \\ &> \delta(\bar{G}). \end{aligned}$$

Case 2. Assume that $(\bar{X} \cup \bar{Y}) \cap (X \cup Y) \neq \emptyset$. Assume, without loss of generality, that there exists a vertex x such that $x \in X$ and $x \in \bar{X}$.

Case 2.1. If $\bar{Y} \cap X \neq \emptyset$, then $Y \subseteq \bar{S}$, since each vertex in X is adjacent to each vertex in Y in $\bar{G}$. Using inequality (1), we obtain $\kappa(\bar{G}) = |\bar{S}| \geq |Y| \geq \delta(G) - \kappa(G) + 1$, and this yields the desired bound.

Case 2.2. If $\bar{Y} \cap X = \emptyset$, then $\bar{Y} \subseteq S$ and therefore $|\bar{Y}| \leq \kappa(G)$. Applying inequality (2), we derive $\kappa(\bar{G}) \geq \delta(\bar{G}) - |\bar{Y}| + 1 \geq \delta(\bar{G}) - \kappa(G) + 1$, and this finally leads to the desired result. □

The following example will show that Theorem 2.6 is best possible, in the sense that $\kappa(G) + \kappa(\bar{G}) \geq \min\{\delta(G), \delta(\bar{G})\} + 2$ is not true in general.

Example 2.7. Let $p \geq 1$ be an integer, and let H be a complete bipartite graph with the partition sets A and B such that $|A| = |B| = 3p$. The graph G is defined as the union of H together with a further vertex x such that $|N_G(x) \cap A| = |N_G(x) \cap B| = p$. It is a simple matter to verify that $\kappa(G) = \delta(G) = 2p$. In the connected graph $\bar{G}$, the vertex x is a cut vertex, and thus $\kappa(\bar{G}) + \kappa(G) = \min\{\delta(G), \delta(\bar{G})\} + 1$, since $\delta(\bar{G}) = 3p - 1 \geq 2p = \delta(G)$.

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